

Gravitation

Chapter 02 / A2 Level / Apollo Physics 9702



Newton's Law of Universal Gravitation

Newton's law of universal gravitation

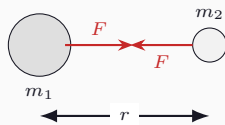
- The force of attraction between two point masses is directly proportional to their product.
- It is inversely proportional to the square of separation.

$$F \propto m_1 m_2, \quad F \propto \frac{1}{r^2}, \quad F \propto \frac{m_1 m_2}{r^2}$$

$$F = \frac{G m_1 m_2}{r^2}$$

Universal gravitational constant

$$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$$



Example 1: two people

$$m_1 = 80 \text{ kg}, \quad m_2 = 85 \text{ kg}, \quad r = 1 \text{ m}$$

$$F = \frac{G m_1 m_2}{r^2} = \frac{6.67 \times 10^{-11} \times 80 \times 85}{1^2}$$

$$F = 4.5 \times 10^{-7} \text{ N}$$

This is very small. **Example 2: person and Earth**

$$M_E = 5.99 \times 10^{24} \text{ kg}, \quad m = 80 \text{ kg}, \quad r = R_E = 6.4 \times 10^6 \text{ m}$$



$$F = \frac{G m M_E}{R_E^2} = \frac{6.67 \times 10^{-11} \times 80 \times 5.99 \times 10^{24}}{(6.4 \times 10^6)^2}$$

$$F = 780 \text{ N} \quad \text{significant}$$

Weight check:

$$W = mg = 80 \times 9.8 = 784 \text{ N}.$$

Attractive forces are written with a negative sign by convention:

$$F = -\frac{G m_1 m_2}{r^2}$$

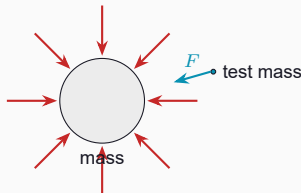
Attractive forces cause objects to lose potential energy and cause them to do work.

Gravitational Field and Field Strength

Gravitational field

A gravitational field is a region in space around a mass where another mass experiences a gravitational force.

- Direction of field lines gives the direction of force on a small mass.
- The gap between field lines shows the strength of field.
- Less gap $\Rightarrow$ strong field; more gap $\Rightarrow$ weak field.



Gravitational field strength

Gravitational field strength is the force per unit mass at a point in a gravitational field.

$$g = \frac{F}{m}.$$

Using $F = \frac{GMm}{r^2}$:

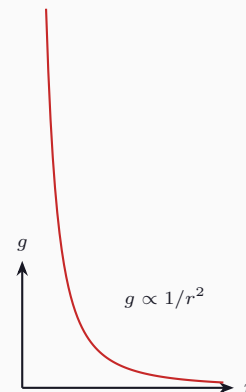
$$g = \frac{GMm/r^2}{m} = \frac{GM}{r^2}.$$

$$g = \frac{F}{m}$$

$$g = \frac{GM}{r^2}$$

- g is a vector quantity.
- g exists till infinity.
- For a point mass: $g \propto 1/r^2$.

$g-r$ graph for a point mass isolated in space



Point mass idea

A uniformly distributed spherical mass can be treated as a point mass at its centre when considered from outside the body.

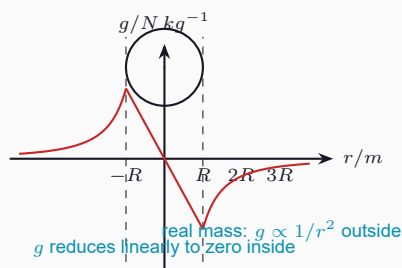


- Presence of the body is experienced outside its surface as if all mass were at the centre.
- No gravitational field or force exists inside a point mass.
- The distance between masses is usually so large compared to their radii that the size of the mass is ignored.

$g - r$ Graphs and Null Points

$g - r$ graph for a real spherical mass

For a real mass, g linearly reduces to zero inside the planet and then follows inverse-square behaviour outside.

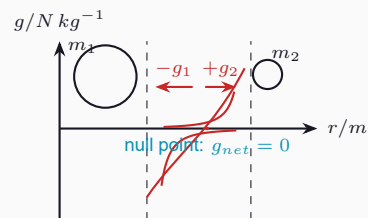


$$r = R \Rightarrow g = g, \quad r = 2R \Rightarrow g = \frac{g}{4}, \quad r = 3R \Rightarrow g = \frac{g}{9}.$$

For a point mass, there is no physical interior; inside a point mass $g = 0$.

$g - r$ graph between two point masses

At a point P between masses, gravitational fields due to the two masses are in opposite directions.



$$g_{net} = -g_1 + g_2$$

$$g_{net} = -\frac{Gm_1}{r^2} + \frac{Gm_2}{(x-r)^2}.$$

For a null point,

$$g_{net} = 0$$

$$0 = -\frac{Gm_1}{r^2} + \frac{Gm_2}{(x-r)^2}$$

$$\frac{Gm_1}{r^2} = \frac{Gm_2}{(x-r)^2}$$

$$\frac{m_1}{m_2} = \frac{r^2}{(x-r)^2}$$

where r is the distance from m_1 to the null point.

Gravitational Potential Energy (U_p)

Meaning of gravitational potential energy

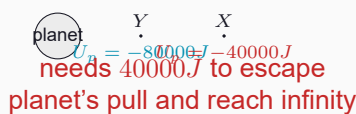
More energy is needed because:

- the object is moved further away from infinity;
- the gravitational field is stronger near the planet.

At infinity:

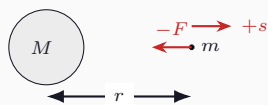
$$U_p = 0$$

A negative potential energy means the object has a deficiency of energy relative to infinity.



Definition

Gravitational potential energy is the work done in moving an object from infinity to a point in a gravitational field.



Using work done against gravity:

Potential energy = work done against gravity = $-F \times s$.

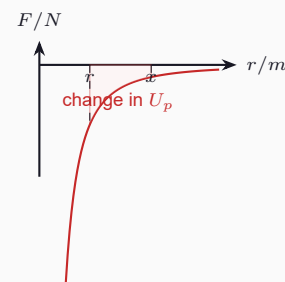
For gravitation:

$$U_p = -\frac{GMm}{r^2} \times r$$

$$U_p = -\frac{GMm}{r}$$

More formally:

$$W = \int F ds = \int \frac{GMm}{r^2} dr = GMm \int r^{-2} dr = -\frac{GMm}{r}.$$



Area under an $F - r$ graph gives the change in gravitational potential energy:

$$\text{Area under } F - r \text{ graph} = \text{change in potential energy}$$

Why mgh is not applicable now; Gravitational Potential

Why mgh is not applicable now?

Change in potential energy near Earth's surface is

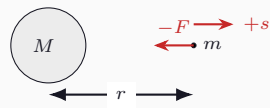
$$\Delta E_p = F \times s = W \times h = mgh.$$

Here g is assumed constant.

- The equation is valid only when gravitational field strength is constant and the object experiences a constant weight force.
- This assumption becomes invalid if the distance travelled is very large and far away from the planet because field strength changes from place to place.

Gravitational potential (ϕ)

It is the work done per unit mass to move an object from infinity to a point in a gravitational field.



$$\phi = \frac{U_p}{m} = \frac{-GMm/r}{m}$$

$$\phi = -\frac{GM}{r}$$

Example:

$$U_p = -100000 \text{ J}, \quad m = 20 \text{ kg}$$

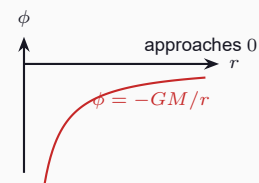
$$\phi = \frac{U_p}{m} = \frac{-100000}{20}$$

$$\phi = -5000 \text{ J kg}^{-1}$$

The closer we are to a planet, the more negative is our potential.

Why is gravitational potential negative?

- Gravitational potential is zero at infinity.
- As a body moves towards the planet, the force is attractive and work is done by the mass.
- Therefore potential decreases and is negative.

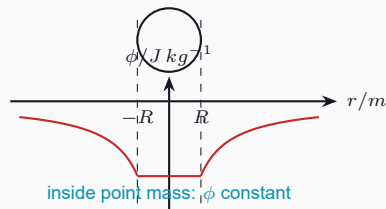


$\phi - r$ Graph and the Link between g and Potential Gradient

$\phi - r$ graph for a point mass isolated in space

For a point mass:

$$\phi = -\frac{GM}{r}.$$



Inside a point mass:

- gravitational field strength is zero;
- force is zero;
- no more work is done;
- ϕ will not decrease, so it is constant.

For outside the mass:

$$\text{gradient} = \frac{\Delta\phi}{\Delta r} = \frac{\Delta\phi}{\Delta x}.$$

$$\phi = -\frac{GM}{r} \Rightarrow \frac{d\phi}{dr} = \frac{GM}{r^2}.$$

By convention, gravitational field strength is the negative of gravitational potential gradient:

$$g = -\frac{\Delta\phi}{\Delta r}$$

Gravitational field strength is negative of gravitational potential gradient

$$\frac{d\phi}{dr} = \frac{GM}{r^2} \Rightarrow g = -\frac{GM}{r^2}$$

for a field directed towards the mass.

Energy Changes from a Potential Graph

Important relationship

$$\text{change in potential energy} = \text{change in kinetic energy}$$

Since $U_p = \phi m$:

$$\Delta U_p = \Delta \phi m.$$

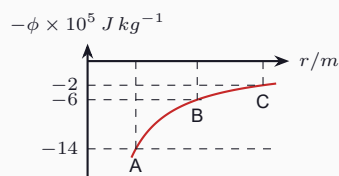
For motion in a gravitational field:

$$\Delta \phi m = \frac{1}{2} m (v^2 - u^2)$$

$$\Delta \phi = \frac{1}{2} (v^2 - u^2)$$

$$(\phi_2 - \phi_1) = \frac{1}{2} (v^2 - u^2).$$

Example graph



(a) A rock falls from rest radially towards a planet from C to A. Calculate its speed at A.

loss in P.E. = gain in K.E.

$$-\Delta \phi m = \frac{1}{2} m (v^2 - u^2)$$

$$-(\phi_2 - \phi_1) = \frac{1}{2} (v^2 - u^2)$$

$$-(-14 - (-2)) \times 10^5 = \frac{1}{2} (v^2 - 0)$$

$$12 \times 10^5 \times 2 = v^2$$

$$v = \sqrt{24 \times 10^5} = 1549.19 \approx 1550 \text{ m s}^{-1}$$

(b) A rock is projected upward radially from A. Determine the minimum speed required for it to reach B.

gain in P.E. = loss in K.E.

$$(\phi_2 - \phi_1) = -\frac{1}{2} (v^2 - u^2)$$

$$(-6 - (-14)) \times 10^5 = -\frac{1}{2} (0 - u^2)$$

$$8 \times 10^5 = \frac{1}{2} u^2 \Rightarrow u = \sqrt{16 \times 10^5}.$$

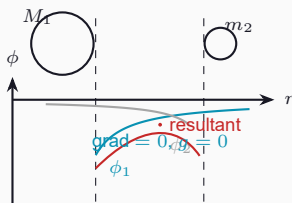
$\phi - r$ Graph between Two Point Masses

For two masses, gravitational potential is a scalar quantity, so it adds algebraically:

$$\phi = \phi_1 + \phi_2$$

$$\phi = -\frac{GM}{r}, \quad \phi \propto \frac{1}{r}.$$

A more negative value of ϕ shows that there are two planets pulling on the object. Hence the object will now need more energy to escape to infinity.



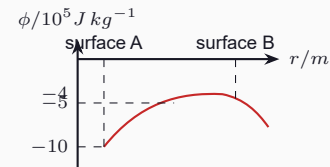
The gravitational field strength is the negative potential gradient:

$$g = -\frac{\Delta\phi}{\Delta r} = -\text{gradient}$$

At the turning point of the resultant potential graph, gradient is zero, so $g = 0$; this is the null point.

Example between two planets

A rock is thrown radially upwards from the surface of planet A towards planet B.



(b) Calculate the speed of the rock at the surface of B if it was thrown from A with a speed of 1400 m s^{-1} .

$$\Delta\phi = -\frac{1}{2}(v^2 - u^2)$$

$$(-5 - (-10)) \times 10^5 = -\frac{1}{2}(v^2 - 1400^2)$$

$$v = 979 \approx 980 \text{ m s}^{-1}$$

(a) Minimum speed at A to reach B:

$$(-4 - (-10)) \times 10^5 = -\frac{1}{2}(0 - u^2)$$

$$6 \times 10^5 = \frac{1}{2}u^2 \Rightarrow u = 1095 \approx 1100 \text{ m s}^{-1}$$

Escape Velocity and Orbital Motion

Escape velocity

The minimum speed of projection required for an object to escape the planet's gravitational field.

loss in K.E. = gain in P.E.

$$-\frac{1}{2}m(v^2 - u^2) = \Delta\phi m$$

$$-\frac{1}{2}(v^2 - u^2) = \phi_f - \phi_i$$

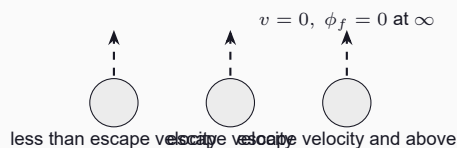
For escape to infinity:

$$v = 0, \quad \phi_f = 0, \quad \phi_i = -\frac{GM}{R}.$$

$$-\frac{1}{2}(0 - u^2) = 0 - \left(-\frac{GM}{R}\right)$$

$$\frac{u^2}{2} = \frac{GM}{R}$$

$$u = \sqrt{\frac{2GM}{R}}$$



For Earth:

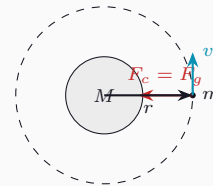
$$M = 6.0 \times 10^{24} \text{ kg}, \quad R = 6.4 \times 10^6 \text{ m}, \quad G = 6.67 \times 10^{-11}.$$

$$u = \sqrt{\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 2}{6.4 \times 10^6}}$$

$$u = 11183 \text{ m s}^{-1} \approx 11 \text{ km s}^{-1}$$

Orbital motion

The gravitational force acts as a centripetal force.



$$F_g = F_c$$

$$\frac{GMm}{r^2} = mr\omega^2.$$

$$\text{Using } \omega = \frac{2\pi}{T}:$$

$$\frac{GMm}{r^2} = mr \left(\frac{2\pi}{T}\right)^2$$

$$GM = r^3 \frac{4\pi^2}{T^2}$$

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

$$T^2 \propto r^3$$

“The square of the time period is directly proportional to the cube of the radius of orbit.”

Example

	Moon A	Moon B
r/m	6.4×10^{10}	7.5×10^{10}
T/days	30	x

$$\frac{30^2}{x^2} = \frac{(6.4 \times 10^{10})^3}{(7.5 \times 10^{10})^3}$$

$$x = 38 \text{ days}$$

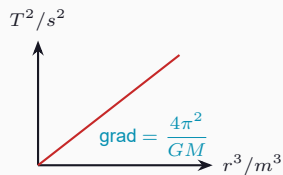
Kepler Graph and Geostationary Satellite

From

$$T^2 = \frac{4\pi^2}{GM} r^3,$$

if a graph of T^2 against r^3 is drawn:

$$y = mx + c, \quad \text{gradient} = \frac{4\pi^2}{GM}.$$

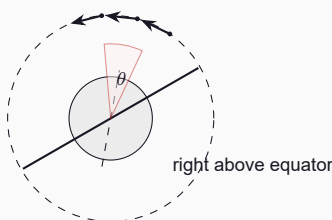


Geostationary satellite

A satellite that has the same angular speed as the Earth and revolves in the same direction as Earth's rotation:

westtoeast.

- $T = 24$ hours.
- It remains right above the same point on Earth's equator.



Speed of geostationary satellite

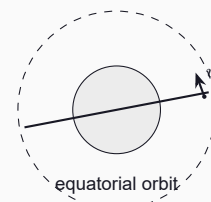
$$v = r\omega = r \left(\frac{2\pi}{T} \right)$$

$$v = 4.23 \times 10^7 \left(\frac{2\pi}{24 \times 3600} \right)$$

$$v = 3075.96 \text{ m s}^{-1} \approx 3100 \text{ m s}^{-1}$$

Visual notes

- The satellite must orbit above the equator.
- It must have the same period as Earth's rotation.
- It must move in the same direction as Earth rotates.



Radius of geostationary orbit

$$G = 6.67 \times 10^{-11}, \quad M = 6.0 \times 10^{24} \text{ kg}, \quad T = 24 \text{ h}.$$

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

$$(24 \times 3600)^2 = \frac{4\pi^2 r^3}{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}$$

$$r = 4.23 \times 10^7 \text{ m}$$

Polar Satellites and Energy of a Satellite

Polar satellite

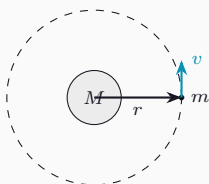
- Satellites that have orbits slightly tilted with respect to the equator.
- They have a smaller radius of orbit, i.e. they travel closer to Earth's surface.
- They act as links between different geostationary satellites.
- They have greater angular speed compared to Earth.

Since $T^2 \propto r^3$,

$$r \downarrow \Rightarrow T \downarrow \Rightarrow \text{speed} \uparrow$$



Energy of a satellite



1. Potential energy

$$E_p = -\frac{GMm}{r}$$

2. Kinetic energy

$$E_k = \frac{1}{2}mv^2.$$

As gravitational force acts as a centripetal force,

$$F_g = F_c, \quad \frac{GMm}{r^2} = \frac{mv^2}{r}.$$

$$\frac{GM}{r} = v^2$$

$$E_k = \frac{1}{2}m \left(\frac{GM}{r} \right)$$

$$E_k = \frac{GMm}{2r}$$

3. Total energy

$$E_T = E_k + E_p$$

$$E_T = \frac{GMm}{2r} + \left(-\frac{GMm}{r} \right)$$

$$E_T = \frac{GMm}{r} \left(\frac{1}{2} - 1 \right)$$

$$E_T = -\frac{GMm}{2r}$$

Final summary

Quantity	Expression
Gravitational force	$F = \frac{GMm}{r^2}$
Field strength	$g = \frac{GM}{r^2}$
Potential energy	$E_p = -\frac{GMm}{r}$
Potential	$\phi = -\frac{GM}{r}$
Escape velocity	$u = \sqrt{\frac{2GM}{R}}$
Orbital period	$T^2 = \frac{4\pi^2 r^3}{GM}$
Kinetic energy in orbit	$E_k = \frac{GMm}{2r}$
Total energy in orbit	$E_T = -\frac{GMm}{2r}$