

Oscillations

Chapter 03 / A2 Level / Apollo Physics 9702



Oscillation and Simple Harmonic Motion

Oscillation

It is a **repetitive motion** that has a time period.

Simple Harmonic Motion (SHM)

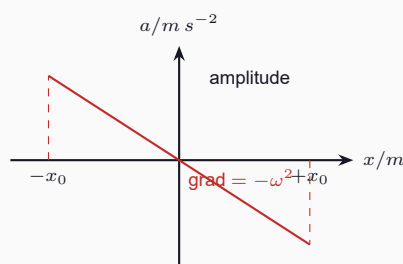
- A periodic motion.
- Acceleration of the body is directly proportional to displacement.
- Acceleration is in the **opposite direction** to displacement.

$$a \propto -x$$

$$a = -\omega^2 x$$

$$\omega = 2\pi f$$

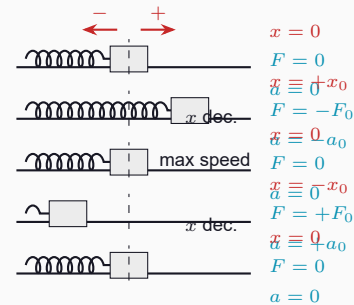
$$\omega = \frac{2\pi}{T}$$



Four features of the SHM graph

1. Straight line.
2. Passing through origin.
3. Negative gradient.
4. Equal amplitudes on both sides.

Spring-mass system under SHM



Hooke's law gives the restoring force:

$$F = -kx.$$

Using Newton's second law:

$$F_{net} = ma, \quad -kx = ma, \quad a = -\frac{k}{m}x.$$

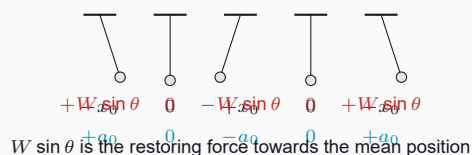
Compare with $a = -\omega^2 x$:

$$\omega^2 = \frac{k}{m}, \quad \left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}.$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where m is mass, k is spring constant and T is time period.

Simple Pendulum under SHM



The component $W \sin \theta$ acts as a **restoring force**; it tends to bring the object towards the mean position.

- As the pendulum swings towards the mean position, θ decreases.
- Therefore $\sin \theta$ decreases.
- Therefore $W \sin \theta$ and acceleration decrease.

For the pendulum:

$$F = W \sin \theta = mg \sin \theta.$$

Using $F = ma$:

$$mg \sin \theta = ma \Rightarrow a = g \sin \theta.$$

The acceleration is towards the mean position, so it is negative relative to displacement. For small angle oscil-

lations:



$$\sin \theta = \frac{x}{L}.$$

Hence

$$a = -g \frac{x}{L} = -\frac{g}{L}x.$$

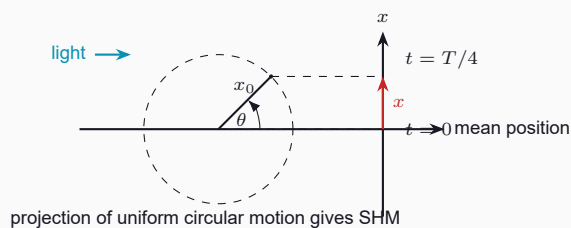
Compare with $a = -\omega^2 x$:

$$\omega^2 = \frac{g}{L}, \quad \left(\frac{2\pi}{T}\right)^2 = \frac{g}{L}.$$

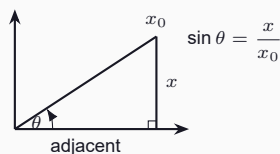
$$T = 2\pi \sqrt{\frac{L}{g}}$$

A simple pendulum observes SHM if it oscillates at a very small amplitude, so that $\theta \approx 0^\circ$ and $W \sin \theta$ is almost horizontal.

Relating Circular Motion to SHM



The vertical projection of a point moving with uniform circular motion performs SHM. If the body starts from the mean position:



$$\sin \theta = \frac{x}{x_0} \Rightarrow x = x_0 \sin \theta.$$

Since $\omega = \frac{\Delta \theta}{\Delta t}$, $\theta = \omega t$.

$$x = x_0 \sin \omega t$$

Velocity

$$\cos \theta = \frac{v}{v_0} \Rightarrow v = v_0 \cos \theta.$$

Since $v_0 = x_0 \omega$,

$$v = v_0 \cos \omega t = x_0 \omega \cos \omega t$$

Acceleration

$$\sin \theta = \frac{a}{a_0} \Rightarrow a = a_0 \sin \theta.$$

Acceleration is in the opposite direction to displacement:

$$a = -a_0 \sin \omega t = -\omega^2 x_0 \sin \omega t$$

If the body starts from maximum displacement:

$$x = x_0 \cos \omega t$$

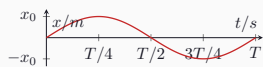
$$v = -v_0 \sin \omega t$$

$$a = -a_0 \cos \omega t$$

Starting condition	Displacement	Velocity / acceleration
Starts at mean	$x = x_0 \sin \omega t$	$v = v_0 \cos \omega t$, $a = -a_0 \sin \omega t$
Starts at extreme	$x = x_0 \cos \omega t$	$v = -v_0 \sin \omega t$, $a = -a_0 \cos \omega t$

Displacement, Velocity and Acceleration Graphs

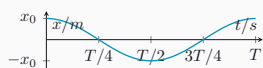
Starting from the mean position



$$x = x_0 \sin \omega t, \quad v = v_0 \cos \omega t, \quad a = -a_0 \sin \omega t.$$



Starting from extreme position



$$x = x_0 \cos \omega t, \quad v = -v_0 \sin \omega t, \quad a = -a_0 \cos \omega t.$$

Relating x and v of a body under SHM

Using

$$x = x_0 \sin \omega t, \quad v = x_0 \omega \cos \omega t,$$

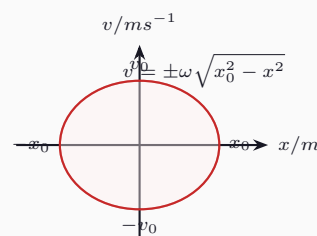
$$\sin^2 \theta + \cos^2 \theta = 1, \quad \left(\frac{x}{x_0} \right)^2 + \left(\frac{v}{x_0 \omega} \right)^2 = 1.$$

$$x^2 \omega^2 + v^2 = x_0^2 \omega^2.$$

$$v = \pm \omega \sqrt{x_0^2 - x^2}$$

- v is velocity at displacement x .
- x_0 is amplitude.
- ω is angular frequency.

Graph of $v = \omega \sqrt{x_0^2 - x^2}$



At $x = 0$ (mean position), $v = \pm v_0$. At $x = \pm x_0$ (extreme positions), $v = 0$.

Energy of an Oscillating Body

1. Kinetic energy

$$K.E. = \frac{1}{2}mv^2, \quad v = \omega\sqrt{x_0^2 - x^2}.$$

$$K.E. = \frac{1}{2}m\omega^2(x_0^2 - x^2)$$

For maximum kinetic energy, $x = 0$:

$$K.E._{max} = \frac{1}{2}m\omega^2x_0^2$$

2. Total energy

$$T.E. = K.E. + P.E.$$

At the mean position, $x = 0$, potential energy is zero and kinetic energy is maximum:

$$T.E. = K.E._{max}.$$

$$T.E. = \frac{1}{2}m\omega^2x_0^2$$

3. Potential energy

Since $T.E. = K.E. + P.E.$,

$$P.E. = T.E. - K.E.$$

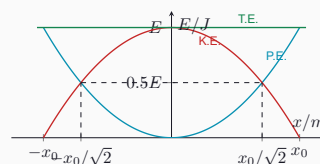
$$P.E. = \frac{1}{2}m\omega^2x_0^2 - \frac{1}{2}m\omega^2(x_0^2 - x^2)$$

$$P.E. = \frac{1}{2}m\omega^2\{x_0^2 - (x_0^2 - x^2)\} = \frac{1}{2}m\omega^2x^2.$$

$$P.E. = \frac{1}{2}m\omega^2x^2$$

At $x = x_0$, potential energy is maximum:

$$P.E._{max} = \frac{1}{2}m\omega^2x_0^2$$



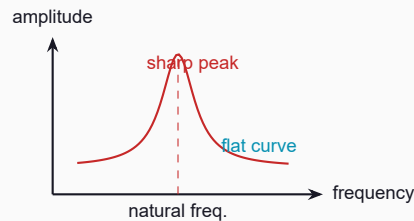
At $x = \pm x_0/\sqrt{2}$, $K.E. = P.E.$

Resonance and Damping

Resonance

Resonance is the increase in amplitude of vibration when the driving frequency becomes equal to the natural frequency of the oscillating body.

- Maximum energy from the external source is absorbed by the vibrating body.
- A sharp peak occurs for light damping.
- A flatter curve occurs for heavier damping.



Uses of resonance

- MRI.
- Ultrasound generation and detection.
- Musical instruments.
- Radio-frequency tuning.

Hazards

- Excessive vibrations of buildings and bridges during earthquakes.
- Vibrations in large rotating machinery, e.g. turbine engines.

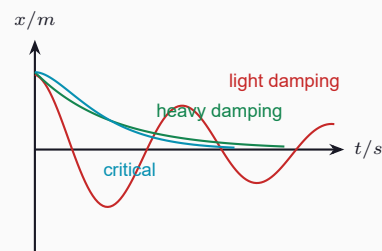
Damping

Damping is the decrease in amplitude of vibration due to loss of energy as heat, caused by a resistive force such as friction or air resistance. **To increase damping:**

1. Increase the surface area of the body by adding feathers, paper or cloth.
2. Oscillate the body in a viscous fluid such as oil.

Types of damping

1. **Light damping:** amplitude decreases gradually in multiple oscillations.
2. **Critical damping:** amplitude decreases to zero in about a quarter of an oscillation and as quickly as possible without oscillating.
3. **Heavy damping:** amplitude decreases to zero gradually in about a quarter of an oscillation.



Worked Example: Vibrating Plate and Cube



A horizontal plate vibrates vertically with frequency 4.5 Hz and amplitude 3.0 mm. A metal cube of mass 5.8 g rests on the plate.

Energy of oscillation

$$E = \frac{1}{2}m\omega^2x_0^2, \quad \omega = 2\pi f.$$

$$E = \frac{1}{2}(5.8 \times 10^{-3})\{2\pi(4.5)\}^2(3.0 \times 10^{-3})^2.$$

$$E = 2.1 \times 10^{-5} \text{ J}$$

Amplitude at loss of contact

At the top of motion, the plate can lose contact with the cube when the downward acceleration reaches g :

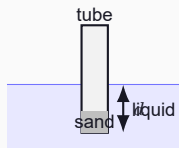
$$a_{max} = \omega^2x_0 = g.$$

$$x_0 = \frac{g}{\omega^2} = \frac{9.81}{(2\pi \times 4.5)^2}.$$

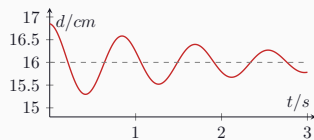
$$x_0 = 1.23 \times 10^{-2} \text{ m} \approx 0.012 \text{ m}$$

If the amplitude is less than this, the cube remains in contact with the plate.

Worked Example: Floating Tube and Damped Oscillations



A tube floats upright in a liquid with equilibrium immersion depth $d = 16$ cm. When displaced and released, d varies with time.



From the graph:

- Mean position: $d = 16$ cm.
- Amplitude: about 0.85 cm.
- Time period: about 0.85 s.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.85} = 7.39 \text{ rad s}^{-1}.$$

Speed at $d = 16.2$ cm

The displacement from equilibrium is

$$x = 16.2 - 16.0 = 0.20 \text{ cm}.$$

Using

$$v = \omega \sqrt{x_0^2 - x^2},$$

$$v = \frac{2\pi}{0.85} \sqrt{(0.85)^2 - (0.20)^2} \text{ cm s}^{-1}.$$

$$v \approx 3.6 \text{ cm s}^{-1}$$

The same speed occurs at the symmetric depth on the other side of equilibrium:

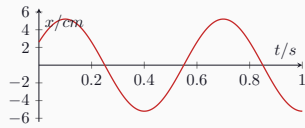
$$d = 15.8 \text{ cm}$$

Damping meaning

Damping is the decrease in amplitude of vibration due to loss of energy as heat caused by a resistive force.

Pendulum Graphs and Kinetic Energy Sketch

A small metal ball suspended from a string performs SHM. The displacement graph gives the period and amplitude.



Acceleration at $x = 2.0 \text{ cm}$:

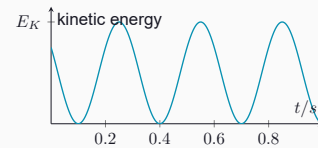
$$a = -\omega^2 x = -\left(\frac{2\pi}{T}\right)^2 x.$$

If $T \approx 0.60 \text{ s}$ and $x = 0.020 \text{ m}$,

$$a = -\left(\frac{2\pi}{0.60}\right)^2 (0.020) \approx -2.2 \text{ m s}^{-2}.$$

Kinetic energy is maximum at mean position and zero

at extreme position. Therefore it has two maxima per oscillation.



- $E_K = 0$ at extreme displacements.
- $E_K = E_{K,max}$ at the mean position.
- Kinetic energy cannot be negative.

Spring-Ball Energy Example

A ball of mass 37 g is held between two fixed points A and B by two stretched helical springs. It oscillates with frequency 3.5 Hz and amplitude 2.8 cm.



Total energy

$$E_T = \frac{1}{2} m \omega^2 x_0^2.$$

$$E_T = \frac{1}{2} \left(\frac{37}{1000} \right) (2\pi \times 3.5)^2 (0.028)^2.$$

$$E_T \approx 0.0070 \text{ J} = 7.0 \text{ mJ}$$

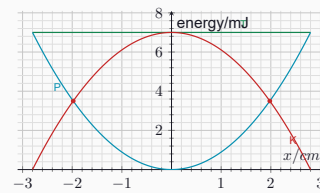
Where $K.E. = P.E.$

$$\frac{1}{2} m \omega^2 (x_0^2 - x^2) = \frac{1}{2} m \omega^2 x^2.$$

$$x_0^2 - x^2 = x^2 \Rightarrow x = \frac{x_0}{\sqrt{2}}.$$

$$x = \frac{2.8}{\sqrt{2}} = 2.0 \text{ cm}$$

Energy sketch against displacement

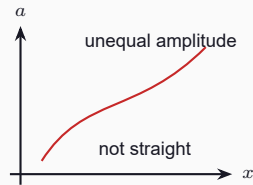


- Total energy is constant.
- Kinetic energy is maximum at $x = 0$.
- Potential energy is maximum at $x = \pm x_0$.
- When the spring system is rotated vertically, gravitational potential energy must also be considered.

Recognising Non-SHM and Damped Block Example

Why a motion may not be SHM

For SHM, the acceleration-displacement graph must be a straight line passing through the origin with negative gradient and equal amplitudes.

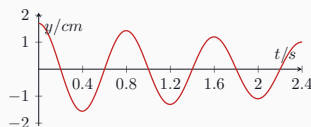


Two reasons the example graph is not SHM:

1. The graph is not a straight line.
2. The amplitudes of the vibration are different on the two sides.

Floating block example

A block floating in a liquid is displaced vertically and released. The displacement-time graph is damped.



If the period from the graph is about 0.80 s,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80} = 7.85 \text{ rad s}^{-1}.$$

If the amplitude is 1.5 cm,

$$a_{max} = \omega^2 x_0 = (7.85)^2 (0.015) = 0.925 \text{ m s}^{-2}.$$

$$a_{max} \approx 0.93 \text{ m s}^{-2}$$

Energy lost in a damped interval is found from

$$\Delta E = \frac{1}{2} m \omega^2 (A_i^2 - A_f^2).$$

For $m = 0.120 \text{ kg}$, $\omega = 7.85 \text{ rad s}^{-1}$ and decreasing amplitudes, the source working gives roughly

$$\Delta E \approx 5.3 \times 10^{-4} \text{ J}$$

Sewing-Machine Needle and Engine Piston Phase

A sewing-machine needle oscillates vertically through a total distance of 22 mm. Therefore the amplitude is

$$x_0 = 11 \text{ mm} = 1.1 \text{ cm}.$$

For SHM:

$$v_{max} = \omega x_0.$$

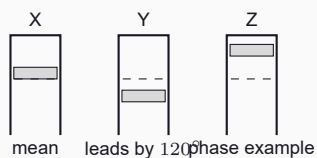
If $\omega = 800 \text{ rad s}^{-1}$ and $x_0 = 1.1 \text{ cm}$,

$$v_{max} = 800 \times 1.1 = 880 \text{ cm s}^{-1}.$$

$$v_{max} = 880 \text{ cm s}^{-1}$$

Phase of pistons

Pistons in an engine may have the same frequency but not be in phase.



If the oscillation of piston Y leads that of piston X by a phase angle $120^\circ = 2\pi/3$, represent positions using cosine:

$$x = x_0 \cos(\omega t + \phi).$$

For a leading piston:

$$x_Y = x_0 \cos(\omega t + 120^\circ).$$

For direction of motion:

$$v = -\omega x_0 \sin(\omega t + \phi).$$

Rules for phase questions

- Same frequency means same period.
- A phase lead means the oscillator is ahead in its cycle.
- Use displacement equation to locate position.
- Use velocity equation to show direction of motion.