APOLLO

Calculus
HL

Mathematics: analysis and approaches
First assessment 2021
135 questions | mark schemes

4. (a) $f'(x) = -2(x-h)$ ***A1***

[1 mark]

(b) $g'(x) = e^{x-2}$ OR $g'(3) = e^{3-2}$ (may be seen anywhere) ***A1***

Note: The derivative of g must be explicitly seen, either in terms of x or 3.

recognizing $f'(3) = g'(3)$ ***(M1)***

$-2(3-h) = e^{3-2}$ $(= e)$

$-6+2h = e$ OR $3-h = -\frac{e}{2}$ ***A1***

Note: The final ***A1*** is dependent on one of the previous marks being awarded.

$h = \frac{e+6}{2}$ ***AG***

[3 marks]

(c) $f(3) = g(3)$ ***(M1)***

$-(3-h)^2 + 2k = e^{3-2} + k$

correct equation in k

EITHER

$-\left(3-\frac{e+6}{2}\right)^2 + 2k = e^{3-2} + k$ ***A1***

$k = e + \left(\frac{6-e-6}{2}\right)^2 \left(= e + \left(\frac{-e}{2}\right)^2\right)$ ***A1***

OR

$k = e + \left(3-\frac{e+6}{2}\right)^2$ ***A1***

$k = e + 9 - 3e - 18 + \frac{e^2+12e+36}{4}$ ***A1***

THEN

$k = e + \frac{e^2}{4}$ ***AG***

[3 marks]
Total [7 marks]

8. attempt to differentiate numerator and denominator ***M1***

$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$$ ***A1A1***

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x=0$ ***(M1)***

$=\frac{2}{3}$ ***A1***

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

12. (a) attempt to use the chain rule ***M1***

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ ***A1***

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ ***A1***

$= -\frac{1}{4\sqrt{(1+x)^3}}$ ***AG***

Note: Award ***M1A0A0*** for $f'(x) = \frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) let $n = 2$

$f''(x) = \left(-\frac{1}{4\sqrt{(1+x)^3}} = \right)\left(-\frac{1}{4}\right)^1 \frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$ ***R1***

Note: Award ***R0*** for not starting at $n = 2$. Award subsequent marks as appropriate.

assume true for $n = k$, (so $f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

consider $n = k+1$

$\text{LHS} = f^{(k+1)}(x) = \frac{d\left(f^{(k)}(x)\right)}{dx}$ ***M1***

$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

EITHER

$\text{RHS} = f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

continued…

Question 12 continued

Note: Award ***A1*** for $\frac{(2k-1)!}{(k-1)!} = \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!} \left(= \frac{2(2k-1)(2k-3)!}{(k-2)!} \right)$

$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

$\left(= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1} \right)$

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$= \left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

OR

Note: The following ***A*** marks can be awarded in any order.

$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-k-1}$

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for isolating $(2k-1)$ correctly.

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(2k-2)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for multiplying top and bottom by $(k-1)$ or $2(k-1)$.

$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

continued…

Question 12 continued

$$=\left(-\frac{1}{4}\right)^{(k+1)-1}\frac{\left(2\left(k+1\right)-3\right)!}{\left(\left(k+1\right)-2\right)!}(1+x)^{\frac{1}{2}-(k+1)}=\text{RHS}$$

THEN

since true for $n=2$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n\in\mathbb{Z}, n\geq 2$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** at least four of the previous marks must have been awarded.

[9 marks]

(c) **METHOD 1**

$h(x)=\sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h'(x)$ ***(M1)***

$h'(x)=\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$ ***A1***

$h''(x)=m\left(\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right)+\frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx}-\frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$ ***A1***

substituting $x=0$ into $h''(x)$ ***M1***

$h''(0)=m^2+\frac{1}{2}m+\frac{1}{2}m-\frac{1}{4}\left(=m^2+m-\frac{1}{4}\right)$ ***A1***

$h(x)=h(0)+xh'(0)+\frac{x^2}{2!}h''(0)+\ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!}=\frac{7}{4}\ \left(\Rightarrow h''(0)=\frac{7}{2}\right)$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 2

EITHER

attempt to find $f(0)$, $f'(0)$, $f''(0)$ ***(M1)***

$f(x) = (1+x)^{\frac{1}{2}}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(0) = -\frac{1}{4}$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$f(x) = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \ldots$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

THEN

$g(x) = 1 + mx + \frac{m^2}{2}x^2 + \ldots$ ***(A1)***

$h(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + mx + \frac{m^2}{2}x^2 + \ldots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8} = \frac{7}{4}$

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 3

$g'(x) = me^{mx}$ and $g''(x) = m^2e^{mx}$ ***(A1)***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + ...$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4}$ $\left(\Rightarrow h''(0) = \frac{7}{2}\right)$

using product rule to find $h'(x)$ and $h''(x)$ ***(M1)***

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h''(x) = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = f(0)g''(0) + 2g'(0)f'(0) + g(0)f''(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right)$ $\left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m + 5)(2m - 3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

Total [20 marks]

4. (a) $\ln\left(x^2-16\right)=0$ ***(M1)***

$e^0=x^2-16\left(=1\right)$

$x^2=17$ OR $x=\pm\sqrt{17}$ ***(A1)***

$a=\sqrt{17}$ ***A1***

[3 marks]

(b) attempt to differentiate (must include $2x$ and/or $\frac{1}{x^2-16}$) ***(M1)***

$f'(x)=\frac{2x}{x^2-16}$ ***A1***

setting their derivative $=\frac{1}{3}$ ***M1***

$\frac{2x}{x^2-16}=\frac{1}{3}$

$x^2-16=6x$ ***OR*** $x^2-6x-16=0$ (or equivalent) ***A1***

valid attempt to solve their quadratic ***(M1)***

$x=8$ ***A1***

Note: Award ***A0*** if the candidate's final answer includes additional solutions (such as $x=-2, 8$).

[6 marks]
Total [9 marks]

9. $u = \sin x \Rightarrow \mathrm{d}u = \cos x \, \mathrm{d}x$ (or equivalent) ***A1***

$= \int \frac{u}{u^2 - u - 2} \mathrm{d}u$ ***A1***

attempt to use partial fractions ***M1***

$\left(\frac{u}{(u+1)(u-2)} \equiv \frac{A}{u+1} + \frac{B}{u-2} \Rightarrow u \equiv A(u-2) + B(u+1) \right)$

Valid attempt to solve for A and B ***(M1)***

$A = \frac{1}{3}$ and $B = \frac{2}{3}$ ***A1***

$\frac{u}{(u+1)(u-2)} \equiv \frac{1}{3(u+1)} + \frac{2}{3(u-2)}$

$\int \left(\frac{1}{3(u+1)} + \frac{2}{3(u-2)} \right) \mathrm{d}u = \frac{1}{3} \ln|u+1| + \frac{2}{3} \ln|u-2| (+C)$ (or equivalent) ***A1***

Note: Condone the absence of $+C$ or lack of moduli here but not in the final answer.

$= \frac{1}{3} \ln|\sin x + 1| + \frac{2}{3} \ln|\sin x - 2| + C$ ***A1***

Note: Condone further simplification of the correct answer.

[7 marks]

Section B

11. (a) $\frac{\mathrm{d}v}{\mathrm{d}t} = -(1+v)$ ***(A1)***

$\int 1\,\mathrm{d}t = \int -\frac{1}{1+v}\,\mathrm{d}v$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$C = \ln(1+v_0)$

$t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow \mathrm{e}^t = \frac{1+v_0}{1+v}$ ***A1***

$\mathrm{e}^t(1+v) = 1+v_0$

$1+v = (1+v_0)\mathrm{e}^{-t}$ ***A1***

$v(t) = (1+v_0)\mathrm{e}^{-t} - 1$ ***AG***

continued…

OR

$t - C = -\ln(1+v) \Rightarrow e^{t-C} = \frac{1}{(1+v)}$

Attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$e^{-C} = \frac{1}{(1+v_0)} \Rightarrow C = \ln(1+v_0)$

$t - \ln(1+v_0) = -\ln(1+v) \Rightarrow t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow e^t = \frac{1+v_0}{1+v}$ ***A1***

$e^t(1+v) = 1+v_0$

$1+v = (1+v_0)e^{-t}$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

OR

$t - C = -\ln(1+v) \Rightarrow e^{-t+C} = 1+v$ ***A1***

$ke^{-t} - 1 = v$

Attempt to find k with initial conditions $t = 0, v = v_0$ ***M1***

$k = 1+v_0$

$e^{-t}(1+v_0) = 1+v$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

continued…

Question 11 continued

(b) (i) recognition that when $t = T, v = 0$ ***M1***

$\left(1+v_0\right)\mathrm{e}^{-T} - 1 = 0 \Rightarrow \mathrm{e}^{-T} = \dfrac{1}{1+v_0}$ ***A1***

$\mathrm{e}^{T} = 1+v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = \mathrm{e}^{T} - 1$ into v and showing that $v = 0$.

(ii) $s(t) = \int v(t)\ \mathrm{d}t\ \left(= \int \left(\left(1+v_0\right)\mathrm{e}^{-t} - 1\right)\mathrm{d}t\right)$ ***(M1)***

$= -\left(1+v_0\right)\mathrm{e}^{-t} - t\left(+D\right)$ ***A1***

$(t = 0, s = 0$ so$)\ D = 1+v_0$ ***A1***

$s(t) = -\left(1+v_0\right)\mathrm{e}^{-t} - t + 1 + v_0$

at $s_{\max}$, $\mathrm{e}^{T} = 1+v_0 \Rightarrow T = \ln\left(1+v_0\right)$

Substituting into $s(t)\left(= -\left(1+v_0\right)\mathrm{e}^{-t} - t + 1 + v_0\right)$ ***M1***

$s_{\max} = -\left(1+v_0\right)\left(\dfrac{1}{1+v_0}\right) - \ln\left(1+v_0\right) + v_0 + 1$ ***A1***

$\left(s_{\max} = v_0 - \ln\left(1+v_0\right)\right)$

[7 marks]

(c) **METHOD 1**

$v\left(T-k\right) = \left(1+v_0\right)\mathrm{e}^{-T}\mathrm{e}^{k} - 1$ ***(M1)***

$= \left(1+v_0\right)\left(\dfrac{1}{1+v_0}\right)\mathrm{e}^{k} - 1$ ***A1***

$= \mathrm{e}^{k} - 1$ ***AG***

[2 marks]

METHOD 2

$v\left(T-k\right) = \left(1+v_0\right)\mathrm{e}^{-(T-k)} - 1$

$= e^{T}\mathrm{e}^{-(T-k)} - 1$ ***M1***

$= e^{T-T+k} - 1$ ***A1***

$= \mathrm{e}^{k} - 1$ ***AG***

[2 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$v(T+k) = (1+v_0)e^{-T}e^{-k} - 1$ ***(A1)***

$= e^{-k} - 1$ ***A1***

[2 marks]

METHOD 2

$v(T+k) = (1+v_0)e^{-(T+k)} - 1$ ***(A1)***

$= e^{T}e^{-(T+k)} - 1$

$= e^{T-T-k} - 1$

$= e^{-k} - 1$ ***A1***

[2 marks]

(e) **METHOD 1**

$v(T-k) + v(T+k) = e^{k} + e^{-k} - 2$ ***A1***

attempt to express as a square ***M1***

$= \left(e^{\frac{k}{2}} - e^{-\frac{k}{2}}\right)^2 (\geq 0)$ ***A1***

so $v(T-k) + v(T+k) \geq 0$ ***AG***

[3 marks]

METHOD 2

$v(T-k) + v(T+k) = e^{k} + e^{-k} - 2$ ***A1***

Attempt to solve $\frac{d}{dk}(e^{k} + e^{-k}) = 0 \ (\Rightarrow k = 0)$ ***M1***

minimum value of 2, (when $k = 0$), hence $e^{k} + e^{-k} \geq 2$ ***R1***

so $v(T-k) + v(T+k) \geq 0$ ***AG***

[3 marks]
Total [20 marks]

4. (a) recognising $v = 0$ ***(M1)***

$t = 6.74416...$

$= 6.74$ (sec) ***A1***

Note: Do not award ***A1*** if additional values are given.

[2 marks]

(b) $\int_0^{10} |v(t)| \mathrm{d}t$ **OR** $-\int_0^{6.74416...} v(t)\,\mathrm{d}t + \int_{6.74416...}^{9.08837...} v(t)\,\mathrm{d}t - \int_{9.08837...}^{10} v(t)\,\mathrm{d}t$ ***(A1)***

$= 37.0968...$

$= 37.1$ (m) ***A1***

[2 marks]

(c) recognising acceleration at $t = 7$ is given by $v'(7)$ ***(M1)***

acceleration $= 5.93430...$

$= 5.93$ (ms^{-2}) ***A1***

[2 marks]

Total [6 marks]

9. (a) $\tan\theta = \frac{50}{y-x}$ OR $\cot\theta = \frac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to t ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50\left(\operatorname{cosec}\theta\right)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

attempt to set speed of B equal to double the speed of A (***M1)***

$2\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50\left(\operatorname{cosec}\theta\right)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\operatorname{cosec}\theta\right)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

$\theta = \arctan 5 \left(= 1.373\ldots = 78.69\ldots°\right)$ OR $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{1}{5}\right)^2 = \frac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\frac{26}{25}\right) \times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]
Total [7 marks]

Section B.

10. (a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $\left(2\ln 45, 2\right)$ (accept $(-2\ln\frac{1}{45}, 2)$ ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -\left(x - 2\ln 45\right)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

continued…

Question 10 continued

(d) (i) $x = \ln 45 + 1 (= 4.81)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$\int_{4.806...}^{5.566...} \left(x - \left(-x + 2\ln 45 + 2\right)\right) \mathrm{d}x + \int_{5.566...}^{7.613...} \left(90\mathrm{e}^{-0.5x} - \left(-x + 2\ln 45 + 2\right)\right) \mathrm{d}x$ ***(A1)(A1)***

Note: Award ***A1*** for one correct integral expression including correct limits and integrand.
Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]
Total [15 marks]

12. (a) $\frac{1}{x(k-x)} \equiv \frac{a}{x} + \frac{b}{k-x}$

$a(k-x) + bx = 1$ ***(A1)***

attempt to compare coefficients OR substitute $x = k$ and $x = 0$ and solve ***(M1)***

$a = \frac{1}{k}$ and $b = \frac{1}{k}$ ***A1***

$f'(x) = \frac{1}{kx} + \frac{1}{k(k-x)}$

[3 marks]

(b) attempt to integrate their $\frac{a}{x} + \frac{b}{k-x}$ ***(M1)***

$f(x) = \frac{1}{k}\int\left(\frac{1}{x} + \frac{1}{k-x}\right)\mathrm{d}x$

$= \frac{1}{k}\left(\ln|x| - \ln|k-x|\right)(+c)\left(= \frac{1}{k}\ln\left|\frac{x}{k-x}\right|(+c)\right)$ ***A1A1***

[3 marks]

Note: Award ***A1*** for each correct term. Award ***A1A0*** for a correct answer without modulus signs. Condone the absence of $+c$.

continued…

Question 12 continued

(c) attempt to separate variables and integrate both sides ***M1***

$$5k\int \frac{1}{P(k-P)}\,\mathrm{d}P=\int 1\,\mathrm{d}t$$

$$5\left(\ln P-\ln\left(k-P\right)\right)=t+c$$ ***A1***

Note: There are variations on this which should be accepted, such as $\frac{1}{k}\left(\ln P-\ln\left(k-P\right)\right)=\frac{1}{5k}t+c$. Subsequent marks for these variations should be awarded as appropriate.

EITHER

attempt to substitute $t=0$, $P=1200$ into an equation involving c ***M1***

$$c=5\left(\ln 1200-\ln\left(k-1200\right)\right)\left(=5\ln\left(\frac{1200}{k-1200}\right)\right)$$ ***A1***

$$5\left(\ln P-\ln\left(k-P\right)\right)=t+5\left(\ln 1200-\ln\left(k-1200\right)\right)$$ ***A1***

$$\ln\left(\frac{P\left(k-1200\right)}{1200\left(k-P\right)}\right)=\frac{t}{5}$$

$$\frac{P\left(k-1200\right)}{1200\left(k-P\right)}=\mathrm{e}^{\frac{t}{5}}$$ ***A1***

OR

$$\ln\left(\frac{P}{k-P}\right)=\frac{t+c}{5}$$

$$\frac{P}{k-P}=A\mathrm{e}^{\frac{t}{5}}$$ ***A1***

attempt to substitute $t=0$, $P=1200$ ***M1***

$$\frac{1200}{k-1200}=A$$ ***A1***

$$\frac{P}{k-P}=\frac{1200\mathrm{e}^{\frac{t}{5}}}{k-1200}$$ ***A1***

continued…

Question 12 continued

THEN

attempt to rearrange and isolate P ***M1***

$Pk - 1200P = 1200k\mathrm{e}^{\frac{t}{5}} - 1200P\mathrm{e}^{\frac{t}{5}}$ OR $Pk\mathrm{e}^{-\frac{t}{5}} - 1200P\mathrm{e}^{-\frac{t}{5}} = 1200k - 1200P$

OR $\frac{k}{P} - 1 = \frac{k - 1200}{1200\mathrm{e}^{\frac{t}{5}}}$

$P\left(k - 1200 + 1200\mathrm{e}^{\frac{t}{5}}\right) = 1200k\mathrm{e}^{\frac{t}{5}}$ OR $P\left(k\mathrm{e}^{-\frac{t}{5}} - 1200\mathrm{e}^{-\frac{t}{5}} + 1200\right) = 1200k$ ***A1***

$P = \frac{1200k}{(k - 1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$ ***AG***

[8 marks]

(d) attempt to substitute $t = 10$, $P = 2400$ ***(M1)***

$2400 = \frac{1200k}{(k - 1200)\mathrm{e}^{-2} + 1200}$ ***(A1)***

$k = 2845.34...$

$k = 2845$ ***A1***

Note: Award ***(M1)(A1)A0*** for any other value of k which rounds to 2850

[3 marks]

(e) attempt to find the maximum of the first derivative graph OR zero of the second derivative graph OR that $P = \frac{k}{2}(= 1422.67...)$ ***(M1)***

$t = 1.57814...$

$= 1.58$ (days) ***A2***

Note: Accept any value which rounds to 1.6.

[3 marks]

Total [20 marks]

9. (a) $1-t+t^2$ ***A1***

Note: Accept $1, -t$ and t^2.

[1 mark]

(b) $\sec x = \dfrac{1}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)} \left(= \left(1-\dfrac{x^2}{2!}+\left(\dfrac{x^4}{4!}(-...)\right)\right)^{-1} \right)$ ***(M1)***

$t = \cos x - 1$ or $\sec x = 1-(\cos x-1)+(\cos x-1)^2$ ***(M1)***

$= 1-\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)\right)+\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)\right)^2$ ***A1***

$= 1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}$ ***A1***

so the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}$ ***AG***

[4 marks]

Note: Condone the absence of '…'

(c) $\arctan 2x = 2x-\dfrac{(2x)^3}{3}+...$

$\lim\limits_{x\to 0}\left(\dfrac{x\arctan 2x}{\sec x-1}\right) = \lim\limits_{x\to 0}\left(\dfrac{x\left(2x-\dfrac{(2x)^3}{3}+...\right)}{\left(1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}\right)-1}\right)$ ***M1***

$= \lim\limits_{x\to 0}\left(\dfrac{2x^2-\dfrac{8x^4}{3}+...}{\dfrac{x^2}{2}+\dfrac{5x^4}{24}}\right)$ ***A1***

$= \lim\limits_{x\to 0}\left(\dfrac{2x^2\left(1-\dfrac{4x^2}{3}\right)}{\dfrac{x^2}{2}\left(1+\dfrac{5x^2}{12}\right)}\right)$

$= 4$ ***A1***

Note: Condone missing 'lim' and errors in higher derivatives.
Do not award ***M1*** unless x is replaced by $2x$ in arctan.

[3 marks]
Total [8 marks]

Section B

11. (a) attempt to use $V = \pi\int_a^b \left(f(x)\right)^2 \mathrm{d}x$ ***(M1)***

$V = \pi\int_0^{\ln 16}\left(\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}\right)^2 \mathrm{d}x \quad \left(V = k^2\pi\int_0^{\ln 16}\frac{\mathrm{e}^x}{\left(1+\mathrm{e}^x\right)^2}\mathrm{d}x\right)$

EITHER

applying integration by recognition ***(M1)***

$= k^2\pi\left[-\frac{1}{1+\mathrm{e}^x}\right]_0^{\ln 16}$ ***A3***

OR

$u = 1+\mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 2$ and when $x = \ln 16$, $u = 17$

$V = k^2\pi\int_2^{17}\frac{1}{u^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$ ***A1***

OR

$u = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 1$ and when $x = \ln 16$, $u = 16$

$V = k^2\pi\int_1^{16}\frac{1}{\left(1+u\right)^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

$= k^2\pi\left(\frac{1}{2}-\frac{1}{17}\right)$ ***A1***

so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

continued…

Question 11 continued

(b) a valid algebraic or graphical attempt to find k ***(M1)***

$k^2 = \frac{300 \times 34}{15\pi}$

$k = 14.7 \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ (as $k \in \mathbb{R}^+$) ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

(c) (i) attempting to find $\text{OA} = f(0) = \frac{k}{2}$

with $k = 14.712...$ $\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{OA} = 7.36 \left(= \sqrt{\frac{170}{\pi}} \right)$ ***A1***

(ii) attempting to find $\text{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712...$ $\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{BC} = 3.46 \left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}} \right)$ ***A1***

[4 marks]

continued…

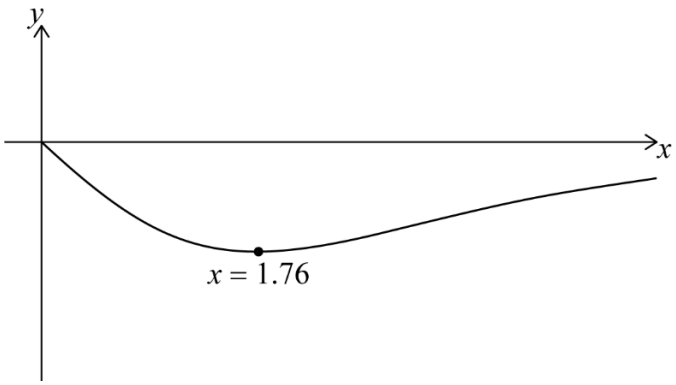
Question 11 continued

(d) (i) **EITHER**

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f'(x) = \frac{ke^{\frac{x}{2}}(1-e^x)}{2(1+e^x)^2}$$



for $x > 0$ graph decreasing to the local minimum ***A1***

before increasing towards the x-axis ***A1***

continued…

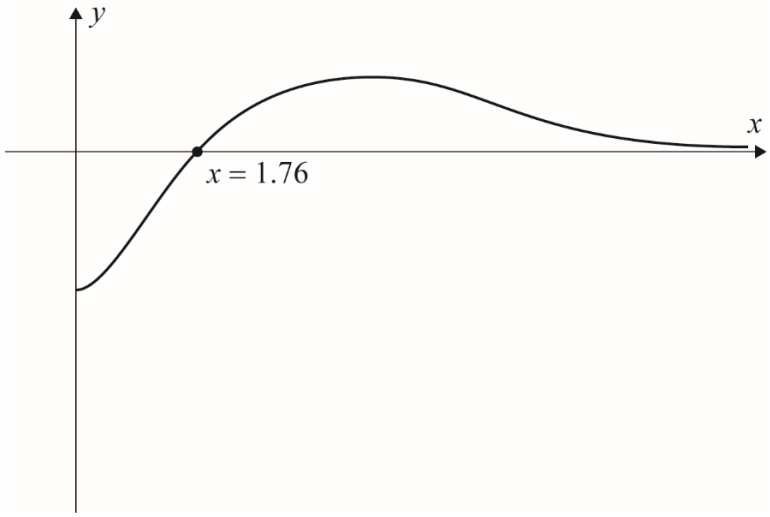
Question 11 continued

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f''(x) = \frac{k\mathrm{e}^{\frac{x}{2}}\left(\mathrm{e}^{2x} - 6\mathrm{e}^{x} + 1\right)}{4\left(1+\mathrm{e}^{x}\right)^{3}}$$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76 \left(= \ln\left(2\sqrt{2}+3\right)\right)$ ***A1***

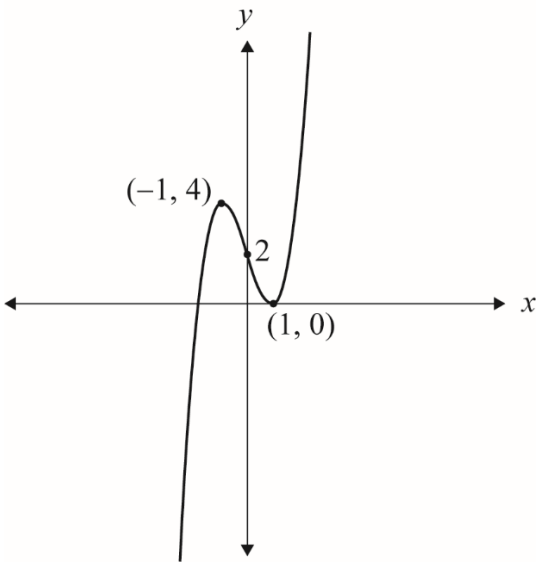
Note: Only award ***A*** marks if either graph is seen.

(ii) attempting to find $f(1.76...)$ ***(M1)***

the cross-sectional radius at this point is $5.20 \left(\sqrt{\frac{85}{\pi}}\right)$ (cm) ***A1***

[6 marks]
Total [18 marks]

1. (a) (i)



$c = 1$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

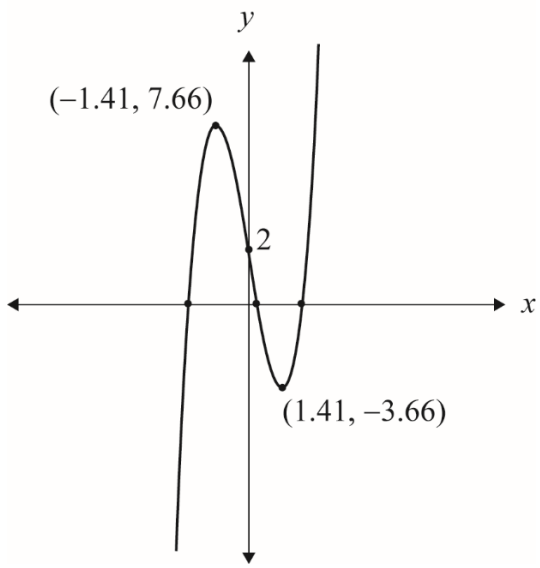
local minimum point correctly labelled ***A1***

[3 marks]

continued…

Question 1 continued

(ii)



$c = 2$: positive cubic with correct *y*-intercept labelled **A1**

local maximum point correctly labelled **A1**

local minimum point correctly labelled **A1**

Note: Accept the following exact answers:

Local maximum point coordinates $\left(-\sqrt{2}, 2+4\sqrt{2}\right)$.

Local minimum point coordinates $\left(\sqrt{2}, 2-4\sqrt{2}\right)$.

[3 marks]

continued…

Question 1 continued

(b) $f'(x) = 3x^2 - 3c$ ***A1***

Note: Accept $3x^2 - 3c$ (an expression).

[1 mark]

(c) (i) $c = 0$ ***A1***

[1 mark]

(ii) considers the number of solutions to their $f'(x) = 0$ ***(M1)***

$3x^2 - 3c = 0$

$c > 0$ ***A1***

[2 marks]

(iii) $c < 0$ ***A1***

Note: The ***(M1)*** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

[1 mark]

(d) attempts to solve their $f'(x)=0$ for x ***(M1)***

$x=\pm\sqrt{c}$ ***(A1)***

Note: Award ***(A1)*** if either $x=-\sqrt{c}$ or $x=\sqrt{c}$ is subsequently considered.
Award the above ***(M1)(A1)*** if this work is seen in part (c).

(i) correctly evaluates $f\left(-\sqrt{c}\right)$ ***A1***

$f\left(-\sqrt{c}\right)=-c^{\frac{3}{2}}+3c^{\frac{3}{2}}+2\ \left(=-c\sqrt{c}+3c\sqrt{c}+2\right)$

the y- coordinate of the local maximum point is $2c^{\frac{3}{2}}+2$ ***AG***

[3 marks]

(ii) correctly evaluates $f\left(\sqrt{c}\right)$ ***A1***

$f\left(\sqrt{c}\right)=c^{\frac{3}{2}}-3c^{\frac{3}{2}}+2\ \left(=c\sqrt{c}-3c\sqrt{c}+2\right)$

the y- coordinate of the local minimum point is $-2c^{\frac{3}{2}}+2$ ***AG***

[1 mark]

continued…

Question 1 continued

(e) (i) the graph of $y = f(x)$ will have one $x-$ axis intercept if

EITHER

$-2c^{\frac{3}{2}} + 2 > 0$ (or equivalent reasoning) ***R1***

OR

the minimum point is above the $x-$ axis ***R1***

Note: Award ***R1*** for a rigorous approach that does not (only) refer to sketched graphs.

THEN

$0 < c < 1$ ***A1***

Note: Condone $c < 1$. The ***A1*** is independent of the ***R1***.

[2 marks]

(ii) the graph of $y = f(x)$ will have two $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 = 0$ (or equivalent reasoning) ***(M1)***

OR

evidence from the graph in part(a)(i) ***(M1)***

THEN

$c = 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(iii) the graph of $y = f(x)$ will have three $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 < 0$ (or equivalent reasoning) ***(M1)***

OR

reasoning from the results in both parts (e)(i) and (e)(ii) ***(M1)***

THEN

$c > 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(f) case 1:

$c \leq 0$ (independent of the value of d) ***A1***

EITHER

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution) ***R1***

OR

$\Rightarrow g'(x) \geq 0$ for $x \in \tilde{}$ ***R1***

OR

the graph of $y = f(x)$ has no local maximum or local minimum points,

hence any vertical translation of this graph ($y = g(x)$) will also have no local maximum or local minimum points ***R1***

THEN

therefore there is only one x- axis intercept ***AG***

Note: Award at most ***A0R1*** if only $c < 0$ is considered.

continued…

Question 1 continued

case 2

$c > 0$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a local minimum point ***(A1)***

Note: Award ***(A1)*** for a correct y- coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point ***(M1)***

EITHER

considers both points above the x- axis or both points below the x- axis

OR

considers either the local minimum point only above the x- axis OR the local maximum point only below the x- axis

THEN

$d > 2c^{\frac{3}{2}}$ (both points above the x- axis) ***A1***

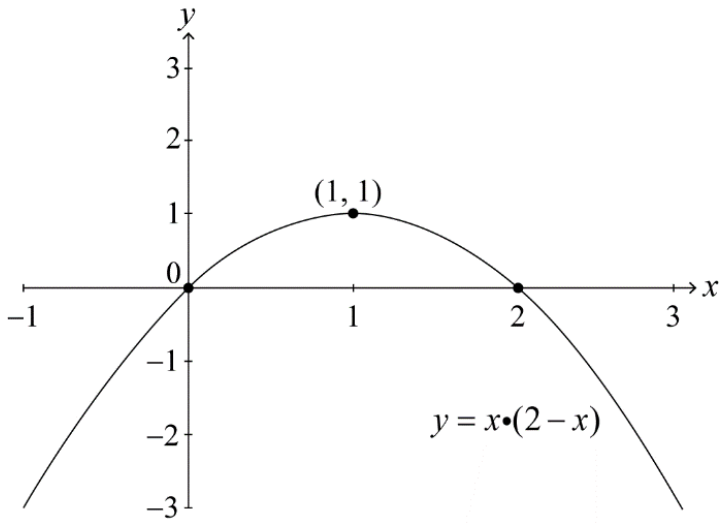
$d < -2c^{\frac{3}{2}}$ (both points below the x- axis) ***A1***

Note: Award at most ***(A1)(M1)A0A0*** for case 2 if $c > 0$ is not clearly stated.

[6 marks]

Total [27 marks]

1. (a)



inverted parabola extended below the x- axis ***A1***

x- axis intercept values $x = 0, 2$ ***A1***

Note: Accept a graph passing through the origin as an indication of $x = 0$.

local maximum at $(1,1)$ ***A1***

Note: Coordinates must be stated to gain the final ***A1***.
Do not accept decimal approximations.

[3 marks]

(b)

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$	1	0	2
$n = 2$ and $n = 4$	1	2	0

A1A1A1A1A1A1

Note: Award ***A1*** for each correct value.

For a table not sufficiently or clearly labelled, assume that their values are in the same order as the table in the question paper and award marks accordingly.

[6 marks]

(c) **METHOD 1**

attempts to use the product rule ***(M1)***

$f_n'(x) = -nx^n(a-x)^{n-1} + nx^{n-1}(a-x)^n$ ***A1A1***

Note: Award ***A1*** for a correct $u\dfrac{\mathrm{d}v}{\mathrm{d}x}$ and ***A1*** for a correct $v\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

EITHER

attempts to factorise $f_n'(x)$ (involving at least one of nx^{n-1} or $(a-x)^{n-1}$) ***(M1)***

$= nx^{n-1}(a-x)^{n-1}\big((a-x)-x\big)$ ***A1***

OR

attempts to express $f_n'(x)$ as the difference of two products with each product containing at least one of nx^{n-1} or $(a-x)^{n-1}$ ***(M1)***

$= (-x)\big(nx^{n-1}\big)(a-x)^{n-1} + (a-x)\big(nx^{n-1}\big)(a-x)^{n-1}$ ***A1***

THEN

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

Note: Award the final ***(M1)A1*** for obtaining any of the following forms:

$f_n'(x) = nx^n(a-x)^n\left(\dfrac{a-x-x}{x(a-x)}\right)$; $f_n'(x) = \dfrac{nx^n(a-x)^n}{x(a-x)}(a-x-x)$;

$f_n'(x) = nx^{n-1}\big((a-x)^n - x(a-x)^{n-1}\big)$;

$f_n'(x) = (a-x)^{n-1}\big(nx^{n-1}(a-x)^n - nx^n\big)$

METHOD 2

$f_n(x) = \big(x(a-x)\big)^n$ ***(M1)***

$= \big(ax-x^2\big)^n$ ***A1***

attempts to use the chain rule ***(M1)***

$f_n'(x) = n(a-2x)\big(ax-x^2\big)^{n-1}$ ***A1A1***

Note: Award ***A1*** for $n(a-2x)$ and ***A1*** for $\left(ax-x^2\right)^{n-1}$.

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

[5 marks]

(d) $x=0,\ x=\frac{a}{2},\ x=a$ ***A2***

Note: Award ***A1*** for either two correct solutions or for obtaining
$x=0,\ x=-a,\ x=-\frac{a}{2}$.
Award ***A0*** otherwise.

[2 marks]

(e) attempts to find an expression for $f_n\left(\frac{a}{2}\right)$ ***(M1)***

$f_n\left(\frac{a}{2}\right)=\left(\frac{a}{2}\right)^n\left(a-\frac{a}{2}\right)^n$

$=\left(\frac{a}{2}\right)^n\left(\frac{a}{2}\right)^n \left(=\left(\frac{a}{2}\right)^{2n}\right), \left(=\left(\left(\frac{a}{2}\right)^n\right)^2\right)$ ***A1***

EITHER

since $a\in\mathbb{R}^+$, $\left(\frac{a}{2}\right)^{2n}>0$ (for $n\in\mathbb{Z}^+, n>1$ and so $f_n\left(\frac{a}{2}\right)>0$) ***R1***

Note: Accept any logically equivalent conditions/statements on a and n.
Award ***R0*** if any conditions/statements specified involving a, n or both are incorrect.

OR

(since $a\in\mathbb{R}^+$), $\frac{a}{2}$ raised to an even power ($2n$) (or equivalent reasoning) is always positive (and so $f_n\left(\frac{a}{2}\right)>0$) ***R1***

Note: The condition $a\in\mathbb{R}^+$ is given in the question. Hence some candidates will assume $a\in\mathbb{R}^+$ and not state it. In these instances, award ***R1*** for a convincing argument.
Accept any logically equivalent conditions/statements on a and n.
Award ***R0*** if any conditions/statements specified involving a, n or both are incorrect.

THEN

so $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ is always above the horizontal axis ***AG***

Note: Do not award ***(M1)A0R1***.

[3 marks]

(f) **METHOD 1**

$f_n'\left(\frac{a}{4}\right) = n\left(\frac{a}{4}\right)^{n-1}\left(a-\frac{a}{2}\right)\left(a-\frac{a}{4}\right)^{n-1} \left(= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1}\right)$ ***A1***

EITHER

$n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1} > 0$ as $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ ***R1***

OR

$n\left(\frac{a}{4}\right)^{n-1}$, $\left(a-\frac{a}{2}\right)$ and $\left(a-\frac{a}{4}\right)^{n-1}$ are all > 0 ***R1***

Note: Do not award ***A0R1***.

Accept equivalent reasoning on correct alternative expressions for $f_n'\left(\frac{a}{4}\right)$ and accept any logically equivalent conditions/statements on a and n.

Exceptions to the above are condone $n > 1$ and condone $n > 0$.

An alternative form for $f_n'\left(\frac{a}{4}\right)$ is $(2n)(3)^{n-1}\left(\frac{a}{4}\right)^{2n-1}$.

THEN

hence $f_n'\left(\frac{a}{4}\right) > 0$ ***AG***

[2 marks]

METHOD 2

$f_n(0) = 0$ and $f_n\left(\frac{a}{2}\right) > 0$ ***A1***

(since f_n is continuous and there are no stationary points between $x = 0$ and $x = \frac{a}{2}$)

the gradient (of the curve) must be positive between $x = 0$ and $x = \frac{a}{2}$ ***R1***

Note: Do not award ***A0R1***.

hence $f_n'\left(\frac{a}{4}\right) > 0$ ***AG***

[2 marks]

(g) (i) $f_n'(-1) = n(-1)^{n-1}(a+2)(a+1)^{n-1}$

for n even:

$n(-1)^{n-1}(=-n) < 0$ (and $(a+2), (a+1)^{n-1}$ are both > 0) ***R1***

$f_n'(-1) < 0$ ***A1***

$f_n'(0) = 0$ and $f_n'\left(\frac{a}{4}\right) > 0$ (seen anywhere) ***A1***

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If n is even, then $n-1$ is odd and hence $(-1)^{n-1} < 0$ $(=-1)$.

Do not award ***R0A1***.

The second ***A1*** is independent of the other two marks.

The ***A*** marks can be awarded for correct descriptions expressed in words.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0) = 0$. The last ***A*** mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last ***A1*** can be awarded for use of a specific case (*e.g.* $n = 2$).

hence $(0,0)$ is a local minimum point ***AG***

[3 marks]

(ii) for n odd:

$n(-1)^{n-1}\,(=n)>0$, (and $(a+2),(a+1)^{n-1}$ are both >0) so $f_n'(-1)>0$ ***R1***

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If n is odd, then $n-1$ is even and hence $(-1)^{n-1}>0\ (=1)$.

$f_n'(0)=0$ and $f_n'\left(\frac{a}{4}\right)>0$ (seen anywhere) ***A1***

Note: The ***A1*** is independent of the ***R1***.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0)=0$. The last *A* mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last *A1* can be awarded for use of a specific case (*e.g.* $n=3$).

hence $(0,0)$ is a point of inflexion with zero gradient ***AG***

[2 marks]

(h) considers the parity of n ***(M1)***

Note: Award ***M1*** for stating at least one specific even value of n.

n must be even (for four solutions) ***A1***

Note: The above 2 marks are independent of the 3 marks below.

$0 < k < \left(\frac{a}{2}\right)^{2n}$ ***A1A1A1***

Note: Award ***A1*** for the correct lower endpoint, ***A1*** for the correct upper endpoint and ***A1*** for strict inequality signs.

The third ***A1*** (strict inequality signs) can only be awarded if ***A1A1*** has been awarded.

For example, award ***A1A1A0*** for $0 \le k \le \left(\frac{a}{2}\right)^{2n}$. Award ***A1A0A0*** for $k > 0$.

Award ***A1A0A0*** for $0 < k < f_n\left(\frac{a}{2}\right)$.

[5 marks]

Total [31 marks]

1. METHOD 1

recognition that $y = \int \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)***

$y = \sin\left(x - \frac{\pi}{4}\right)(+c)$ ***(A1)***

substitute both x and y values into their integrated expression including c (***M1)***

$2 = \sin\frac{\pi}{2} + c$

$c = 1$

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

METHOD 2

$\int_{2}^{y} \mathrm{d}y = \int_{\frac{3\pi}{4}}^{x} \cos\left(x - \frac{\pi}{4}\right)\mathrm{d}x$ ***(M1)(A1)***

$y - 2 = \sin\left(x - \frac{\pi}{4}\right) - \sin\frac{\pi}{2}$ ***A1***

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

5. (a) $f'(4) = 6$ ***A1***

[1 mark]

(b) $f(4) = 6 \times 4 - 1 = 23$ ***A1***

[1 mark]

(c) $h(4) = f\left(g(4)\right)$ ***(M1)***

$h(4) = f(4^2 - 3 \times 4) = f(4)$

$h(4) = 23$ ***A1***

[2 marks]

(d) attempt to use chain rule to find h' ***(M1)***

$f'\left(g\left(x\right)\right) \times g'\left(x\right)$ **OR** $\left(x^2 - 3x\right)' \times f'\left(x^2 - 3x\right)$

$h'(4) = \left(2 \times 4 - 3\right) f'\left(4^2 - 3 \times 4\right)$ ***A1***

$= 30$

$y - 23 = 30\left(x - 4\right)$ OR $y = 30x - 97$ ***A1***

[3 marks]

Total [7 marks]

8. $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x}=\frac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$\left(\mathrm{e}^{\int\frac{2}{x}\mathrm{d}x}=\mathrm{e}^{2\ln x}\right)=x^2$ ***(A1)***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=\ln 2x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right)=\ln 2x$

$x^2y=\int\ln 2x\ \mathrm{d}x$

attempt to use integration by parts ***(M1)***

$x^2y=x\ln 2x-x\left(+c\right)$ ***A1***

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{c}{x^2}$

substituting $x=\frac{1}{2}, y=4$ into an integrated equation involving c ***M1***

$4=0-2+4c$

$\Rightarrow c=\frac{3}{2}$

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{3}{2x^2}$ ***A1***

[7 marks]

10. (a) (i) valid approach to find turning point ($v' = 0$, $-\frac{b}{2a}$, average of roots) ***(M1)***

$4 - 6t = 0$ OR $-\frac{4}{2(-3)}$ OR $\frac{-\frac{2}{3}+2}{2}$

$t = \frac{2}{3}$ (s) ***A1***

(ii) attempt to integrate v ***(M1)***

$\int v \, \mathrm{d}t = \int \left(4 + 4t - 3t^2\right) \mathrm{d}t = 4t + 2t^2 - t^3 (+c)$ ***A1A1***

Note: Award ***A1*** for $4t + 2t^2$, ***A1*** for $-t^3$.

attempt to substitute their t into their solution for the integral ***(M1)***

distance $= 4\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$

$= \frac{8}{3} + \frac{8}{9} - \frac{8}{27}$ (or equivalent) ***A1***

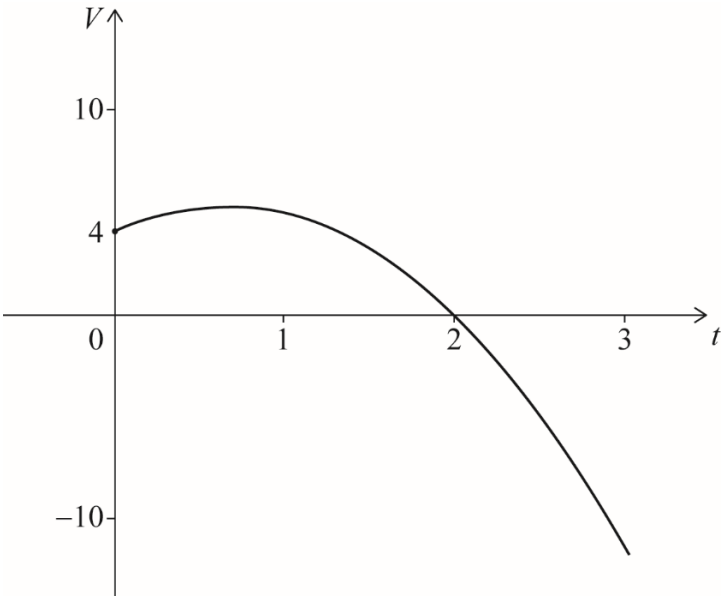
$= \frac{88}{27}$ (m) ***AG***

[7 marks]

continue…

Question 10 continued.

(b)



valid approach to solve $4+4t-3t^2=0$ (may be seen in part (a)) ***(M1)***

$(2-t)(2+3t)$ OR $\frac{-4\pm\sqrt{16+48}}{-6}$

correct x- intercept on the graph at $t=2$ ***A1***

Note: The following two ***A*** marks may only be awarded if the shape is a concave down parabola. These two marks are independent of each other and the ***(M1)***.

correct domain from 0 to 3 starting at $(0,4)$ ***A1***

Note: The 3 must be clearly indicated.

vertex in approximately correct place for $t=\frac{2}{3}$ and $v>4$ ***A1***

[4 marks]

continue…

Question 10 continued.

(c) recognising to integrate between 0 and 2, or 2 and 3 OR $\int_0^3 \left|4+4t-3t^2\right| \, \mathrm{d}t$ ***(M1)***

$\int_0^2 \left(4+4t-3t^2\right) \, \mathrm{d}t$

$=8$ ***A1***

$\int_2^3 \left(4+4t-3t^2\right) \, \mathrm{d}t$

$=-5$ ***A1***

valid approach to sum the two areas (seen anywhere) ***(M1)***

$\int_0^2 v \, \mathrm{d}t - \int_2^3 v \, \mathrm{d}t$ OR $\int_0^2 v \, \mathrm{d}t + \left|\int_2^3 v \, \mathrm{d}t\right|$

total distance travelled $=13$ (m) ***A1***

[5 marks]

Total [16 marks]

11. (a) For $n=1$

LHS: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

RHS: $\left(x^2+2(1)x+1(1-1)\right)\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

so true for $n=1$

now assume true for $n=k$; i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left[x^2+2kx+k\left(k-1\right)\right]\mathrm{e}^x$ ***M1***

Note: Do not award ***M1*** for statements such as ''let $n=k$''. Subsequent marks can still be awarded.

attempt to differentiate the RHS ***M1***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x^2+2kx+k\left(k-1\right)\right]\mathrm{e}^x\right)$

$=\left(2x+2k\right)\mathrm{e}^x+\left(x^2+2kx+k\left(k-1\right)\right)\mathrm{e}^x$ ***A1***

$=\left[x^2+2\left(k+1\right)x+k\left(k+1\right)\right]\mathrm{e}^x$ ***A1***

so true for $n=k$ implies true for $n=k+1$

therefore $n=1$ true and $n=k$ true $\Rightarrow$ $n=k+1$ true

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Award **R1** only if three of the previous four marks have been awarded

[7 marks]

continue…

Question 11 continued.

(b) **METHOD 1**

attempt to use $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n\left(n-1\right)\right]\mathrm{e}^x$ ***(M1)***

Note: For $x=0$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)_{|x=0}=n\left(n-1\right)$ may be seen.

$f\left(0\right)=0,\ f'\left(0\right)=0,\ f''\left(0\right)=2,\ f'''\left(0\right)=6,\ f^{(4)}\left(0\right)=12$

use of $f\left(x\right)=f\left(0\right)+xf'\left(0\right)+\frac{x^2}{2!}f''\left(0\right)+\frac{x^3}{3!}f'''\left(0\right)+\frac{x^4}{4!}f^{(4)}\left(0\right)+\ldots$ ***(M1)***

$\Rightarrow f\left(x\right)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

METHOD 2

'$x^2\times$ Maclaurin series of e^x' ***(M1)***

$x^2\left(1+x+\frac{x^2}{2!}+\ldots\right)$ ***(A1)***

$\Rightarrow f\left(x\right)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

continue…

Question 11 continued.

(c) **METHOD 1**

attempt to substitute $x^2e^x \approx x^2 + x^3 + \frac{1}{2}x^4$ into $\frac{\left(x^2e^x - x^2\right)^3}{x^9}$ ***M1***

$$\frac{\left(x^2e^x - x^2\right)^3}{x^9} \approx \frac{\left(x^2 + x^3 + \frac{1}{2}x^4(+\ldots) - x^2\right)^3}{x^9}$$ ***(A1)***

EITHER

$$= \frac{\left(x^3 + \frac{1}{2}x^4 + \ldots\right)^3}{x^9}$$ ***A1***

$$= \frac{x^9(+\text{higher order terms})}{x^9}$$

OR

$$\left(\frac{x^3 + \frac{1}{2}x^4(+\ldots)}{x^3}\right)^3$$ ***A1***

$$\left(1 + \frac{1}{2}x(+\ldots)\right)^3$$

THEN

$= 1$ (+ higher order terms)

So $\lim_{x \to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = 1$ ***A1***

[4 marks]

continue…

Question 11 continued.

METHOD 2

$$\lim_{x\to 0}\left[\frac{\left(x^2\mathrm{e}^x - x^2\right)^3}{x^9}\right] = \lim_{x\to 0}\left(\frac{x^2\mathrm{e}^x - x^2}{x^3}\right)^3$$ ***M1***

$$= \lim_{x\to 0}\left(\frac{\mathrm{e}^x - 1}{x}\right)^3$$ ***(A1)***

attempt to use L'Hôpital's rule ***M1***

$$= \lim_{x\to 0}\left(\frac{\mathrm{e}^x - 0}{1}\right)^3$$

$$= \left[\lim_{x\to 0}\mathrm{e}^x\right]^3$$

$$= 1$$ ***A1***

[4 marks]

Total [14 marks]

$\left(\quad\right)^3$

8. (a) **METHOD 1**

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$u = xy$, $v = \ln(xy)$, $\frac{du}{dx} = x\frac{dy}{dx} + y$, $\frac{dv}{dx} = \left(x\frac{dy}{dx} + y\right)\frac{1}{xy}$

$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$ ***A1***

Note: Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining $\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + x\frac{dy}{dx}\ln(xy) + y\ln(xy)\right]$.

$\frac{dy}{dx} = 1 - \left[\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))$ ***A1***

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 2

$y = x - xy\ln x - xy\ln y$

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$\frac{dy}{dx} = 1 - \left(\frac{xy}{x} + \left(x\frac{dy}{dx} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\ln y\right)$ ***A1***

or equivalent to the above, for example

$\frac{dy}{dx} = 1 - \left(x\ln x\frac{dy}{dx} + (1 + \ln x)y\right) - \left(y\ln y + x\left(\ln y\frac{dy}{dx} + \frac{dy}{dx}\right)\right)$

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$ ***A1***

or equivalent to the above, for example

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 3

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$u = x\ln(xy),\ v = y,\ \dfrac{du}{dx} = \ln(xy) + \left(x\dfrac{dy}{dx} + y\right)\dfrac{x}{xy},\ \dfrac{dv}{dx} = \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = 1 - \left(x\dfrac{dy}{dx}\ln(xy) + y\ln(xy) + \dfrac{xy}{xy}\left(x\dfrac{dy}{dx} + y\right)\right)$ ***A1***

$\dfrac{dy}{dx} = 1 - x\dfrac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$ ***A1***

$\dfrac{dy}{dx} + \left(x\dfrac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 4

lets $w = xy$ and attempts to find $\dfrac{dy}{dx}$ where $y = x - w\ln w$ ***M1***

$\dfrac{dy}{dx} = 1 - \left(\dfrac{dw}{dx} + \dfrac{dw}{dx}\ln w\right)\ \left(= 1 - \dfrac{dw}{dx}(1 + \ln w)\right)$ ***A1***

$\dfrac{dw}{dx} = x\dfrac{dy}{dx} + y$ ***A1***

$\dfrac{dy}{dx} = 1 - \left(x\dfrac{dy}{dx} + y + \left(x\dfrac{dy}{dx} + y\right)\ln(xy)\right)\ \left(= 1 - \left(x\dfrac{dy}{dx} + y\right)(1 + \ln(xy))\right)$

$\dfrac{dy}{dx} + \left(x\dfrac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

[3 marks]

(b) **METHOD 1**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$ ***A1***

substitutes $x=1$ and their non-zero value of y into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(xy))=1$ ***(M1)***

$2\frac{\mathrm{d}y}{\mathrm{d}x}=0\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}=0\right)$ ***A1***

equation of the tangent is $y=1$ ***A1***

METHOD 2

substitutes $x=1$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(xy))=1$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$

EITHER

correctly substitutes $\ln y=\frac{1-y}{y}$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\frac{1}{y}\right)=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=0\ (y=1)$ ***A1***

OR

correctly substitutes $y+y\ln y=1$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}(2+\ln y)=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=0\ (y=1)$ ***A1***

THEN

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$

equation of the tangent is $y=1$ ***A1***

[5 marks]

Total [8 marks]

Section B

2. (a) (i) **METHOD 1**

$\frac{dy}{dt} = y$

$\int \frac{dy}{y} = \int dt$ ***(M1)***

$\ln y = t + c$ **OR** $\ln|y| = t + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ and ***A1*** for t and c.

$y = Ae^t$ ***AG***

METHOD 2

rearranging to $\frac{dy}{dt} - y = 0$ AND multiplying by integrating factor e^{-t} ***M1***

$ye^{-t} = A$ ***A1A1***

$y = Ae^t$ ***AG***

[3 marks]

(ii) substituting $y = Ae^t$ into differential equation in x ***M1***

$\frac{dx}{dt} = x - Ae^t$

$\frac{dx}{dt} - x = -Ae^t$ ***AG***

[1 mark]

(iii) integrating factor (IF) is $e^{\int -1 dt}$ ***(M1)***

$= e^{-t}$ ***(A1)***

$e^{-t}\frac{dx}{dt} - xe^{-t} = -A$

$xe^{-t} = -At + D$ (***A1)***

$x = (-At + D)e^t$ ***A1***

Note: The first constant must be A**,** and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[4 marks]

(b) (i) $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

EITHER

$= -x + y + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$= \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

OR

$= -x + y + (-x + y)$ ***(M1)***

$= 2(-x + y)$ ***A1***

THEN

$= 2\frac{\mathrm{d}y}{\mathrm{d}t}$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}Y}{\mathrm{d}t} = 2Y$ ***A1***

$\int \frac{\mathrm{d}Y}{Y} = \int 2\mathrm{d}t$ ***M1***

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ ***A1***

$Y = B\mathrm{e}^{2t}$ ***AG***

[3 marks]

(iii) $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$

$y = \int B\mathrm{e}^{2t}\,\mathrm{d}t$ ***M1***

$y = \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

Note: The first constant must be B, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[2 marks]

(iv) **METHOD 1**

substituting $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$ and their (iii) into $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ ***M1(M1)***

$B\mathrm{e}^{2t} = -x + \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

Note: Follow through from incorrect part (iii) cannot be awarded if it does not lead to the ***AG*.**

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -\frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}(x\mathrm{e}^{-t})}{\mathrm{d}t} = -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}$ ***M1***

$x\mathrm{e}^{-t} = \int -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}\ \mathrm{d}t$

$x\mathrm{e}^{-t} = -\frac{B}{2}\mathrm{e}^{t} + C\mathrm{e}^{-t} + D$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C + D\mathrm{e}^{t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y \Rightarrow B\mathrm{e}^{2t} = \frac{B}{2}\mathrm{e}^{2t} - C - D\mathrm{e}^{t} + \frac{B}{2}\mathrm{e}^{2t} + C \Rightarrow D = 0$ ***M1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

[3 marks]

(c) (i) $\frac{\mathrm{d}y}{\mathrm{d}t} = -4x + y$

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ seen anywhere ***M1***

METHOD 1

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4(x - y) + \frac{\mathrm{d}y}{\mathrm{d}t}$

attempt to eliminate x ***M1***

$= -4\left(\frac{1}{4}\left(y - \frac{\mathrm{d}y}{\mathrm{d}t}\right) - y\right) + \frac{\mathrm{d}y}{\mathrm{d}t}$

$= 2\frac{\mathrm{d}y}{\mathrm{d}t} + 3y$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$ ***AG***

METHOD 2

rewriting LHS in terms of x and y ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ ***A1***

$= 0$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda\mathrm{e}^{\lambda t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F\lambda^2\mathrm{e}^{\lambda t}$ ***(A1)***

$F\lambda^2\mathrm{e}^{\lambda t} - 2F\lambda\mathrm{e}^{\lambda t} - 3F\mathrm{e}^{\lambda t} = 0$ ***(M1)***

$\lambda^2 - 2\lambda - 3 = 0$ (since $\mathrm{e}^{\lambda t} \neq 0$) ***A1***

λ_1 and λ_2 are 3 and -1 (either order) ***A1***

[4 marks]

(iii) **METHOD 1**

$y = F\mathrm{e}^{3t} + G\mathrm{e}^{-t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = 3F\mathrm{e}^{3t} - G\mathrm{e}^{-t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t}$ ***(A1)(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t} - 2\left(3F\mathrm{e}^{3t} - G\mathrm{e}^{-t}\right) - 3\left(F\mathrm{e}^{3t} + G\mathrm{e}^{-t}\right)$ ***M1***

$= 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t} - 6F\mathrm{e}^{3t} + 2G\mathrm{e}^{-t} - 3F\mathrm{e}^{3t} - 3G\mathrm{e}^{-t}$ ***A1***

$= 0$ ***AG***

METHOD 2

$y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda_1\mathrm{e}^{\lambda_1 t} + G\lambda_2\mathrm{e}^{\lambda_2 t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F\lambda_1^2\mathrm{e}^{\lambda_1 t} + G\lambda_2^2\mathrm{e}^{\lambda_2 t}$ ***(A1)(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = F\lambda_1^2\mathrm{e}^{\lambda_1 t} + G\lambda_2^2\mathrm{e}^{\lambda_2 t} - 2\left(F\lambda_1\mathrm{e}^{\lambda_1 t} + G\lambda_2\mathrm{e}^{\lambda_2 t}\right) - 3\left(F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}\right)$ ***M1***

$= F\mathrm{e}^{\lambda_1 t}\left(\lambda^2 - 2\lambda - 3\right) + G\mathrm{e}^{\lambda_2 t}\left(\lambda^2 - 2\lambda - 3\right)$ ***A1***

$= 0$ ***AG***

[4 marks]

Total [30 marks]

3. attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}}\mathrm{d}x = \int \frac{2}{\sqrt{u}}\mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

Total [5 marks]

10. (a) attempt to use quotient rule ***(M1)***

correct substitution into quotient rule

$f'(x) = \dfrac{5kx\left(\dfrac{1}{5x}\right) - k\ln 5x}{(kx)^2}$ (or equivalent) ***A1***

$= \dfrac{k - k\ln 5x}{k^2x^2}, (k \in \mathbb{R}^+)$ ***A1***

$= \dfrac{1 - \ln 5x}{kx^2}$ ***AG***

[3 marks]

(b) $f'(x) = 0$ ***M1***

$\dfrac{1 - \ln 5x}{kx^2} = 0$

$\ln 5x = 1$ ***(A1)***

$x = \dfrac{e}{5}$ ***A1***

[3 marks]

(c) $f''(x) = 0$ ***M1***

$\dfrac{2\ln 5x - 3}{kx^3} = 0$

$\ln 5x = \dfrac{3}{2}$ ***A1***

$5x = e^{\frac{3}{2}}$ ***A1***

so the point of inflexion occurs at $x = \dfrac{1}{5}e^{\frac{3}{2}}$ ***AG***

[3 marks]

continued…

Question 10 continued

(d) attempt to integrate ***(M1)***

$u = \ln 5x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{\ln 5x}{kx}\mathrm{d}x = \frac{1}{k}\int u\ \mathrm{d}u$ ***(A1)***

EITHER

$= \frac{u^2}{2k}$ ***A1***

so $\frac{1}{k}\int_1^{\frac{3}{2}} u\ \mathrm{d}u = \left[\frac{u^2}{2k}\right]_1^{\frac{3}{2}}$ ***A1***

OR

$= \frac{(\ln 5x)^2}{2k}$ ***A1***

so $\int_{\frac{1}{5}}^{\frac{1}{5}\mathrm{e}^{\frac{3}{2}}} \frac{\ln 5x}{kx}\mathrm{d}x = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{1}{5}}^{\frac{1}{5}\mathrm{e}^{\frac{3}{2}}}$ ***A1***

THEN

$= \frac{1}{2k}\left(\frac{9}{4} - 1\right)$

$= \frac{5}{8k}$ ***A1***

setting **their** expression for area equal to 3 ***M1***

$\frac{5}{8k} = 3$

$k = \frac{5}{24}$ ***A1***

[7 marks]

Total [16 marks]

12. (a) attempting to use the chain rule to find the first derivative ***M1***

$f'(x) = (\cos x)\mathrm{e}^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x) = \mathrm{e}^{\sin x}\left(\cos^2 x - \sin x\right)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ ***M1***

$f(0) = 1;\ f'(0) = (\cos 0)\mathrm{e}^{\sin 0} = 1;\ f''(0) = \mathrm{e}^{\sin 0}\left(\cos^2 0 - \sin 0\right) = 1$ ***A1***

substitution into the Maclaurin formula $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$ ***M1***

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1 + x + \frac{x^2}{2}$ ***A1***

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ ***M1***

$f'''(x) = (\cos x)\mathrm{e}^{\sin x}\left(\cos^2 x - \sin x\right) - (\cos x)\mathrm{e}^{\sin x}\left(2\sin x + 1\right)$ (or equivalent) ***A2***

substituting $x = 0$ into **their** $f'''(x)$ ***M1***

$f'''(0) = 1(1-0) - 1(0+1) = 0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$\mathrm{e}^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \ldots$

substituting Maclaurin series for $\sin x$ ***M1***

$\mathrm{e}^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^3}{3!} + \ldots$ ***A1***

coefficient of x^3 is $-\frac{1}{3!} + \frac{1}{3!} = 0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

continued…

Question 12 continued

(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + ...$ ***A1***

substituting $\left(e^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(e^{3x} - 1\right) = \left(e^{3x} - 1\right) - \frac{\left(e^{3x} - 1\right)^3}{3} + \frac{\left(e^{3x} - 1\right)^5}{5} - ...$$

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right)^3}{3} + ...$$ ***A1***

selecting correct terms from above ***M1***

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$$

$$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + ...}{3x + \frac{9x^2}{2} + ...}$$ ***A1***

$$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + ...}{3 + \frac{9x}{2} + ...}$$

$$= \frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x \to 0} \frac{(\cos x)e^{\sin x}}{\frac{3e^{3x}}{1 + \left(e^{3x} - 1\right)^2}}$$ (or equivalent) ***A1***

$$= \frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]

5. (a) use of a graph to find the coordinates of the local minimum ***(M1)***

$s = -16.513...$ ***(A1)***

maximum distance is $16.5\,\text{cm}$ (to the left of O) ***A1***

[3 marks]

(b) attempt to find time when particle changes direction *eg* considering the first maximum on the graph of s or the first t – intercept on the graph of s'. ***(M1)***

$t = 1.51986...$ ***(A1)***

attempt to find the gradient of s' for **their** value of t, $s''(1.51986...)$ ***(M1)***

$= -8.92\ \ (\text{cm/s}^2)$ ***A1***

[4 marks]

Total [7 marks]

11. (a) ***METHOD 1***

using $I(t) = e^{\int P(t)dt}$ ***M1***

$e^{\int \frac{1}{t+1}dt}$

$= e^{\ln(t+1)}$ ***A1***

$= t+1$ ***AG***

METHOD 2

attempting product rule differentiation on $\frac{d}{dt}(x(t+1))$ ***M1***

$\frac{d}{dt}(x(t+1)) = \frac{dx}{dt}(t+1) + x$

$= (t+1)\left(\frac{dx}{dt} + \frac{x}{t+1}\right)$ ***A1***

so $t+1$ is an integrating factor for this differential equation ***AG***

[2 marks]

continued…

Question 11 continued

(b) attempting to multiply through by $(t+1)$ and rearrange to give ***(M1)***

$(t+1)\dfrac{\mathrm{d}x}{\mathrm{d}t}+x=10(t+1)\mathrm{e}^{-\frac{t}{4}}$ ***A1***

$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right)=10(t+1)\mathrm{e}^{-\frac{t}{4}}$

$x(t+1)=\int 10(t+1)\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t$ ***A1***

attempting to integrate the RHS by parts ***M1***

$=-40(t+1)\mathrm{e}^{-\frac{t}{4}}+40\int \mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t$

$=-40(t+1)\mathrm{e}^{-\frac{t}{4}}-160\mathrm{e}^{-\frac{t}{4}}+C$ ***A1***

Note: Condone the absence of C .

EITHER

substituting $t=0, x=0 \Rightarrow C=200$ ***M1***

$x=\dfrac{-40(t+1)\mathrm{e}^{-\frac{t}{4}}-160\mathrm{e}^{-\frac{t}{4}}+200}{t+1}$ ***A1***

using $-40\mathrm{e}^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)\mathrm{e}^{-\frac{t}{4}}$ and $-160\mathrm{e}^{-\frac{t}{4}}$ ***M1***

OR

using $-40\mathrm{e}^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)\mathrm{e}^{-\frac{t}{4}}$ and $-160\mathrm{e}^{-\frac{t}{4}}$ giving

$x(t+1)=-40\mathrm{e}^{-\frac{t}{4}}(t+5)+C$ (or equivalent) ***M1A1***

substituting $t=0,\ x=0 \Rightarrow C=200$ ***M1***

THEN

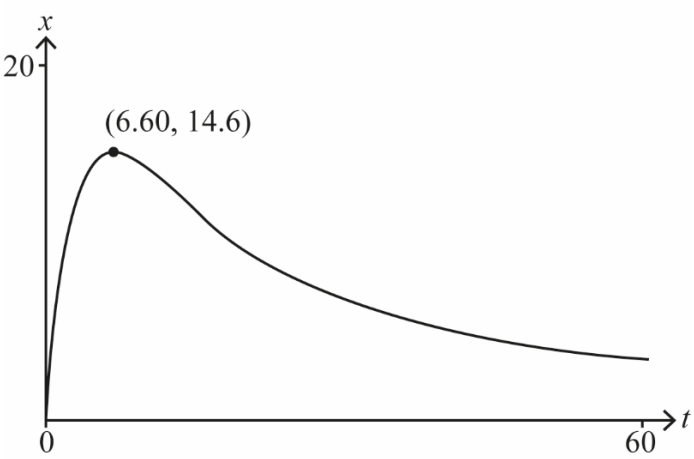
$x(t)=\dfrac{200-40\mathrm{e}^{-\frac{t}{4}}(t+5)}{t+1}$ ***AG***

[8 marks]

continued…

Question 11 continued

(c)



graph starts at the origin and has a local maximum (coordinates not required) ***A1***
sketched for $0 \leq t \leq 60$ ***A1***
correct concavity for $0 \leq t \leq 60$ ***A1***
maximum amount of salt is 14.6 (grams) at $t = 6.60$ (minutes) ***A1A1***

[5 marks]

(d) using an appropriate graph or equation (first or second derivative) ***M1***
amount of salt is decreasing most rapidly at $t = 12.9$ (minutes) ***A1***

[2 marks]

(e) **EITHER**

attempting to form an integral representing the amount of salt that left the tank ***M1***

$$\int_0^{60} \frac{x(t)}{t+1}\,\mathrm{d}t$$

$$\int_0^{60} \frac{200 - 40\mathrm{e}^{-\frac{t}{4}}(t+5)}{(t+1)^2}\,\mathrm{d}t$$ ***A1***

OR

attempting to form an integral representing the amount of salt that entered the tank minus the amount of salt in the tank at $t = 60$ (minutes) ***M1***

amount of salt that left the tank is $\int_0^{60} 10\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t - x(60)$ ***A1***

THEN

$= 36.7$ (grams) ***A2***

[4 marks]

Total [21 marks]

1. $\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = \int \left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}x$ ***(A1)***

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = 3x-10x^{\frac{1}{2}}\ (+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times 3-(3-10)$

$=4$ ***A1***

[5 marks]

4. evidence of using product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x-1)\times\left(k\mathrm{e}^{kx}\right)+2\times\mathrm{e}^{kx} \quad \left(=\mathrm{e}^{kx}(2kx-k+2)\right)$ ***A1***

correct working for one of (seen anywhere) ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1 \Rightarrow k\mathrm{e}^{k}+2\mathrm{e}^{k}$

OR

slope of tangent is $5\mathrm{e}^{k}$

their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1$ equals the slope of $y=5\mathrm{e}^{k}x \left(=5\mathrm{e}^{k}\right)$ (seen anywhere) ***(M1)***

$k\mathrm{e}^{k}+2\mathrm{e}^{k}=5\mathrm{e}^{k}$

$k=3$ ***A1***

[5 marks]

12. (a) **METHOD 1**

recognition of both known series ***(M1)***

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \ldots$$ ***(A1)***

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$ ***A1***

[4 marks]

METHOD 2

$f(x) = e^x \sin x$

$f'(x) = e^x \cos x + e^x \sin x$ ***A1***

$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \left(= 2e^x \cos x\right)$

$f'''(x) = 2e^x \cos x - 2e^x \sin x$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x(\cos x - \sin x)$ ***A1***

substitute $x = 0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$$e^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \ldots$$

$$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$ ***A1***

[4 marks]

continued…

Question 12 continued

(b) $e^{x^2}\sin\left(x^2\right) = x^2 + x^4 + \frac{1}{3}x^6 + ...$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$$\int_0^1 e^{x^2}\sin(x^2)\,dx \approx \int_0^1 \left(x^2 + x^4 + \frac{1}{3}x^6 \right) dx$$

Note: Condone absence of limits up to this stage.

$$= \left[\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{21} \right]_0^1$$ ***A1***

$$= \frac{61}{105}$$ ***A1***

[4 marks]

continued…

Question 12 continued

(c) (i) attempt to use product rule at least once ***M1***

$g'(x) = e^x \cos x - e^x \sin x$ ***A1***

$g''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \left(= -2e^x \sin x\right)$ ***A1***

EITHER

$2\left(g'(x) - g(x)\right) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right) = -2e^x \sin x$ ***A1***

OR

$g''(x) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right)$ ***A1***

THEN

$g''(x) = 2\left(g'(x) - g(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-2e^x \sin x$.

(ii) $g'''(x) = 2\left(g''(x) - g'(x)\right)$ ***A1***

$g^{(4)}(x) = 2\left(g'''(x) - g''(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-4e^x \cos x$.

[5 marks]

continued…

Question 12 continued

(d) attempt to substitute $x = 0$ into a derivative ***(M1)***

$g(0) = 1,\ g'(0) = 1,\ g''(0) = 0$ ***A1***

$g'''(0) = -2,\ g^{(4)}(0) = -4$ ***(A1)***

attempt to substitute into Maclaurin formula ***(M1)***

$g(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 + ... \left(= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + ... \right)$ ***A1***

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) **METHOD 1**

$\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3} = \lim_{x \to 0} \frac{\left(1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + ...\right) - 1 - x}{x^3}$ ***M1***

$= \lim_{x \to 0} \left(-\frac{1}{3} - \frac{1}{6}x + ... \right)$ ***(A1)***

$= -\frac{1}{3}$ ***A1***

Note: Condone the omission of $+...$ in their working.

continued…

Question 12 continued

METHOD 2

$\lim_{x\to 0}\frac{e^x\cos x-1-x}{x^3}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$=\lim_{x\to 0}\frac{e^x\cos x-e^x\sin x-1}{3x^2}\left(=\lim_{x\to 0}\frac{g'(x)-1}{3x^2}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x}{6x}\left(=\lim_{x\to 0}\frac{g''(x)}{6x}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x-2e^x\cos x}{6}\left(=\lim_{x\to 0}\frac{g'''(x)}{6}\right)$ ***A1***

$=-\frac{1}{3}$ ***A1***

[3 marks]

Total [21 marks]

6. (a) attempts to replace x with $-x$ ***M1***

$f(-x) = -x\sqrt{1-(-x)^2}$

$= -x\sqrt{1-(-x)^2}\ (= -f(x))$ ***A1***

Note: Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal.

Note: Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by $180°$ about the origin **or** the graph is invariant after a reflection in the y- axis and then in the x- axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempts both product rule and chain rule differentiation to find $f'(x)$ ***M1***

$f'(x) = x \times \frac{1}{2} \times (-2x) \times (1-x^2)^{-\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} \times 1 \left(= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right)$ ***A1***

$= \frac{1-2x^2}{\sqrt{1-x^2}}$

sets their $f'(x) = 0$ ***M1***

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to evaluate $f(x)$ at least at one of their $f'(x) = 0$ roots.

$a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ***A1***

Note: Award ***A1*** for $-\frac{1}{2} \le y \le \frac{1}{2}$.

[6 marks]

Total [8 marks]

7. METHOD 1

$u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \ \mathrm{d}x$ ***(A1)***

attempts to express the integral in terms of u ***M1***

$\int_1^2 u^{n-1} \mathrm{d}u$ ***A1***

$= \frac{1}{n}\left[u^n\right]_1^2 \quad \left(= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}\right)$ ***A1***

Note: Condone the absence of or incorrect limits up to this point.

$= \frac{2^n - 1^n}{n}$ ***M1***

$= \frac{2^n - 1}{n}$ ***A1***

Note: Award ***M1*** for correct substitution of **their** limits for u into their antiderivative for u (or given limits for x into their antiderivative for x).

METHOD 2

$\int \sec^n x \tan x \ \mathrm{d}x = \int \sec^{n-1} x \sec x \tan x \ \mathrm{d}x$ ***(A1)***

applies integration by inspection ***(M1)***

$= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}$ ***A2***

Note: Award ***A2*** if the limits are not stated.

$= \frac{1}{n}\left(\sec^n \frac{\pi}{3} - \sec^n 0\right)$ ***M1***

Note: Award ***M1*** for correct substitution into their antiderivative.

$= \frac{2^n - 1}{n}$ ***A1***

[6 marks]

4. (a) recognizing at rest $v = 0$ ***(M1)***

$t = 3.34692...$

$t = 3.35$ (seconds) ***A1***

Note: Award ***(M1)A0*** for any other solution to $v = 0$ eg $t = -0.205$ or $t = 6.08$.

[2 marks]

(b) recognizing particle changes direction when $v = 0$ OR when $t = 3.34692...$ ***(M1)***

$a = -4.71439...$

$a = -4.71\,(\text{ms}^{-2})$ ***A2***

[3 marks]

(c) distance travelled = $\int_0^6 |v| \, dt$ OR

$$\int_0^{3.34...} \left(e^{\sin(t)} + 4\sin(t)\right) dt - \int_{3.34...}^6 \left(e^{\sin(t)} + 4\sin(t)\right) dt \ (= 14.3104... + 6.44300...)$$ ***(A1)***

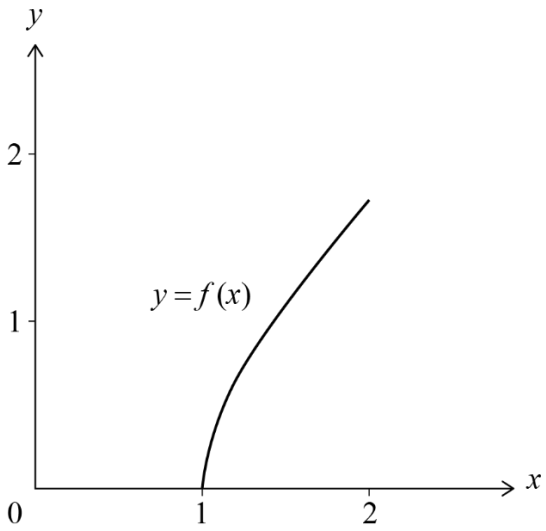
$= 20.7534...$

= 20.8 (metres) ***A1***

[2 marks]

Total [7 marks]

10. (a)



correct shape (concave down) within the given domain $1 \leq x \leq 2$ ***A1***

$(1,0)$ and $\left(2,\sqrt{3}\right)\left(=(2,1.73)\right)$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

continued…

Question 10 continued

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $\left[0, \sqrt{3}\right] \left(= [0, 1.73]\right)$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1, 2]$ ***A1***

[5 marks]

continued…

Question 10 continued

(c) (i) attempt to substitute $x = \sqrt{y^2 + 1}$ into the correct volume formula ***(M1)***

$V = \pi \int_0^h \left(\sqrt{y^2 + 1}\right)^2 \, \mathrm{d}y \left(= \pi \int_0^h \left(y^2 + 1\right) \, \mathrm{d}y \right)$ ***A1***

$= \pi \left[\frac{1}{3} y^3 + y \right]_0^h$ ***A1***

$= \pi \left(\frac{1}{3} h^3 + h \right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g. $\pi \int_0^h \left(\sqrt{x^2 + 1}\right)^2 \mathrm{d}x$

(ii) attempt to substitute $h = \sqrt{3} \ (= 1.732...)$ into V ***(M1)***

$V = 10.8828...$

$V = 10.9$ (m³) $\left(= 2\sqrt{3}\pi\right)$ (m³) ***A1***

[5 marks]

continued…

Question 10 continued

(d) **METHOD 1**

$\text{time} = \frac{10.8828...}{0.4}\left(= \frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$= 27.207...$

$= 27.2\left(= 5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

continued…

Question 10 continued

(e) attempt to find the height of the tank when $V = 5.4414... \left(= \sqrt{3}\pi\right)$ ***(M1)***

$$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414... \left(= \sqrt{3}\pi\right)$$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi(h^2+1)} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ OR $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.4$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.4}{\pi\left(1.1818...^2+1\right)} = 0.053124...$$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

Total [20 marks]

12. (a) attempt to use Euler's method ***(M1)***

$x_{n+1} = x_n + 0.1;\ \ y_{n+1} = y_n + 0.1 \times \frac{\mathrm{d}y}{\mathrm{d}x}$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{x^2}$

correct intermediate y-values ***(A1)(A1)***

$3.7, 4.63140..., 5.92098..., 7.79542...$

Note: ***A1*** for any two correct y-values seen

$y = 10.6958...$

$y = 10.7$ ***A1***

Note: For the final ***A1,*** the value 10.7 must be the last value in a table or a list, or be given as a final answer, not just embedded in a table which has further lines.

[4 marks]

(b) $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

replacing y with vx and $\frac{\mathrm{d}y}{\mathrm{d}x}$ with $v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x^2 \Rightarrow x^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = v^2x^2 - 2x^2$ ***A1***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - 2$ (since $x > 0$)

$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - v - 2$ ***AG***

[3 marks]

continued…

Question 12 continued

(c) (i) attempt to separate variables v and x ***(M1)***

$$\int \frac{\mathrm{d}v}{v^2 - v - 2} = \int \frac{\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}v}{(v-2)(v+1)} = \int \frac{\mathrm{d}x}{x}$$ ***(A1)***

attempt to express in partial fraction form ***M1***

$$\frac{1}{(v-2)(v+1)} \equiv \frac{A}{v-2} + \frac{B}{v+1}$$

$$\frac{1}{(v-2)(v+1)} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)$$ ***A1***

$$\frac{1}{3}\int\left(\frac{1}{v-2} - \frac{1}{v+1}\right)\mathrm{d}v = \int \frac{\mathrm{d}x}{x}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| (+c)$$ ***A1***

Note: Condone absence of modulus signs throughout.

EITHER

attempt to find c using $x = 1, y = 3, v = 3$ ***M1***

$$c = \frac{1}{3}\ln\frac{1}{4}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| + \frac{1}{3}\ln\frac{1}{4}$$

expressing both sides as a single logarithm ***(M1)***

$$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(\frac{|x|^3}{4}\right)$$

continued…

Question 12 continued

OR

expressing both sides as a single logarithm ***(M1)***

$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(A|x|^3\right)$

attempt to find A using $x=1, y=3, v=3$ ***M1***

$A = \frac{1}{4}$

THEN

$\left|\frac{v-2}{v+1}\right| = \frac{1}{4}x^3$ (since $x>0$)

substitute $v = \frac{y}{x}$ (seen anywhere) ***M1***

$\frac{\frac{y}{x}-2}{\frac{y}{x}+1} = \frac{1}{4}x^3$ (since $y>2x$)

$\left(\Rightarrow \frac{y-2x}{y+x} = \frac{1}{4}x^3\right)$

attempt to make y the subject ***M1***

$y - \frac{x^3y}{4} = 2x + \frac{x^4}{4}$ ***A1***

$y = \frac{8x+x^4}{4-x^3}$ ***AG***

continued…

Question 12 continued

(ii) actual value at $y(1.5) = 27.3$ ***A1***

(iii) gradient changes rapidly (during the interval considered) OR

the curve has a vertical asymptote at $x = \sqrt[3]{4} \left(= 1.5874...\right)$ ***R1***

[12 marks]

Total [19 marks]

2. METHOD 1

recognises that $g(x) = \int \left(3x^2 + 5e^x\right) dx$ ***(M1)***

$g(x) = x^3 + 5e^x (+C)$ ***(A1)(A1)***

Note: Award ***A1*** for each integrated term.

substitutes $x = 0$ and $y = 4$ into their integrated function (must involve $+C$) ***(M1)***

$4 = 0 + 5 + C \Rightarrow C = -1$

$g(x) = x^3 + 5e^x - 1$ ***A1***

METHOD 2

attempts to write both sides in the form of a definite integral ***(M1)***

$\int_0^x g'(t)\, dt = \int_0^x \left(3t^2 + 5e^t\right) dt$ ***(A1)***

$g(x) - 4 = x^3 + 5e^x - 5e^0$ ***(A1)(A1)***

Note: Award ***A1*** for $g(x) - 4$ and ***A1*** for $x^3 + 5e^x - 5e^0$.

$g(x) = x^3 + 5e^x - 1$ ***A1***

[5 marks]

5. (a) recognises the need to find the value of t when $v = 0$ ***(M1)***

$t = 1.5707... \left(= \frac{\pi}{2} \right)$

$t = 1.57 \left(= \frac{\pi}{2} \right)$ (s) ***A1***

[2 marks]

(b) recognises that $a(t) = v'(t)$ ***(M1)***

$t_1 = 2.2627...$, $t_2 = 2.9573...$

$t_1 = 2.26$, $t_2 = 2.96$ (s) ***A1A1***

Note: Award ***M1A1A0*** if the two correct answers are given with additional values outside $0 \le t \le 3$.

[3 marks]

(c) speed is greatest at $t = 3$ ***(A1)***

$a = -1.8377...$

$a = -1.84$ (m s^{-2}) ***A1***

[2 marks]

Total [7 marks]

6. attempts to express x^2 in terms of y ***(M1)***

$V = \pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y$ ***A1***

Note: Correct limits are required.

Attempts to solve $\pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y = 285$ for h ***(M1)***

Note: Award ***M1*** for attempting to solve $36\pi\left(\frac{h^3}{48}-\frac{h^2}{4}+\frac{8}{3}\right)=285$ or equivalent for h.

$h = 0.7926...$

$h = 0.793$ (cm) ***A2***

[5 marks]

7. (a) (as $\lim_{x\to 0} x^2 = 0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$\Rightarrow \lim_{x\to 0}\left(\arctan\left(\cos x\right) - k\right) = 0$ ***M1***

$\arctan 1 - k = 0 \ \left(k = \arctan 1\right)$ ***A1***

so $k = \frac{\pi}{4}$ ***AG***

Note: Award ***M1A0*** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

continued…

Question 7 continued

(b) $\lim\limits_{x\to 0}\dfrac{\arctan(\cos x)-\frac{\pi}{4}}{x^2}\left(=\dfrac{0}{0}\right)$

$=\lim\limits_{x\to 0}\dfrac{\frac{-\sin x}{1+\cos^2 x}}{2x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

recognises to apply l'Hôpital's rule again ***(M1)***

$=\lim\limits_{x\to 0}\dfrac{\frac{-\sin x}{1+\cos^2 x}}{2x}\left(=\dfrac{0}{0}\right)$

Note: Award ***M0*** if their limit is not the indeterminate form $\frac{0}{0}$.

EITHER

$=\lim\limits_{x\to 0}\dfrac{\frac{-\cos x(1+\cos^2 x)-2\sin^2 x\cos x}{(1+\cos^2 x)^2}}{2}$ ***A1A1***

Note: Award ***A1*** for a correct first term in the numerator and ***A1*** for a correct second term in the numerator.

OR

$\lim\limits_{x\to 0}\dfrac{-\cos x}{2(1+\cos^2 x)-4x\sin x\cos x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

THEN

substitutes $x=0$ into the correct expression to evaluate the limit ***A1***

Note: The final ***A1*** is dependent on all previous marks.

$=-\dfrac{1}{4}$ ***AG***

[6 marks]

Total [8 marks]

10. (a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t + 6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t + 6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t + 6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

continued…

Question 10 continued

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required ***(M1)***

attempts to solve $h_A'(t) = h_B'(t)$ for t ***(M1)***

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ ***(A1)***

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ ***(A1)***

attempts to calculate the total amount of time ***(M1)***

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) ***A1***

[6 marks]

Total [15 marks]

12. (a) rate of growth (change) of the (marsupial) population (with respect to time) ***A1***

[1 mark]

Note: Do not accept growth (change) in the (marsupials) population per year.

(b) **METHOD 1**

attempts implicit differentiation on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - \frac{kP^2}{N}$ by expanding $kP\left(1-\frac{P}{N}\right)$ ***(M1)***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t} - 2\frac{kP}{N}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***A1A1***

$= k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1-\frac{2P}{N}\right)$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ and so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ ***AG***

METHOD 2

attempts implicit differentiation (product rule) on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1-\frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)\frac{\mathrm{d}P}{\mathrm{d}t}\right)$ ***A1***

substitutes $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ into their $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\left(kP\left(1-\frac{P}{N}\right)\right)\left(1-\frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)kP\left(1-\frac{P}{N}\right)\right)$

$= k^2P\left(1-\frac{P}{N}\right)^2 - k^2P\left(1-\frac{P}{N}\right)\left(\frac{P}{N}\right)$

$= k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{P}{N}-\frac{P}{N}\right)$ ***A1***

so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = 0 \Rightarrow k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right) = 0$ ***(M1)***

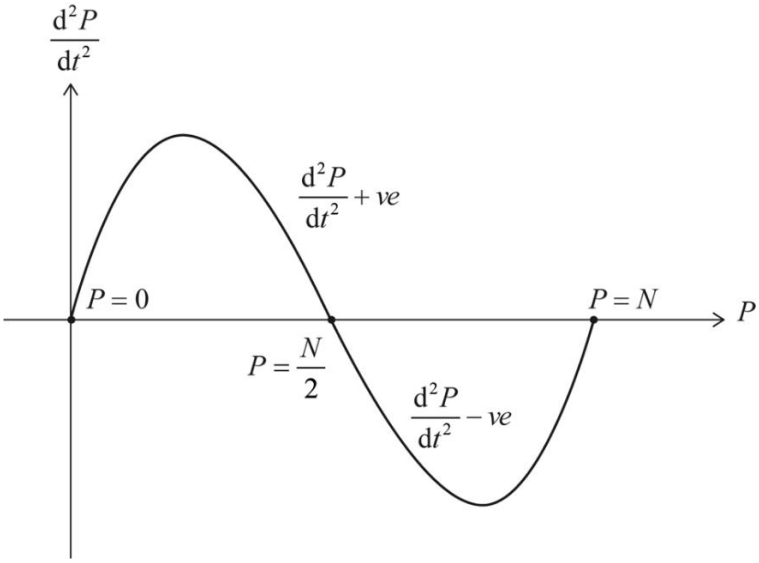
$P = 0, \frac{N}{2}, N$ ***A2***

Note: Award ***A1*** for $P = \frac{N}{2}$ only.

uses the second derivative to show that concavity changes at $P = \frac{N}{2}$ or the first derivative to show a local maximum at $P = \frac{N}{2}$ ***M1***

EITHER

a clearly labelled correct sketch of $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ versus P showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***



OR

a correct and clearly labelled sign diagram (table) showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

continued…

Question 12 continued

OR

for example, $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = \frac{3k^2N}{32}(>0)$ with $P = \frac{N}{4}$ and $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = -\frac{3k^2N}{32}(<0)$ with $P = \frac{3N}{4}$

showing $P = \frac{N}{2}$ corresponds to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

so the population is increasing at its maximum rate when $P = \frac{N}{2}$ ***AG***

[5 marks]

(d) substitutes $P = \frac{N}{2}$ into $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***(M1)***

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{N}{2}\right)\left(1 - \frac{\frac{N}{2}}{N}\right)$$

the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ is $\frac{kN}{4}$ ***A1***

[2 marks]

continued…

Question 12 continued

(e) **METHOD 1**

attempts to separate variables ***M1***

$$\int \frac{N}{P(N-P)}\mathrm{d}P = \int k\ \mathrm{d}t$$

attempts to write $\frac{N}{P(N-P)}$ in partial fractions form ***M1***

$$\frac{N}{P(N-P)} \equiv \frac{A}{P} + \frac{B}{(N-P)} \Rightarrow N \equiv A(N-P) + BP$$

$A=1,\ B=1$ ***A1***

$$\frac{N}{P(N-P)} \equiv \frac{1}{P} + \frac{1}{(N-P)}$$

$$\int\left(\frac{1}{P} + \frac{1}{(N-P)}\right)\mathrm{d}P = \int k\ \mathrm{d}t$$

$$\Rightarrow \ln P - \ln(N-P) = kt\,(+C)$$ ***A1A1***

Note: Award ***A1*** for $-\ln(N-P)$ and ***A1*** for $\ln P$ and $kt\,(+C)$. Absolute value signs are not required.

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$ and so $C = \ln P_0 - \ln(N-P_0)$

$$kt = \ln\left(\frac{P}{N-P}\right) - \ln\left(\frac{P_0}{N-P_0}\right)\left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

so $kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 2

attempts to separate variables ***M1***

$$\int \frac{1}{P\left(1-\frac{P}{N}\right)} \mathrm{d}P = \int k \ \mathrm{d}t$$

attempts to write $\frac{1}{P\left(1-\frac{P}{N}\right)}$ in partial fractions form ***M1***

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{A}{P} + \frac{B}{1-\frac{P}{N}} \Rightarrow 1 \equiv A\left(1-\frac{P}{N}\right) + BP$$

$A=1,\ B=\frac{1}{N}$ ***A1***

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)}$$

$$\int \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)} \mathrm{d}P = \int k \ \mathrm{d}t$$

$$\Rightarrow \ln P - \ln\left(1-\frac{P}{N}\right) = kt\left(+C\right)$$ ***A1A1***

Note: Award ***A1*** for $-\ln\left(1-\frac{P}{N}\right)$ and ***A1*** for $\ln P$ and $kt\left(+C\right)$. Absolute value signs are not required.

continued…

Question 12 continued

$$\ln\left(\frac{P}{1-\frac{P}{N}}\right) = kt + C \Rightarrow \ln\left(\frac{NP}{N-P}\right) = kt + C$$

attempts to find C in terms of N and P_0 ***M1***

when $t = 0$, $P = P_0$ and so $C = \ln\left(\frac{NP_0}{N-P_0}\right)$

$$kt = \ln\left(\frac{NP}{N-P}\right) - \ln\left(\frac{NP_0}{N-P_0}\right)\left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

$$kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 3

lets $u = \frac{1}{P}$ and forms $\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***M1***

multiplies both sides of the differential equation by $-\frac{1}{P^2}$ and makes the above substitutions ***M1***

$$-\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{1}{N} - \frac{1}{P}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = k\left(\frac{1}{N} - u\right)$$

$\frac{\mathrm{d}u}{\mathrm{d}t} + ku = \frac{k}{N}$ (linear first-order DE) ***A1***

$\mathrm{IF} = \mathrm{e}^{\int k\,\mathrm{d}t} = \mathrm{e}^{kt} \Rightarrow \mathrm{e}^{kt}\frac{\mathrm{d}u}{\mathrm{d}t} + k\mathrm{e}^{kt}u = \frac{k}{N}\mathrm{e}^{kt}$ ***(M1)***

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\mathrm{e}^{kt}\right) = \frac{k}{N}\mathrm{e}^{kt}$$

$u\mathrm{e}^{kt} = \frac{1}{N}\mathrm{e}^{kt}(+C) \quad \left(\frac{1}{P}\mathrm{e}^{kt} = \frac{1}{N}\mathrm{e}^{kt}(+C)\right)$ ***A1***

attempts to find C in terms of N and P_0 ***M1***

when $t = 0$, $P = P_0$, $u = \frac{1}{P_0}$ and so $C = \frac{1}{P_0} - \frac{1}{N}\left(= \frac{N - P_0}{NP_0}\right)$

$$\mathrm{e}^{kt}\left(\frac{N - P}{NP}\right) = \frac{N - P_0}{NP_0}$$

$\mathrm{e}^{kt} = \left(\frac{P}{N - P}\right)\left(\frac{N - P_0}{P_0}\right)$ ***A1***

$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right)$ ***AG***

[7 marks]

continued…

Question 12 continued

(f) substitutes $t = 10$, $P = 3P_0$ and $N = 4P_0$ into $kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right)$ ***M1***

$10k = \ln 3\left(\frac{4P_0 - P_0}{4P_0 - 3P_0}\right)\ \left(= \ln 9\right)$

$k = 0.220\ \left(= \frac{1}{10}\ln 9, = \frac{1}{5}\ln 3\right)$ ***A1***

[2 marks]

Total [21 marks]

2. (a) (i) $4-\mathrm{i}$ ***A1***

[1 mark]

(ii) $\text{mean} = \frac{1}{2}(4+\mathrm{i}+4-\mathrm{i})$ ***A1***

$= 4$ ***AG***

[1 mark]

(b) **METHOD 1**

attempts product rule differentiation ***(M1)***

Note: Award ***(M1)*** for attempting to express $f(x)$ as $f(x) = x^3 - 9x^2 + 25x - 17$.

$f'(x) = (x-1)(2x-8) + x^2 - 8x + 17 \ \left(f'(x) = 3x^2 - 18x + 25\right)$ ***A1***

$f'(4) = 1$ ***A1***

Note: Where $f'(x)$ is correct, award ***A1*** for solving $f'(x) = 1$ and obtaining $x = 4$.

EITHER

$y - 3 = 1(x-4)$ ***A1***

OR

$y = x + c$

$3 = 4 + c \Rightarrow c = -1$ ***A1***

OR

states the gradient of $y = x - 1$ is also 1 and verifies that $(4, 3)$ lies on the line $y = x - 1$ ***A1***

THEN

so $y = x - 1$ is the tangent to the curve at $\mathrm{A}(4, 3)$ ***AG***

Note: Award a maximum of ***(M0)A0A1A1*** to a candidate who does not attempt to find $f'(x)$.

continued…

Question 2 continued

METHOD 2

sets $f(x) = x - 1$ to form $x - 1 = (x - 1)(x^2 - 8x + 17)$ ***(M1)***

EITHER

$(x - 1)(x^2 - 8x + 16) = 0$ $(x^3 - 9x^2 + 24x - 16 = 0)$ ***A1***

attempts to solve a correct cubic equation ***(M1)***

$(x - 1)(x - 4)^2 = 0 \Rightarrow x = 1, 4$

OR

recognises that $x \neq 1$ and forms $x^2 - 8x + 17 = 1$ $(x^2 - 8x + 16 = 0)$ ***A1***

attempts to solve a correct quadratic equation ***(M1)***

$(x - 4)^2 = 0 \Rightarrow x = 4$

THEN

$x = 4$ is a double root ***R1***

so $y = x - 1$ is the tangent to the curve at $\mathrm{A}(4, 3)$ ***AG***

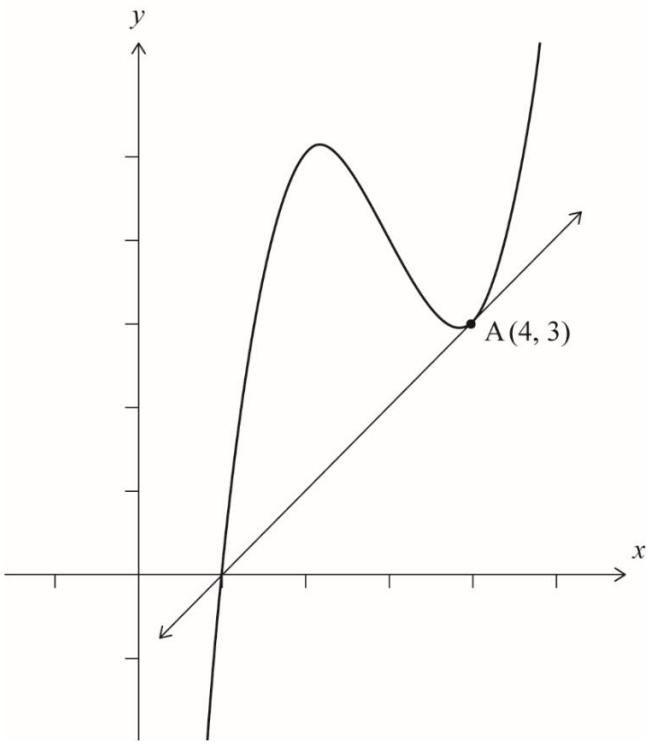
Note: Candidates using this method are not required to verify that $y = 3$.

[4 marks]

continued…

Question 2 continued

(c)



a positive cubic with an x- intercept $(x=1)$, and a local maximum and local minimum in the first quadrant both positioned to the left of A ***A1***

Note: As the local minimum and point A are very close to each other, condone graphs that seem to show these points coinciding.
For the point of tangency, accept labels such as A, $(4,3)$ or the point labelled from both axes. Coordinates are not required.

a correct sketch of the tangent passing through A and crossing the x-axis at the same point ($x=1$) as the curve ***A1***

Note: Award ***A1A0*** if both graphs cross the x- axis at distinctly different points.

[2 marks]

continued…

Question 2 continued

(d) (i) **EITHER**

$g'(x) = (x-r)(2x-2a) + x^2 - 2ax + a^2 + b^2$ ***(M1)A1***

OR

$g(x) = x^3 - (2a+r)x^2 + (a^2+b^2+2ar)x - (a^2+b^2)r$

attempts to find $g'(x)$ ***M1***

$g'(x) = 3x^2 - 2(2a+r)x + a^2 + b^2 + 2ar$

$= 2x^2 - 2(a+r)x + 2ar + x^2 - 2ax + a^2 + b^2$ ***A1***

$\left(= 2(x^2 - ax - rx + ar) + x^2 - 2ax + a^2 + b^2\right)$

THEN

$g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) **METHOD 1**

$g(a) = b^2(a-r)$ ***(A1)***

$g'(a) = b^2$ ***(A1)***

attempts to substitute their $g(a)$ and $g'(a)$ into $y - g(a) = g'(a)(x-a)$ ***M1***

$y - b^2(a-r) = b^2(x-a)$

EITHER

$y = b^2(x-r)$ $\left(y = b^2x - b^2r\right)$ ***A1***

sets $y = 0$ so $b^2(x-r) = 0$ ***M1***

$b > 0 \Rightarrow x = r$ OR $b \neq 0 \Rightarrow x = r$ ***R1***

OR

sets $y = 0$ so $-b^2(a-r) = b^2(x-a)$ ***M1***

$b > 0$ OR $b \neq 0$ $\Rightarrow -(a-r) = x-a$ ***R1***

$x = r$ ***A1***

THEN

so the tangent intersects the x- axis at the point $\mathrm{R}(r, 0)$ ***AG***

continued…

Question 2 continued

METHOD 2

$g'(a) = b^2$ ***(A1)***

$g(a) = b^2(a - r)$ ***(A1)***

attempts to substitute their $g(a)$ and $g'(a)$ into $y = g'(a)x + c$ and attempts to find c ***M1***

$c = -b^2 r$

EITHER

$y = b^2(x - r)$ $(y = b^2 x - b^2 r)$ ***A1***

sets $y = 0$ so $b^2(x - r) = 0$ ***M1***

$b > 0 \Rightarrow x = r$ OR $b \neq 0 \Rightarrow x = r$ ***R1***

OR

sets $y = 0$ so $b^2(x - r) = 0$ ***M1***

$b > 0$ OR $b \neq 0$ $\Rightarrow x - r = 0$ ***R1***

$x = r$ ***A1***

METHOD 3

$g'(a) = b^2$ ***(A1)***

the line through $\mathrm{R}(r, 0)$ parallel to the tangent at A has equation

$y = b^2(x - r)$ ***A1***

sets $g(x) = b^2(x - r)$ to form $b^2(x - r) = (x - r)(x^2 - 2ax + a^2 + b^2)$ ***M1***

$b^2 = x^2 - 2ax + a^2 + b^2$, $(x \neq r)$ ***A1***

$(x - a)^2 = 0$ ***A1***

since there is a double root ($x = a$), this parallel line through

$\mathrm{R}(r, 0)$ is the required tangent at A ***R1***

[6 marks]

continued…

Question 2 continued

(e) **EITHER**

$g'(a) = b^2 \Rightarrow b = \sqrt{g'(a)}$ (since $b > 0$) ***R1***

Note: Accept $b = \pm\sqrt{g'(a)}$.

OR

$(a \pm b\text{i} =) a \pm \text{i}\sqrt{b^2}$ and $g'(a) = b^2$ ***R1***

THEN

hence the complex roots can be expressed as $a \pm \text{i}\sqrt{g'(a)}$ ***AG***

[1 mark]

(f) (i) $b = 4$ (seen anywhere) ***A1***

EITHER

attempts to find the gradient of the tangent in terms of a and equates to 16 ***(M1)***

OR

substitutes $r = -2, x = a$ and $y = 80$ to form $80 = (a - (-2))(a^2 - 2a^2 + a^2 + 16)$ ***(M1)***

OR

substitutes $r = -2, x = a$ and $y = 80$ into $y = 16(x - r)$ ***(M1)***

THEN

$\frac{80}{a+2} = 16 \Rightarrow a = 3$

roots are -2 (seen anywhere) and $3 \pm 4\text{i}$ ***A1A1***

Note: Award ***A1*** for -2 and ***A1*** for $3 \pm 4\text{i}$. Do not accept coordinates.

[4 marks]

(ii) $(3, -4)$ ***A1***

Note: Accept "$x = 3$ and $y = -4$".
Do not award ***A1FT*** for $(a, -4)$.

[1 mark]

continued…

Question 2 continued

(g) (i) $g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$

attempts to find $g''(x)$ ***M1***

$g''(x) = 2(x-a) + 2(x-r) + 2x - 2a \ (= 6x - 2r - 4a)$

sets $g''(x) = 0$ and correctly solves for x ***A1***

for example, obtaining $x - r + 2(x-a) = 0$ leading to $3x = 2a + r$

so $x = \frac{1}{3}(2a + r)$ ***AG***

Note: Do not award ***A1*** if the answer does not lead to the ***AG***.

[2 marks]

(ii) point P is $\frac{2}{3}$ of the horizontal distance (way) from point R to point A ***A1***

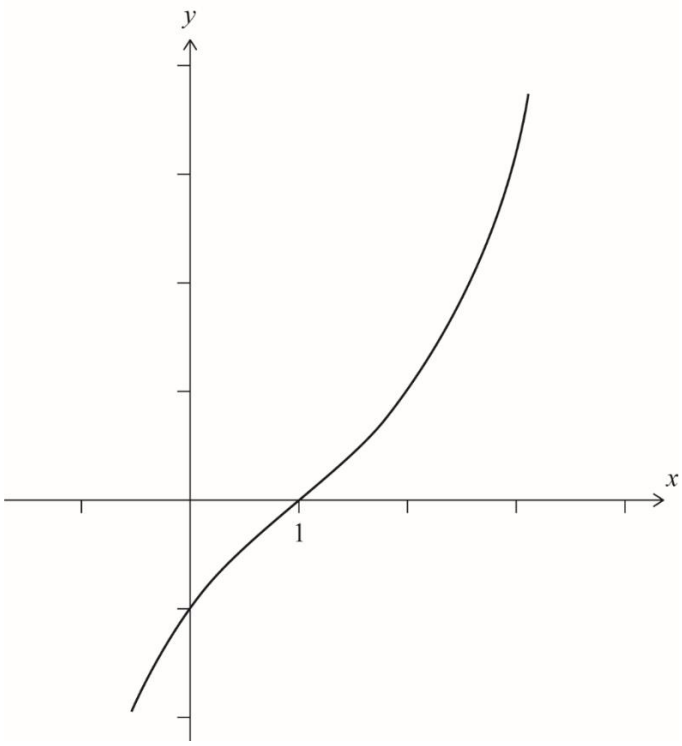
Note: Accept equivalent numerical statements or a clearly labelled diagram displaying the numerical relationship.
Award ***A0*** for non-numerical statements such as "P is between R and A, closer to A".

[1 mark]

continued…

Question 2 continued

(h) (i) $y = (x-1)(x^2-2x+5)$ ***(A1)***



a positive cubic with no stationary points and a non-stationary point of inflexion at $x = 1$ ***A1***

Note: Graphs may appear approximately linear. Award this ***A1*** if a change of concavity either side of $x = 1$ is apparent.
Coordinates are not required and the y-intercept need not be indicated.

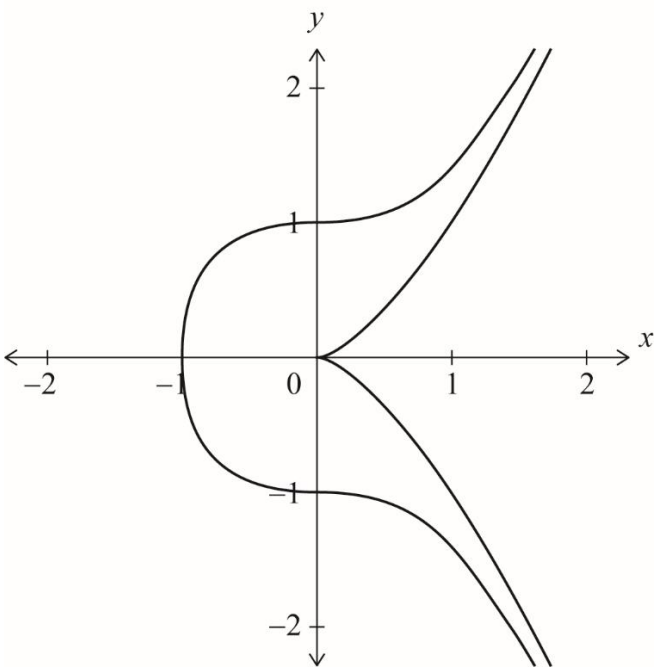
[2 marks]

(ii) $(r, 0)$ ***A1***

[1 mark]

Total [28 marks]

1. (a) (i)



approximately symmetric about the x-axis graph of $y^2 = x^3$ ***A1***

including cusp/sharp point at (0, 0) ***A1***

[2 marks]

(ii) approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts ***A1***

some indication of position of intersections at $x = -1,\ y = \pm 1$ ***A1***

[2 marks]

Note: Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct. Award ***A1A0A0A0*** if only one correct branch of both curves are seen.
Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0.***

continued…

Question 1 continued

(b) (i) $(0, 1)$ and $(0, -1)$ ***A1***

[1 mark]

(ii) Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different x-axis intercepts (one goes through the origin, and the other does not)

graphs have different y-axis intercepts ***A1***

Note: Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent.

[1 mark]

continued…

Question 1 continued

(c) Any **two** from:

as $x \to \infty$, $y \to \pm\infty$

as $x \to \infty$, $y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have x intercepts at $x = -\sqrt[3]{b}$

they have y intercepts at $y = (\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or x-axis) is a line of symmetry

they all have the same line of symmetry $(y = 0)$

they have one x-axis intercept

they have two y-axis intercepts

they have two points of inflexion

at x-axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at x-axis intercepts ***A1A1***

Note: The last example is the only valid answer for things "not" present. Do not credit an answer of "they are all symmetrical" without some reference to the line of symmetry.
Note: Do not allow same/ similar shape or equivalent.
Note: In accordance with marking rules, mark their first two responses and ignore any subsequent.

[2 marks]

continued…

Question 1 continued

(d) (i) **METHOD 1**

attempt to differentiate implicitly ***M1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+1}{2y}$ OR $(\pm)2\sqrt{x^3+x}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2+1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

METHOD 2

attempt to use chain rule $y = (\pm)\sqrt{x^3+x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+1)$ ***A1A1***

Note: Award ***A1*** for $(\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}$, ***A1*** for $(3x^2+1)$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

[3 marks]

(ii) **EITHER**

local minima/maxima occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$1+3x^2 = 0$ has no (real) solutions (or equivalent) ***R1***

OR

$\left(x^2 \geq 0 \Rightarrow\right)\ 3x^2+1 > 0$, so $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ ***R1***

THEN

so, no local minima/maxima exist ***AG***

[1 mark]

continued…

Question 1 continued

(e) **EITHER**

attempt to use quotient rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\pm)\frac{12x\sqrt{x+x^3} - (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)}{4(x+x^3)}$ ***A1A1***

Note: Award ***A1*** for correct $12x\sqrt{x+x^3}$ and correct denominator, ***A1*** for correct $-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$.

Note: Future ***A*** marks may be awarded if the denominator is missing or incorrect.

stating or using $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ (may be seen anywhere) ***(M1)***

$12x\sqrt{x+x^3} = (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$

OR

attempt to use product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2}\left(3x^2+1\right)\left(-\frac{1}{2}\right)\left(3x^2+1\right)\left(x^3+x\right)^{-\frac{3}{2}} + 3x\left(x^3+x\right)^{-\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct first term, ***A1*** for correct second term.

setting $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ ***(M1)***

continued…

Question 1 continued

OR

attempts implicit differentiation on $2y\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+1$ ***M1***

$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x$ ***A1***

recognizes that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{3x}$

$(\pm)\frac{3x^2+1}{2\sqrt{x^3+x}}=(\pm)\sqrt{3x}$ ***(A1)***

THEN

$12x(x+x^3)=(1+3x^2)^2$

$12x^2+12x^4=9x^4+6x^2+1$

$3x^4+6x^2-1=0$ ***A1***

attempt to use quadratic formula or equivalent ***(M1)***

$x^2=\frac{-6\pm\sqrt{48}}{6}$

$(x>0\Rightarrow)\,x=\sqrt{\frac{2\sqrt{3}-3}{3}}\ \ (p=2,\ q=-3,\ r=3)$ ***A1***

Note: Accept any integer multiple of p, q and r (e.g. $4, -6$ and 6).

[7 marks]

continued…

Question 1 continued

(f) (i) attempt to find tangent line through $(-1,-1)$ ***(M1)***

$y+1=-\frac{3}{2}(x+1)$ OR $y=-1.5x-2.5$ ***A1***

[2 marks]

(ii) attempt to solve simultaneously with $y^2=x^3+2$ ***(M1)***

Note: The ***M1*** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. (4.25, -8.875)).

obtain $\left(\frac{17}{4},-\frac{71}{8}\right)$ ***A1***

[2 marks]

continued…

Question 1 continued

(g) attempt to find equation of $[\mathrm{QS}]$ ***(M1)***

$\frac{y-1}{x+1} = -\frac{79}{42} \left(= -1.88095\ldots\right)$ ***(A1)***

solve simultaneously with $y^2 = x^3 + 2$ ***(M1)***

$x = 0.28798\ldots \left(= \frac{127}{441}\right)$ ***A1***

$y = -1.4226\ldots \left(= \frac{13175}{9261}\right)$ ***A1***

$(0.288, -1.42)$

OR

attempt to find vector equation of $[\mathrm{QS}]$ ***(M1)***

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{21}{4} \\ -\frac{79}{8} \end{pmatrix}$ ***(A1)***

$x = -1 + \frac{21}{4}\lambda$

$y = 1 - \frac{79}{8}\lambda$

attempt to solve $\left(1 - \frac{79}{8}\lambda\right)^2 = \left(-1 + \frac{21}{4}\lambda\right)^3 + 2$ ***(M1)***

$\lambda = 0.2453\ldots$

$x = 0.28798\ldots \left(= \frac{127}{441}\right)$ ***A1***

$y = -1.4226\ldots \left(= \frac{13175}{9261}\right)$ ***A1***

$(0.288, -1.42)$

[5 marks]

[Total 28 marks]

1. $g'(x) = 2x\mathrm{e}^{x^2+1}$ ***(A2)***

substitute $x = -1$ into **their** derivative ***(M1)***

$g'(-1) = -2\mathrm{e}^2$ ***A1***

Note: Award ***A0M0A0*** in cases where candidate's incorrect derivative is $g'(x) = \mathrm{e}^{x^2+1}$.

[4 marks]

4. recognizing need to integrate ***(M1)***

$\int \frac{6x}{x^2+1}\mathrm{d}x$ OR $u = x^2+1$ OR $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\int \frac{3}{u}\mathrm{d}u$ OR $3\int \frac{2x}{x^2+1}\mathrm{d}x$ ***(A1)***

$= 3\ln\left(x^2+1\right)(+c)$ or $3\ln u\,(+c)$ ***A1***

correct substitution of $x = 1$ and $f(x) = 5$ or $x = 1$ and $u = 2$ into equation

using **their** integrated expression (must involve *c*) ***(M1)***

$5 = 3\ln 2 + c$

$f(x) = 3\ln\left(x^2+1\right) + 5 - 3\ln 2$ $\left(= 3\ln\left(x^2+1\right) + 5 - \ln 8 = 3\ln\left(\frac{x^2+1}{2}\right) + 5\right)$ (or equivalent) ***A1***

Note: Accept the use of the modulus sign in working and the final answer.

[5 marks]

7. attempt at implicit differentiation, including use of the product rule ***(M1)***

EITHER

$\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2+\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}=8x$ ***A1A1A1***

Note: Award ***A1*** for each of $\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2$, $\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

OR

$x^2y^2+y^4=4x^2$

$2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+4y^3\frac{\mathrm{d}y}{\mathrm{d}x}=8x$ ***A1A1A1***

Note: Award ***A1*** for each of $2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

THEN

at a local maximum or minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***(M1)***

$2xy^2=8x$

$x=0$ or $y^2=4\left(\Rightarrow y=\pm2\right)$ ***A1***

Note: Award ***A0*** for $x=0$ or $y=2$

since $x>0$ and $-2<y<2$ there are no solutions ***R1***

hence there are no local maximum or minimum points ***AG***

[7 marks]

9. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{xy}$

let $y = vx$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2 - 2x^2}{vx^2}$ ***(M1)***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-2}{v}$ (***A1)***

$\Rightarrow \int v\mathrm{d}v = -\int \frac{2}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \frac{v^2}{2} = -2\ln|x|(+c)$ ***A1***

Note: Condone the absence of the modulus sign up to this point.

$\Rightarrow \frac{y^2}{2x^2} = -2\ln|x| + c$ ***A1***

attempt to substitute $x = 1, y = 2$ into their integrated expression to find c ***M1***

$\Rightarrow 2 = -2\ln|1| + c \Rightarrow c = 2$

$\Rightarrow \frac{y^2}{2x^2} = -2\ln|x| + 2$

$\Rightarrow y^2 = 2x^2\left(-2\ln|x| + 2\right)\left(= 4x^2\left(1 - \ln|x|\right)\right)$ ***A1***

[8 marks]

continued…

Question 9 continued

(b) attempt to set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ in the differential equation ***(M1)***

$y = \sqrt{2}x$ and $y = -\sqrt{2}x$ or $m = \pm\sqrt{2}$ ***A1***

[2 marks]

Total [10 marks]

Section B

2. (a) $(0.708519..., 0.639580...)$

$(0.709, 0.640)$ $(x = 0.709,\ y = 0.640)$ ***A1A1***

[2 marks]

(b) $1.09885...$

$x = 1.10$ (accept $(1.10, 0)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 |f(x)|\,dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^{2} f(x)\,dx$ OR $\int_{1.09885...}^{2} |f(x)|\,dx$ OR $4.17527\ldots$ ***(A1)***

$\int_0^{1.09885...} f(x)\,dx - \int_{1.09885...}^{2} f(x)\,dx$ OR $0.435901\ldots + 4.17527\ldots$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

Total [6 marks]

8. EITHER

$\left(\frac{\mathrm{d}V}{\mathrm{d}h} =\right) 10\pi h - \pi h^2$ ***(A1)***

Note: This ***A1*** may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h} = 76.5616...$.

attempt to use chain rule to find a relationship between $\frac{\mathrm{d}h}{\mathrm{d}t}$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} \left(= \frac{1}{\left(\frac{\mathrm{d}V}{\mathrm{d}h}\right)} \times \frac{\mathrm{d}V}{\mathrm{d}t} \right)$

OR

attempt to differentiate $V = 5\pi h^2 - \frac{1}{3}\pi h^3$ throughout with respect to t ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t} = 10\pi h \frac{\mathrm{d}h}{\mathrm{d}t} - \pi h^2 \frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

continued…

Question 8 continued

THEN

$\left(10\pi h - \pi h^2\right)\frac{\mathrm{d}h}{\mathrm{d}t} = 2$ OR $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{10\pi h - \pi h^2}$ ***(A1)***

Note: Award this ***A1*** if the correct expression is seen with their h already substituted.

attempt to solve $200 = 5\pi h^2 - \frac{1}{3}\pi h^3$ ***(M1)***

$h = 4.20648...$ ***(A1)***

Note: This ***(M1)(A1)*** can be awarded independently of all previous marks, and may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h} = 76.5616...$
Ignore extra values of h -3.24 and 14.0.

$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.0261227...$

$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.0261\left(\mathrm{cms}^{-1}\right)$ ***A1***

[6 marks]

11. (a) attempt to use product rule ***(M1)***

$f'(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ ***(M1)***

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$\pi\int_0^{\frac{4}{3}} \left(e^{2x}(3x-4)\right)^2 dx$ ***(A1)***

Note: This ***(A1)*** can be awarded if the dx is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

continued…

Question 11 continued

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 (= -e^2)$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1) (= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 (= 2e^2)$ ***A1***

[5 marks]

Total [15 marks]

1. (a) **EITHER**

$S_n = \frac{n}{2}\left(2\times 1+\left(n-1\right)\times 1\right)$ ***A1***

OR

$u_1 = 1$ and either $u_n = n$ or $d = 1$ stated explicitly ***A1***

OR

$1+2+...+n$ (or equivalent) stated explicitly ***A1***

THEN

$S_n = \frac{n}{2}\left(1+n\right)$ ***AG***

Note: Award ***A0*** for a numerical verification.

[1 mark]

(b) (i) 14 ***A1***

[1 mark]

(ii) $a_1 + a_2 + a_3 = 1$ ***A1***

$2a_1 + 4a_2 + 8a_3 = 5$ ***A1***

$3a_1 + 9a_2 + 27a_3 = 14$ ***A1***

Note: For the third ***A*** mark, award ***A1FT*** for $3a_1 + 9a_2 + 27a_3 = p$ where p is their answer to part (b) (i).

[3 marks]

continued...

Question 1 continued

(iii) $a_1 = \frac{1}{6}\ (= 0.166666... \approx 0.167)$, $a_2 = \frac{1}{2}\ (= 0.5)$,

$a_3 = \frac{1}{3}\ (= 0.333333... \approx 0.333)$ ***A2***

Note: Award ***A1*** if only two of a_1, a_2, a_3 are correct.
Only award ***FT*** if three linear equations, each in a_1, a_2 and a_3 are stated in part (b) (ii) or (iii).
Award ***A2FT*** for their a_1, a_2 and a_3.
Award ***A1FT*** for their a_1, a_2 and $a_3 = 0$.

[2 marks]

(c) $f'(x) = 1 + 2x + 3x^2 + ... + nx^{n-1}$ ***A1***

Note: Award ***A1*** for $f'(x) = \sum_{i=1}^{n} ix^{i-1}$.

$\Rightarrow x f'(x) = x + 2x^2 + 3x^3 + ... + nx^n$ ***AG***

[1 mark]

continued...

Question 1 continued

(d) (i) **METHOD 1**

$f_2(x) = xf_1'(x)$

$f_1'(x) = 1^2 + 2^2 x + \left(3^2 x^2\right) + \ldots + n^2 x^{n-1} \ \left(= 1 + 4x + \left(9x^2\right) + \ldots + n^2 x^{n-1}\right)$ ***A1***

Note: Award ***A1*** for

$xf_1'(x) = x\left(1^2 + 2^2 x + \left(3^2 x^2\right) + \ldots + n^2 x^{n-1}\right) \ \left(= x\left(1 + 4x + \left(9x^2\right) + \ldots + n^2 x^{n-1}\right)\right)$

.

$xf_1'(x) = 1^2 x + 2^2 x^2 + \left(3^2 x^3\right) + \ldots + n^2 x^n \ \left(= x + 4x^2 + \left(9x^3\right) + \ldots + n^2 x^n\right)$ ***A1***

Note: Award ***A1*** for $f_1'(x) = \sum_{i=1}^{n} i^2 x^{i-1}$ and ***A1*** for $xf_1'(x) = x\sum_{i=1}^{n} i^2 x^{i-1}$.

The second ***A1*** is dependent on the first ***A1***.

Award a maximum of ***A0A1*** if a general term is not considered.

$= \sum_{i=1}^{n} i^2 x^i$ ***AG***

METHOD 2

$f_2(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(xf'(x)\right)$

$= x\left(f'(x) + xf''(x)\right) \ \left(= xf'(x) + x^2 f''(x)\right)$ ***A1***

$= x\sum_{i=1}^{n} i x^{i-1} + x^2 \sum_{i=1}^{n} i(i-1)x^{i-2}$

$= \sum_{i=1}^{n} i x^i + \sum_{i=1}^{n} i(i-1)x^i \ \left(= \sum_{i=1}^{n}\left(i + i^2 - i\right)x^i\right)$ ***A1***

$= \sum_{i=1}^{n} i^2 x^i$ ***AG***

[2 marks]

continued...

Question 1 continued

(ii) consider $q = 1$

$f_1(x) = x + 2x^2 + ... + nx^n$ (reference to part (c)) and $f_1(x) = \sum_{i=1}^{n} ix^i$ ***R1***

assume true for $q = k$, ($f_k(x) = \sum_{i=1}^{n} i^k x^i$) ***M1***

Note: Do not award ***M1*** for statements such as "let $q = k$" or "$q = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $q = k + 1$

$f_{k+1}(x) = xf_k'(x)$ ***M1***

$= x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ **OR** $x(1 + 2^{k+1}x + 3^{k+1}x^2 + ... + n^{k+1}x^{n-1})$ ***A1***

Note: Award the above ***M1*** if $f_{k+1}(x) = x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ or $xf_k'(x) = x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ (or equivalent) is stated.

$= \sum_{i=1}^{n} i^{k+1}x^i$ **OR** $x + 2^{k+1}x^2 + 3^{k+1}x^3 + ... + n^{k+1}x^n$ ***A1***

since true for $q = 1$ and true for $q = k + 1$ if true for $q = k$, hence true for all q $(\in \mathbb{Z}^+)$ ***R1***

Note: To obtain the final ***R1***, three of the previous five marks must have been awarded.

[6 marks]

(iii) $f_q(1) = 1^q + 2^q + 3^q + ... + n^q$

$= \sum_{i=1}^{n} i^q \left(= \sum_{i=1}^{n} 1^i i^q \right)$ ***A1***

[1 mark]

continued...

Question 1 continued

(e) uses $S_n = \frac{u_1\left(r^n - 1\right)}{r-1}$ with $r = x$ and $u_1 = 1$ ***M1***

clear indication there are $(n+1)$ terms ***R1***

$f(x) = \frac{x^{n+1} - 1}{x - 1}$ ***AG***

[2 marks]

(f) **METHOD 1**

$f_1(x) = xf'(x)$

$= x\frac{(x-1)(n+1)x^n - 1 \times \left(x^{n+1} - 1\right)}{(x-1)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to use the quotient or the product rule to find $f'(x)$.

$= x\frac{(nx + x - n - 1)x^n - \left(x^{n+1} - 1\right)}{(x-1)^2}\left(= x\frac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2}\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$= \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$ ***AG***

METHOD 2

attempts to form $(x-1)f_1(x)$ ***M1***

$(x-1)f_1(x) = nx^{n+1} - \left(x + x^2 + x^3 + \ldots + x^n\right)$

$f_1(x) = \frac{1}{x-1}\left(nx^{n+1} - \left(\frac{x^{n+1} - 1}{x-1} - 1\right)\right)$ ***A1***

$f_1(x) = \frac{1}{x-1}\left(\frac{nx^{n+1}(x-1) - x^{n+1} + x}{x-1}\right)\left(= \frac{1}{x-1}\left(\frac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{x-1}\right)\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$$f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$ ***AG***

[3 marks]

(g) (i) $\lim_{x \to 1} f_1(x) = \frac{n - (n+1) + 1}{0} \left(= \frac{0}{0} \right)$ ***R1***

Note: Only award ***R1*** for sufficient simplification of the numerator, for example, as shown above.
Do not award ***R1*** if $\lim_{x \to 1}$ is not referred to or stated.

[1 mark]

continued...

Question 1 continued

(ii) attempts to differentiate both the numerator and the denominator ***M1***

$$\lim_{x\to 1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}$$ ***A1***

Note: Award ***A1*** for $\left(\left(\lim_{x\to 1}\right)\frac{n(n+2)x^{n+1}-n(n+1)x^n-(n+1)x^n+1}{2(x-1)}\right)$. This form can be used in subsequent work.

(l'Hôpital's rule applies again since)

$$\lim_{x\to 1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}=\frac{0}{0}$$ ***R1***

Note: Do not award ***R1*** if $\lim_{x\to 1}$ is not referred to or stated.
Subsequent marks are independent of this ***R*** mark.

attempts to differentiate both the numerator and the denominator ***M1***

$$\lim_{x\to 1}\frac{n(n+2)(n+1)x^n-n(n+1)^2x^{n-1}}{2}$$

$$=\frac{n(n+2)(n+1)-n(n+1)^2}{2}\quad\left(=\frac{n^3+3n^2+2n-\left(n^3+2n^2+n\right)}{2}\right)$$

$$=\frac{n(n+1)\left((n+2)-(n+1)\right)}{2}\quad\left(=\frac{n^2+n}{2}\right)$$ ***A1***

$$=\frac{1}{2}n(n+1)$$ ***AG***

[5 marks]

Total [28 marks]

2. (a) **EITHER**

$A = 2\pi\int_0^3 2x\sqrt{1+2^2}\ \mathrm{d}x \left(= 4\sqrt{5}\pi\int_0^3 x\,\mathrm{d}x\right)$ ***(A1)***

$= 2\pi\sqrt{5}\ \left[x^2\right]_0^3\ \left(= 2\pi\sqrt{5}\left(3^2 - 0^2\right)\right)$ ***A1***

OR

$A = 2\pi m\sqrt{1+m^2}\ \left[\frac{x^2}{2}\right]_0^h\ \left(= 2\pi m\sqrt{1+m^2}\left(\frac{h^2}{2}\right)\right)$ ***(A1)***

$= 2\pi(2)\sqrt{5}\ \left[\frac{x^2}{2}\right]_0^3\ \left(= 2\pi(2)\sqrt{5}\left(\frac{3^2}{2}\right)\right)$ ***A1***

THEN

$= 18\sqrt{5}\,\pi$ ***AG***

[2 marks]

(b) (i) $r = mh$ ***A1***

[1 mark]

(ii) $l = \sqrt{h^2 + r^2}$ ***(M1)***

$l = \sqrt{h^2 + h^2m^2}\ (= h\sqrt{1+m^2})$ ***A1***

[2 marks]

continued...

Question 2 continued

(iii) $A = 2\pi\int_0^h mx\sqrt{1+m^2}\ \mathrm{d}x$ ***(A1)***

$= 2\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$ (***M1)***

Note: Award ***(M1)*** for $(c)\,\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$.

At least one of the above two lines needs to be seen.

$= \pi h^2 m\sqrt{1+m^2}\ \left(= \pi hm \times \sqrt{\left(h^2+h^2m^2\right)}\right)$ ***A1***

$= \pi rl$ ***AG***

Note: Award as above if $\frac{l}{h} = \sqrt{1+m^2}$ is used, for example.

[3 marks]

(c) **METHOD 1**

attempts to use the chain rule ***(M1)***

Note: Award ***(M1)*** for $\frac{\mathrm{d}y}{\mathrm{d}x} = (c)\,x\left(r^2-x^2\right)^{-\frac{1}{2}}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2-x^2\right)^{-\frac{1}{2}}\ (-2x)\ \left(= -x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***A1***

[2 marks]

continued...

Question 2 continued

(d) **EITHER**

attempts to substitute $y = \sqrt{r^2 - x^2}$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into A ***(M1)***

$$A = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \left(-x\left(r^2 - x^2\right)^{-\frac{1}{2}} \right)^2} \, \mathrm{d}x$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, \mathrm{d}x$$ ***A1***

OR

attempts to substitute y and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y into A ***(M1)***

$$A = 2\pi \int_{-r}^{r} y \sqrt{1 + \left(-\frac{x}{y} \right)^2} \, \mathrm{d}x$$

$$= 2\pi \int_{-r}^{r} y \sqrt{1 + \frac{x^2}{y^2}} \, \mathrm{d}x \left(= 2\pi \int_{-r}^{r} \sqrt{x^2 + y^2} \, \mathrm{d}x \right)$$ ***A1***

THEN

attempts to perform valid algebraic simplification to form a definite integral in terms of r only ***M1***

$$= 2\pi \int_{-r}^{r} r \, \mathrm{d}x$$

$$= 2\pi r \left[x \right]_{-r}^{r} \left(= 2\pi r\left(r - \left(-r\right)\right) \right)$$ ***A1***

$$= 4\pi r^2$$ ***AG***

Note: Award marks as above for $A = 4\pi \int_{0}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \left(-x\left(r^2 - x^2\right)^{-\frac{1}{2}} \right)^2} \, \mathrm{d}x$.

[4 marks]

continued...

Question 2 continued

(e) (i) **EITHER**

horizontal stretch ***A1***

factor $\frac{1}{k}$ ***A1***

OR

horizontal compression ***A1***

factor k (invariant line y- axis) ***A1***

Note: Award ***A1A1*** as above for correct alternative descriptions.

For example, dilation by a factor of $\frac{1}{k}$ from the y- axis.

[2 marks]

(ii) $\pm\frac{r}{k}$ ***A1***

Note: Award ***A0*** for $\frac{r}{k}$ only and ***A0*** for $-\frac{r}{k}$ only.

[1 mark]

continued...

Question 2 continued

(iii) **METHOD 1**

attempts to use the chain rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2 - (kx)^2\right)^{-\frac{1}{2}} \times \left(-k^2 2x\right)$ $\left(= -k^2 x\left(r^2 - (kx)^2\right)^{-\frac{1}{2}} \left(= \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}\right)\right)$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - k^2 x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{k^2 x}{y}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}$ ***A1***

[2 marks]

continued...

Question 2 continued

(iv) **EITHER**

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2}\sqrt{1 + \frac{k^4x^2}{r^2 - k^2x^2}}\ \mathrm{d}x \qquad \textbf{(A1)}$$

Note: Award ***(A1)*** for the correct substitution of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of $r^2 - k^2x^2$ and then cancelling $r^2 - k^2x^2$ ***(M1)***

OR

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \left(-\frac{k^2x}{y}\right)^2}\ \mathrm{d}x \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \frac{k^4x^2}{y^2}}\ \mathrm{d}x\right) \qquad \textbf{(A1)}$$

Note: Award ***(A1)*** for the correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of y^2 and cancelling y^2 ***(M1)***

THEN

$$= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2 + k^4x^2}\ \mathrm{d}x \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 + \left(k^4 - k^2\right)x^2}\mathrm{d}x\right) \qquad \textbf{A1A1}$$

Note: Award ***A1*** for correct limits (seen anywhere) and ***A1*** for $p\left(x\right)$ correct.
The above ***A1*** for correct limits is independent of the ***(M1)***.

[4 marks]

continued...

Question 2 continued

(v) $r = 6378\ (\text{km})$ ***(A1)***

$k = 1.00330...\left(= \frac{6378}{6357} = \frac{2126}{2119}\right)$ ***(A1)***

attempts to form a definite integral for surface area ***(M1)***

$$= 2\pi \int_{-6357}^{6357} \sqrt{6378^2 - \left(\frac{6378}{6357}\right)^2 x^2 + \left(\frac{6378}{6357}\right)^4 x^2}\ \mathrm{d}x$$

$$= 510064226.3...$$

$$= 5.101 \times 10^8\ (\text{km}^2)$$ ***A1***

Note: Award ***A0A1FT(M1)A0*** for using $r = 6357\ (\text{km})$ and $k = 0.996707...$ leading to an answer of $5.089 \times 10^8\ (\text{km}^2)$.

[4 marks]

Total [27 marks]

6. $A = \frac{1}{2}x^2 \sin\frac{\pi}{3}$ OR $A = \frac{1}{2}x^2 \sin 60^\circ$ OR triangle height $h = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}$ $\left(= \frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$= \frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(= \frac{\sqrt{3}}{4}x^2\right)$ ***A1***

Note: Award ***A1*** for $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\sqrt{3}}{4} \times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} \times \sin\frac{\pi}{3} \times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ ***(A1)***

$= \frac{2\sqrt{3}}{4} \times 5\sqrt{3} \times 4$

$\frac{\mathrm{d}A}{\mathrm{d}t} = 30\,(\mathrm{cm^2 s^{-1}})$ ***A1***

Note: Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

9. METHOD 1 (subtracting volumes)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2h}{2}-\frac{h^3}{24}$

recognition that the volume of the ring is $\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi R^2h$ where $R\neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi\left(r^2-\frac{h^2}{4}\right)h$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{r^2h}{2}-\frac{h^3}{24}\right)-\pi r^2h+\frac{\pi h^3}{4}=\pi$ OR $\frac{h^3}{4}-\frac{h^3}{12}=1$ (or equivalent)

OR

recognition that the volume of the ring is $\pi\int\left(\left(r^2-y^2\right)-\left(r^2-\frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$\frac{h^2}{4}y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{h^3}{8}-\frac{h^3}{24}$

correct equation ***(A1)***

$2\pi\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=\pi$ OR $2\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=1$ (or equivalent)

THEN

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi R^2h$ where $R\neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi\left(r^2-\frac{h^2}{4}\right)h\ \left(=\frac{4}{3}\pi r^3-\pi\right)$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}\right)+\pi r^2h-\frac{\pi h^3}{4}=\frac{4}{3}\pi r^3-\pi$ OR $\frac{h^3}{12}-\frac{h^3}{4}=-1$ (or equivalent)

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$2\pi\int x\sqrt{r^2-x^2}\,\mathrm{d}x$

correct limits *r* and $\sqrt{r^2-\frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$-\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression ***(M1)***

$-\frac{1}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$

correct equation ***(A1)***

$2\times\frac{-2\pi}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)=\pi$ OR $2\left(\frac{2\pi}{3}\times\frac{h^3}{8}\right)=\pi$

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

Section B

11. (a) $L = \text{AC} + \text{CB}$

$\dfrac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\alpha \left(\Rightarrow \text{AC} = \dfrac{\frac{3}{4}}{\cos\alpha} \Rightarrow \text{AC} = \dfrac{3}{4}\sec\alpha \right)$ ***A1***

$\dfrac{6}{\text{CB}} = \sin\alpha \left(\Rightarrow \text{CB} = \dfrac{6}{\sin\alpha} \Rightarrow \text{CB} = 6\text{cosec}\,\alpha \right)$ ***A1***

so $L = \dfrac{3}{4}\sec\alpha + 6\text{cosec}\,\alpha$ ***AG***

[2 marks]

(b) (i) $\dfrac{\text{d}L}{\text{d}\alpha} = \dfrac{3}{4}\sec\alpha\tan\alpha - 6\text{cosec}\,\alpha\cot\alpha$ ***A1***

(ii) attempt to write $\dfrac{\text{d}L}{\text{d}\alpha}$ in terms of $\sin\alpha, \cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\dfrac{\text{d}L}{\text{d}\alpha} = \dfrac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \dfrac{6\cos\alpha}{\sin^2\alpha}$ OR $\dfrac{\text{d}L}{\text{d}\alpha} = \dfrac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \dfrac{6}{\sin\alpha\tan\alpha} \left(= \dfrac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha} \right)$

$\dfrac{\text{d}L}{\text{d}\alpha} = 0 \Rightarrow \dfrac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\dfrac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

[5 marks]

continued…

Question 11 continued

(c) (i) attempt to use product rule (at least once) ***(M1)***

$$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$. Allow unsimplified correct answer.

$$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\alpha\tan^2\alpha + \frac{3}{4}\sec^3\alpha + 6\operatorname{cosec}\alpha\cot^2\alpha + 6\operatorname{cosec}^3\alpha\right)$$

(ii) attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha = \arctan 2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$$\sec\alpha = \sqrt{5} \text{ OR } \operatorname{cosec}\alpha = \frac{\sqrt{5}}{2} \text{ OR } \cot\alpha = \frac{1}{2} \text{ OR } \cos\alpha = \frac{1}{\sqrt{5}} \text{ OR } \sin\alpha = \frac{2}{\sqrt{5}}$$

Note: ***M1A1*** may be seen in part (d).

$$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3 \text{ (or equivalent)}$$ ***A2***

$$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$$

Note: Award ***A1*** for only two or three correct terms.
Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5}$$ ***AG***

[7 marks]

continued…

Question 11 continued

(d) (i) $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

(ii) $\left(L_{\min} =\right)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

(e) $\left(11.25 =\right)\frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [19 marks]

4. $A = \int_0^c \frac{x}{x^2+2}\,\mathrm{d}x$

EITHER

attempts to integrate by inspection or substitution using $u = x^2 + 2$ or $u = x^2$ ***(M1)***

Note: If candidate simply states $u = x^2 + 2$ or $u = x^2$, but does not attempt to integrate, do not award the ***(M1)*** .

Note: If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k \ln u$ or $k \ln(u+2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln(u+2)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^c$ ***A1***

Note: Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

Note: Award the ***(M1)*** only for expressions of the form $k \ln\left(x^2+2\right)$.

$\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^c$ ***A1***

Note: Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln\left(c^2+2\right) - \ln 2\right)\left(= \ln 3\right)$ OR $\frac{1}{2}\ln\left(c^2+2\right) - \frac{1}{2}\ln 2\left(= \ln 3\right)$

continued…

Question 4 continued

correctly applies at least one log law to their expression ***(M1)***

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)(=\ln 3)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}(=\ln 3)$ OR $\ln\left(\frac{c^2+2}{2}\right)=\ln 9$

OR $\ln(c^2+2)-\ln 2=\ln 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}(=\ln 3)$ OR $\ln\frac{\sqrt{c^2+2}}{\sqrt{2}}(=\ln 3)$

Note: Condone the absence of $\ln 3$ up to this stage.

$\frac{c^2+2}{2}=9$ OR $\sqrt{\frac{c^2+2}{2}}=3$ ***A1***

$c^2=16$

$c=4$ ***A1***

Note: Award ***A0*** for $c=\pm 4$ as a final answer.

Total [6 marks]

10. (a) $y^2 = 9 - x^2$ OR $y = \pm\sqrt{9-x^2}$ ***A1***

(since $y > 0$) $\Rightarrow y = \sqrt{9-x^2}$ ***AG***

[1 mark]

(b) $b = 2y \left(= 2\sqrt{9-x^2}\right)$ or $h = x + 3$ ***(A1)***

attempts to substitute their base expression and height expression into $A = \frac{1}{2}bh$ ***(M1)***

$A = \sqrt{9-x^2}(x+3)$ (or equivalent) $\left(= \frac{2(x+3)\sqrt{9-x^2}}{2} = x\sqrt{9-x^2} + 3\sqrt{9-x^2}\right)$ ***A1***

[3 marks]

(c) **METHOD 1**

attempts to use the product rule to find $\frac{\mathrm{d}A}{\mathrm{d}x}$ ***(M1)***

attempts to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^2}$ ***(M1)***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} = \right) \sqrt{9-x^2} + (3+x)\left(\frac{1}{2}\right)\left(9-x^2\right)^{-\frac{1}{2}}(-2x) \left(= \sqrt{9-x^2} - \frac{x^2+3x}{\sqrt{9-x^2}}\right)$ ***A1***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} = \right) \frac{9-x^2}{\sqrt{9-x^2}} - \frac{x^2+3x}{\sqrt{9-x^2}} \left(= \frac{9-x^2-\left(x^2+3x\right)}{\sqrt{9-x^2}}\right)$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

continued…

Question 10 continued

METHOD 2

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{\mathrm{d}A}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}x}$

attempts to find $\frac{\mathrm{d}A}{\mathrm{d}y}$ where $A=y(x+3)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y^2=9-x^2$ ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}y}=y\frac{\mathrm{d}x}{\mathrm{d}y}+x+3$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$ (or equivalent) ***A1***

substitutes their $\frac{\mathrm{d}A}{\mathrm{d}y}$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into $\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{\mathrm{d}A}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}x}=\left(y\left(-\frac{y}{x}\right)+x+3\right)\left(-\frac{x}{y}\right)$ (or equivalent)

$=\frac{9-x^2-x^2-3x}{\sqrt{9-x^2}}$ (or equivalent) ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

[4 marks]

continued…

Question 10 continued

(d) $\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \left(\frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}} = 0 \right)$ ***(M1)***

attempts to solve $9 - 3x - 2x^2 = 0$ (or equivalent) ***(M1)***

$-(2x - 3)(x + 3)(= 0)$ OR $x = \frac{3 \pm \sqrt{(-3)^2 - 4(-2)(9)}}{2(-2)}$ (or equivalent) ***(A1)***

$x = \frac{3}{2}$ ***A1***

Note: Award the above ***A1*** if $x = -3$ is also given.

substitutes their value of x into either $y = \sqrt{9 - x^2}$ or $y = -\sqrt{9 - x^2}$ ***(M1)***

Note: Do not award the above ***(M1)*** if $x \leq 0$.

$y = -\sqrt{9 - \left(\frac{3}{2}\right)^2}$

$= -\frac{\sqrt{27}}{2} \left(= -\frac{3\sqrt{3}}{2}, = -\sqrt{\frac{27}{4}}, = -\sqrt{6.75} \right)$ ***A1***

[6 marks]

Total [14 marks]

12. (a) let $t = \sqrt{x}$ ***M1***

$t^2 = x \Rightarrow 2t\ \mathrm{d}t = \mathrm{d}x$ (or equivalent) ***A1***

so $\int \cos\sqrt{x}\ \mathrm{d}x = 2\int t\cos t\ \mathrm{d}t$ ***A1***

attempts integration by parts ***(M1)***

$u = 2t,\ \mathrm{d}v = \cos t\ \mathrm{d}t,\ \mathrm{d}u = 2\ \mathrm{d}t,\ v = \sin t$

$2\int t\cos t\ \mathrm{d}t = 2t\sin t - 2\int \sin t\ \mathrm{d}t$ ***(A1)***

$= 2t\sin t + 2\cos t + C$ ***A1***

substitution of $t = \sqrt{x}\ \Rightarrow \int \cos\sqrt{x}\ \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ ***AG***

[6 marks]

(b) $x_{n+1} = \dfrac{\left(2\left(n+1\right)-1\right)^2\pi^2}{4}\ \left(= \dfrac{\left(2n+1\right)^2\pi^2}{4}\right)$ ***A1***

[1 mark]

continued…

Question 12 continued

(c) area of R_n is $\left|\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x\right|$ ***(M1)***

Note: Modulus may be seen at a later stage.

$$= \left|\left[2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}\right]_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}}\right| \quad \textbf{A1}$$

Note: Condone $+C$ at this stage.

attempts to substitute their limits into their integrated expression ***(M1)***

$$= 2\left|\frac{(2n+1)\pi}{2}\times\sin\frac{(2n+1)\pi}{2} + \cos\frac{(2n+1)\pi}{2} - \left(\frac{(2n-1)\pi}{2}\times\sin\frac{(2n-1)\pi}{2} + \cos\frac{(2n-1)\pi}{2}\right)\right|$$

A1

$$= 2\left|(-1)^n\frac{(2n+1)\pi}{2} - \left((-1)^{n+1}\frac{(2n-1)\pi}{2}\right)\right| \text{ (or equivalent)} \quad \textbf{A1}$$

$$= 2\left|(-1)^n\frac{(2n+1)\pi}{2} + (-1)^n\frac{(2n-1)\pi}{2}\right| \quad \textbf{A1}$$

$$= 2\left|(-1)^n\frac{4n\pi}{2}\right|$$

$$= 4n\pi \quad \textbf{A1}$$

Note: Award a maximum of (***M1)A1M1A1A1A0A0*** for only attempting to calculate $\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x$, and not applying the modulus.

[7 marks]

continued…

Question 12 continued

(d) **EITHER**

attempts to find $(d =) R_{n+1} - R_n$ ***M1***

$(d =) 4(n+1)\pi - 4n\pi$

$= 4\pi$ ***A1***

Note: Award ***M0*** for consideration of special cases for example R_3 and R_2.
Accept $d = k\pi$.

which is a constant (common difference is 4π) ***R1***

OR

an arithmetic sequence is of the form $u_n = dn + c$ $(u_n = dn + u_1 - d)$ ***M1***

attempts to compare $u_n = dn + c$ $(u_n = dn + u_1 - d)$ and $R_n = 4n\pi$ ***M1***

$d = 4\pi$ and $c = 0$ $(u_1 - d = 0)$ ***A1***

Note: Accept $d = k\pi$.

THEN

so the areas of the regions form an arithmetic sequence ***AG***

[3 marks]

Total [17 marks]

12.

Note: Penalise only once for an answer not given to six significant figures in parts (a), c(ii) and (d)(ii).

(a) attempt to use Euler's method ***(M1)***

$$y_{n+1} = y_n + 0.03\left(\frac{x_n{}^2 y_n - y_n}{x_n{}^2 + 1}\right), y_1 = 3 + 0.03\left(\frac{0-3}{0+1}\right)$$

$y_1 = 2.91$ ***(A1)***

at least one **correct** further intermediate value given to at least 3 significant figures ***(A1)***

y_0	3
y_1	2.91
y_2	2.82285...
y_3	2.73877...
y_4	2.65793...
y_5	2.58046...

$y(0.15) \approx y_5 = 2.58046160...$

$= 2.58046$ ***A1***

Note: Award final ***A1*** for the correct answer seen as the last line in a table.
If the table goes beyond this value and the correct answer is not explicitly identified award maximum ***(M1)(A1)(A1)A0***

[4 marks]

continued…

Question 12 continued

(b) (i) $\frac{dy}{dx} = -3$ ***A1***

(ii) **METHOD 1**

attempt to use quotient (or product) rule on

$\frac{dy}{dx} = \frac{x^2y - y}{x^2 + 1}\left(= \left(x^2 + 1\right)^{-1}\left(x^2y - y\right)\right)$ ***(M1)***

attempt to use product rule and implicit differentiation on x^2y ***(M1)***

$\frac{d}{dx}\left(x^2y - y\right) = x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}$ (seen anywhere) ***A1***

$= 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$)

$$\frac{d^2y}{dx^2} = \frac{\left(x^2 + 1\right)\left(x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}\right) - \left(x^2y - y\right)(2x)}{\left(x^2 + 1\right)^2}$$

OR $\frac{d^2y}{dx^2} = \left(x^2 + 1\right)^{-1}\left(x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}\right) - 2x\left(x^2 + 1\right)^{-2}\left(x^2y - y\right)$ ***A1***

$\frac{d^2y}{dx^2} = 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$) ***AG***

METHOD 2

$\left(x^2 + 1\right)\frac{dy}{dx} = x^2y - y$ ***A1***

attempt to use product rule and implicit differentiation ***(M1)***

$2x\frac{dy}{dx} + \left(x^2 + 1\right)\frac{d^2y}{dx^2} = x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$\frac{d^2y}{dx^2} = 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$) ***AG***

[5 marks]

continued…

Question 12 continued

(c) (i) $3-3x+\frac{x^2}{2!}(3)+\frac{x^3}{3!}(9)+...\left(=3-3x+\frac{3}{2}x^2+\frac{3}{2}x^3+...\right)$ ***A1A1***

Note: Award ***A1*** for first three terms, ***A1*** for fourth term

(ii) $y(0.15)=2.58881125...$

$=2.58881$ ***A1***

[3 marks]

continued…

Question 12 continued

(d) (i) **EITHER**

attempt to separate variables ***M1***

$\int \frac{1}{y}\mathrm{d}y = \int \frac{x^2-1}{x^2+1}\mathrm{d}x$ OR $\int \frac{1}{y}\mathrm{d}y = \int \left(1-\frac{2}{x^2+1}\right)\mathrm{d}x$

$\ln y = x - 2\arctan x + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ or $\ln|y|$, ***A1*** for $x - 2\arctan x$.
Condone missing $+c$ at this stage.

OR

attempt to use integrating factor ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(1-\frac{2}{x^2+1}\right) = 0$

IF $= \mathrm{e}^{\int -\left(1-\frac{2}{x^2+1}\right)\mathrm{d}x} = \mathrm{e}^{-x+2\arctan x}$ ***A1***

$\mathrm{e}^{-x+2\arctan x}\frac{\mathrm{d}y}{\mathrm{d}x} - y\mathrm{e}^{-x+2\arctan x}\left(1-\frac{2}{x^2+1}\right) = 0$

$\mathrm{e}^{-x+2\arctan x}y = A$ ***A1***

THEN

$\ln y = x - 2\arctan x + c$ OR $y = A\mathrm{e}^{x-2\arctan x}$

attempt to find c or A using $x = 0, y = 3$ ***(M1)***

$\ln 3 = 0 - 2\arctan 0 + c$ OR $3 = A\mathrm{e}^{0-2\arctan 0}$

$c = \ln 3$ OR $A = 3$ ***(A1)***

Note: This ***A1*** should not be awarded if a correct value of c or A is preceded by incorrect working.

$y = \mathrm{e}^{x-2\arctan x+\ln 3}\left(=3\mathrm{e}^{x-2\arctan x}\right)$ ***A1***

continued…

Question 12 continued

(ii) $y(0.15) = 2.58786288...$

$= 2.58786$ ***A1***

[7 marks]

(e) the graph of $y = f(x)$ is concave up OR $\frac{d^2y}{dx^2} > 0$ (for $0 \leq x < 1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

hence tangents used (in Euler's method) give an underestimate,

so the approximate value for y when $x = 1.5$ is less than the actual value. ***R1***

[2 marks]

Note: ***R1*** is dependent on ***A1***, as well as reference to tangents, in words or on a diagram.

Total [21 marks]

5. (a) recognize that acceleration is zero when $v'(t) = 0$ OR at a local maximum on the graph of v ***(M1)***

$t_1 = 0.394791...$

$t_1 = 0.395 \left(= \arctan\left(\frac{5}{12}\right)\right)$ (seconds) ***A1***

[2 marks]

(b) recognition that $v = 0$ ***(M1)***

sketch OR $t = 4.71238...$ OR $t = 10.9955...$

$t_2 = 10.9955...$

$t_2 = 11.0 \left(= \frac{7\pi}{2}\right)$ ***A1***

[2 marks]

(c) $\int_{t_1}^{t_2} |v| \, \mathrm{d}t$ OR $\int_{0.394791...}^{10.9955...} |v| \, \mathrm{d}t$ OR $\int_{0.394791...}^{4.71238...} v \, \mathrm{d}t + \int_{4.71238...}^{10.9955...} |v| \, \mathrm{d}t \left(= 6.53806... + 1.29313...\right)$

OR $\int_{0.394791...}^{4.71238...} v \, \mathrm{d}t - \int_{4.71238...}^{10.9955...} v \, \mathrm{d}t \left(= 6.53806... - \left(-1.29313...\right)\right)$ ***(A1)***

distance = $7.83118\ldots$

= 7.83 (m) ***A1***

[2 marks]

Total [6 marks]

7. attempts to express x in terms of $\tan y$ ***(M1)***

$x = \tan y + 2$ ***(A1)***

let V be the volume of the solid

correctly uses $V = \pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

Note: Award ***M0*** for $V = \pi\int_a^b \left(\arctan(x-2)\right)^2 dy$

$V = \pi\int_0^{\frac{\pi}{3}} \left(\tan y + 2\right)^2 \mathrm{d}y \left(= \pi \int_0^{1.0472...} \left(\tan y + 2\right)^2 \mathrm{d}y \right)$ ***(A1)***

$= 24.0213...$

$= 24.0 \ \left(= \pi\left(4\ln 2 + \pi + \sqrt{3}\right)\right)$ ***A1***

Note: GDC in degrees gives13.3

[5 marks]

12. (a) attempts to use $y_1 = y_0 + h \times f(x_0, y_0)$ ***(M1)***

$$y_1 = 2 + 0.1 \times \frac{1^2 + 3(2)^2}{2}$$

$$= 2.65$$ ***A1***

Note: Award ***(M1)A0*** for 2.35.

[2 marks]

(b) let $y = vx$ ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$ ***(A1)***

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x^2 + 3v^2x^2}{vx^2}$$ ***(M1)***

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1 + 2v^2}{v}$$ ***A1***

attempt to separate variables x and v ***M1***

$$\int \frac{v}{2v^2 + 1}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$$

$$\frac{1}{4}\ln(2v^2 + 1) = \ln x + C$$ ***A1***

Note: Condone the absence of C to this stage.

continued…

Question 12 continued

EITHER

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+C$$

when $x=1,\ y=2\Rightarrow C=\frac{1}{4}\ln 9$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+\frac{1}{4}\ln 9$$

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=\sqrt{3}x$$

OR

$$\ln\left(\frac{2y^2}{x^2+1}\right)=\ln\left(x^4\right)+\ln C$$

$$\frac{2y^2}{x^2}+1=Cx^4$$

when $x=1,\ y=2\Rightarrow C=9$ ***M1***

THEN

$$\frac{2y^2}{x^2}+1=9x^4$$ ***A1***

$$y=x\sqrt{\frac{9x^4-1}{2}}$$ ***AG***

[8 marks]

continued…

Question 12 continued

(c) $y = 2.71422...$

$y = 2.71$ ***A1***

[1 mark]

(d) **EITHER**

the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ is concave up (for $1 \leq x \leq 1.1$) ***A1***

OR

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$ (for $1 \leq x \leq 1.1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

THEN

hence the tangent drawn using Euler's method gives an underestimate of the true value, so the value of y when $x = 1.1$ is greater than the approximate value found in part (a) ***R1***

Note: Only award ***R1*** if there is reference to tangent (in words or in a diagram).

[2 marks]

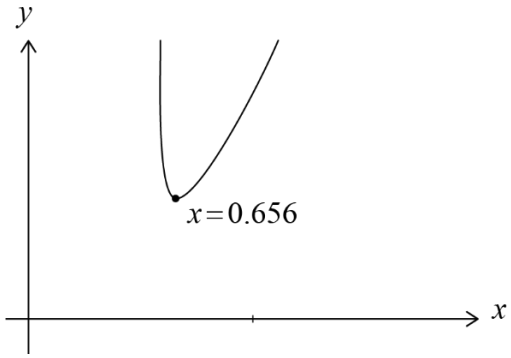
continued…

Question 12 continued

(e) **EITHER**

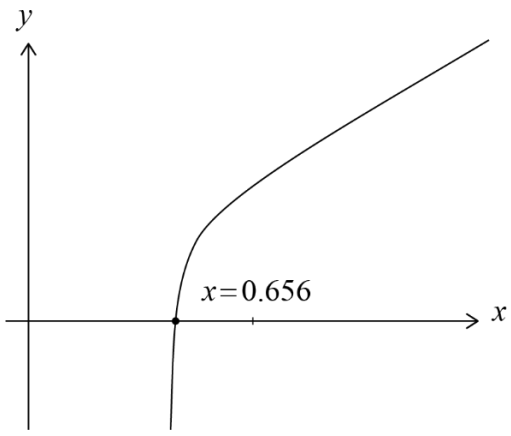
Note: Award the first ***A*** mark for a correct graph seen in part (d).

correct graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***



OR

correct graph of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***A1***



THEN

$x = 0.655996...$

$x = 0.656$ ***A1***

[2 marks]

continued…

Question 12 continued

(f) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ (seen anywhere) ***(A1)***

Note: Award ***(A1)*** for equivalent answers (seen anywhere) such as $\frac{-x^4+x^2y^2+6y^4}{x^2y^3}=0$ or $-x^4+x^2y^2+6y^4=0$.

EITHER

divides $-x^4+x^2y^2+6y^4(=0)$ through by y^4 ***(M1)***

$-\frac{x^4}{y^4}+\frac{x^2}{y^2}+6(=0)$

$m=\frac{y}{x}\Rightarrow-\frac{1}{m^4}+\frac{1}{m^2}+6(=0)$ ***(M1)***

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$ ***A1***

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

OR

divides $-x^4+x^2y^2+6y^4(=0)$ through by x^2y^2 ***(M1)***

$-\frac{x^2}{y^2}+1+6\frac{y^2}{x^2}(=0)$

$m=\frac{y}{x}\Rightarrow-\frac{1}{m^2}+1+6m^2(=0)$ ***(M1)***

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$ ***A1***

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

continued…

Question 12 continued

OR

attempts to factorize $-x^4 + x^2y^2 + 6y^4 (= 0)$ ***(M1)***

$-\left(x^2 - 3y^2\right)\left(x^2 + 2y^2\right)(= 0)$ ***A1***

attempts to solve their factorized equation ***(M1)***

$\Rightarrow y = \pm\frac{1}{\sqrt{3}}x \left(y^2 = -\frac{1}{2}x^2 \right)$

THEN

$y = \frac{1}{\sqrt{3}}x$ (and so $m = \frac{1}{\sqrt{3}}$) ***A1***

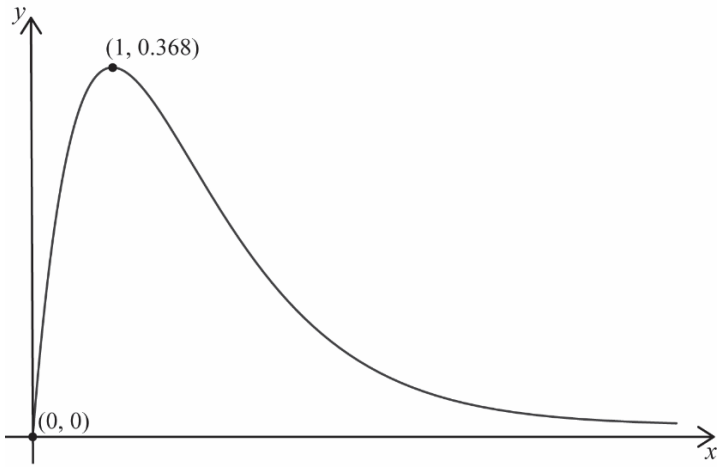
as $x > 0, y > 0$ (or equivalent reasoning/justification) ***R1***

Note: Award ***R1*** for $y = \frac{1}{\sqrt{3}}x$ (and so $m = \frac{1}{\sqrt{3}}$) as $y \neq -\frac{1}{\sqrt{3}}x$ and $x^2 + 2y^2 = 0$ for $x = 0$ and $y = 0$ only.

[6 marks]

Total [21 marks]

1. (a)



A1 for $(1,\ 0.368)$ or $\left(1,\ \frac{1}{\mathrm{e}}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through,the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

continued…

Question 1 continued

(b) $\int_0^b x\mathrm{e}^{-x}\,\mathrm{d}x$ ***(A1)***

Note: Award (***A1)*** for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$=\left[-x\mathrm{e}^{-x}\right]_0^b+\int_0^b \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for each part (including the correct sign with each)

$=\left[-x\mathrm{e}^{-x}\right]_0^b-\left[\mathrm{e}^{-x}\right]_0^b$ ***A1***

Note: Award ***A1*** for correct second term. Condone absence of limits to this point

attempt to substitute limits ***M1***

$=-b\mathrm{e}^{-b}-\mathrm{e}^{-b}+1$ ***A1***

$=\dfrac{\mathrm{e}^b-b-1}{\mathrm{e}^b}$ ***AG***

[6 marks]

(c) (i) $\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^b-b-1}{\mathrm{e}^b}=\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^b-1}{\mathrm{e}^b}$ ***A1***

Note: Award ***A1*** for correct quotient. Condone absence of limit.

$\left(=\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^b}{\mathrm{e}^b}\right)=1$ ***A1***

[2 marks]

(ii) $\left(\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x=\right)\ 1$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) (i) correct integral ***(M1)***

Note: Award ***M1*** for correct integrand with limits from 0 to a larger number.

24 ***A1***

[2 marks]

(ii) 120 ***A1***

Note: The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) $A_n = n!$ ***A1***

[1 mark]

continued…

Question 1 continued

(f)

Note: Accept starting at $n = 0$, throughout this proof.

$n = 1$

$A_1 = 1$ and $1! = 1$ ***M1A1***

Note: Award ***M1*** for considering the case where $n = 1$, and ***A1*** if it is clear that both $A_1 = 1$ and $1! = 1$ have been considered.

so true for $n = 1$

assume true for $n = k$, ($A_k = \int_0^\infty x^k \mathrm{e}^{-x}\,\mathrm{d}x = k!$) ***M1***

Note: Award ***M0*** for statements such as "let $n = k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

when $n = k + 1$

attempt to integrate by parts ***M1***

Note: To obtain the ***M1***, a minimum of an expression +/- an integral must be seen.

$\int_0^\infty x^{k+1}\mathrm{e}^{-x}\,\mathrm{d}x = \left[-x^{k+1}\mathrm{e}^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1***

$(k+1)\int_0^\infty x^k \mathrm{e}^{-x}\,\mathrm{d}x$ simplified to $(k+1)\,k!$ seen ***A1***

$= 0 + (k+1)\,k!$

Note: Condone omission of the zero.

$= (k+1)!$ ***A1***

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$. ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[8 marks]

Total [25 marks]

2. (a) $x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b) (i) $(x_1 =)\ 6$ ***A1***

Note: Award the ***A1*** if 6 seen in part (ii).

[1 mark]

(ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at $(6, 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

(c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

continued…

Question 2 continued

(d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(ii) $x_1 + x_2 + ... + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times ... \times x_n)^{\frac{1}{n}} \leq \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times ... \times x_n \leq \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n(S) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) (i) $M_3(12) = 4^3 = 64$ ***A1***

[1 mark]

(ii) $M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(iii) $M_5(12) = 2.4^5 = 79.6 \ \ (79.6262...)$ ***A1***

[1 mark]

continued…

Question 2 continued

(f) considering $M_n(12)$ for higher values of n

$P(12) = 81$ ***A1***

$n = 4$
A1

Note: Award ***A0A0*** for $P(12) = 82.6$ and $n = 4.41$.

[2 marks]

(g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n = 7$ ***A1***

$P(20) = \left(\frac{20}{7}\right)^7 = 1550 \ (1554.260...)$ ***A1***

Note: Award ***(M1)A0A1*** for $n = 7.36$ and $P(20) = 1570$.

[3 marks]

continued…

Question 2 continued

(h) **EITHER**

$\ln(g(x)) = x\left(\ln(S) - \ln x\right)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g'(x)}{g(x)} = \ln S - \ln x - x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g'(x)}{g(x)} = \ln\frac{S}{x} + \left(x\frac{x}{S}\times\frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g\left(x\right) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g'\left(x\right) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x\times\frac{x}{S}\times\left(-Sx^{-2}\right) + \ln\left(\frac{S}{x}\right)\right]$

$= \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right) - 1\right]$ ***A1***

THEN

$g'(x) = \left(\ln\frac{S}{x} - 1\right)g(x)$

$g(x) \neq 0$

$g'(x) = 0 \Rightarrow \ln\frac{S}{x} - 1 = 0$ ***M1***

$x = \frac{S}{\mathrm{e}}\ (0.368S,\ 0.36789...S)$ ***A1***

[6 marks]

continued…

Question 2 continued

(i) $\ln\left(g(x)\right) = x\ln\left(\frac{S}{x}\right) \Rightarrow \ln\left(g(x)\right) = \ln\left(\frac{S}{x}\right)^x$ ***M1***

$g(x) = \left(\frac{S}{x}\right)^x$ ***A1***

$= M_x(S)$ for $x \in \mathbb{Z}^+$ ***AG***

[2 marks]

(j) $\frac{100}{\mathrm{e}} = 36.8$ ***M1***

$\left(\frac{100}{36}\right)^{36} = 9.3996...\times10^{15}$ **AND** $\left(\frac{100}{37}\right)^{37} = 9.47406...\times10^{15}$ ***R1***

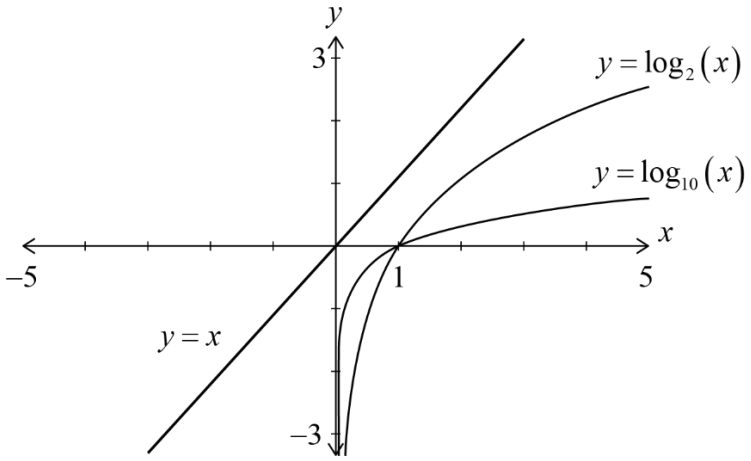
largest possible product is 9.47×10^{15} $(9.47406...\times10^{15})$ ***A1***

Note: Award ***A1*** independently of the ***R1*** (but not independently of the ***M1***).

[3 marks]

Total [30 marks]

1. (a)



clearly labelled graphs of $y = \log_2 x$ and $y = \log_{10} x$ with correct domain, asymptotic behaviour and concavity evident ***A1***

correct relative positions of the two log graphs both above and below the x-axis ***A1***

$(1, 0)$ indicated (coordinates not required) ***A1***

correct graph of $y = x$ ***A1***

[4 marks]

continued…

Question 1 continued

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x-\ln x\right)$

$=1-\frac{1}{x}$ ***A1***

attempts to solve their $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ for x ***(M1)***

$1-\frac{1}{x}=0\Rightarrow x=1$

(when $x=1$,) $x-\ln x=1$ ***A1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-\frac{1}{x}\right)$

$=\frac{1}{x^2}$ ***A1***

$\frac{1}{x^2}>0$ (when $x=1$) ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1*** for either '$1>0$' or a graph of $y=\frac{1}{x^2}>0$ or 'the graph of $y=x-\ln x$ is concave-up'. Do not award ***R1*** if the second derivative is incorrect.

continued…

Question 1 continued

OR

for $(0<)x<1$, $1-\frac{1}{x}<0$ ***R1***

for $x>1$, $1-\frac{1}{x}>0$ ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1R1*** for either a clearly labelled sign diagram/table (accept correct numerical values) or the graph of $y=1-\frac{1}{x}$ with sign change in gradient indicated.

Note: Award a maximum of ***A0(M1)A1A0R1*** or ***A0(M1)A1R0R1*** if no symbolic derivatives are seen.

[5 marks]

(c)

EITHER

$x-\ln x \geq 1 \ \left(x \in \mathbb{R}^+\right)$ ***R1***

OR

$x-\ln x > 0 \ \left(x \in \mathbb{R}^+\right)$ ***R1***

THEN

so $x > \ln x$ ***AG***

[1 mark]

continued…

Question 1 continued

(d)

Interval	Number of intersection points
$0 < a < 1$	$p = 1$
$1 < a < 1.4$	$q = 2$
$1.5 < a < 2$	$r = 0$

A1A2A1

Note: Award ***A1*** for $p = 1$, ***A2*** for $q = 2$ and ***A1*** for $r = 0$.

[4 marks]

(e) **METHOD 1**

EITHER

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

continued…

Question 1 continued

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\frac{\ln x}{\ln a} = \frac{1}{\ln a}$ OR $\ln x = 1$ OR $\left(e^{\frac{1}{x}}\right)^x = x$ for x ***(M1)***

$x = e$

coordinates of P are (e, e) (accept $x = e$, $y = e$) ***A1A1***

attempts to solve $\frac{1}{\ln a} = e$ OR $\log_a e = e$ for a analytically ***(M1)***

$\ln a = \frac{1}{e}$ OR $a^e = e$

$a = e^{\frac{1}{e}}$ ***A1***

continued…

Question 1 continued

METHOD 2

EITHER

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\log_a\left(\frac{1}{\ln a}\right) = \frac{1}{\ln a}$ for a ***(M1)***

EITHER

$\frac{\ln\left(\frac{1}{\ln a}\right)}{\ln a} = \frac{1}{\ln a} \Rightarrow \ln\left(\frac{1}{\ln a}\right) = 1$

OR

for example, writes $a^{\log_a\left(\frac{1}{\ln a}\right)} = a^{\frac{1}{\ln a}}$ and then attempts to apply appropriate index/log laws to both sides: $\ln a = \frac{\log_a a}{\log_a \mathrm{e}}$ and so $\frac{1}{\ln a} = \log_a \mathrm{e}$

$a^{\frac{1}{\ln a}} = a^{\log_a \mathrm{e}} = \mathrm{e}$

continued…

Question 1 continued

THEN

attempts to solve $\frac{1}{\ln a} = \mathrm{e}$ OR $\log_a \mathrm{e} = \mathrm{e}$ for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ OR $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

$x = \frac{1}{\ln \mathrm{e}^{\frac{1}{\mathrm{e}}}} = \frac{1}{\frac{1}{\mathrm{e}}}$

coordinates of P are $\left(\mathrm{e}, \mathrm{e}\right)$ (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

continued…

Question 1 continued

METHOD 3

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

(equation of the tangent at (x_1, y_1) is) $y = \frac{1}{x_1 \ln a}(x - x_1) + \frac{\ln x_1}{\ln a}$ (or equivalent) ***A1***

compares this equation with $y = x$ and attempts to form at least one of the following ***M1***

$\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$

attempts to solve $\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$ for x_1 ***(M1)***

$x_1 = \mathrm{e}$

coordinates of P are (e, e) (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

attempts to solve $\frac{1}{\mathrm{e} \ln a} = 1$ (or equivalent) for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ **OR** $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

[8 marks]

continued…

Question 1 continued

(f) (i) $1 < a < e^{\frac{1}{e}}$ ***A1***

Note: Award ***A0*** for $a < e^{\frac{1}{e}}$.

[1 mark]

(ii) $a > e^{\frac{1}{e}}$ ***A1***

Note: Only award ***FT*** for $1.4 < a < 1.5$. If the value of a is not exact, e.g. 1.44, award at most ***A0A1*** in part (f) for a consistent approximate endpoint value.
If a value of a is not found in part (e), award at most ***A0A1*** in part (f) for a consistent approximate endpoint value provided that $1.4 < a < 1.5$.

[1 mark]

Total [24 marks]

8. (a) **METHOD 1**

attempt to integrate by parts ***(M1)***

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ \mathrm{d}x$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ ***(M1)***

$\int x \ln x \ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$ ***A1***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ \mathrm{d}x = x \ln x - x$ **)**

attempt to integrate by parts ***(M1)***

$u = x \ln x$, $\mathrm{d}v = \ln x \mathrm{d}x$ ***(M1)***

$\int x \ln x (\ln x) \ \mathrm{d}x = x \ln x (x \ln x - x) - \int (\ln x + 1)(x \ln x - x) \mathrm{d}x$ ***A1***

$= x \ln x (x \ln x - x) - \int x(\ln x)^2 \ \mathrm{d}x + \int x \mathrm{d}x$ ***A1***

$2I = x \ln x (x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x\ \mathrm{d}x$ **)**

$\int x\ln x\ \mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$

attempt to integrate by parts ***(M1)***

$u = \ln x\ ,\ \mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$\int (x\ln x)\ln x\ \mathrm{d}x = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)\mathrm{d}x$ ***A1***

$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int\left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)\mathrm{d}x$ ***A1***

$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$ ***A1***

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$ ***A1***

$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x)=\dfrac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$=\dfrac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$=\dfrac{\sin^2(kx)}{x^2}\left(=f(x)\right)$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ (***M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim_{x\to 0}\left(f\left(x\right)\right)=\lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$=\lim_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+...\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+...\right)$ ***A1***

$\lim_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{k^2x^2\left(+...\right)}{x^2}\right)$ ***M1***

$=\lim_{x\to 0}\left(k^2+\text{terms in }x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits **(*M1*)**

$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = \lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{k\cos kx}{1}\right) = k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx) = kx(+...)$

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{kx+...}{x}\right) = \lim_{x\to 0}(k + \text{terms in } x) = k$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = k\times k = k^2$ ***A1***

$k^2 = 16 \Rightarrow k = 4(k>0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)\mathrm{e}^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}), \left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right), (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***

For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x} - (4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4\mathrm{e} < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4\mathrm{e} < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{\mathrm{e}} > 0$ so minimum ***R1***

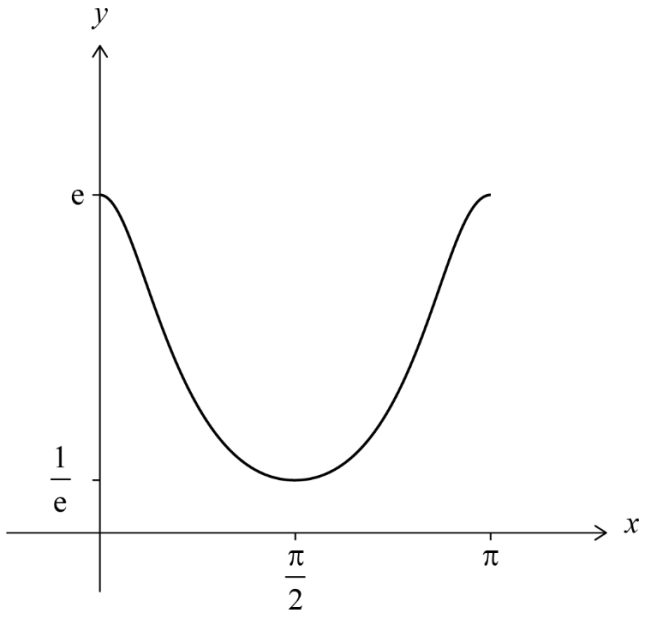
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0,e),(\pi,e)$ and minimum point $\left(\frac{\pi}{2},\frac{1}{e}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + ...$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + ...;\ e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + ...$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + ...\right)\left(1 + \frac{2}{3}x^4 + ...\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x) \approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x \approx \mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

8. (a) **METHOD 1**

attempt to integrate by parts ***(M1)***

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ \mathrm{d}x$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ ***(M1)***

$\int x \ln x \ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$ ***A1***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ \mathrm{d}x = x \ln x - x$ **)**

attempt to integrate by parts ***(M1)***

$u = x \ln x$, $\mathrm{d}v = \ln x \mathrm{d}x$ ***(M1)***

$\int x \ln x(\ln x) \ \mathrm{d}x = x \ln x(x \ln x - x) - \int (\ln x + 1)(x \ln x - x)\mathrm{d}x$ ***A1***

$= x \ln x(x \ln x - x) - \int x(\ln x)^2 \ \mathrm{d}x + \int x \mathrm{d}x$ ***A1***

$2I = x \ln x(x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x\ \mathrm{d}x$ **)**

$$\int x\ln x\ \mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts ***(M1)***

$u = \ln x\ ,\ \mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$$\int (x\ln x)\ln x\ \mathrm{d}x = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x) = \frac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$= \frac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$= \frac{\sin^2(kx)}{x^2} (= f(x))$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ (***M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim_{x\to 0}\left(f\left(x\right)\right)=\lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$=\lim_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+...\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+...\right)$ ***A1***

$\lim_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{k^2x^2\left(+...\right)}{x^2}\right)$ ***M1***

$=\lim_{x\to 0}\left(k^2+\text{terms in }x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits **(*M1*)**

$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = \lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{k\cos kx}{1}\right) = k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx) = kx(+...)$

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{kx+...}{x}\right) = \lim_{x\to 0}(k + \text{terms in } x) = k$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = k\times k = k^2$ ***A1***

$k^2 = 16 \Rightarrow k = 4(k > 0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)\mathrm{e}^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}), \left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right), (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***

For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x} - (4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4\mathrm{e} < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4\mathrm{e} < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{\mathrm{e}} > 0$ so minimum ***R1***

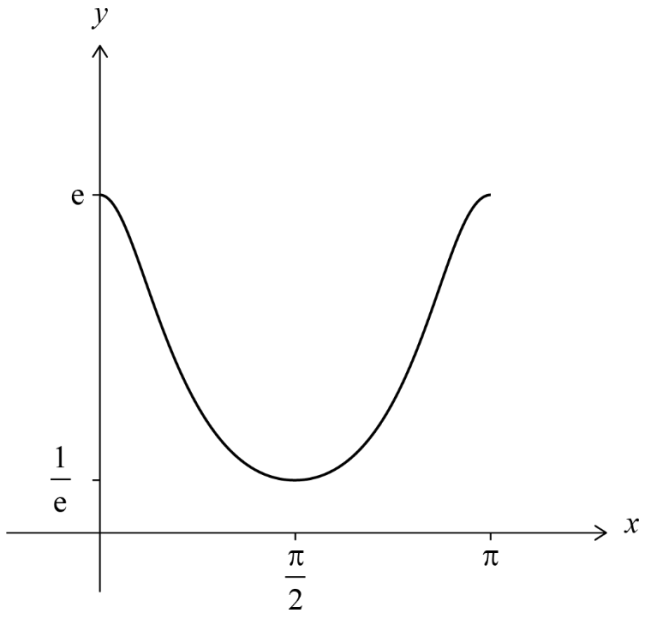
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0,e),(\pi,e)$ and minimum point $\left(\frac{\pi}{2},\frac{1}{e}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + ...$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + ...;\ e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + ...$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + ...\right)\left(1 + \frac{2}{3}x^4 + ...\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x) \approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x \approx \mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

4. (a) recognizing at rest when $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ OR s is a minimum ***(M1)***

$q=5.73553...$

$=5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q=5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...}|v(t)|\,\mathrm{d}t$ OR $\int_0^{5.73...}\left|\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right|\mathrm{d}t$ OR $\left|\int_0^{5.73...}v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...}v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of dt.

Only accept $\left|\int_0^q v(t)\,\mathrm{d}t\right|$ if their value of q does not result in the particle changing direction in the first q seconds.

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: Special Cases:

Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q=1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ $(3.37644...$ from 3sf).

Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...}v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302...-(-4.3)$

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x = 0$, $y = 2$ to find the value of A ***M1***

$2 = A\mathrm{e}^{0} \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $|y-4| = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{dy}{dx} + \frac{1}{10}y = \frac{4}{10}$ so IF $e^{\int \frac{1}{10}dx} = e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}\frac{dy}{dx} + \frac{1}{10}e^{\frac{1}{10}x}y = \frac{4}{10}e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x}(+c)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(2e^0 = 4e^0 + c \Rightarrow) c = -2$ ***A1***

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x} - 2$

$y = 4 - 2e^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $= 2.0980199... - \left(4 - 2e^{-0.05}\right) = 0.000478749...$

$= 0.000479 \left(= 4.79 \times 10^{-4}\right)$ ***A1***

Note: Accept $0.000459 \left(= 4.59 \times 10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

4. (a) recognizing at rest when $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ OR s is a minimum ***(M1)***

$q=5.73553...$

$=5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q=5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...}|v(t)|\,\mathrm{d}t$ OR $\int_0^{5.73...}\left|\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right|\mathrm{d}t$ OR $\left|\int_0^{5.73...}v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...}v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of dt.

Only accept $\left|\int_0^q v(t)\,\mathrm{d}t\right|$ if their value of q does not result in the particle changing direction in the first q seconds.

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: Special Cases:

Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q=1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ $(3.37644...$ from 3sf).

Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...}v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302...-(-4.3)$

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
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$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x = 0$, $y = 2$ to find the value of A ***M1***

$2 = A\mathrm{e}^{0} \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $|y-4| = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{dy}{dx} + \frac{1}{10}y = \frac{4}{10}$ so IF $e^{\int \frac{1}{10}dx} = e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}\frac{dy}{dx} + \frac{1}{10}e^{\frac{1}{10}x}y = \frac{4}{10}e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x}(+c)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(2e^0 = 4e^0 + c \Rightarrow) c = -2$ ***A1***

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x} - 2$

$y = 4 - 2e^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $= 2.0980199... - \left(4 - 2e^{-0.05}\right) = 0.000478749...$

$= 0.000479 \left(= 4.79 \times 10^{-4}\right)$ ***A1***

Note: Accept $0.000459 \left(= 4.59 \times 10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

2. (a) (i) $\frac{y}{x}$ ***A1***

[1 mark]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(\frac{y}{x}\right)} \left(= -\frac{1}{m}\right)$ ***A1***

Note: Award ***A1*** for responses such as 'the gradient is the negative (opposite) reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).

Award ***A1*** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.

Do not award ***FT*** from part (a) (i).

so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***AG***

[1 mark]

Question 2 continued

(iii) attempts to separate variables x and y ***(M1)***

$$\int y \, \mathrm{d}y = -\int x \, \mathrm{d}x$$

Note: Award ***(M1)*** for $y \, \mathrm{d}y = -x \, \mathrm{d}x$.

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$ ***A1***

Note: Award ***A1*** for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.

Award ***A0*** for $\frac{y^2}{2} = -\frac{x^2}{2}$.

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

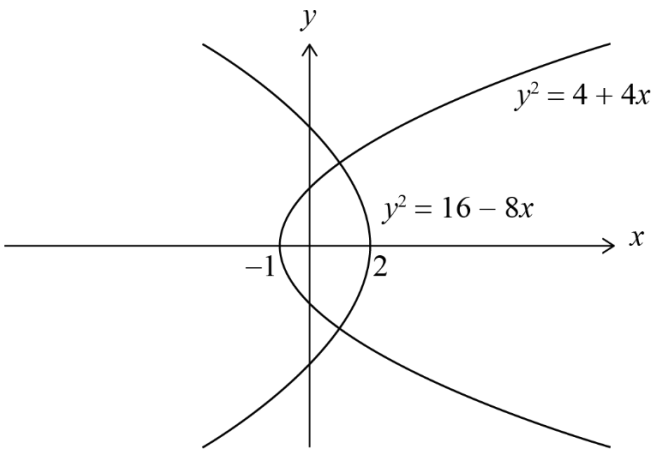
$\Rightarrow x^2 + y^2 = k$ (where $k = 2c$) ***AG***

[2 marks]

continued…

Question 2 continued

(b)



two parabolic shaped curves with approximately correct shape/position
(e.g. two intersection points, in first and fourth quadrant) ***A1A1***

x-intercepts -1 and 2 ***A1***

[3 marks]

continued…

Question 2 continued

(c) at intersection, $4a^2 - 4ax = 4b^2 + 4bx$

$4a^2 - 4b^2 = 4ax + 4bx$ ($a^2 - b^2 = ax + bx$, $4a^2 - 4b^2 - 4ax - 4bx = 0$) ***A1***

attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of $a^2 - b^2 - ax - bx = 0$ (or equivalent) ***(M1)***

$(a+b)(a-b) = (a+b)x$

Note: Accept alternative forms such as $4(a+b)(a-b) = 4(a+b)x$ or $(a+b)((a-b)-x) = 0$.

recognition that $a + b > 0$ (or equivalent, eg. $a > 0, b > 0$) (allows division by $a + b$) ***R1***

Note: Subsequent marks are not dependent on this ***R1***.

$x = a - b$ ***A1***

Note: As $x = a - b$ is an ***AG***, only award the above ***A1*** if $a^2 - b^2 = (a+b)(a-b)$ has been used.

substitutes $x = a - b$ into either $y^2 = 4a^2 - 4ax$ or $y^2 = 4b^2 + 4bx$ and attempts to simplify ***(M1)***

$y^2 = 4a^2 - 4a(a-b)$ OR $y^2 = 4b^2 + 4b(a-b)$

$y^2 = 4a^2 - 4a^2 + 4ab \Rightarrow y = \pm 2\sqrt{ab}$ ***A1***

so $\mathrm{M}\left(a-b, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b, -2\sqrt{ab}\right)$ ***AG***

[6 marks]

continued…

Question 2 continued

(d) **METHOD 1**

attempts implicit differentiation on either curve ***(M1)***

$\frac{dy}{dx} = -\frac{4a}{2y}$ (or equivalent) and $\frac{dy}{dx} = \frac{4b}{2y}$ (or equivalent) ***A1***

substitutes $y = 2\sqrt{ab}$ into either $\frac{dy}{dx} = -\frac{4a}{2y}$ or $\frac{dy}{dx} = \frac{4b}{2y}$ ***(M1)***

$\frac{dy}{dx} = -\sqrt{\frac{a}{b}}$ $\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{dy}{dx} = \sqrt{\frac{b}{a}}$ $\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

OR

eg. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

continued…

Question 2 continued

METHOD 2

attempts chain rule differentiation on either $y = \sqrt{4a^2 - 4ax}$ or $y = \sqrt{4b^2 + 4bx}$ ***(M1)***

$\frac{dy}{dx} = -\frac{2a}{\sqrt{4a^2 - 4ax}}$ (or equivalent) and $\frac{dy}{dx} = \frac{2b}{\sqrt{4b^2 + 4bx}}$ (or equivalent) ***A1***

substitutes $x = a - b$ into either $\frac{dy}{dx} = -\frac{2a}{\sqrt{4a^2 - 4ax}}$ or $\frac{dy}{dx} = \frac{2b}{\sqrt{4b^2 + 4bx}}$ ***(M1)***

$\frac{dy}{dx} = -\sqrt{\frac{a}{b}}$ $\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{dy}{dx} = \sqrt{\frac{b}{a}}$ $\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

OR

eg. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

[5 marks]

continued…

Question 2 continued

(e) (i) $g(x, y) = \dfrac{-\frac{x}{y} + \tan\frac{\pi}{4}}{1 - \left(-\frac{x}{y}\right)\tan\frac{\pi}{4}}$ ***(A1)***

$g(x, y) = \dfrac{-\frac{x}{y} + 1}{1 + \frac{x}{y}} \left(= \dfrac{\frac{-x+y}{y}}{\frac{y+x}{y}} \right)$ ***A1***

so $g(x, y) = \dfrac{y-x}{y+x}$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) let $y = vx$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x} =\right) \frac{vx - x}{vx + x} \left(= \frac{v-1}{v+1}\right)$ ***(A1)***

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single fraction in v ***(M1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{v+1}$ (or equivalent) ***(A1)***

attempts to separate variables x and v ***(M1)***

$\int \frac{v+1}{v^2+1}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$ (or equivalent)

$\frac{1}{2}\ln\left(v^2+1\right) + \arctan v = -\ln|x|\left(+d\right)$ (or equivalent) ***A1A1***

$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\frac{y}{x} + \ln|x| = d$ (or equivalent) ***A1***

[9 marks]

continued…

Question 2 continued

(f) **METHOD 1**

$$g(x,\ y)=\frac{\frac{1}{\tan\alpha}f\left(x,y\right)+1}{\frac{1}{\tan\alpha}-f\left(x,y\right)}$$ ***M1***

EITHER

as $\alpha\to\frac{\pi}{2}$, $\frac{1}{\tan\alpha}\to 0$, (hence $g(x,\ y)\to-\frac{1}{f(x,\ y)}$) ***R1***

OR

as $\alpha\to\frac{\pi}{2}$, $\tan\alpha\to\infty$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

THEN

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

METHOD 2

uses either $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$ or $\frac{1}{\tan\alpha}=\frac{\cos\alpha}{\sin\alpha}$ to form $g(x,\ y)=\frac{\cos\alpha f\left(x,y\right)+\sin\alpha}{\cos\alpha-\sin\alpha f\left(x,y\right)}$ ***M1***

as $\alpha\to\frac{\pi}{2}$, $\cos\alpha\to 0$ and $\sin\alpha\to 1$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

[2 marks]

Total [31 marks]

4. (a) recognising $\cos x = 2\sin x \cos x$ ***(M1)***

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) ***(A1)***

x - coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ ***A1***

Note: Award ***(M1)(A1)A0*** for solutions of 30° and 150°.

[3 marks]

(b) **METHOD 1**

attempt to integrate $\pm(\cos x - \sin 2x)$ ***(M1)***

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - \sin 2x)\,dx$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - 2\sin x \cos x)\,dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ OR $= \left[\sin x - \sin^2 x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for $\pm$ correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract ***M1***

$= \left(\sin\left(\frac{5\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi)\right)$ OR

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(1 - \frac{1}{2}\right)$ OR $= \left(\frac{1}{2} - \frac{1}{4}\right) - (1 - 1)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B.
Accept work done in degrees.

continued…

Question 4 continued

METHOD 2

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x = \left[\sin x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ and $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x = \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) ***M1***

$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)$ and $-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)$

attempt to subtract the two integrals in either order (seen anywhere) ***(M1)***

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)\right)$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x$

$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \quad \left(= -\frac{1}{4}\right)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B.

Accept work done in degrees.

[4 marks]

Total [7 marks]

6. METHOD 1

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1.***

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}\left(f(x)\right)^2\mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

EITHER

attempt to use integration by substitution ***M1***

$\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin(u)\mathrm{d}u$

Note: Award ***M1*** for $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$=\left[-\frac{\pi}{2}\cos(u)\right]_0^{\frac{\pi}{4}}$ ***A1***

OR

attempt to integrate by inspection ***(M1)***

$\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}2x\sin\left(x^2\right)\mathrm{d}x$ OR $\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}\sin\left(x^2\right)\mathrm{d}\left(x^2\right)$

$=\left[-\frac{\pi}{2}\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ ***A1***

Note: Condone incorrect or absent limits for ***M1***.
The correct limits may be seen or implied by later work for the ***A1***.

THEN

$=\left(-\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\right)-\left(-\frac{\pi}{2}\cos(0)\right)$ (or equivalent) ***(A1)***

$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{2}$ OR $-\frac{\pi\sqrt{2}}{4}+\frac{\pi}{2}$ OR $\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}}+1\right)$ OR $\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$ ***A1***

$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

continued…

Question 6 continued

METHOD 2

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1***.

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}\left(f(x)\right)^2\,\mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

attempt to use integration by substitution ***M1***

$u=\cos\left(x^2\right)\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=-2x\sin\left(x^2\right)$

Note: Award ***M1*** for $u=\cos\left(x^2\right)$

$=-\frac{\pi}{2}\int_{-\frac{1}{\sqrt{2}}}^{-1}\mathrm{d}u$

$=\left[-\frac{\pi}{2}u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$ (or equivalent) ***A1A1***

Note: Condone incorrect or absent limits for ***M1***.

A1 for $-\frac{\pi}{2}u$ and ***A1*** for both correct limits.

$=\frac{\pi}{2}-\frac{\pi}{2\sqrt{2}}$ OR $\frac{\pi}{2}-\frac{\pi\sqrt{2}}{4}$ OR $\frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)$ OR $\frac{\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)$ ***A1***

$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

Total [6 marks]

8. (a) (i) $\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \ldots \left(= x^2 - \frac{x^6}{6} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

(ii) **METHOD 1**

attempt to square their series for $\sin\left(x^2\right)$ ***(M1)***

$\left(\sin\left(x^2\right)\right)^2 = \left(x^2 - \frac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\frac{x^6}{3!}\right)^2 + \ldots.$

$= x^4 - \frac{2x^8}{3!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\left(x^2\right) = \frac{1 - \cos\left(2x^2\right)}{2}$ ***(M1)***

$\sin^2\left(x^2\right) = \frac{1}{2}\left(1 - \left(1 - \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^4}{4!}\right)\right)$

$= x^4 - \frac{8x^8}{4!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[5 marks]

continued…

Question 8 continued

(b) **METHOD 1**

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=\frac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

METHOD 2

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=2x\sin\left(2x^2\right)$ ***(M1)***

$=2x\left(2x^2-\frac{\left(2x^2\right)^3}{3!}+\ldots\right)$

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

METHOD 3

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$=4x\left(x^2-\frac{x^6}{3!}+\ldots\right)\left(1-\frac{x^4}{2!}+\ldots\right)$ ***(A1)***

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

continued…

Question 8 continued

METHOD 4

recognition that $2x\cos\left(x^2\right)=\frac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$=2\left(x^2-\frac{x^6}{3!}+\ldots\right)\left(2x-\frac{6x^5}{2!}+\ldots\right)$

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

[2 marks]

Total [7 marks]

11. (a) attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division ***M1***

$3(-1)^3+5(-1)^2+(-1)-1=0$ OR

	3	5	1	-1
-1		-3	-2	1
	3	2	-1	0

OR $x+1\overline{)3x^3+5x^2+x-1}$ ***A1***

$$\begin{array}{r} 3x^2+2x-1 \\ \hline 3x^3+3x^2 \\ \hline 2x^2+x \\ 2x^2+2x \\ \hline -x-1 \\ -x-1 \\ \hline 0 \end{array}$$

[2 marks]

(b) attempt to divide $P(x)$ by $(x+1)$ e.g. using long division or synthetic division ***(M1)***

$P(x)=(x+1)(3x^2+2x-1)$ ***(A1)***

$=(x+1)(x+1)(3x-1)\left(=(x+1)^2(3x-1)\right)$ ***A1***

[3 marks]

(c) $\dfrac{1}{(x+1)(2x+1)}\equiv\dfrac{A}{x+1}+\dfrac{B}{2x+1}\Rightarrow 1\equiv A(2x+1)+B(x+1)$

attempt to equate both coefficients OR substitute two values eg -1 and $-\dfrac{1}{2}$ ***(M1)***

$2A+B=0$ and $A+B=1$ OR $1=-A$ and $1=\dfrac{1}{2}B$

$A=-1$ and $B=2$ ***A1A1***

Note: Award ***A1*** for each value.

$$\frac{1}{(x+1)(2x+1)}=-\frac{1}{x+1}+\frac{2}{2x+1}$$

[3 marks]

continued…

Question 11 continued

(d) $\dfrac{1}{(x+1)(x+1)(2x+1)}$

$= \dfrac{1}{(x+1)}\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)$ ***(A1)***

$= -\dfrac{1}{(x+1)^2}+\dfrac{2}{(2x+1)(x+1)}\left(= -\dfrac{1}{(x+1)^2}+2\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)\right)$ ***A1***

$= \dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}$ ***AG***

Note: Award ***A1A0*** for follow through from incorrect values in part (c).

[2 marks]

(e) attempt to integrate at least one term in $\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)$ ***(M1)***

$\int\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)\mathrm{d}x$

$= 2\ln|2x+1|-2\ln|x+1|+\dfrac{1}{x+1}(+c)$ ***A1A1A1***

Note: Award ***A1*** for each correct term.
Award a maximum of ***M1A1A0A1*** if modulus signs are omitted.
Condone the absence of $+c$.

[4 marks]

(f) (i) **METHOD 1**

attempt to cancel factors and substitute $x=-1$ ***(M1)***

$\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1}\left(\dfrac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim\limits_{x\to -1}\left(\dfrac{3x-1}{2x+1}\right) = \dfrac{3(-1)-1}{2(-1)+1}$

$= 4$ ***A1***

continued…

Question 11 continued

METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x = -1$ ***(M1)***

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x\to -1}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x\to -1}\left(\frac{18x+10}{12x+10}\right) = \frac{18(-1)+10}{12(-1)+10}$$

$= 4$ ***A1***

(ii) **METHOD 1**

attempt to consider coefficients of x^3 or divide all terms by x^3

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(\frac{3x^3+\ldots}{2x^3+\ldots}\right) \text{ or } \lim_{x\to \infty}\left(\frac{3+\text{terms which tend to } 0}{2+\text{terms which tend to } 0}\right)$$

$= \frac{3}{2}$ ***A1***

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim_{x\to \infty}\left(\frac{3x-1}{2x+1}\right) \text{ or } \lim_{x\to \infty}\left(\frac{3-\frac{1}{x}}{2+\frac{1}{x}}\right)$$

$= \frac{3}{2}$ ***A1***

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x\to \infty}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x\to \infty}\left(\frac{18x+10}{12x+10}\right) = \lim_{x\to \infty}\left(\frac{18}{12}\right)$$

$= \frac{3}{2}$ ***A1***

Note: If the ***M1*** has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

Total [17 marks]

8.

Note: To award full marks limit notation $\lim_{x\to 0}$ must be seen at least once in their working. If no limit notation is seen but otherwise all correct, do not award the final ***A1.***

$$\lim_{x\to 0}\frac{4\sec^4 x\tan x+2\sin x\cos x}{4x^3-2x}$$ ***A1A1***

Note: Award ***A1*** for numerator and ***A1*** for denominator.

$$=\lim_{x\to 0}\frac{16\sec^4 x\tan^2 x+4\sec^6 x-2\sin^2 x+2\cos^2 x}{12x^2-2}$$ ***M1A1A1***

Note: Award ***M1 for second use of*** l'Hôpital's rule providing their expression is in indeterminate form as $x\to 0$ and providing there is **no third attempt** at using l'Hôpital's Rule.

$$=-3$$ ***A1***

[6 marks]

12. (a) $n=1: LHS = f^{(1)}(x) = -\frac{1}{2}\times -a(1-ax)^{-\frac{3}{2}}\left(=\frac{a}{2}(1-ax)^{-\frac{3}{2}}\right)$ ***A1***

$RHS = \frac{a(1)!(1-ax)^{-\frac{3}{2}}}{2^1(0)!}$ therefore true for $n=1$ ***R1***

assume (that the result is) true for $n=k$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "assume that $n=k$ is true". The assumption of truth must be clear.

$$f^{(k)}(x) = \frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}}}{2^{2k-1}(k-1)!}$$

attempt to differentiate the right-hand side with respect to x: ***M1***

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$$

$$= \frac{-(2k+1)\times -a}{2}\times\frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}-1}}{2^{2k-1}(k-1)!}$$ (or equivalent) ***A1***

attempt to multiply top and bottom by $2k$ ***M1***

$$= \frac{(2k+1)}{2}\times\frac{a^{k+1}(2k-1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k-1}(k-1)!}\times\frac{2k}{2k}$$

$$= \frac{a^{k+1}(2k+1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k+1}(k)!}$$ ***A1***

hence if the result is true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$ it is true for all $n\in\mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, at least five of the previous marks must have been awarded.

[8 marks]

continued…

Question 12 continued.

(b) $f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \ldots$

$f''(x) = \frac{a^2(3)!(1-ax)^{-\frac{5}{2}}}{2^3(1)!}$ OR $\frac{3}{4}a^2(1-ax)^{-\frac{5}{2}}$ OR $f''(0) = \frac{a^2(3)!}{2^3}$ ***A1***

$f(0) = 1,\ f'(0) = \frac{a}{2},\ f''(0) = \frac{6a^2}{8}$ ***A1***

$f(x) = 1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + \ldots.$ ***AG***

[2 marks]

continued…

Question 12 continued.

(c) attempt to use $a=2$ or $a=4$ in the expansion ***M1***

$$(1-2x)^{-\frac{1}{2}} = 1+\frac{2x}{2}+\frac{3\times 4x^2}{8}\left(=1+x+\frac{3}{2}x^2+\ldots\right)$$

$$(1-4x)^{-\frac{1}{2}} = 1+\frac{4x}{2}+\frac{3\times 16x^2}{8}\left(=1+2x+6x^2+\ldots\right)$$ ***A1***

Note: Award ***A1*** for at least one correct.

attempt to multiply their two expansions together ***M1***

$$\left(1+x+\frac{3}{2}x^2+\ldots\right)\left(1+2x+6x^2+\ldots\right)=1+2x+6x^2+x+2x^2+\frac{3}{2}x^2+\ldots$$

$$=1+3x+\frac{19}{2}x^2+\ldots \text{ OR } \frac{2+4x+12x^2+2x+4x^2+3x^2+\ldots}{2}$$ ***A1***

$$(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$$ ***AG***

[4 marks]

(d) $|x|<\frac{1}{4}$ ***A1***

[1 mark]

continued…

Question 12 continued.

(e)

$\left(\frac{2+6x+19x^2}{2}\right)=\frac{2+0.6+0.19}{2}\left(=\frac{279}{200}\right)$ (or equivalent) ***A1***

attempt to substitute $x=\frac{1}{10}$ into $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}$ ***(M1)***

$\left((1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}=\right)\frac{10}{\sqrt{48}}$ (or equivalent) ***A1***

$\frac{10}{4\sqrt{3}}\approx\frac{279}{200}$ (or equivalent in terms of $\sqrt{3}$) ***A1***

$\frac{1}{\sqrt{3}}\approx\frac{279}{500}$

$\sqrt{3}\approx\frac{500}{279}$ ***A1***

[5 marks]

Total [20 marks]

4. (a) recognition that velocity is zero ***(M1)***

$v = 2\sin(0.5t) + 0.3t - 2 = 0$

$t = 1.68694\ldots$

$t = 1.69$ ***A1***

[2 marks]

(b) recognition that $v > 0$ ***(M1)***

$1.68694... < t < 6.11857...$

$1.69 < t < 6.12$ ***A1***

[2 marks]

(c) attempt to substitute into the total displacement formula (condone missing or incorrect limits, and absence of $\mathrm{d}t$) ***(M1)***

$\int_0^{10}(2\sin(0.5t) + 0.3t - 2)\,\mathrm{d}t$ OR $\int_0^{10} v(t)\,\mathrm{d}t$

$= -2.13464...$

$= -2.13\,(\mathrm{m})$ ***A1***

Note: Award ***(M1)A0*** if -2.13 is followed by 2.13.

[2 marks]

Total [6 marks]

7. $\frac{\mathrm{d}y}{\mathrm{d}x} - y = x$

recognition that an integrating factor is required ***(M1)***

$\mathrm{e}^{\int P(x)dx} \left(= \mathrm{e}^{\int -1dx}\right)$

$= \mathrm{e}^{-x}$ ***(A1)***

$\mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{-x}y = x\mathrm{e}^{-x}$

$\mathrm{e}^{-x}y\left(= \int x\mathrm{e}^{-x}\mathrm{d}x\right)$ ***A1***

attempt to integrate right hand side using integration by parts ***(M1)***

$\left(\mathrm{e}^{-x}y =\right) - x\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\mathrm{d}x$

$= -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$ ***A1***

$y = -x - 1 + c\mathrm{e}^{x}$

substitute initial values $y = 2$ and $x = 0$ into an integrated expression involving c ***(M1)***

$2 = -1 + c \Rightarrow c = 3$

$y = 3\mathrm{e}^{x} - x - 1$ ***A1***

[7 marks]

11. (a) attempt to use implicit differentiation ***M1***

$\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{e}^{x+y}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

attempt to expand brackets and collect $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms on the same side ***M1***

$\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}=2x-\mathrm{e}^{x+y}$ OR $\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^{x+y}=2x$ OR $\left(\mathrm{e}^{x+y}-2y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2x-\mathrm{e}^{x+y}$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-\mathrm{e}^{x+y}}{\mathrm{e}^{x+y}-2y}$ ***AG***

[5 marks]

(b) (i) recognition that $2x-\mathrm{e}^{x+y}=0$ at P and Q ***(M1)***

$\Rightarrow x+y=\ln(2x)$ ***A1***

recognition that $\mathrm{e}^{x+y}=x^2+y^2$ at P and Q ***(M1)***

$\Rightarrow 2x=x^2+\left(\ln(2x)-x\right)^2$ or equivalent ***A1***

$2x^2+\left(\ln(2x)\right)^2-2x\ln(2x)-2x=0$ ***AG***

continued…

Question 11 continued

(ii) $x = 0.331077\ldots$ or $1.84273\ldots$

$x = 0.331$ or 1.84 ***A1A1***

attempt to use $x + y = \ln(2x)$ OR $e^{x+y} = x^2 + y^2$ to find y-coordinates ***(M1)***

$y = -0.743332\ldots$, $y = -0.538335...$

coordinates $(0.331, -0.743)$ and $(1.84, -0.538)$ ***A1A1***

Note: If no working shown, award ***A1A1(M1)A0A0*** for $(0.33, -0.74)$ and $(1.8, -0.54)$ (2 sf)

[9 marks]

(c) coordinates $(-0.743, 0.331)$ and $(-0.538, 1.84)$ ***A1***

Note: Do not award ***FT*** from (b) if only one coordinate pair is given.

[1 mark]

(d) setting $\frac{2x - e^{x+y}}{e^{x+y} - 2y} = -1$ OR recognition that the point lies on line of symmetry ***(M1)***

$2x - e^{x+y} = 2y - e^{x+y}$

$y = x$ ***(A1)***

attempt to substitute $y = x$ into $e^{x+y} = x^2 + y^2$ ***(M1)***

$e^{2x} = 2x^2$ OR $e^{2y} = 2y^2$

$x = -0.451$, $y = -0.451$ ***A1***

coordinates $(-0.451, -0.451)$

Note: If no working shown, award ***M1A1(M1)A0*** for $x = -0.45$, $y = -0.45$ (2 sf)

[4 marks]

Total [19 marks]

1. (a) attempts to find an intersection point ***(M1)***

$a = -0.916562...$ or $b = 0$

$a = -0.917$, $b = 0$ ***A1A1***

[3 marks]

(b) let A be the area of the region

EITHER

attempts to form the required integral involving subtraction (in any order). Accept absence of limits or incorrect limits. Accept absence of dx. ***(M1)***

OR

shows a graph with the required area shaded ***(M1)***

THEN

$A = \left(\int_{a}^{b} \left(f(x) - g(x) \right) \mathrm{d}x \right) = \int_{-0.916562...}^{0} \left(1 - x^2 - \mathrm{e}^{2x} \right) \mathrm{d}x$ (or equivalent) ***(A1)***

$A = 0.239855...$

$A = 0.240$ ***A1***

[3 marks]

Total [6 marks]

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4. (a) $v = -0.996114...$

$v = -0.996$ (ms^{-1}) ***A1***

[1 mark]

(b)

considers $v'(t) = 0$ ***(M1)***

$t = 0.405833...$

$v_{\max} = 1.18230...$

$v_{\max} = 1.18$ (ms^{-1}) ***A1***

[2 marks]

(c) recognizes that the particle changes direction when $v = 0$ ***(M1)***

Note: Award ***(M1)*** for $t = 1.65840...$ seen.

finds acceleration for their value of *t* for which $v(t) = 0$ ***(M1)***

$v'(1.65840...)$

$a = -2.53487...$

$a = -2.53$ (ms^{-2}) ***A1***

[3 marks]

Total [6 marks]

6. attempts to solve $\left(V=\right)\frac{4}{3}\pi r^3=20$ for r ***(M1)***

$r=1.68389...\left(=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}\right)$ (seen anywhere) ***A1***

attempts to use the chain rule ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ OR $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\mathrm{d}r}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ (or equivalent)

EITHER

$\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}\left(=4\pi\left(1.68389...\right)^2\frac{\mathrm{d}r}{\mathrm{d}t}\right)\left(=4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)$ ***A1***

OR

$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{4\pi r^2}\frac{\mathrm{d}V}{\mathrm{d}t}\left(=\frac{1}{4\pi\left(1.68389...\right)^2}\frac{\mathrm{d}V}{\mathrm{d}t}\right)\left(=\frac{1}{4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}}\frac{\mathrm{d}V}{\mathrm{d}t}\right)$ ***A1***

THEN

attempts to find $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $\frac{\mathrm{d}V}{\mathrm{d}t}=5$ ***(M1)***

$\frac{\mathrm{d}r}{\mathrm{d}t}=0.140324...$

$\frac{\mathrm{d}r}{\mathrm{d}t}=0.140\left(=\frac{5}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}\right)$ (cm s^{-1}) (accept 0.14) ***A1***

[6 marks]

7. attempts to express x (or x^2) in terms of y ***(M1)***

Note: Only award ***(M1)*** if base e is applied to both sides.

$x = \mathrm{e}^{\frac{y}{4}} + 2 \left(x^2 = \left(\mathrm{e}^{\frac{y}{4}} + 2 \right)^2 = \mathrm{e}^{\frac{y}{2}} + 4\mathrm{e}^{\frac{y}{4}} + 4 \right)$ ***(A1)***

let V be the volume of the solid formed

forms a definite integral of the form $\pi \int_c^d x^2 \mathrm{d}y$ with their expression for x^2 in terms of y ***(M1)***

$V = \pi \int_0^4 \left(\mathrm{e}^{\frac{y}{4}} + 2 \right)^2 \mathrm{d}y \left(= \pi \int_0^4 \left(\mathrm{e}^{\frac{y}{2}} + 4\mathrm{e}^{\frac{y}{4}} + 4 \right) \mathrm{d}y \right)$ ***(A1)***

$= 176.779...$

$= 177 \left(= 2\pi\left(\mathrm{e}^2 + 8\mathrm{e} - 1 \right) \right)$ (cubic units) ***A1***

[5 marks]

9. recognizes that $x=1 \Rightarrow ax^2+bx+c=0$ ***(M1)***

$a+b+c=0$ (seen anywhere) ***A1***

passes through $(2,1)$ so:

$1=\dfrac{2-4}{4a+2b+c}$ $(4a+2b+c=-2)$ (seen anywhere) ***A1***

local minimum point at $(2,1)$ so:

attempts to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ using quotient or product rule ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(ax^2+bx+c\right)-\left(x-4\right)\left(2ax+b\right)}{\left(ax^2+bx+c\right)^2}$$

substitutes $x=2$ into the numerator of their $\dfrac{\mathrm{d}y}{\mathrm{d}x}(=0)$ ***(M1)***

$(4a+2b+c)-(2-4)(4a+b)=0$ $\left(\dfrac{(4a+2b+c)-(2-4)(4a+b)}{(4a+2b+c)^2}=0\right)$ ***A1***

$(12a+4b+c=0)$

Note: An incorrect numerator may lead to a correct equation.
In this instance, award ***A0*** here and do not award the final ***A*** mark.

attempts to solve their 3 linear equations in a,b and c ***(M1)***

$a=3, b=-11$ and $c=8$ ***A1***

Note: Three linear equations and a value for each of a,b and c need to be seen to gain the last ***M*** mark.

Note*:* The last ***M*** mark is dependent on an equation formed from the numerator of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

[8 marks]

SECTION B

12. (a) attempts to use Euler's method ***(M1)***

$x_{n+1} = x_n + \frac{\pi}{12}$; $y_{n+1} = y_n + \frac{\pi}{12} \times \frac{\mathrm{d}y}{\mathrm{d}x}$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = y\operatorname{cosec} 2x + \sqrt{\tan x}$

$y_1 = 1.25281... \left(= \frac{\pi}{4} + \frac{\pi}{12}\left(\frac{\pi}{4} + 1\right)\right)$ ***(A1)***

$y_2 = 1.97608...$

$y = 1.98$ ***A1***

[3 marks]

(b) attempts chain rule differentiation with multiplication of two derivatives ***(M1)***

$u = \cot x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\operatorname{cosec}^2 x$ and $y = \frac{1}{2}\ln u \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2u}$ OR

$u = \frac{\cos x}{\sin x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sin^2 x}$ and $y = \frac{1}{2}\ln u \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2u}$ OR

$u = \sqrt{\cot(x)} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{\cot(x)}} \times -\operatorname{cosec}^2 x$ and $\frac{dy}{du} = \frac{1}{\sqrt{\cot(x)}}$ OR

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln f(x)\right) = \frac{f'(x)}{2f(x)}$

THEN

$= -\frac{\operatorname{cosec}^2 x}{2\cot x}$ OR $-\frac{1}{\sin^2 x} \times \frac{\sin x}{2\cos x}$ ***A1***

$= -\frac{1}{2\sin x \cos x}$ ***(A1)***

$= -\frac{1}{\sin 2x}$ ***A1***

$= -\operatorname{cosec} 2x$ ***AG***

[4 marks]

continued…

Question 12 continued.

(c) **METHOD 1**

attempts to use $I(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}$ ***(M1)***

$\mathrm{e}^{\int -\operatorname{cosec} 2x\,\mathrm{d}x}$ ***A1***

$= \mathrm{e}^{\frac{1}{2}\ln(\cot x)}$ (***A1)***

$= \mathrm{e}^{\ln\left(\sqrt{\cot x}\right)}$ ***A1***

$= \sqrt{\cot x}$ ***AG***

METHOD 2

attempts product rule differentiation on $\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{\cot x}\right)$ ***M1***

$= \frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{\cot x} - \frac{y\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$ ***A1***

$= \sqrt{\cot x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\operatorname{cosec}^2 x}{2\cot x}\right)$ ***A1***

$= \sqrt{\cot x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec} 2x\right)$ ***A1***

so $\sqrt{\cot x}$ is an integrating factor ***AG***

[4 marks]

continued…

Question 12 continued.

(d) $\sqrt{\cot x}\dfrac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec} 2x\sqrt{\cot x} = \sqrt{\tan x}\sqrt{\cot x}$ (or equivalent) ***(M1)***

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{\cot x}\right) = 1$ ***(A1)***

$y\sqrt{\cot x} = \int 1\,\mathrm{d}x$ ***A1***

$y\sqrt{\cot x} = x(+C)$ or equivalent ***A1***

substitutes $x = \dfrac{\pi}{4}, y = \dfrac{\pi}{4} \Rightarrow C = 0$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = x\sqrt{\tan x}$ ***AG***

[5 marks]

(e) (i) $y = 2.52878...$

$y = 2.53$ ***A1***

(ii) the gradient changes substantially (in the neighbourhood of $x = \dfrac{5\pi}{12}$) ***R1***

Note: Award ***R0*** for saying the gradient is very large at $x = \dfrac{5\pi}{12}$

(iii) **EITHER**

the curve is concave up (over the interval) ***A1***

OR

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ (over the interval) ***A1***

[3 marks]

continued…

Question 12 continued.

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = y\operatorname{cosec} 2x + \sqrt{\tan x}\ \left(= x\sqrt{\tan x}\operatorname{cosec} 2x + \sqrt{\tan x}\right)$

$\operatorname{cosec} 2x, \sqrt{\tan x} > 0$ (for $0 < x < \frac{\pi}{2}$) ***R1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} > 0$ ***A1***

so the curve has a positive gradient for $0 < x < \frac{\pi}{2}$ ***AG***

Note: Do not award ***R0A1***.

[2 marks]

Total [21 marks]

2. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 + 3$ ***(M1)(A1)***

Note: Award ***M1*** for at least two correct terms.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 18x$ ***A1***

[3 marks]

(b) valid attempt to find x-coordinates (e.g. solving $12x^2 - 18x = 0$ or graphing $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 + 3$ ***(M1)***

$x = 0, 1.5 \left(\frac{3}{2}\right)$ ***(A1)***

point B is $(0, 0)$ ***A1***

point C is $(1.5, -0.5625)$ $\left((1.5, -0.563), \left(\frac{3}{2}, -\frac{9}{16}\right)\right)$ ***A1***

Note: Award ***M0A0A1A0*** for an unsupported answer of point B is $(0, 0)$.

[4 marks]

(c) y-intercept $= 0$ (as line passes through $(0, 0)$) ***R1***

Note: Award ***R1*** for correctly substituting point B or point C to show that $y = 0$.

Award ***R1*** for $y = -0.375x + 0$.

gradient $= \frac{-0.5625}{1.5}$ $\left(= \frac{-\frac{9}{16}}{\frac{3}{2}}\right)$ ***A1***

Note: Award ***A0FT*** if their answer to (b) doesn't lead to the given answer, but condone $\frac{-0.563}{1.5}$.

$= -0.375$

so equation is $y = -0.375x$ ***AG***

Note: Award at most ***A1R0*** for working backwards to verify points B and C lie on the given line.

[2 marks]

continued…

Question 2 continued

(d) $x^4 - 3x^3 + 3x = -0.375x$ ***(M1)***

2.427 ***A1***

Note: Correct answer must be given to three decimal places for the ***A1*** to be awarded.

Award ***(M1)A1*** for $(2.427, -0.910)$.

Award ***(M1)A0*** for an unsupported answer of 2.43.

[2 marks]

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 3mx^2 + n$ (***A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 6mx$ (***A1)***

$6x(2x - m) = 0$

$x = 0, \frac{m}{2}$ ***A1***

Note: Accept $0m$ in place of 0.

[3 marks]

(f) (i) point B is $(0, 0)$ ***A1***

[1 mark]

(ii) $y_c = \left(\frac{m}{2}\right)^4 - m\left(\frac{m}{2}\right)^3 + n\left(\frac{m}{2}\right) \quad \left(= \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}\right)$ ***(A1)***

$= -\frac{m^4}{16} + \frac{nm}{2}$ OR $\frac{8nm - m^4}{16}$ OR $\frac{m}{2}\left(n - \frac{m^3}{8}\right)$ OR equivalent simplification ***A1***

Point C is $\left(\frac{m}{2}, -\frac{m^4}{16} + \frac{nm}{2}\right)$

Note: Award the second ***A1*** if the simplified y-coordinate is seen here or in part (g).

[2 marks]

continued…

Question 2 continued

(g) attempt to divide their y_C by their x_C OR substitute into $\frac{y_2 - y_1}{x_2 - x_1}$ ***(M1)***

e.g. $\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \div \frac{m}{2}$

$\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \times \frac{2}{m}$ **OR** equivalent manipulation leading to given answer ***A1***

Note: Award ***A0 FT*** if their answer does not lead to the ***AG***.

$= -\frac{m^3}{8} + n$ and y-intercept $= 0$,

so equation is $y = \left(-\frac{m^3}{8} + n\right)x$ ***AG***

Note: Award at most ***M1A0*** for working backwards to verify points B and C lie on the given line.

[2 marks]

(h) **METHOD 1**

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \frac{m^3}{8}x = 0$

recognizing their $\left(x - x_C\right)$ is a factor of this expression ***(M1)***

$x\left(x - \frac{m}{2}\right)\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)\ (= 0)$ OR equivalent ***A1M1A1***

Note: Award ***A1*** for $x\left(x - \frac{m}{2}\right)$ **,** ***M1A1*** for $\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)$.

If a candidate divides by x without justification, do not award the first ***A1***, but all subsequent marks can still be awarded.

continued…

Question 2 continued

use of quadratic formula to find roots, x_A (and x_D), of the quadratic ***M1***

$x_A = \dfrac{\dfrac{m}{2} - \sqrt{\dfrac{m^2}{4} + m^2}}{2}$ OR equivalent ***A1***

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$x_A = \dfrac{m}{4} - \dfrac{m}{4}\sqrt{5}$ ***AG***

Note: Award ***(M1)(M0)A0M0A0M0A0*** for attempting to verify the given answer satisfies $x^4 - mx^3 + nx = -\dfrac{m^3}{8}x + nx$.

METHOD 2

$x^4 - mx^3 + nx = -\dfrac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \dfrac{m^3}{8}x = 0$

$x\left(x^3 - mx^2 + \dfrac{m^3}{8}\right) = 0$

Recognise that their x_C is a root of this equation ***(M1)***

Attempt to find sum and product of roots ***(M1)***

$x_A + x_D + \dfrac{m}{2} = m$ AND $x_A \times x_D \times \dfrac{m}{2} = -\dfrac{m^3}{8}$ ***A1***

$x_A \times \left(\dfrac{m}{2} - x_A\right) \times \dfrac{m}{2} = -\dfrac{m^3}{8}$

$x_A{}^2 - \dfrac{m}{2}x_A - \dfrac{m^2}{4} = 0$ ***A1***

use of quadratic formula to find roots, x_A (and x_D), of the quadratic ***M1***

continued…

Question 2 continued

$$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2}$$ OR equivalent ***A1***

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$$ ***AG***

[7 marks]

(i) $$\frac{0 - \left(\frac{m}{4} - \frac{m}{4}\sqrt{5}\right)}{\frac{m}{2} - 0}$$ ***(A1)***

$$= \frac{\sqrt{5} - 1}{2}$$ ***A1***

Note: Answer must be exact.

[2 marks]

Total [28 marks]

1. (a) (i) attempts chain rule differentiation to find $f'(x)$ ***(M1)***

$f'(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3}\right)$ ***A1***

Note: Award ***(M1)*** for attempting chain rule differentiation on $(4-4x+x^2)^{-1}$ or attempting quotient rule differentiation on $\frac{1}{(4-4x+x^2)} \left(= \frac{1}{(2-x)^2}\right)$.

Award ***A1*** for $f'(x) = \frac{2}{(2-x)^3} \left(= (-1)(-2)(2-x)^{-3}\right)$.

[2 marks]

(ii) $g'(x) = 2x$ ***(A1)***

$f'(x)g'(x) = \left(2(2-x)^{-3}\right)(2x) \left(= \frac{2(2x)}{(2-x)^3}\right)$ (or equivalent) ***A1***

$= \frac{4x}{(2-x)^3}$ ***AG***

[2 marks]

continued…

Question 1 continued.

(iii)

Note: Award a maximum of ***(M1)A1(M1)A0FT*** from parts (a) (i) and (ii).

substitutes $f(x), g(x)$ and their $g'(x), f'(x)$ into the given expression ***(M1)***

EITHER

$f(x)g'(x)+g(x)f'(x)=2x(2-x)^{-2}+2x^2(2-x)^{-3}$ ***A1***

Note: Award ***A1*** if $f(x)g'(x)=2x(2-x)^{-2}$ and $g(x)f'(x)=2x^2(2-x)^{-3}$ are stated separately.

attempts to factorise their expression ***(M1)***

$=2x(2-x)^{-3}\left((2-x)+x\right)$ ***A1***

OR

$f(x)g'(x)+g(x)f'(x)=\frac{2x}{(2-x)^2}+\frac{2x^2}{(2-x)^3}$ ***A1***

Note: Award ***A1*** if $f(x)g'(x)=\frac{2x}{(2-x)^2}$ and $g(x)f'(x)=\frac{2x^2}{(2-x)^3}$ are stated separately.

attempts to form an expression with a common denominator ***(M1)***

Note: Award ***(M1)*** for $(2-x)^2(2-x)^3$ as a common denominator.

$=\frac{2x(2-x)}{(2-x)^3}+\frac{2x^2}{(2-x)^3}\left(=\frac{4x-2x^2+2x^2}{(2-x)^3}\right)$ ***A1***

THEN

$=\frac{4x}{(2-x)^3}$

AG

Note: Award marks as appropriate for attempting to find the derivative of $f(x)g(x)=\frac{x^2}{(2-x)^2}$ (or equivalent).

[4 marks]

continued…

Question 1 continued.

(b) **METHOD 1**

$f'(x)g'(x)-g(x)f'(x)=f(x)g'(x)$ ***(A1)***

$\left(f'(x)g'(x)-g(x)f'(x)-f(x)g'(x)=0\right)$

$f'(x)\left(g'(x)-g(x)\right)=f(x)g'(x)\ \ \left(f'(x)\left(g(x)-g'(x)\right)=-f(x)g'(x)\right)$ ***A1***

$\dfrac{f'(x)}{f(x)}=\dfrac{g'(x)}{g'(x)-g(x)}$ ***AG***

Note: Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 2

$g'(x)=\dfrac{f'(x)g'(x)}{f(x)}-\dfrac{g(x)f'(x)}{f(x)}$ ***(A1)***

$g'(x)=\dfrac{f'(x)}{f(x)}\left(g'(x)-g(x)\right)$ ***A1***

$\dfrac{f'(x)}{f(x)}=\dfrac{g'(x)}{g'(x)-g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

continued…

Question 1 continued.

METHOD 3

$g'(x) = \frac{f(x)g'(x)}{f'(x)} + g(x)$ ***(A1)***

$\frac{f(x)}{f'(x)} = \frac{g'(x) - g(x)}{g'(x)}$ ***A1***

$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 4

$\frac{f(x)}{f'(x)} + \frac{g(x)}{g'(x)} = 1$ ***(A1)***

$\frac{f(x)}{f'(x)} = \frac{g'(x) - g(x)}{g'(x)}$ ***A1***

$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

[2 marks]

continued…

Question 1 continued.

(c) **METHOD 1**

Note: Condone the absence of '$\mathrm{d}x$' and the modulus sign throughout.
Only award the second ***A*** mark if the constant of integration has been dealt with correctly.

EITHER

$$\ln f(x) = \int \frac{g'(x)}{g'(x) - g(x)} \mathrm{d}x \ (+C)$$ ***A1***

Note: Condone the absence of '$+C$' when awarding the first ***A*** mark.

$$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} \mathrm{e}^{C} \quad \left(f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x + C\right)} \right)$$ ***A1***

Note: Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$, where $A = \mathrm{e}^{C}$.

OR

$$\ln f(x)(+C) = \int \frac{g'(x)}{g'(x) - g(x)} \mathrm{d}x$$ ***A1***

Note: Condone the absence of '$+C$' when awarding the first ***A*** mark.

$$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} \mathrm{e}^{-C} \quad \left(f(x)\mathrm{e}^{C} = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}, f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x - C\right)} \right)$$ ***A1***

Note: Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$, where $A = \mathrm{e}^{-C}$.

THEN

$$f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$$ ***AG***

continued…

Question 1 continued.

METHOD 2

Note: Condone the absence of '$\mathrm{d}x$' throughout.

$$f'(x)-\frac{g'(x)}{g'(x)-g(x)}f(x)=0$$

Integrating factor: $\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$ ***A1***

Note: Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=C$.

$$\frac{d}{dx}\left[f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}\right]=0\Rightarrow f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=A$$ ***A1***

Note: Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=C$.

$$f(x)=A\mathrm{e}^{\left(\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$$ ***AG***

[2 marks]

continued…

Question 1 continued.

(d) $g'(x) = xe^x + e^x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\frac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$= \frac{xe^x + e^x}{e^x} \left(= \frac{e^x(x+1)}{e^x} \right)$

$= x + 1$ (as $e^x \neq 0$) ***(A1)***

attempts to integrate their $\frac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$\int (x+1)\,dx = \frac{1}{2}x^2 + x\,(+C)$

$f(x) = e^{\left(\frac{1}{2}x^2 + x\right)}$ ***A1***

Note: Award ***A0*** for $f(x) = e^{\left(\frac{1}{2}x^2 + x + C\right)}$ (or equivalent expressed with an arbitrary constant).

[5 marks]

continued…

Question 1 continued.

(e) $g'(x) = \cos x - \sin x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\dfrac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$= \dfrac{\cos x - \sin x}{\cos x - \sin x - \sin x - \cos x} \left(= \dfrac{\sin x - \cos x}{2\sin x} \right)$

$= \dfrac{1}{2} - \dfrac{1}{2}\cot x$ (as $\sin x \neq 0$) OR $= \dfrac{1}{2} - \dfrac{1}{2}\dfrac{\cos x}{\sin x}$ (as $\sin x \neq 0$) ***A1***

$f(x) = \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2}\cot x\right) \mathrm{d}x}$

attempts to find the indefinite integral of $(\pm k)\cot x$ OR $(\pm k)\dfrac{\cos x}{\sin x}$ ***(M1)***

Note: As $|\sin x| = \sin x$ for $0 < x < \pi$, condone the presence or omission of the modulus sign throughout a candidate's solution.
Condone the presence of an arbitrary constant except when awarding the final ***A*** mark.

$\int \left(\dfrac{1}{2} - \dfrac{1}{2}\cot x \right) \mathrm{d}x = \dfrac{x}{2} - \dfrac{1}{2}\ln|\sin x| (+C) \left(= \dfrac{1}{2}\left(x - \ln|\sin x| (+C)\right) \right)$ ***A1***

$f(x) = \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{-\frac{1}{2}\ln|\sin x|} \left(\mathrm{e}^{C}\right)$

$= \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{\ln\sqrt{\frac{1}{\sin x}}} \left(\mathrm{e}^{C}\right) \left(= \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{\frac{1}{2}\ln\left(\frac{1}{\sin x}\right)} \left(\mathrm{e}^{C}\right), = \sqrt{\mathrm{e}^{x - \ln(\sin x)}} \left(\mathrm{e}^{C}\right) \right)$ ***A1***

$= \mathrm{e}^{\frac{x}{2}} \sqrt{\dfrac{1}{\sin x}}$

$= \sqrt{\mathrm{e}^{x} \operatorname{cosec} x} \left(= \sqrt{\dfrac{\mathrm{e}^{x}}{\sin x}} \right)$ (where $h(x) = \dfrac{1}{\sin x}$) ***A1***

[7 marks]

Total [24 marks]

7. (a) attempt to find $f(0)=\sqrt{2}, f\left(\frac{\pi}{4}\right)=1$ or $f\left(\frac{\pi}{2}\right)=\sqrt{2}$ or sketch of graph ***(M1)***

$1 \leq f(x) \leq \sqrt{2}$ ***A1A1***

Note: Award ***A1A0*** for strong inequality seen. Allow equivalent, and/or interval notation.

[3 marks]

(b) consider $\pi\int_0^{\frac{\pi}{2}} \sec^2\left(x-\frac{\pi}{4}\right) \mathrm{d}x$ ***M1***

Note: For the ***M1,*** condone incorrect or missing limits and omission of π.

$= \pi\left[\tan\left(x-\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \pi\left[\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)\right]$ ***(A1)***

$= \pi\left(1-\left(-1\right)\right)$

$= 2\pi$ ***A1***

[4 marks]

Total [7 marks]

10. (a) outer curved surface area is $2\pi(4r)h$ AND inner curved surface area is $2\pi rh$ ***(A1)***

area of each base (top and bottom) is $\pi(4r)^2 - \pi r^2$ ***(A1)***

$S = 2\left[\pi(4r)^2 - \pi r^2\right] + 2\pi(4r)h + 2\pi rh$ ***A1***

$= 30\pi r^2 + 10\pi rh$ ***AG***

[3 marks]

(b) $30\pi r^2 + 10\pi rh = 240\pi$

attempt to solve their equation for h or rh in terms of r (must isolate h or rh) ***(M1)***

$h = \dfrac{240 - 30r^2}{10r}\left(= \dfrac{24 - 3r^2}{r}\right)$ OR $rh = \dfrac{240 - 30r^2}{10}\left(= 24 - 3r^2\right)$ (or equivalent)

A1

uses volume = large cylinder – small cylinder ***(M1)***

$V = \pi(4r)^2 h - \pi r^2 h \quad \left(= 16\pi r^2 h - \pi r^2 h = 15\pi r^2 h\right)$ ***A1***

attempt to substitute in for h or rh ***(M1)***

$V = 15\pi r^2\left(\dfrac{24 - 3r^2}{r}\right)$ OR $V = 15\pi r\left(\dfrac{240 - 30r^2}{10}\right)\left(= 15\pi r\left(24 - 3r^2\right)\right)$ OR

$384\pi r - 48\pi r^3 - 24\pi r + 3\pi r^3$ ***A1***

$= 360\pi r - 45\pi r^3$ ***AG***

[6 marks]

(c) $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 360\pi - 135\pi r^2$ ***A1A1***

[2 marks]

continued…

Question 10 continued.

(d) **METHOD 1** (working with r)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi r^2=0$

$r^2=\frac{360}{135}\left(=\frac{8}{3}\right)$

$r=\sqrt{\frac{360}{135}}\left(=\sqrt{\frac{8}{3}}\right)$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

METHOD 2 (working with $p\sqrt{\frac{2}{3}}$)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi\left(p\sqrt{\frac{2}{3}}\right)^2=0$

$360-90p^2=0$

$p^2=4$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

[3 marks]

continued…

Question 10 continued.

(e) attempt to substitute their value of r into $V = 360\pi r - 45\pi r^3$ ***M1***

$$V = 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times \left(2\sqrt{\frac{2}{3}}\right)^3$$

$$= 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times 8 \times \frac{2}{3} \times \sqrt{\frac{2}{3}}$$

$$\left(= 720\pi\sqrt{\frac{2}{3}} - 240\pi\sqrt{\frac{2}{3}}\right) \quad \textbf{(A1)}$$

$$= 480\pi\sqrt{\frac{2}{3}} \quad \textbf{A1}$$

[3 marks]

Total [17 marks]

11. (a) **EITHER**

attempt to use l'Hôpital's ***(M1)***

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\frac{2e^{2x}}{2e^{2x}}$$ ***A1***

OR

attempt to divide each term by e^{2x} ***(M1)***

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)$$ ***A1***

THEN

$=1$ ***AG***

[2 marks]

(b) (i) attempt to use quotient rule or product rule ***M1***

$$\frac{dy}{dx}=\frac{\left(e^{2x}+1\right)2e^{2x}-\left(e^{2x}-1\right)2e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for second term in numerator. If denominator is incorrect award ***A1A0.***

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***AG***

(ii) attempt to substitute for y and express as a single fraction ***M1***

$$1-y^2=1-\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}$$

$$=\frac{\left(e^{2x}+1\right)^2-\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}\text{ or equivalent}$$ ***(A1)***

$$=\frac{e^{4x}+2e^{2x}+1-\left(e^{4x}-2e^{2x}+1\right)}{\left(e^{2x}+1\right)^2}\quad\text{OR}\quad\frac{2e^{2x}\times 2}{\left(e^{2x}+1\right)^2}$$ ***A1***

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***AG***

[6 marks]

continued…

Question 11 continued.

(c) (i) attempt to use implicit differentiation ***M1***

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} =\right) -2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***

$= -2y\left(1-y^2\right)$ ***A1***

$= 2y^3 - 2y$ ***AG***

(ii) $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3} =\right)\left(6y^2-2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***

$= \left(6y^2-2\right)\left(1-y^2\right)\left(= 8y^2 - 6y^4 - 2\right)$ ***A1***

[5 marks]

(d) attempt to evaluate at least three of y, y', y'', y''' at $x = 0$ ***(M1)***

$y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = -2$ ***A1***

attempt to use Maclaurin series $y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$

M1

$y \approx 0 + 1x + 0x^2 + \frac{(-2)}{3!}x^3 + \ldots$

$y \approx x - \frac{x^3}{3} + \ldots$ ***A1***

[4 marks]

Total [17 marks]

10. (a) (i) $a = 0.358$ (exact); $b = 30.5$ (exact answer is 30.52) ***A1A1***

Note: Award ***A1A0*** if the values of a and b are interchanged or not labeled.

(ii) a represents the (average) rate of increase (change) in population (0.358 millions of people per year). (or equivalent) ***R1***

[3 marks]

(b) It is unreliable because 2030 is outside the range of data (extrapolation). ***A1***

[1 mark]

(c) (i) attempt to find $B(100)$ ***(M1)***

$55.1633\ldots$

55.2 million OR $55{,}200{,}000$ ***A1***

(ii) The annual growth rate of the population is 0.5%. ***A1***

Note: Description must include some reference to annual rate.

[3 marks]

(d) $54.6094\ldots$

54.6 million OR $54{,}600{,}000$ ***A1***

[1 mark]

(e) consideration of the difference function $C(t) - B(t)$ or $B(t) - C(t)$ or $|C(t) - B(t)|$ ***(M1)***

evidence of finding the maximum (or minimum) of this function. ***(M1)***

$t = 58.6283\ldots$

2058 (accept 2059) ***A1***

[3 marks]

continued…

Question 10 continued.

(f) (i) $0.242876\ldots$
$B'(75) = 0.243$ ***A1***

(ii) $0.184941...$
$C'(75) = 0.185$ ***A1***

[2 marks]

(g) $B'(75) > C'(75)$ (or equivalent in words) ***A1***

the population in Benoit's model is increasing at a faster rate than in Cecilia's model (in 2075) (or equivalent) ***R1***

Note: Do not award ***A0R1***.

[2 marks]

Total [15 marks]

1. $f'(x) = 4x^2 - 16$ ***A1***

sets their derivative equal to zero ***(M1)***

$4x^2 - 16 = 0$, $(x = \pm 2)$

$p = 2$ (accept $x = 2$) ***A1***

substitutes their **positive** *p* into $f(x)$ ***(M1)***

$y = \frac{4\left(2^3\right)}{3} - 16(2)\left(= \frac{32}{3} - 32 = -\frac{64}{3}\right)$

$q = -\frac{64}{3}$ (accept $y = \frac{-64}{3}$) ***A1***

Total [5 marks]

3. METHOD 1

attempt to set up integral $e^x - \left(-e^x\right) = 2e^x$ or e^x and then double ***(M1)***

$\int \left(e^x - \left(-e^x\right)\right)dx$ OR $2\int e^x\,dx$

$= 2\int_{-1}^{1} e^x\,dx$

$= 2\left[e^x\right]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into their integrated function and subtract ***(M1)***

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

METHOD 2

$\int_{-1}^{1} e^x\,dx = \left[e^x\right]_{-1}^{1}$ and $\int_{-1}^{1} -e^x\,dx = \left[-e^x\right]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into both their integrated functions and subtract ***(M1)***

$e^1 - e^{-1}$ and $-e^1 - (-e^{-1})$

subtracts their two integrals in correct order ***(M1)***

$e^1 - e^{-1} - \left(-e^1 + e^{-1}\right)$

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

Total [4 marks]

9. $y = vx,\ \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{x - vx}{x + vx}\left(= \frac{1 - v}{1 + v}\right)$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{1 - v}{1 + v}$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1 - v}{1 + v} - v\left(= \frac{1 - 2v - v^2}{1 + v}\right)$

attempt to separate variables and form two integrals ***(M1)***

$\int \frac{1 + v}{1 - 2v - v^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$

$\int \frac{1}{x}\mathrm{d}x = \ln x + c$ ***A1***

use of substitution or inspection to integrate $\frac{1 + v}{1 - 2v - v^2}$ or equivalent ***(M1)***

$u = 1 - 2v - v^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}v} = -2 - 2v = -2(1 + v)$

$\int \frac{1 + v}{1 - 2v - v^2}\mathrm{d}v = -\frac{1}{2}\ln\left|1 - 2v - v^2\right|$ OR $-\frac{1}{2}\ln\left|v^2 + 2v - 1\right| = \ln|x| + c$ ***A1***

$-\frac{1}{2}\ln\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| - \ln|x| = c$ OR $-\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| - \ln|x| = c$

EITHER

attempt to substitute $x = 2$ and either $y = 0$ or $v = 0$ to find a constant c ***(M1)***

$c = -\ln 2$

$-\frac{1}{2}\ln\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| - \ln|x| = -\ln 2$ OR $-\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| - \ln|x| = -\ln 2$ ***A1***

OR

$x^2\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| = A$ OR $x^2\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = A$ ***A1***

attempt to substitute $x = 2$ either $y = 0$ or $v = 0$ to find a constant A ***(M1)***

THEN

$x^2\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| = 4$ OR $x^2\left(1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right) = 4$ ***A1***

checking boundary values confirms $x^2 - 2xy - y^2 = 4$ ***AG***

continued…

Question 9 continued.

Note: Condone absence of absolute value signs even if removed incorrectly until the final ***A1*** mark where they must be seen or have been removed to form a correct equation.

Total [8 marks]

Section B

12. (a) attempt to use integration by parts on $f_n(x) = \cos^{n-1} x \cos x$ ***M1***

$u = \cos^{n-1} x,\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = -(n-1)\cos^{n-2} x \sin x,\ v = \sin x$

$\int f_n(x)\mathrm{d}x = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \mathrm{d}x$ ***A1A1***

Note: ***A1*** for each term with correct signs

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1 - \cos^2 x\right)\mathrm{d}x$ ***A1***

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \mathrm{d}x - (n-1)\int \cos^n x \mathrm{d}x$ ***AG***

[4 marks]

(b) attempt to rearrange the equation given in part (a) to collect terms in $\int f_n(x)\mathrm{d}x$ or $\int \cos^n x \mathrm{d}x$ ***M1***

$\int f_n(x)\mathrm{d}x + (n-1)\int f_n(x)\mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)\mathrm{d}x$ OR

$\int \cos^n x \mathrm{d}x + (n-1)\int \cos^n x \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \mathrm{d}x$

$n\int f_n(x)\mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)\mathrm{d}x$ OR

$n\int \cos^n x \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \mathrm{d}x$ ***A1***

$\int f_n(x)\mathrm{d}x = \dfrac{1}{n}\cos^{n-1} x \sin x + \dfrac{n-1}{n}\int f_{n-2}(x)\mathrm{d}x$ ***AG***

[2 marks]

continued…

Question 12 continued.

(c) attempt to use equation from part (a) to reduce the power of $\cos x$ ***(M1)***

$$\int \cos^4 x\mathrm{d}x = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\mathrm{d}x$$

EITHER

attempt to use equation from part (a) again to reduce the power of $\cos x$ ***(M1)***

$$\int \cos^4 x\mathrm{d}x = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\mathrm{d}x\right)$$ ***(A1)***

OR

attempt to use double angle formula to rewrite $\cos^2 x$ in terms of $\cos 2x$ ***(M1)***

$$\int \cos^4 x\mathrm{d}x = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int\left(\frac{\cos 2x+1}{2}\right)\mathrm{d}x$$ ***(A1)***

THEN

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + c$$ ***A1***

[4 marks]

(d) attempt to use the formula for volume of revolution using π and $\left(\cos^2 x\right)^2$ ***(M1)***

$$\text{volume} = \int \pi\left(\cos^2 x\right)^2\mathrm{d}x$$

$$\text{volume} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2\mathrm{d}x \quad \text{OR} \quad 2\int_{0}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2\mathrm{d}x$$ ***(A1)***

Note: Condone omission $\mathrm{d}x$ for the ***A1***.

attempt to substitute correct limits into their (c) and subtract ***(M1)***

$$\pi\left(\frac{1}{4}\cos^3\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3}{8}\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - \pi\left(\frac{1}{4}\cos^3\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{3\pi^2}{8}$$ ***A1***

Note: ***FT*** marks may be awarded for a final answer of $r\pi^2$ based on non-zero values of p, q and r.

[4 marks]

continued…

Question 12 continued.

(e) (i) **METHOD 1**

attempt to raise Maclaurin expansion for $\cos x$ to the power of n ***(M1)***

$$f_n(x) = (\cos x)^n = \left(1 - \frac{x^2}{2} + \ldots\right)^n$$

$$= 1 - \frac{nx^2}{2} + \ldots$$ ***A2***

METHOD 2

attempt to differentiate $f_n(x)$ twice ***(M1)***

$f_n'(x) = -n\cos^{n-1} x \sin x,\ f_n''(x) = -n\cos^n x + n(n-1)\cos^{n-2} x \sin^2 x$ ***A1***

$f_n(0) = 1,\ f_n'(0) = 0,\ f_n''(0) = -n$

$$f_n(x) = 1 - \frac{nx^2}{2} + \ldots$$ ***A1***

(ii) **METHOD 1**

attempt to use Maclaurin expansion for $f_n(x)$ ***(M1)***

$$\lim_{x\to 0} \frac{f_n(x) - 1}{x^2} = \lim_{x\to 0} \frac{-\frac{nx^2}{2} + \ldots}{x^2} \left(= \lim_{x\to 0}\left(-\frac{n}{2} + \text{ powers of } x\right)\right)$$

$$= -\frac{n}{2}$$ ***A1***

METHOD 2

attempt to use l'Hôpital's rule twice on $\frac{\cos^n(x) - 1}{x^2}$ ***(M1)***

$$\lim_{x\to 0} \frac{\cos^n(x) - 1}{x^2} = \lim_{x\to 0} \frac{-n\cos^{n-1} x \sin x}{2x} = \lim_{x\to 0} \frac{n(n-1)\cos^{n-2} x \sin^2 x - n\cos^n x}{2}$$

$$= -\frac{n}{2}$$ ***A1***

Note: Do not award ***FT*** marks for an expression that does not involve n.

[5 marks]

Total [19 marks]

6.

(a) correct substitution into distance formula ***A1***

$l^2 = x^2 \ln x + 4 - x^2 + x^2$ OR $\sqrt{(x-0)^2 + \left(\sqrt{x^2 \ln x + 4 - x^2} - 0\right)^2}$ OR

$\sqrt{x^2 + x^2 \ln x + 4 - x^2}$

$l = \sqrt{x^2 \ln x + 4}$ ***AG***

[1 mark]

(b) recognising $\frac{\mathrm{d}l}{\mathrm{d}x} = 0$ (seen anywhere) ***(M1)***

EITHER

attempt to use chain rule with l ***(M1)***

$\frac{1}{2}\left(x^2 \ln x + 4\right)^{-\frac{1}{2}} \times \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x + 4\right)$

attempt to use product rule with $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x + 4\right)$ ***(M1)***

$\frac{1}{2}\left(x^2 \ln x + 4\right)^{-\frac{1}{2}} \times \left[x^2 \times \frac{1}{x} + \ln x \times 2x\right]$ ***A1***

OR

recognising to minimise $x^2 \ln x + 4$ ***(M1)***

attempt to use product rule ***(M1)***

$x^2 \times \frac{1}{x} + \ln x \times 2x$ ***A1***

THEN

$x + 2x \ln x = 0$ (or equivalent) ***A1***

$\Rightarrow \ln x = -\frac{1}{2}$

$x = \mathrm{e}^{-\frac{1}{2}} \left(= \frac{1}{\sqrt{\mathrm{e}}}\right)$ ***A1***

Note: Award ***A0*** for including $x = 0$ in the final answer.

[6 marks]

Total [7 marks]

7. $x^2 = r^4 - y^4$ (seen anywhere) (***A1)***

substituting their expression for x^2 into $\int_a^b \pi x^2 \mathrm{d}y$ ***(M1)***

$\pi\int_{-r}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ **OR** $2\pi\int_{0}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ (***A1)***

$= \pi\left[r^4 y - \frac{y^5}{5}\right]_{-r}^{r}$ **OR** $2\pi\left[r^4 y - \frac{y^5}{5}\right]_{0}^{r}$ ***A1***

Note: Award ***A1*** for $r^4 y - \frac{y^5}{5}$ seen.

$= \pi\left(\left(r^5 - \frac{r^5}{5}\right) - \left(-r^5 + \frac{r^5}{5}\right)\right)$ **OR** $2\pi\left(r^5 - \frac{r^5}{5}\right)$ ***(A1)***

$= \frac{8}{5}\pi\mathrm{r}^5$ ***A1***

Total [6 marks]

9. METHOD 1

$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***R1***

Note: Subsequent marks are independent of this ***R*** mark.

$=\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}\ \left(=\lim_{x\to 0}\left(1+\frac{x\cos x}{\sin x}\right)\right)$ ***A1***

EITHER

Applying l'Hôpital's rule again (since $\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=\frac{0}{0}$) ***(M1)***

Note: Award ***(M0)*** if their limit is not the indeterminate form $\frac{0}{0}$.

$\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{\cos x}$ ***A1***

OR

$=\lim_{x\to 0}\left(1+\frac{\cos x}{\left(\frac{\sin x}{x}\right)}\right)$ ***A1***

recognizing fundamental trig limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ ***R1***

THEN

substituting $x=0$ ***(M1)***

$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$ ***A1***

continued…

Question 9 continued

METHOD 2

multiplying numerator and denominator by $1+\cos x$ ***M1***

$$\lim_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$$

$$\lim_{x\to 0}\left(\frac{x\sin x\left(1+\cos x\right)}{1-\cos^2 x}\right)$$

$$\lim_{x\to 0}\left(\frac{x\left(1+\cos x\right)}{\sin x}\right)$$ ***A1***

$$=\lim_{x\to 0}\left(\frac{1+\cos x}{\left(\frac{\sin x}{x}\right)}\right)$$ ***(A1)***

recognizing fundamental trig limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ OR $\lim_{x\to 0}\frac{x}{\sin x}=1$ ***(R1)***

substituting $x=0$ ***(M1)***

$$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$$ ***A1***

continued…

Question 9 continued

METHOD 3

expressing either $x\sin x$ or $1-\cos x$ as a Maclaurin series with at least two terms ***(M1)***

$$\lim_{x\to 0}\left(\frac{x\left(x-\frac{x^3}{3!}\left(+\frac{x^5}{5!}\ldots\right)\right)}{\frac{x^2}{2!}-\frac{x^4}{4!}\left(+\frac{x^6}{6!}+\ldots\right)}\right)$$ ***A1***

recognizing numerator and denominator are multiples of x^2 ***(M1)***

$$\lim_{x\to 0}\left(\frac{1-\frac{x^2}{3!}\left(+\frac{x^4}{5!}-\ldots\right)}{\frac{1}{2!}-\frac{x^2}{4!}\left(+\frac{x^4}{6!}+\ldots\right)}\right)$$ ***A1***

substituting $x=0$ ***(M1)***

$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$ ***A1***

METHOD 4

Expressing in terms of half angles ***M1***

$$\lim_{x\to 0}\frac{2x\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$ ***A1A1***

$$=\lim_{x\to 0}\frac{x\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=2\lim_{x\to 0}\frac{\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}$$ ***(M1)***

Using $\lim_{x\to 0}\frac{x}{\tan x}=1$ ***M1***

=2 ***A1***

[6 marks]

Section B

3. METHOD 1

attempt to find change in population using a definite integral (***M1)***

$t = 4$ at the start of 2026 (seen anywhere) ***(A1)***

$\int_0^4 -104000\mathrm{e}^{-0.0145t}\mathrm{d}t$ (***A1)***

$= -404165.8...$ (***A1)***

attempt to add initial population to their change in population from a definite integral (***M1)***

population at the start of 2026 $= 6.78\times10^6 - 404165.8...$

$= 6375834.1...$

$= 6380000\left(= 6.38\times10^6\right)$ ***A1***

METHOD 2

attempt to find population using an indefinite integral (***M1)***

$P = \int -104000\mathrm{e}^{-0.0145t}\mathrm{d}t$

$\frac{-104000\mathrm{e}^{-0.0145t}}{-0.0145} + c\left(= 7172413.7...\mathrm{e}^{-0.0145t} + c\right)$ ***(A1)***

attempt to substitute $t = 0, P = 6.78\times10^6$ into equation with c. ***(M1)***

$6.78\times10^6 = 7172413.7... + c \Rightarrow c = -392413.7...$

$P = 7172413.7...\mathrm{e}^{-0.0145t} - 392413.7...$ ***(A1)***

$t = 4$ at the start of 2026 (seen anywhere) ***(A1)***

population at the start of 2026 $7172413.7...\mathrm{e}^{-0.0145(4)} - 392413.7...$

$= 6375834.1...$

$= 6380000\left(= 6.38\times10^6\right)$ ***A1***

Total [6 marks]

5. (a) recognition of the need to differentiate **(*M1*)**

$h'(x) = \frac{\pi}{50}\left(-15\sin\left(\frac{\pi x}{50}\right)\right)\left(= -\frac{15\pi}{50}\sin\left(\frac{\pi x}{50}\right) = -\frac{3\pi}{10}\sin\left(\frac{\pi x}{50}\right)\right)$ **A1A1**

$h'(k) = -\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right)\left(= -\frac{3\pi}{10}\sin\left(\frac{\pi k}{50}\right)\right)$

Note: Award ***A1*** for $-15\sin\left(\frac{\pi k}{50}\right)$ and ***A1*** for factor of $\frac{\pi}{50}$.

Award ***A1A0*** for a correct expression with additional terms or additional factors.

[3 marks]

(b) recognition that gradient of tangent = $-\tan\left(\frac{\pi}{8}\right)$ OR $\tan\left(\frac{7\pi}{8}\right)$ **(*M1*)**

Note: Accept $\tan\left(\frac{\pi}{8}\right)$ OR $-\tan\left(\frac{7\pi}{8}\right)$ for the (***M1***)

setting their $h'(k)$ equal to $-\tan\left(\frac{\pi}{8}\right)$ OR $\tan\left(\frac{7\pi}{8}\right)(= -0.414213...)$ **(*A1*)**

$-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right) = -\tan\left(\frac{\pi}{8}\right)$ OR $-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right) = \tan\left(\frac{7\pi}{8}\right)$

$k = 7.24211...$, $k = 42.7578...$

$k = 7.24$, $k = 42.8$ ***A1***

[3 marks]
Total [6 marks]

9. (a) attempt to use Euler's method with a step of 0.25 ***(M1)***

$\left(x_{n+1} = x_n + 0.25\right), y_{n+1} = y_n + 0.25\left(\frac{2x_n}{{x_n}^2 + y_n}\right)$

$\left(y_0 = 0\right)$
$y_1 = 0.5$
$y_2 = 0.803030...$ ***(A1)***
$y_3 = 1.04868...$

Note: Award ***(A1)*** for at least two correct intermediate values given to 3sf.

If $x = 2, y = 1.26152...$

$y = 1.26$ ***A1***

[3 marks]

(b) (i) The estimate will be an overestimate ***A1***

because the tangents to the curve lie above the curve (the curve is concave down). ***R1***

(ii) The gradient of the curve (at $(1, 0)$) is positive (so the method works in the direction of the upper part of the curve). ***R1***

[3 marks]

Total [6 marks]

Section B

3.

(a) one critical value (seen anywhere) ***(A1)***

recognizing that for f to be decreasing $f' < 0$ ***(M1)***

$x = -1.73554...$ or $x = 0.517999...$

$x \leq -1.74$ and $x \geq 0.518$ OR $x < -1.74$ and $x > 0.518$ ***A1***

[3 marks]

(b) correct critical value $x = -0.405465... \left(= \ln \frac{2}{3} \right)$ (seen anywhere) ***(A1)***

attempt to find maximum of f' OR recognising that $f''(x) > 0$ OR sketching graph of f'' OR $f''(x) = 2 - 3e^x$ ***(M1)***

concave-up for $x < \ln \frac{2}{3}$ OR $x < -0.405$ ***A1***

[3 marks]

Total [6 marks]

5.

(a) recognition that maximum speed occurs when $|v|$ is greatest ***(M1)***

one correct coordinate for minimum $(4.71238..., -2.71828...)$

$2.72\ (=e)\ \ (\text{ms}^{-1})$ ***A1***

Note: Award ***(M1)A0*** for the answer 1, from working with degrees.

[2 marks]

(b) substitution of limits into correct formula ***(M1)***

$\int_0^5 |v| \, dt$

$3.84591...$

$3.85\ (\text{metres})$ ***A1***

[2 marks]

(c) $v = 0$ (seen anywhere) ***(M1)***

$t = 2.35619... \left(= \frac{3\pi}{4}\right)$ ***(A1)***

$a = 0.986137...$

$a = 0.986 \left(= 2e^{-\frac{1}{\sqrt{2}}}\right) \ (\text{ms}^{-2})$ ***A1***

[3 marks]

Total [7 marks]

11. (a) **METHOD 1**

$\frac{{}^{15}C_5 \times {}^{10}C_5 \times {}^{5}C_5}{3!}\left(=\frac{3003\times 252\times 1}{6}=\frac{756756}{6}\right)$ OR $\frac{15!}{5!\times 5!\times 5!\times 3!}$ ***(A1)(A1)***

Note: Award ***A1*** for ${}^{15}C_5 \times {}^{10}C_5 \left(=756756\right)$ OR $\frac{15!}{5!\times 5!\times 5!}$ and ***A1*** for dividing by $3!\left(=6\right)$.

$=126126$ ***A1***

METHOD 2

${}^{14}C_4 \times {}^{9}C_4 \times {}^{4}C_4$ ***(A1)(A1)***

Note: Award ***A1*** for ${}^{14}C_4$ and ***A1*** for ${}^{14}C_4 \times {}^{9}C_4$.

If a random student comes out and picks a team of 4 on their turn.

$=126126$ ***A1***

[3 marks]

(b) attempt to use integration by parts ***(M1)***

$u=t$ and $\frac{\mathrm{d}v}{\mathrm{d}t}=\mathrm{e}^{-3t}$

$\int \mathrm{e}^{-3t}dt=-\frac{1}{3}\mathrm{e}^{-3t}$ (seen anywhere) ***A1***

$\int_0^a f(t)\,\mathrm{d}t = k\left\{\left[t\left(-\frac{1}{3}\mathrm{e}^{-3t}\right)\right]_0^a-\int_0^a -\frac{1}{3}\mathrm{e}^{-3t}dt\right\}$

$=k\left\{\left[t\left(-\frac{1}{3}\mathrm{e}^{-3t}\right)\right]_0^a+\frac{1}{3}\left[-\frac{1}{3}\mathrm{e}^{-3t}\right]_0^a\right\}$ ***A1***

Note: Condone the absence of k and limits up to this stage.

continued…

Question 11 continued.

$$= k\left\{\left(-\frac{1}{3}a\mathrm{e}^{-3a} - 0\right) + \frac{1}{3}\left(-\frac{1}{3}\mathrm{e}^{-3a} + \frac{1}{3}\right)\right\}$$

$$= k\left(-\frac{1}{3}a\mathrm{e}^{-3a} - \frac{1}{9}\mathrm{e}^{-3a} + \frac{1}{9}\right) \quad \textbf{A1}$$

$$= \frac{k}{9}\left[1 - (3a+1)\mathrm{e}^{-3a}\right] \quad \textbf{AG}$$

[4 marks]

(c) (i) $\lim\limits_{x\to\infty}\dfrac{3x+1}{\mathrm{e}^{3x}} = \lim\limits_{x\to\infty}\dfrac{3}{3\mathrm{e}^{3x}}\left(= \lim\limits_{x\to\infty}\dfrac{1}{\mathrm{e}^{3x}}\right)$ ***A1***

Note: This first ***A1*** must be seen.

$= 0$ ***A1***

(ii) $\lim\limits_{a\to\infty}\dfrac{k}{9}\left[1-(3a+1)\mathrm{e}^{-3a}\right] = \dfrac{k}{9}\left(1 - \lim\limits_{a\to\infty}\dfrac{3a+1}{\mathrm{e}^{3a}}\right)$

$= \dfrac{k}{9}(1-0)$ ***(A1)***

recognising area of probability density function $= 1$ (seen anywhere) ***(M1)***

$\dfrac{k}{9} = 1$

$k = 9$ ***A1***

[5 marks]

(d) recognition that the integral or their expression equals 0.5 for $a = m$ ***(M1)***

$\int_0^m f(t)\,dt = 0.5$ OR $1-(3m+1)\mathrm{e}^{-3m} = 0.5$

$0.559448\ldots$ OR $33.566939\ldots$

median = 0.559 (min) OR 33.6 (sec) ***A1***

Note: ***FT*** can only be awarded for a positive value of k.

Award ***(M1)(A1)*** for a final answer of 0.56 with no working shown.

[2 marks]

Total [14 marks]

12. (a) **METHOD 1**

attempt to differentiate implicitly ***(M1)***

$8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 24 + 4\frac{\mathrm{d}y}{\mathrm{d}x}(= 0)$ ***A1A1***

Note: Award ***A1*** for $8x - 24$ and ***A1*** for $2y\frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}y}{\mathrm{d}x}$.

Award ***A1A0*** for $8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 24 + 4\frac{\mathrm{d}y}{\mathrm{d}x} = -20$ (or equivalent).

The implicit differentiation must be seen to award the above ***A*** marks.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{24 - 8x}{2y + 4}\ \left(= \frac{12 - 4x}{y + 2}\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(3 - x)}{y + 2}$ ***AG***

METHOD 2

$\frac{d}{dx}\left(4x^2 + y^2 - 24x + 4y + 20\right) = 8x - 24$ ***A1***

$\frac{d}{dy}\left(4x^2 + y^2 - 24x + 4y + 20\right) = 2y + 4$ ***A1***

$\frac{dy}{dx} = -\frac{\frac{dC}{dx}}{\frac{dC}{dy}}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{24 - 8x}{2y + 4}\ \left(= \frac{12 - 4x}{y + 2}\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(3 - x)}{y + 2}$ ***AG***

[4 marks]

continued…

Question 12 continued.

(b) recognising boundaries of domain when $\frac{\mathrm{d}y}{\mathrm{d}x}$ undefined ***(M1)***

$y+2=0$

$y=-2$ ***(A1)***

substituting their y value into curve equation and attempt to solve for x ***(M1)***

$4x^2+(-2)^2-24x+4(-2)+20=0 \quad (x^2-6x+4=0)$

$x=\frac{6\pm\sqrt{20}}{2}$ OR $(x-3)^2-5=0$

$3-\sqrt{5}\le x\le 3+\sqrt{5}$ (accept $a=5$) ***A1***

Note: Award a maximum of ***M1A1M1A0*** for a final answer of $0.764\le x\le 5.24$.

[4 marks]

(c) $x_{\mathrm{A}}=3$ ***A1***

substituting their x value into curve equation ***(M1)***

$4(3)^2+y^2-24(3)+4y+20=0$

$2.472135\ldots$OR $-2+2\sqrt{5}$

$y_{\mathrm{A}}=2.47$ OR $y_{\mathrm{A}}=-2+2\sqrt{5}$ ***A1***

$A(3,2.47)$ OR $A\left(3,-2+2\sqrt{5}\right)$

Note: Award ***A1(M1)A1*** for a final answer of $A(3,2.5)$ with no working shown.

[3 marks]

(d) $4x^2+(mx)^2-24x+4(mx)+20=0 \quad \left(\left(4+m^2\right)x^2+(4m-24)x+20=0\right)$ ***(A1)***

$(4m-24)^2-4\left(4+m^2\right)(20)$ ***A1***

recognising discriminant = 0 (seen anywhere) ***M1***

both $m=-4$ and $m=1$ ***A1***

[4 marks]

continued…

Question 12 continued.

(e) **METHOD 1**

attempt to form an equation in y by substituting $x = -\frac{1}{4}y$ ***(M1)***

$4\left(-\frac{1}{4}y\right)^2 + y^2 - 24\left(-\frac{1}{4}y\right) + 4y + 20 = 0$ ***(A1)***

$y_B = -4$ ***A1***

Note: Award ***(M1)A1A0*** if an extra value is given for y_B.

METHOD 2

attempt to form an equation in x by substituting $y = -4x$ ***(M1)***

$4x^2 + (-4x)^2 - 24x + 4(-4x) + 20 (= 0)$

$x = 1$ ***(A1)***

$y_B = -4$ ***A1***

Note: Award ***(M1)A1A0*** if an extra value is given for y_B.

[3 marks]

(f) attempt to solve for x by quadratic formula or completing the square ***(M1)***

$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4)(y^2 + 4y + 20)}}{2 \times 4}$ OR $4(x-3)^2 - 36 = -y^2 - 4y - 20$

$x = \frac{24 - \sqrt{256 - 64y - 16y^2}}{8}$ $\left(x = 3 - \sqrt{\frac{16 - 4y - y^2}{4}}\right)$ ***A1***

Note: Condone $\pm$ at this stage.

$V = \pi \int_{-4}^{-2+2\sqrt{5}} \left(\frac{24 - \sqrt{256 - 64y - 16y^2}}{8}\right)^2 \mathrm{d}y \left(= \pi \int_{-4}^{2.47\ldots} \left(3 - \sqrt{\frac{16 - 4y - y^2}{4}}\right)^2 \mathrm{d}y\right)$ ***A1***

$29.708027\ldots$

volume = 29.7 ***A1***

Note: Award ***(M1)A1A1*** for a final answer of 30 with no working shown.

[4 marks]

Total [22 marks]

5. (a) use of quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3\times 2\times (2x+a)^2(x+5)^2 - 2(2x+a)^3(x+5)}{\left((x+5)^2\right)^2}$$ ***(A1)***

$$f'(x) = \frac{6(2x+a)^2(x+5)^2 - 2(2x+a)^3(x+5)}{(x+5)^4} \left(= \frac{2(x-a+15)(2x+a)^2}{(x+5)^3}\right)$$ ***A1***

OR

$$f'(x) = 3\times 2\times (2x+a)^2(x+5)^{-2} - 2(2x+a)^3(x+5)^{-3}$$ ***(A1)***

$$f'(x) = 6(2x+a)^2(x+5)^{-2} - 2(2x+a)^3(x+5)^{-3} \left(= 2(x-a+15)(2x+a)^2(x+5)^{-3}\right)$$ ***A1***

[3 marks]

(b) recognizing $f'(1)$ is equal to $\tan 70^\circ$ (=2.74747) ***(M1)***

$\frac{6(2+a)^2 6^2 - 2(2+a)^3\times 6}{6^4} = \tan 70^\circ$ **OR** $6(2+a)^2 6^{-2} - 2(2+a)^3\times 6^{-3} = \tan 70^\circ$ ***(A1)***

$2.72844\ldots,\ 14.96968\ldots$
$a = 2.73, 15.0$ ***A1A1***

[4 marks]

Total [7 marks]

9. (a) Let $\hat{\text{APG}} = \alpha$

$\theta = \hat{\text{BPG}} - \hat{\text{APG}}$ (may be seen in diagram) ***(M1)***

$\tan(\theta + \alpha) = \frac{x+2}{6}$ $\left(\Rightarrow \theta + \alpha = \arctan\left(\frac{x+2}{6}\right)\right)$ AND $\tan\alpha = \frac{x}{6}$ $\left(\Rightarrow \alpha = \arctan\left(\frac{x}{6}\right)\right)$ ***A1***

$\theta = \arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right)$

AG

[2 marks]

(b) **METHOD 1**

setting $\theta = 0.178$ ***(M1)***

$\arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right) = 0.178$

$x = 4.63047...$ ***(A1)***

Note: The following three marks are independent of the first two marks.

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\left(\frac{x+2}{6}\right)\right) = \frac{6}{(x+2)^2+36}$ OR $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\left(\frac{x}{6}\right)\right) = \frac{6}{x^2+36}$ (seen anywhere) ***(A1)***

attempt to use the chain rule to differentiate $\arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right)$ with respect to t ***(M1)***

$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} =\right) \frac{6}{(x+2)^2+36} \times \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{6}{x^2+36} \times \frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} =\right) -0.0294199... \times \frac{\mathrm{d}x}{\mathrm{d}t}$ (or equivalent) ***(A1)***

substituting their x and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 12.5$ ***(M1)***

$12.5 = -0.0294199... \times \frac{\mathrm{d}x}{\mathrm{d}t}$

continued…

Question 9 continued

OR

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{12}{x^2+2x+36}\right)=-\frac{12(2x+2)}{\left(x^2+2x+36\right)^2}$ (seen anywhere) ***(A1)***

attempt to use the chain rule to differentiate $\tan\theta=\frac{12}{x^2+2x+36}$ with respect to t ***(M1)***

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{12(2x+2)}{\left(x^2+2x+36\right)^2}\frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-0.0303721...\times\frac{\mathrm{d}x}{\mathrm{d}t}$ (or equivalent) ***(A1)***

substituting $\theta=0.178$, their x and $\frac{\mathrm{d}\theta}{\mathrm{d}t}=12.5$ ***(M1)***

$\sec^2(0.178)\times12.5=-0.0303721...\times\frac{\mathrm{d}x}{\mathrm{d}t}$

THEN

$\frac{\mathrm{d}x}{\mathrm{d}t}=-424.881...$

$(\text{speed}=)\ 425\ (\text{km hr}^{-1})$ ***A1***

continued…

Question 9 continued

METHOD 2

setting $\theta = 0.178$ ***(M1)***

$$\arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right) = 0.178$$

$x = 4.63047...$ ***(A1)***

substitutes their value of x to find $\frac{\mathrm{d}\theta}{\mathrm{d}x}$ when $\theta = 0.178$ ***(M1)***

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = -0.0294199...$$

recognition that chain rule required (seen anywhere) ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ (or equivalent, must include x, t and θ)

substituting their value for $\frac{\mathrm{d}\theta}{\mathrm{d}x}\left(= -0.0294199...\right)$ and $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} =\right) 12.5$ ***(M1)***

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t} =\right) \frac{1}{-0.0294199...} \times 12.5$$

$= -424.881...$

$(\text{speed} =)\ 425\ (\text{km hr}^{-1})$ ***A2***

[7 marks]

Total [9 marks]

Section B

10. (a) (i) $7.43076\ldots$

$v(1) = 7.43 \ (\text{m s}^{-1})$ ***A1***

(ii) equating $v(t)$ and 5 ***(M1)***

$0.348114\ldots$

0.348 (seconds) ***A1***

[3 marks]

(b) recognizing $a = v'(t)$ ***(M1)***

$0.590930...$

0.591 (seconds) ***A1***

[2 marks]

(c) (i) considering large values of t ***(M1)***

$\left(\lim_{t \to \infty}(v(t)) =\right) 8.14 \ (\text{ms}^{-1})$ ***A1***

(ii) **EITHER**

the race lasts a finite time (e.g. it ends after (Fiona) crosses the line) ***R1***

OR

the graph approaches the limiting value but Fiona will never attain that speed ***R1***

OR

Fiona cannot maintain that speed for a long period of time ***R1***

Note: Award ***R1*** for a valid reason that does not contradict their answer to part (c)(i).

[3 marks]

continued…

Question 10 continued

(d) Fiona takes t_f seconds to travel $200\ \text{m}$

EITHER

recognizing distance travelled by either athlete in the first t seconds is

$\int_0^t v(t)\,\mathrm{d}t$ **OR** $\int_0^t w(t)\,\mathrm{d}t$ (seen anywhere) ***(M1)***

equating distance travelled by Fiona to 200 (m) ***(M1)***

$$\int_0^{t_f} v(t)\,\mathrm{d}t = 200$$

OR

attempt to integrate $v(t)$ ***(M1)***

$$s(t) = 8.14\sqrt{t^2 + 0.2} - 3.64031...$$

equating 200 to *their* $s(t)$ (must include a constant of integration) ***(M1)***

$$8.14\sqrt{t^2 + 0.2} - 3.64031... = 200$$

THEN

$t_f = 25.0132...$ (accept 25) ***(A1)***

recognition that a definite integral of $w(t)$ with 0 and *their* t_f is required ***(M1)***

(Lucy's distance =) $\int_0^{25.0132...} w(t)\,\mathrm{d}t\ \ (= 195.772...)$

distance from finishing line = 200 – *their* Lucy's distance ***(M1)***

$4.22788\ldots$

$4.23\ (\text{m})$ ***A1***

[6 marks]

Total [14 marks]

12. (a) attempt to use integration by parts ***(M1)***

$\left(x^2-5\right)\mathrm{e}^x-2\int x\mathrm{e}^x\,\mathrm{d}x$ **OR** $-5\mathrm{e}^x+x^2\mathrm{e}^x-2\int x\mathrm{e}^x\,\mathrm{d}x$ ***A1***

attempt to use integration by parts a second time ***(M1)***

$\left(x^2-5\right)\mathrm{e}^x-2\left(x\mathrm{e}^x-\int \mathrm{e}^x\,\mathrm{d}x\right)$ **OR** $-5\mathrm{e}^x+x^2\mathrm{e}^x-2\left(x\mathrm{e}^x-\int \mathrm{e}^x\,\mathrm{d}x\right)$ ***A1***

$\left(x^2-5\right)\mathrm{e}^x-2\left(x\mathrm{e}^x-\mathrm{e}^x\right)$ ***(A1)***

$\left(x^2-2x-3\right)\mathrm{e}^x+c$ OR $\left(x^2-5\right)\mathrm{e}^x-2\left(x\mathrm{e}^x-\mathrm{e}^x\right)+c$ (or equivalent) ***A1***

[6 marks]

(b) rearranging to $\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2-5$ **AND** attempt to find integrating factor ***(M1)***

$\mathrm{e}^{\int \mathrm{d}x}=\mathrm{e}^x$ (seen anywhere) ***A1***

$\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^x y=\left(x^2-5\right)\mathrm{e}^x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x y\right)=\left(x^2-5\right)\mathrm{e}^x$ **OR** $\mathrm{e}^x y=\int\left(x^2-5\right)\mathrm{e}^x\,\mathrm{d}x$ ***M1***

$\mathrm{e}^x y=\left(x^2-2x-3\right)\mathrm{e}^x+C$ **OR** $\mathrm{e}^x y=x^2\mathrm{e}^x-2x\mathrm{e}^x-3\mathrm{e}^x+C$ ***A1***

$y=x^2-2x-3+C\mathrm{e}^{-x}$ ***AG***

[4 marks]

continued…

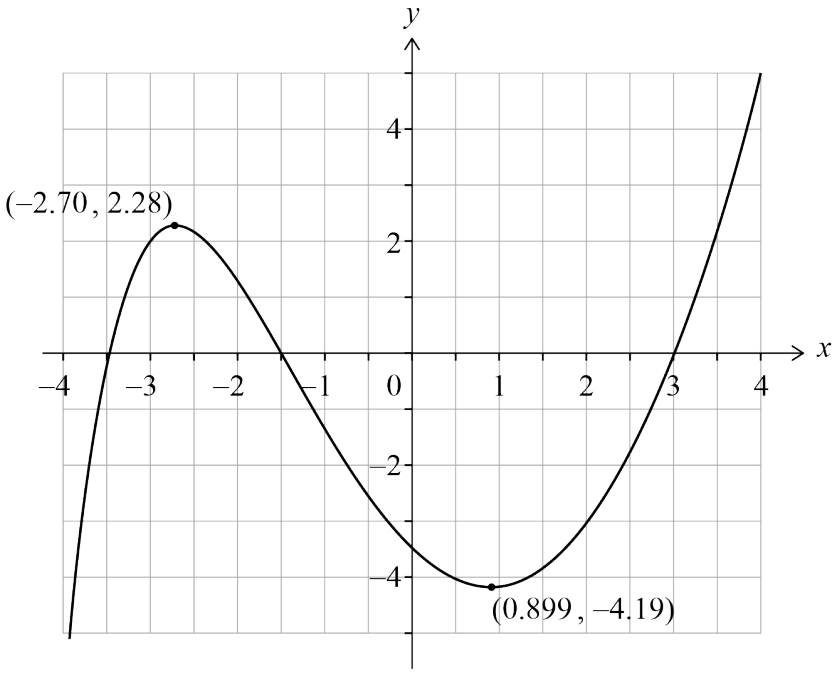
Question 12 continued

(c) substituting $x = -3, y = 2$ into $y = x^2 - 2x - 3 + Ce^{-x}$ ***(M1)***

$2 = 9 + 6 - 3 + Ce^3$

$\Rightarrow C = -0.497870... \left(= -10e^{-3}\right)$ ***A1***

$y = x^2 - 2x - 3 - 0.498e^{-x}$ (accept $y = x^2 - 2x - 3 - 10e^{-x-3}$)



A1A1A1

Note: Award marks as follows:

A1 for approximately correct shape AND x-intercepts in approximately correct place (exactly two negative and one positive x-intercepts)

A1 for maximum in approximately correct position and labelled $(-2.70, 2.28)$ AND minimum in approximately correct position and labelled $(0.899, -4.19)$

A1 for domain.

[5 marks]

continued…

Question 12 continued

(d) substituting -3 and p **OR** -3 and q into $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

gradient of tangent at $(-3,\ p)$ is $4-p$ and at $(-3,\ q)$ is $4-q$

finding equation of tangents ***(M1)***

$y-p=(4-p)(x+3)$ **AND** $y-q=(4-q)(x+3)$ ***A1***

solving their equations simultaneously ***(M1)***

$q-p=(x+3)\big(4-p-(4-q)\big)\ \big(=(x+3)(q-p)\big)$

$x+3=1$ (since $p\neq q$)

$x=-2$ ***A1***

$y-p=4-p$ **OR** $y-q=4-q$

$y=4$ ***A1***

hence all tangents intersect at $(-2,\ 4)$ ***AG***

[6 marks]

Total [21 marks]

2. (a) $g'(x) = m$ OR $g''(x) = 0$ (seen anywhere) ***(A1)***

Note: Award ***A0*** for $g''(x) = 0$ obtained from an incorrect $g'(x)$, for example, $g'(x) = 1$ OR $g'(x) = 0$.

EITHER

$k(x) = \dfrac{|0|}{\left(1+m^2\right)^{\frac{3}{2}}}$ OR $k(x) = \dfrac{0}{\left(1+m^2\right)^{\frac{3}{2}}}$ OR $k(x) = \dfrac{0}{\sqrt{1+3m^2+3m^4+m^6}}$ ***A1***

OR

the numerator of $k(x)$ is zero and the denominator is non-zero ***A1***

THEN

$\Rightarrow k(x) = 0$ ***AG***

[2 marks]

(b) (i) $x = -\dfrac{b}{2a}$ ***A1***

Note: Do not award ***A1*** if $a = 0$ is included as a solution.

[1 mark]

(ii) substitutes their value of x into $k(x)$ ***(M1)***

$$k\left(-\frac{b}{2a}\right) = \frac{2|a|}{\left(1+\left(2a\left(-\frac{b}{2a}\right)+b\right)^2\right)^{\frac{3}{2}}} \quad \left(= \frac{2|a|}{\left(1+(-b+b)^2\right)^{\frac{3}{2}}}\right)$$

$= 2|a|$ ***A1***

Note: Award ***(M1)A0*** if the modulus sign is not considered.

[2 marks]

continued…

Question 2 continued.

(iii) **EITHER**

$\lim_{x \to \infty} k(x) = 0$ ***A1***

OR

the curvature (of a quadratic) approaches zero as $x \to \infty$ ***A1***

THEN

for large positive values of x, quadratic functions are close to being straight ***R1***

Note: Award ***R1*** for similar statements such as 'for large positive values of x, a quadratic function behaves like a linear function.

[2 marks]

(iv) **C** ($k_{\max}$ of $p = k_{\max}$ of q) ***A1***

$k_{\max}$ of p is $(2 \times |-2|) = 4$ and $k_{\max}$ of q is $(2 \times |2|) = 4$ ***R1***

Note: Accept arguments such as '$k_{\max} = 2|a|$ and both quadratics have $|a| = 2$.
Do not award ***A1R0***.
If modulus signs were omitted in part (b)(ii), award ***FT*** in part (b)(iv) for an answer of **B** with appropriate values shown.

[2 marks]

(c) (i) sets $k'(x) = 0$ and attempts to solve for x ***(M1)***

$x^2 = \frac{1}{2}$

$x = \frac{1}{\sqrt{2}}$ ***A1***

Note: Award ***A0*** for $x = \pm\frac{1}{\sqrt{2}} (= \pm 0.707)$. Award ***A0*** for $x = 0.707$.

[2 marks]

continued…

Question 2 continued.

(ii) substitutes their value of x into $k(x)$ ***(M1)***

Note: Award a maximum of ***(M1)(A0)A0A0*** if their value of x is a decimal.
Award ***(M1)(A0)A0A0*** for $(0.707, 0.385)$.

$$k\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above denominator.

$$\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{3}{2}}$$ ***(A1)***

$= \frac{3\sqrt{3}}{2\sqrt{2}}$ (or equivalent unsimplified fractional surd e.g. $\sqrt{\frac{27}{8}}$) ***A1***

$$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}} \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}\right)$$

$= \frac{2}{3\sqrt{3}}$ (or equivalent surd e.g. $\frac{\sqrt{12}}{9}$) ***A1***

$k_{\max} = \frac{2\sqrt{3}}{9}$ ***AG***

[4 marks]

continued…

Question 2 continued.

(d) (i) **EITHER**

sets $k'(x) = 0$ and attempts to solve for e^{2x} ***(M1)***

$e^{2x} = \frac{1}{2}$ ***A1***

$e^{x} = \frac{1}{\sqrt{2}}$

substitutes their values for e^{2x} and e^{x} into $k(x)$ ***(M1)***

OR

attempts to solve $k'(x) = 0$ for x ***(M1)***

$x = \frac{1}{2}\ln\frac{1}{2}\ \left(= \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2\right)$ ***A1***

Note: Award a maximum of ***(M1)(A0)(M0)A0A0*** if their value of x is a decimal. Award ***(M1)(A0)(M0)A0A0*** for $(-0.347, 0.385)$.

substitutes their value of x into $k(x)$ ***(M1)***

THEN

for example, $\left(k\left(\ln\frac{1}{\sqrt{2}}\right) = \right)\frac{\frac{1}{\sqrt{2}}}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}\left(= \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{\frac{3}{2}}}\right)$ ***A1***

$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}}\ \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}\right)$ ***A1***

$= \frac{2}{3\sqrt{3}}$

$k_{\max} = \frac{2\sqrt{3}}{9}$ ***AG***

Note: After obtaining the penultimate ***A1*** above, candidates may state $k_{\max} = \frac{2\sqrt{3}}{9}$ as per correct work seen in part (c) (ii). In such cases, award the final ***A1***.

[5 marks]

continued…

Question 2 continued.

(ii) **EITHER**

w and v are inverse functions of each other (and as such are equally 'curved') (or equivalent) ***R1***

OR

the graphs of w and v are reflections of each other in the line $y = x$ ***R1***

OR

w and v have the same shape (or equivalent) ***R1***

[1 mark]

(e) (i) **METHOD 1**

attempts to use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(-2x)\left(r^2 - x^2\right)^{-\frac{1}{2}} \left(= -\frac{x}{\sqrt{r^2 - x^2}}, = -\frac{x}{y}\right)$ ***A1***

attempts to use the quotient or product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-y + x\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2} \left(= \frac{x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\right)$ ***A1***

correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ to form $= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} \left(= \frac{x}{y^2}\left(-\frac{x}{y}\right) - \frac{1}{y}\right)$ ***A1***

$= \frac{-y^2 - x^2}{y^3} \left(= -\frac{y^2 + x^2}{y^3}\right)$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

continued…

Question 2 continued.

METHOD 2

attempts to use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(-2x)\left(r^2 - x^2\right)^{-\frac{1}{2}} \left(= -\frac{x}{\sqrt{r^2 - x^2}}\right)$ ***A1***

attempts to use the quotient or product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***(M1)***

$= \frac{-\left(r^2 - x^2\right)^{\frac{1}{2}} - x^2\left(r^2 - x^2\right)^{-\frac{1}{2}}}{r^2 - x^2} \left(= -x^2\left(r^2 - x^2\right)^{-\frac{3}{2}} - \left(r^2 - x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

$= -\frac{r^2}{\left(r^2 - x^2\right)^{\frac{3}{2}}} \left(= \frac{-\left(r^2 - x^2\right) - x^2}{\left(r^2 - x^2\right)^{\frac{3}{2}}}, = -\frac{r^2}{\left(y^2\right)^{\frac{3}{2}}}\right)$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

Candidates may express their $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of r and y instead of r and x.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

continued…

Question 2 continued.

METHOD 3

attempts to use implicit differentiation on $y^2 = r^2 - x^2$ ***(M1)***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \left(y\frac{\mathrm{d}y}{\mathrm{d}x} = -x\right)$ ***A1***

attempts to use the product rule and implicit differentiation ***(M1)***

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$ (or equivalent) ***A1***

correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ to form $\left(-\frac{x}{y}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{y}\left(-1 - \frac{x^2}{y^2}\right)$

$= \frac{-y^2 - x^2}{y^3} \quad \left(= -\frac{y^2 + x^2}{y^3}\right)$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

[6 marks]

continued…

Question 2 continued.

(ii) **METHOD 1**

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left|-\frac{r^2}{y^3}\right|}{\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left|-\frac{r^2}{y^3}\right| = \frac{r^2}{y^3}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{y}\right)^2\right)$ as $\frac{y^2+x^2}{y^2}$ ***M1***

denominator is $\left(\frac{r^2}{y^2}\right)^{\frac{3}{2}}\left(=\frac{r^3}{y^3}\right)$ ***A1***

Note: Award the following ***A*** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{r^2}{y^3}}{\frac{r^3}{y^3}} \left(k = \frac{r^2}{y^3} \times \frac{y^3}{r^3}\right) \Rightarrow k = \frac{1}{r}$$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left|-\frac{r^2}{y^3}\right| = \frac{r^2}{y^3}$ is not considered.

continued…

Question 2 continued.

METHOD 2

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left| -\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \right|}{\left(1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left| -\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \right| = \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)$ as $\frac{r^2}{r^2-x^2}$ ***M1***

denominator is $\left(\frac{r^2}{r^2-x^2}\right)^{\frac{3}{2}}$ ***A1***

Note: Award the following ***A*** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}}{\frac{r^3}{\left(r^2-x^2\right)^{\frac{3}{2}}}} \quad \left(k = \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \times \frac{\left(r^2-x^2\right)^{\frac{3}{2}}}{r^3} \Rightarrow k = \frac{1}{r} \right)$$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left| -\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \right| = \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}$ is not considered.

continued…

Question 2 continued.

METHOD 3

substitutes their $\frac{dy}{dx}$ and the given $\frac{d^2y}{dx^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left| -\frac{x^2+y^2}{y^3} \right|}{\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left| -\frac{x^2+y^2}{y^3} \right| = \frac{x^2+y^2}{y^3}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}$ as $\left(\frac{y^2+x^2}{y^2}\right)^{\frac{3}{2}}$ ***M1***

Note: Award the following ***A1A1*** for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{x^2+y^2}{y^3}}{\left(\frac{x^2+y^2}{y^2}\right)^{\frac{3}{2}}} \left(= \frac{\frac{x^2+y^2}{y^3}}{\frac{\left(x^2+y^2\right)^{\frac{3}{2}}}{y^3}} \right)$$

$$= \frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}$$ ***A1***

$$k = \frac{1}{\left(r^2\right)^{\frac{1}{2}}} \Rightarrow k = \frac{1}{r}$$ ***A1***

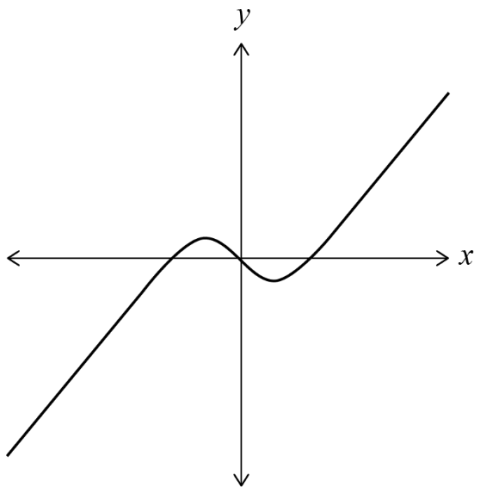
which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left| -\frac{x^2+y^2}{y^3} \right| = \frac{x^2+y^2}{y^3}$ is not considered.

[5 marks]

Total [32 marks]

1. (a) (i)



general shape of curve passing through $(0,0)$ ***A1***

[1 mark]

(ii) $(-4,0),(0,0),(4,0)$ ***A1***

Note: Award ***A0*** if additional points are given.

[1 mark]

(iii) Max $(-1.94,1.20)(-1.94347...,1.20113...)$ ***A1***

Min $(1.94,-1.20)$ $(1.94347...,-1.20113...)$ ***A1***

[2 marks]

continued

Question 1 continued.

(b) **EITHER**

odd ***A1***

since $f(-x)=\dfrac{(-x)\left((-x)^2-16\right)}{(-x)^2+16}=\dfrac{-x\left(x^2-16\right)}{x^2+16}=-f(x)$ (or equivalent) ***R1***

OR

odd ***A1***

since graph of f has rotational symmetry order 2 (of 180 degrees) about $(0,0)$ ***R1***

OR

odd ***A1***

since graph has reflection symmetry over the x-axis followed by reflection symmetry over the y-axis (or vice versa) ***R1***

Note: Do not award ***A1R0.***

[2 marks]

(c) attempt to use quotient rule or product rule ***(M1)***

$$f'(x)=\frac{\left(x^2+A\right)\left(3x^2-A\right)-2x\left(x^3-Ax\right)}{\left(x^2+A\right)^2}\quad\left(f'(x)=\frac{2x^2+\left(x^2-A\right)}{x^2+A}-\frac{2x^2\left(x^2-A\right)}{\left(x^2+A\right)^2}\right)$$

(or equivalent) ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

Note: If product rule used, award ***A1*** for each correct term.

Note: Award ***(M1)*** for an attempt to use quotient rule or product rule with $A=16$ or any other constant. Award no further marks.

$$f'\left(\sqrt{A}\right)\left(=\frac{(2A)(2A)-0}{(2A)^2}\right)=1$$ ***A1***

hence independent of A ***AG***

[4 marks]

continued

Question 1 continued.

(d) (i) **METHOD 1**

$x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 + A\right) - 2Ax}{x^2 + A} \left(\equiv \frac{x^3 + Ax - 2Ax}{x^2 + A} \right)$ ***M1***

$\equiv \frac{x^3 - Ax}{x^2 + A} \left(\equiv \frac{x\left(x^2 + A - 2A\right)}{x^2 + A} \right)$ ***A1***

$x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$ ***AG***

Note: Award no marks for numerical approaches.

METHOD 2

EITHER

attempt to do long division on $\frac{x\left(x^2 - A\right)}{x^2 + A}$ or $\frac{x^3 - Ax}{x^2 + A}$ ***M1***

quotient of x and remainder of $-2Ax$ ***A1***

$x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$ ***AG***

OR

$\frac{x^3}{x^2 + A} - \frac{Ax}{x^2 + A} \equiv \frac{x^3 + Ax}{x^2 + A} - \frac{Ax + Ax}{x^2 + A}$ ***M1***

$\equiv \frac{x\left(x^2 + A\right)}{\left(x^2 + A\right)} - \frac{2Ax}{\left(x^2 + A\right)}$ ***A1***

$\equiv x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$ ***AG***

Note: Award no marks for numerical approaches.

[2 marks]

continued

Question 1 continued.

(ii) hence oblique asymptote is $y = x$ (since $\frac{2Ax}{x^2 + A} \to 0$ as $x \to \pm\infty$) ***A1***

Note: Award ***A0*** if not given as an equation.

[1 mark]

(iii) $\left(\sqrt{A}, 0\right)$ OR $\left(-\sqrt{A}, 0\right)$ ***A1***

[1 mark]

continued

Question 1 continued.

(e)

Note: Award no marks for attempts at integration that do not involve partial fractions.

attempt to write $\frac{1}{1-v^2}$ in the form $\frac{B}{1+v}+\frac{C}{1-v}$ ***(M1)***

$B=\frac{1}{2}$ and $C=\frac{1}{2}$ ***A1***

$$\int \frac{1}{1-v^2}\,\mathrm{d}v = \int \frac{\frac{1}{2}}{1+v}+\frac{\frac{1}{2}}{1-v}\,\mathrm{d}v$$

attempt to integrate their partial fractions ***(M1)***

EITHER

$\frac{1}{2}\ln|1+v|-\frac{1}{2}\ln|1-v|+\frac{1}{2}\ln A$ ***A1A1***

Note: Award ***A1*** for first two terms. The moduli must be present for this mark. Award ***A1*** for $\frac{1}{2}\ln A$.

OR

$\frac{1}{2}\ln|1+v|-\frac{1}{2}\ln|1-v|(+c)$ ***A1***

$=\frac{1}{2}\ln\left|\frac{\mathrm{e}^{2c}(1+v)}{1-v}\right|$ where $\mathrm{e}^{2c}=A$ ***A1***

Note: The moduli must be present for each of these marks.

THEN

$=\frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$ ***AG***

[5 marks]

continued

Question 1 continued.

(f) attempt to separate variables and integrate ***(M1)***

$$\int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}v}{1-v^2}$$

$$\ln|x| = \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right| \quad \textbf{A1}$$

attempt to use power rule of logs ***(M1)***

$$\ln|x| = \ln\left|\frac{A(1+v)}{1-v}\right|^{\frac{1}{2}} \text{ OR } \ln|x|^2 = \ln\left|\frac{A(1+v)}{1-v}\right|$$

$$|x| = \left|\frac{A(1+v)}{1-v}\right|^{\frac{1}{2}} \text{ OR } x^2 = \left|\frac{A(1+v)}{1-v}\right| \quad \textbf{A1}$$

$$x^2 = \left|\frac{A\left(1+\frac{y}{x}\right)}{1-\frac{y}{x}}\right| \quad \textbf{A1}$$

$$x^2 = \left|\frac{A(x+y)}{x-y}\right| \quad \textbf{AG}$$

Note: Award at most ***(M1)A1(M1)A1A0*** for incorrect use or omission of moduli.

Note: The substitution of $\frac{y}{x}$ may be seen earlier.

[5 marks]

continued

Question 1 continued.

(g) $x^2 = \frac{A(x+y)}{x-y}$ ***(A1)***

attempt to clear denominator ***M1***

$x^2(x-y) = A(x+y)$

$x^3 - x^2y = Ax + Ay$

attempt to isolate y ***M1***

$x^3 - Ax = x^2y + Ay$

$x(x^2 - A) = y(x^2 + A)$ $\left(x^3 - Ax = y(x^2 + A)\right)$ ***A1***

$y = \frac{x(x^2 - A)}{x^2 + A}$ ***AG***

Note: The y must be factored out for the final ***A1***.

[4 marks]

Total [28 marks]

2. (a) (i) 0 ***A1***

[1 mark]

(ii) 0 ***A1***

[1 mark]

(b) attempts to expand a product of two of the side lengths ***(M1)***

EITHER

$V = x\left(4x^2 - 2(a+b)x + ab\right)$ $\left(V = x\left(4x^2 - 2ax - 2bx + ab\right)\right)$ ***A1***

OR

$V = (ax - 2x^2)(b - 2x)$

$= 4x^3 - 2ax^2 - 2bx^2 + abx$ ***A1***

OR

$V = (a - 2x)(bx - 2x^2)$

$= 4x^3 - 2ax^2 - 2bx^2 + abx$ ***A1***

THEN

$V = 4x^3 - 2(a+b)x^2 + abx$ ***AG***

[2 marks]

continued…

Question 2 continued

(c) attempts to find $\frac{\mathrm{d}V}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 4(a+b)x + ab$ ***A1***

Note: Award ***(M1)*** for obtaining the equivalent of two correct terms.

EITHER

attempts to solve $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ for x using the quadratic formula ***(M1)***

$x = \frac{-\left(-4(a+b)\right) \pm \sqrt{\left(-4(a+b)\right)^2 - 4(12)(ab)}}{24}$ ***(A1)***

Note: Award ***A0*** for incorrect expressions, for example:

$x = \frac{-\left(-4(a+b)\right) \pm \sqrt{-4(a+b)^2 - 4(12)(ab)}}{24}$

attempts to expand brackets within the square root sign and/or to take a common factor of 4 ***(M1)***

$x = \frac{-\left(-4(a+b)\right) \pm \sqrt{16a^2 + 16b^2 + 32ab - 48ab}}{24}$ ***A1***

or $x = \frac{4(a+b) \pm 4\sqrt{a^2 + 2ab + b^2 - 3ab}}{24}$

OR

attempt to substitute either of the given x values into their $\frac{\mathrm{d}V}{\mathrm{d}x}$ ***M1***

correct manipulation with each solution to reduce each expression to zero ***A1A1***

$\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ is a quadratic equation and hence these are the only two solutions ***R1***

THEN

$x = \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6}$ ***AG***

[6 marks]

continued…

Question 2 continued

(d) attempts to find their $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24x - 4(a+b)$ ***A1***

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} < 0$ ***R1***

Note: Award ***R1*** for equivalent statement in words.

substitutes $x = \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$ into their $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ ***(M1)***

Note: Award ***(M1)*** for substituting $x = \frac{(a+b)\pm\sqrt{a^2-ab+b^2}}{6}$

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24\left(\frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}\right) - 4(a+b)$

$= -4\sqrt{a^2-ab+b^2}\ (<0)$ ***A1***

so $x_m = \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$ gives a local maximum of V ***AG***

[5 marks]

(e) the maximum volume will occur at either the endpoints or at the local maximum. ***R1***

Note: Award ***R1*** for an answer which refers to the volume being 0 at both endpoints ($x = 0$ and $x = \frac{a}{2}$) and the volume of the box at $x_m > 0$

hence the maximum volume of the box occurs at x_m / the local maximum is a global maximum. ***AG***

[1 mark]

continued…

Question 2 continued

(f) $2st - s^2 < t^2 - s^2$ ***A1***

$t^2 > 2st$ and since $t \in \mathbb{Z}^+$ (dividing both sides by t) ***R1***

Note: Award ***R1*** for equivalent reasoning, $t > 0$ or t is positive

$\Rightarrow t > 2s$ ***AG***

[2 marks]

(g) attempts to substitute for each of a, b and $\sqrt{a^2 - ab + b^2}$ in terms of s and t ***(M1)***

$(x_m =) \dfrac{\left(2st - s^2\right) + \left(t^2 - s^2\right) - \left(s^2 - st + t^2\right)}{6}$ ***A1***

$= \dfrac{-3s^2 + 3st}{6}$ ***A1***

$x_m = \dfrac{s(t-s)}{2}$ ***AG***

[3 marks]

(h) (i) **METHOD 1**

when s is even, x_m is an integer ***R1***

hence V_m is a product of integers (a, $b \in \mathbb{Z}^+$) ***R1***

hence V_m is an integer ***AG***

METHOD 2

$V_m = \dfrac{s(t-s)}{2}\left(2st - s^2 - 2\left(\dfrac{s(t-s)}{2}\right)\right)\left(t^2 - s^2 - 2\left(\dfrac{s(t-s)}{2}\right)\right)$ (or equivalent) ***A1***

Note: Award ***A1*** for $V_m = \dfrac{s(t-s)}{2}\left(a - 2\left(\dfrac{s(t-s)}{2}\right)\right)\left(b - 2\left(\dfrac{s(t-s)}{2}\right)\right)$

when s is even, $\dfrac{s(t-s)}{2}$ is an integer **AND**

hence V_m is a product of integers (a, $b \in \mathbb{Z}^+$) ***R1***

hence V_m is an integer ***AG***

[2 marks]

continued…

Question 2 continued

(ii) when s is odd **AND** t is odd ($s(t-s)$ is even) ***A1***

x_m and V_m are both integers ***AG***

[1 mark]

(iii) the following table shows some possible sets of values for s, t, a, b, x_m and V_m

s	t	a	b	x_m	V_m
1	3	5	8	1	18
1	5	9	24	2	200
2	5	16	21	3	450
2	6	20	32	4	1152
2	7	24	45	5	2450
2	8	28	60	6	4608
3	7	33	40	6	3528
4	10	64	84	12	28800
…	…	…	…	…	…

attempts to use values of s and t to find

EITHER x_m **OR** a and b **OR** V_m ***(M1)***

a corresponding correct x_m value ***A1***

corresponding correct a and b values ***(A1)***

a corresponding correct V_m value ***A1***

Note: Award at most ***(M1)A0A0A0*** if the values of s and t do not satisfy the conditions: s, $t \in \mathbb{Z}^+$, $t > 2s$, and s is even / s and t are odd.

The three ***A*** marks can be awarded in any order.

The ***A1*** for x_m and the ***A1*** for a and b can be awarded independently.

Some candidates may determine in part (h) that $V_m = \frac{s^2t^2(t-s)^2}{2}$.

[4 marks]

Total [28 marks]

3. (a) (i) $c = 9$ ***A1***

(ii) attempt to substitute (3,0) into $y = ax^2 + c$ with *their* c ***(M1)***

$9a + 9 = 0$

$a = -1$ ***A1***

[3 marks]

(b) **METHOD 1** (single integral)

attempt to subtract areas (seen anywhere) ***(M1)***

$\int_0^3 \left(9 - x^2\right) - \left(9 - 3x\right)\ \mathrm{d}x \quad \left(= \int_0^3 3x - x^2\ \mathrm{d}x\right)$ OR $\int_0^3 \left(9 - x^2\right)\mathrm{d}x - \text{triangle area}$

correct integration $\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$ **OR** $\left[9x - \frac{1}{3}x^3\right]_0^3$ ***(A1)***

attempt to substitute their limits into their integrated expression (condone absence of the 0 substitution) ***(M1)***

$\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3 - \left(\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3\right)$ OR $9(3) - \frac{1}{3}(3)^3$

$\frac{27}{2} - 9$ OR $18 - \frac{27}{2}$

Area $= \frac{9}{2}$ (square units) ***A1***

continued…

Question 3 continued.

METHOD 2 (evaluating two integrals separately)

$\int_0^3 \left(9-x^2\right)\,\mathrm{d}x = \left[9x-\frac{1}{3}x^3\right]_0^3$ **OR** $\int_0^3 (9-3x)\,\mathrm{d}x = \left[9x-\frac{3}{2}x^2\right]_0^3$ ***(A1)***

Note: Award ***A1*** for correct integration. Ignore absence of limits.

attempt to substitute their limits into both of their integrated expressions (condone absence of the 0 substitution) ***(M1)***

$9(3)-\frac{1}{3}(3)^3-\left(9(0)-\frac{1}{3}(0)^3\right)$ and $9(3)-\frac{3}{2}(3)^2$

attempt to subtract the two integrals in any order (seen anywhere) ***(M1)***

$9(3)-\frac{1}{3}(3)^3-\left(9(3)-\frac{3}{2}(3)^2\right)$ OR $\int\left(9-x^2\right)\mathrm{d}x-\int(9-3x)\mathrm{d}x$

$18-\frac{27}{2}$

Area $=\frac{9}{2}$ (square units) ***A1***

[4 marks]

Total [7 marks]

5. (a) (i) **METHOD 1**

$f'(x) = 4(1-x)^{-5}$ ***A1***

substitution of x=0 into derivative ***M1***

$4(1-0)^{-5}$ OR $f'(0) = 4$

$a = 4$ ***AG***

Note: Do not award ***M1*** for $1+4x$ seen without clear evidence of a substitution.

METHOD 2

attempt to create binomial expansion to at least two terms ***M1***

$1+\binom{-4}{1}x+\left(\binom{-4}{2}(x)^2+...\right)$

correct expansion

$1+(-4)(-x)+\left(\frac{(-4)(-5)}{2}(-x)^2+...\right)$ ***A1***

$a = 4$ ***AG***

Note: Do not award ***A1*** for $1+4x$ seen without clear evidence of correct working.

(ii) **METHOD 1**

$f''(x) = (-5)\times(4)(1-x)^6\times(-1),\ \ 20(1-x)^{-6}$ ***(A1)***

$(f''(0) =)\,20$ ***(A1)***

$\frac{20}{2!}$

$b = 10$ ***A1***

METHOD 2

$\frac{(-4)(-5)}{2!}(-x)^2$ (may be seen in their expansion in (i) ***(A1)***

$\frac{20}{2}$ ***(A1)***

$b = 10$ ***A1***

[5 marks]

continued…

Question 5 continued.

(b) attempt to substitute 0.1 into the expansion ***(M1)***

1.52

1520 (dollars) ***A1***

[2 marks]

Total [7 marks]

7. (a) attempt to separate variables ***(M1)***

$$\int \frac{\mathrm{d}y}{y^2} = \int \cos^2 x \ \mathrm{d}x$$

attempt to use an identity for $\cos^2 x$, may be seen above ***(M1)***

$$\frac{1}{2}(1+\cos 2x)$$

$$-y^{-1} = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$ ***A1A1***

Note: Award ***A1*** for LHS and award ***A1*** for RHS. Award ***A1A0*** for at least one correct side and missing constant of integration.

$$-y^{-1} = \frac{2x+\sin 2x}{4} + C_1 \quad \textbf{OR} \quad y = \frac{-4}{2x+\sin 2x + C_2}$$ ***(A1)***

Note: Condone solutions that do not deal correctly with the constant(s) of integration.

$a = -4$ ***A1***

[6 marks]

(b) attempt to substitute initial conditions with *their* a. ***(M1)***

$$1 = \frac{-4}{\pi + 0 + b}$$

$b = -4 - \pi$ ***A1***

[2 marks]

Total [8 marks]

9. (a) recognition that $f'(x) = 0$ (seen anywhere) ***(M1)***

attempt to solve quadratic equation (condone not seeing $=0$) ***(M1)***

$a = -5,\ b = 1$ ***A1***

[3 marks]

(b) (local) maximum ***A1***

($f'(x)$/gradient) changes from positive to negative (at $x = a$)

OR $f(x)$ changes from increasing to decreasing (at $x = a$)

OR $f''(a) = -18$ (accept $f''(a) < 0$) ***R1***

Note: Do not award ***A0R1***.

[2 marks]

(c) $f''(x) = 6x + 12$ ***(A1)***

$6c + 12 = 0$ ***(A1)***

$c = -2$ ***A1***

[3 marks]

(d) yes (there is a point of inflexion at $x = c$) ***A1***

($f''(x)$/second derivative) changes sign at $x = c$

OR concavity (of f) changes at $x = c$

OR $f'(x)$/gradient decreasing to the left of c, and increasing to the right of c ***R1***

[2 marks]

Note: Award ***R0*** if their only reason is that $f''(c) = 0$. Do not award ***A0R1***.

continued…

Question 9 continued.

(e) recognize to find $\int f'(x)\mathrm{d}x$ ***(M1)***

$x^3+6x^2-15x(+C)$ ***(A1)***

substitution of $(-2,36)$ into their integrated expression (must contain $+C$) ***(M1)***

$36=(-2)^3+6(-2)^2-15(-2)+C$

$C=-10$ ***A1***

$f(x)=x^3+6x^2-15x-10$

[4 marks]

Total [14 marks]

10. (a) (i) attempt to create binomial expansion to four terms with $n=-1$ ***(M1)***

$$1+\binom{-1}{1}x^2+\binom{-1}{2}\left(x^2\right)^2+\binom{-1}{3}\left(x^2\right)^2+\ldots$$

$$1-x^2+\frac{(-1)(-2)}{2}\left(x^2\right)^2+\frac{(-1)(-2)(-3)}{6}\left(x^2\right)^3$$ ***A1A1***

Note: Award ***A1*** for the third term and ***A1*** for the fourth term.

$$1-x^2+x^4-x^6$$ ***AG***

[3 marks]

(ii) $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7\left(+C\right)$ ***A2***

[2 marks]

Note: Award ***A1A0*** for any three correct terms.

(b) (i) attempt to create binomial expansion the term in x^4 with $n=-\frac{1}{2}$ ***(M1)***

$$1+\binom{-\frac{1}{2}}{1}x^2+\binom{-\frac{1}{2}}{2}\left(x^2\right)^2+\ldots$$

$$1-\frac{1}{2}\left(-x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2$$ ***(A1)(A1)***

Note: Award ***(A1)*** for the first two terms and ***(A1)*** for the third term.

$$1+\frac{1}{2}x^2+\frac{3}{8}x^4$$ ***A1***

(ii) $x+\frac{1}{6}x^3+\frac{3}{40}x^5\left(+C\right)$ ***A1***

[5 marks]

continued…

Question 10 continued.

(c) $\arcsin x = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ (seen anywhere) ***(A1)***

attempt to substitute $x = \frac{1}{2}$ into their antiderivative ***(M1)***

$\left(\arcsin\frac{1}{2} \approx\right)\frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5$, $\frac{1}{2} + \frac{1}{48} + \frac{3}{1280}$ ***A1***

$k = 3$ (accept $\frac{25}{48} + \frac{3}{1280}$) ***A1***

[4 marks]

(d) $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (seen anywhere) ***(A1)***

equating their value of $\arcsin\left(\frac{1}{2}\right)$ to their expression from (c) ***(M1)***

$\frac{\pi}{6} \approx \frac{25}{48} + \frac{3}{1280}$, $\pi \approx 6 \times \left(\frac{25}{48} + \frac{3}{1280}\right)$

$\pi \approx \frac{25}{8} + \frac{18}{1280}$, $\frac{4000}{1280} + \frac{18}{1280}$ ***(A1)***

$\pi \approx \frac{2009}{640}$ (accept $p = 2009$) ***A1***

Note: Condone use of "$=$" instead of "$\approx$".

[4 marks]

Total [18 marks]

6. $x\dfrac{\mathrm{d}y}{\mathrm{d}x}+3y=\dfrac{\operatorname{cosec}^2 x}{x}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{3}{x}y=\dfrac{\operatorname{cosec}^2 x}{x^2}$ ***(A1)***

IF $=\mathrm{e}^{\int\frac{3}{x}\mathrm{d}x}\left(=x^3\right)$ ***(A1)***

attempt to multiply through by their IF: ***(M1)***

$x^3\dfrac{\mathrm{d}y}{\mathrm{d}x}+3x^2y=x\operatorname{cosec}^2 x$

Note: Award first 3 marks if candidates directly multiply through the DE by x^2.

attempt to write LHS as an exact differential ***(M1)***

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^3y\right)=x\operatorname{cosec}^2 x$

$x^3y=-x\cot x+\ln\left|\sin x\right|+c$ ***A1***

$y=\dfrac{-x\cot x+\ln\left|\sin x\right|+c}{x^3}$

substituting $x=\dfrac{\pi}{2}$, $y=\dfrac{8}{\pi^2}$ into their integrated expression ***(M1)***

$\Rightarrow c=\pi$

$y=\dfrac{\pi-x\cot x+\ln\left|\sin x\right|}{x^3}$ ***A1***

Note: Do not penalise omission of absolute value signs as for $0<x<\pi$, $\sin x$ is always positive.

[7 marks]

8. (a) attempt to sum times for both sections using $T = \frac{D}{S}$ ***(M1)***

$T = \frac{400\sec\theta}{0.8} + \frac{1000 - 400\tan\theta}{1.2}$ ***A1***

$T = 500\sec\theta + \frac{2500 - 1000\tan\theta}{3}$ ***AG***

[2 marks]

(b) attempt to differentiate T $\left(\sec\theta\tan\theta \text{ or } \sec^2\theta \text{ seen}\right)$ ***M1***

$\left(\frac{\mathrm{d}T}{\mathrm{d}\theta} =\right) 500\sec\theta\tan\theta - \frac{1000}{3}\sec^2\theta$ ***A1A1***

[3 marks]

(c) set their $\frac{\mathrm{d}T}{\mathrm{d}\theta} = 0$ and attempt to solve ***(M1)***

$500\sin\theta - \frac{1000}{3} = 0$

$\sin\theta = \frac{2}{3}$ ***A1***

$\tan\theta = \frac{2}{\sqrt{5}}$ OR $\sec\theta = \frac{3}{\sqrt{5}}$ or equivalent ***A1***

$\mathrm{PX} = 400\tan\theta$ (seen anywhere) OR $\mathrm{PX} = \frac{800}{\sqrt{5}}$ ***A1***

$= 160\sqrt{5}$ ***AG***

[4 marks]

Total [9 marks]

Section B

9. (a)

Note: In this question, the marks are independent and may be awarded in either order.

finding correct discriminant $k^2 - 26$ ***A1***

recognition that discriminant $\left(b^2 - 4ac\right) < 0$ ***M1***

$b^2 - 4ac < 0$ or $k^2 - 26 < 0$ or $k^2 < 26$

(since $k \in \mathbb{Z}^+$) $k = 5$ ***AG***

[2 marks]

(b) (i) $x = -5$ (must be an equation) ***A1***

(ii) attempt to substitute their value of x (from axis of symmetry) into $\frac{1}{2}x^2 + 5x + 13$

OR attempt to use completing the square ***(M1)***

$(y =) \frac{1}{2}(-5)^2 + 5(-5) + 13 \left(= \frac{1}{2}\right)$ OR $(y =) \frac{1}{2}(x+5)^2 + \frac{1}{2}$

$\left(-5, \frac{1}{2}\right)$ ***A1***

[3 marks]

(c) at $x = -3,\ y = \frac{5}{2}$ (seen anywhere) ***A1***

$\frac{dy}{dx} = x + 5$ ***A1***

at $x = -3$, $\frac{dy}{dx} = 2$ ***A1***

normal gradient is negative reciprocal of their $\frac{dy}{dx}$ at $x = -3$ ***(M1)***

gradient of normal $= -\frac{1}{2}$

$y - \frac{5}{2} = -\frac{1}{2}(x+3)$ OR $\frac{5}{2} = -\frac{1}{2}(-3) + c \Rightarrow c = 1$ ***A1***

$y = -\frac{1}{2}x + 1$ ***AG***

[5 marks]

continued…

Question 9 continued

(d) attempt to describe shaded area as the difference of two integrals or one integral and the area of a trapezium ***(M1)***

$(A=)\ \int_{-3}^{0}\left(\frac{1}{2}x^2+5x+13\right)\mathrm{d}x-\int_{-3}^{0}\left(-\frac{1}{2}x+1\right)\mathrm{d}x$ OR $\int_{-3}^{0}\left(\frac{1}{2}x^2+\frac{11}{2}x+12\right)\mathrm{d}x$

OR $\int_{-3}^{0}\left(\frac{1}{2}x^2+5x+13\right)\mathrm{d}x-\frac{1}{2}\left(\frac{5}{2}+1\right)3$

correct integration ***A2***

$\left[\frac{x^3}{6}+\frac{5x^2}{2}+13x\right]_{-3}^{0}-\left[-\frac{x^2}{4}+x\right]_{-3}^{0}$ OR $\left[\frac{x^3}{6}+\frac{11x^2}{4}+12x\right]_{-3}^{0}$ OR $\left[\frac{x^3}{6}+\frac{5x^2}{2}+13x\right]_{-3}^{0}-\frac{21}{4}$

Note: If not completely correct, award ***A1*** for at least two terms correctly integrated.

substituting $(0$ and$)\ -3$ into their integrated expression and subtracting ***(M1)***

$0-\left(\frac{(-3)^3}{6}+\frac{11(-3)^2}{4}+12(-3)\right)$ OR $0-\left(\frac{(-3)^3}{6}+\frac{5(-3)^2}{2}+13(-3)\right)-\frac{21}{4}$ (or equivalent)

$=\frac{63}{4}$ ***A1***

[5 marks]
Total [15 marks]

$\begin{pmatrix} -1 \\ \end{pmatrix} \quad \begin{pmatrix} 7 \\ \end{pmatrix}$

8. attempt to write at least one function in terms of y ***(M1)***

$x^2 = y^2 + 16$ **OR** $x^2 = 25y - 50$ (or equivalent) ***(A1)***

one correct expression for volume

$\pi\int_0^3 y^2 + 16 \, \mathrm{d}y$ **OR** $\pi\int_2^3 25y - 50 \, \mathrm{d}y$ ***(A1)***

attempt to subtract two volumes with limits for y ***(M1)***

$\pi\int_0^3 \left(y^2 + 16\right) \mathrm{d}y - \pi\int_2^3 \left(25y - 50\right) \mathrm{d}y$ ***(A1)***

Note: Award at most ***M1A1A1M0A0A0*** for a candidate who writes $\pi\int_0^3 \left(y^2 + 16\right) - \left(25y - 50\right) \mathrm{d}y$.

$179.0707... - 39.269908$

$139.8008\ldots$

$V = 140$ (cubic units) $\left(= \frac{89}{2}\pi\right)$ ***A1***

Note: Ignore absence of π throughout if their final answer is correct. Otherwise award at most ***M1A1A1M1A0A0.***

Total [6 marks]

9. attempt to find a relationship between x and θ using cosine rule ***(M1)***

$x^2 = 15^2 + 20^2 - 2(15)(20)\cos\theta,\ \ x = \sqrt{625 - 600\cos\theta}$

attempt to relate the rates (seen anywhere) ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{1}{2}(625 - 600\cos\theta)^{-\frac{1}{2}} \times 600\sin\theta$ ***(A1)***

OR

Attempt to differentiate implicitly with respect to t ***(M1)***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} = 600\sin\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}x}{\mathrm{dt}} = \frac{1}{2}(625 - 600\cos\theta)^{-\frac{1}{2}} \times 600\sin\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***(A1)***

OR

Using GDC to find $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 13.835110...$ ***(A1)***

attempt to find $\cos\theta$ or $\sin\theta$ or θ when area is 140 ***(M1)***

$\sin\theta = \frac{14}{15}$ or $\cos\theta = \frac{\sqrt{29}}{15}$ or $\theta = 1.20358...$ (seen anywhere) ***(A1)***

(accept $68.9605...^\circ$)

$\frac{1}{2}\left(625 - 600\left(\frac{\sqrt{29}}{15}\right)\right)^{\left(-\frac{1}{2}\right)}\left(600 \times \frac{280}{300}\right) \times \left(-\frac{\pi}{60}\right)$ (***A1)***

$-0.724402...$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -0.724\ \left(\mathrm{cm\,min^{-1}}\right)$ ***A1***

Note: Award **(M1)(M1)(A1)(M1)(A1)(A1)A0** for $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.724\ \left(\mathrm{cm\,min^{-1}}\right)$.

Total [7 marks]

Section B

10. (a) (i) speed of airplane $= 60\ (\text{ms}^{-1})$ ***A1***

(ii) speed of car $= 0\ (\text{ms}^{-1})$ ***A1***

[2 marks]

(b) (i) recognizing $v_{\text{air}} = v_{\text{car}}$ when speeds are the same ***(M1)***

$6.35564...$

$t = 6.36\ (\text{s})$ ***A1***

(ii) $31.7782\ldots$

speed $= 31.8\ (\text{ms}^{-1})$ ***A1***

[3 marks]

(c) let $s_{\text{air}}(t)$ be the displacement of the airplane and $s_{\text{car}}(t)$ be the displacement of the car at time t

recognizing to integrate either velocity function ***(M1)***

$\int 60\text{e}^{-0.1t}\ \text{d}t,\ \int 5t\ \text{d}t$

correct integration ***(A1)(A1)***

$-600\text{e}^{-0.1t},\ \ \frac{5}{2}t^2\ (+C)$ (award ***A1*** for each term)

recognize to subtract displacements (seen anywhere) ***(M1)***

EITHER

substituting initial condition into their $d(t)$ ***(M1)***

$100 = -600\text{e}^{-1(0)} - \frac{5}{2}\times 0^2 + C$ OR $-100 = 600\text{e}^{-1(0)} + \frac{5}{2}\times 0^2 + C$ (or equivalent)

OR

substituting initial condition into their $s_{\text{air}}(t)$ ***(M1)***

$\left(s_{\text{car}}(0) = 0 \Rightarrow\right) 100 = -600\text{e}^{-1(0)} + C$

continued…

Question 10 continued.

THEN

$C = 700$ (or -700) ***(A1)***

$d(t) = -600\mathrm{e}^{-0.1t} - \frac{5}{2}t^2 + 700$ (accept $d(t) = \left|600\mathrm{e}^{-0.1t} + \frac{5}{2}t^2 - 700\right|$) ***A1***

[7 marks]

(d) recognizing to solve their $d(t) = 0$ ***(M1)***

$15.0586...$

$t = 15.1$ (s) ***A1***

[2 marks]

(e) attempt to find the distance travelled by the car ***(M1)***

$\int_0^{15.0586} 5t \, \mathrm{d}t$, $\frac{5}{2}(15.0586)^2$ ***(A1)***

$566.904\ldots$

567 (m) (accept 570 (m) from 3 sf answer in (d)) ***A1***

Note: Award ***(M1)(A1FT)A0***, where appropriate, if the candidate has an incorrect $d(t)$ and/or incorrect value of t which leads to their distance travelled by the car to be less than 100 (m).

For example, where the candidate finds $d(t) = 600\mathrm{e}^{-0.1t} - \frac{5}{2}t^2 - 500$ in part (c), award ***(M1)(A1FT)A0*** to 7.08, or 7.06 from using 3 sf value.

[3 marks]

Total [17 marks]

5. attempt to rearrange $f(x) = e^{\frac{3x}{4}}$ to get an expression for x in terms of y OR find $f^{-1}(x)$ ***(M1)***

$x = \frac{4}{3}\ln y$ OR $f^{-1}(x) = \frac{4}{3}\ln x$ ***(A1)***

attempt to use $\pi\int x^2 dy$ OR $\pi\int \left(f^{-1}(x)\right)^2 dx$ for their rearrangement of $y = e^{\frac{3x}{4}}$ ***(M1)***

$\pi\int \left(\frac{4}{3}\ln y\right)^2 dy$ OR $\pi\int \left(\frac{4}{3}\ln x\right)^2 dx$

$\pi\int_1^8 \left(\frac{4}{3}\ln y\right)^2 dy$ OR $\pi\int_1^8 \left(\frac{4}{3}\ln x\right)^2 dx$ ***(A1)***

$= 85.6(85.5716...)\left(= \frac{16\pi}{9}\left(8(\ln 8)^2 - 16\ln 8 + 14\right)\right)$ ***A1***

[5 marks]

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10. (a) (i) attempt to find a maximum point ***(M1)***

$(x =) 0.718442...$ or $(y =) 1.80645\ldots$ or $(x =) 3.71844...$ or $(y =) 1.19181\ldots$ etc

$1.80645\ldots$

(maximum displacement) $= 1.81$ (cm) ***A1***

(ii) recognition that $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ ***(M1)***

recognition that the gradient at $t = 0$ represents the maximum velocity ***(M1)***

$v = \frac{\mathrm{d}s}{\mathrm{d}t}$ at $t = 0$

$v = 4.18879025...\left(\frac{4\pi}{3}\right)$

$v = 4.19\left(\frac{4\pi}{3}\right)$ (cms^{-1}) ***A1***

[5 marks]

(b) (i) attempt to find a minimum point ***(M1)***

$(x =) 2.21844...$ or $(y =) -1.46729...$ OR $(x =) 5.21844...$ OR $(y =) -0.968053\ldots$

$-1.46729...$

-1.47 ***A1***

(ii) $(t = 3.5)\ s = 1.06620...$

$s = 1.07$ ***A1***

[3 marks]

continued…

Question 10 continued

(c) **METHOD 1**

attempt to sum **distances** from O involving maximum, minimum and distance at $t = 3.5$ ***(M1)***

$1.80645\ldots \times 2 + 1.46729... \times 2 + 1.06620...$ ***(A1)***

$= 7.61369...$

$7.61\,(\text{cm})$ ***A1***

METHOD 2

attempt to integrate the absolute value of $\frac{\mathrm{d}s}{\mathrm{d}t}$ ***(M1)***

$\int_0^{3.5} \left| \frac{d}{dt}\left(2^{1-\frac{t}{5}} \sin\left(\frac{2\pi t}{3}\right)\right)\right| \mathrm{d}t$ ***(A1)***

$= 7.61369...$

$7.61\,(\text{cm})$ ***A1***

[3 marks]

(d) $(T =) 1.5$ ***A1***

[1 mark]

continued…

Question 10 continued

(e)

(i) attempt to divide two of $u_1, u_2, u_3 ...$ ***(M1)***

$\frac{u_2}{u_1}\left(=\frac{1.46729...}{1.80645...}\right)$ or $\frac{u_3}{u_2}\left(=\frac{1.19181...}{1.46729...}\right)$ or $r^2=\frac{u_3}{u_1}\left(=\frac{1.19181...}{1.80645...}\right)$

$0.812252...$

$(r=)0.812$ ***A1***

(ii) attempt to use the sum to infinity of the GP (***M1)***

$S_\infty=\frac{1.80645...}{1-0.812252...}(=9.62167...)$

recognition that the total distance is double their S_∞ (***M1)***

$2\times S_\infty=2\times\frac{1.80645...}{1-0.812252...}$ OR $2\times 9.62167..$

$19.2433...$

$19.2(\text{cm})$ ***A1***

Note: Award (***M1)***(***M1)A0*** for values of r that lead to an answer that is smaller than 2 x first maximum displacement (3.6129).

[5 marks]

Total [17 marks]

12. (a) (i) $\frac{dy}{dx} = -1.7$ ***A1***

(ii) attempt to use Euler's formula with step length 0.2 ***(M1)***

step	x	y
0	1	1.5
1	1.2	1.16
2	1.4	1.07491524

$1.5 + 0.2 \times -1.7$ or 1.16 ***(A1)***

1.07 ***A1***

Note: Accept 1.075.

(iii) change (decrease) in the y values is decreasing. OR the gradient is increasing as the gradient at 1.2 is $-0.425423\ldots$ ***R1***

concave up ***A1***

Note: Do not award ***R0A1.***
Do not award ***FT*** from either part (i) or part (ii).

[6 marks]

continued…

Question 12 continued

(b) (i) **METHOD 1**

$\left(\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\right)x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ (seen anywhere) ***A1***

attempt to differentiate either the LHS or the RHS implicitly using the product rule ***M1***

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y)\frac{\mathrm{d}y}{\mathrm{d}x}=2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y-6y\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***A1A1***

Note: Award **A1** for LHS and **A1** for RHS.

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2x+y-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-(x+7y)\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***AG***

METHOD 2

$\left(\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\right)x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ (seen anywhere) ***A1***

attempt to rearrange into the form $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{x^2+xy-3y^2}{x^2+xy}$ and attempt to differentiate the RHS implicitly using quotient rule ***M1***

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\frac{\left(x^2+xy\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y-6y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(x^2+xy-3y^2\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)}{\left(x^2+xy\right)^2}$$ ***A1***

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\frac{\left(x^2+xy\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y-6y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(\left(x^2+xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)}{\left(x^2+xy\right)^2}$$ ***A1***

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2x+y-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-(x+7y)\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***AG***

continued…

Question 12 continued

(ii) attempt to substitute their $\frac{dy}{dx}$ and $x = 1, y = \frac{3}{2}$ ***(M1)***

$$\frac{5}{2}\frac{d^2y}{dx^2} = \frac{7}{2} - (-1.7)^2 - \frac{23}{2}(-1.7)$$

$8.064\left(\frac{1008}{125}\right)$ *OR* 8.06 ***A1***

Note: Award ***(M1)A0*** for 8.1 seen with no working.

[6 marks]

(c) attempt to make the numerators of common denominator equivalent ***(M1)***

$$1 + v = A(1 + 2v) + B(1 - 2v)$$

$A = \frac{3}{4}(0.75), B = \frac{1}{4}(0.25)$ ***A1***

[2 marks]

continued…

Question 12 continued

(d) attempt to replace y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ by substitution of $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

$\left(x^2 + x^2v\right)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = x^2 + vx^2 - 3x^2v^2$

$\left(1+v\right)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = 1 + v - 3v^2$ or $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - \frac{3v^2}{1+v}$ ***A1***

$v\left(1+v\right) + (1+v)x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v - 3v^2$

$(1+v)x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v - 3v^2 - v - v^2$

$(1+v)x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - 4v^2$

$\int \frac{1+v}{1-4v^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***A1***

hence $\ln x = -\frac{3}{8}\ln\left|1-2v\right| + \frac{1}{8}\ln\left|1+2v\right|(+c)$ $\left(= -\frac{3}{8}\ln\left(2v-1\right) + \frac{1}{8}\ln\left(2v+1\right)(+c)\right)$ ***A1A1***

substitution of $x = 1,\ v = 1.5$ OR $x = 1, y = 1.5$ to find a value of c ***(M1)***

$c = \frac{1}{8}\ln 2\ \left(0.086643\ldots\right)$

$8\ln x + 3\ln\left(\frac{2y}{x} - 1\right) = \ln\left(1 + \frac{2y}{x}\right) + \ln 2$ or equivalent ***A1***

$x^8\left(\frac{2y}{x} - 1\right)^3 = 2\left(1 + \frac{2y}{x}\right)$ ***A1***

$x^6\left(2y - x\right)^3 = 2\left(x + 2y\right)$ ***AG***

[8 marks]

Total [22 marks]

1. (a) (i) **METHOD 1**

attempts to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}$ AND $m_N = \frac{-1}{-\frac{1}{t^2}}$ (or equivalent) ***A1***

Note: Accept $m_1 m_2 = -1$ (or equivalent) seen anywhere.
Award ***A1*** for $m_N = x^2$ and as $x = t$, then $m_N = t^2$.

so gradient of N is t^2 ***AG***

METHOD 2

uses $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{t^2}}{1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}$ AND $m_N = \frac{-1}{-\frac{1}{t^2}}$ (or equivalent) ***A1***

Note: Accept $m_1 m_2 = -1$ (or equivalent) seen anywhere.
Award ***A1*** for $m_N = x^2$ and as $x = t$, then $m_N = t^2$.

so gradient of N is t^2 ***AG***

[2 marks]

continued…

Question 1 continued.

(ii) **EITHER**

$y-\frac{1}{t}=t^2\left(x-t\right)$ OR $y=t^2\left(x-t\right)+\frac{1}{t}$ ***A1***

OR

$\frac{1}{t}=t^2\left(t\right)+c\Rightarrow c=\frac{1}{t}-t^3$ ***A1***

OR

$t^2x-y-\left(t^2\left(t\right)-\frac{1}{t}\right)=0$ OR $t^2x-y=t^2\left(t\right)-\frac{1}{t}$ ***A1***

THEN

leading to $y=t^2x+\frac{1}{t}-t^3$ ***AG***

[1 mark]

(b) (i) $m=t^2\left(m^2=t^4\right)$ OR $c=\frac{1}{t}-t^3$ OR $c=\frac{1}{t}\left(1-t^4\right)$ ***(A1)***

$t=\pm\sqrt{m}$ (seen anywhere) ***A1***

substitutes one of their $t=\sqrt{m}$ OR $t=-\sqrt{m}$ into $c=\frac{1}{t}-t^3$ OR $c=\frac{1}{t}\left(1-t^4\right)$ ***M1***

$c=\pm\frac{1}{\sqrt{m}}\left(1-m^2\right)$ ***A1***

so either $c=\frac{1}{\sqrt{m}}\left(1-m^2\right)$ OR $c=\frac{1}{\sqrt{m}}\left(m^2-1\right)$ ***AG***

[4 marks]

(ii) $m\in\mathbb{R}^+$ (or equivalent) ***A1***

[1 mark]

continued…

Question 1 continued.

(c) (i) $m = 1$ ***A1***

[1 mark]

(ii) $m \in \mathbb{R}^+, m \neq 1$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $0 < m < 1$ and ***A1*** for $m > 1$.

[2 marks]

(d) (i) a correct sketch of $y = \frac{1}{x}$ ***A1***

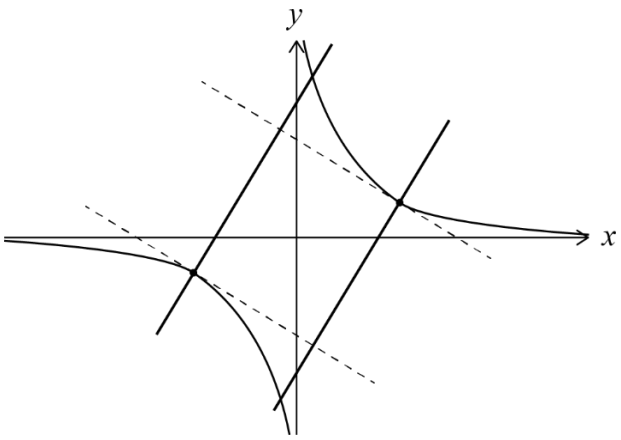
[1 mark]

(ii) a line normal to H such that $c \neq 0$ ***A1***

a second parallel line which is normal to H such that $c \neq 0$ ***A1***

Note: For normal lines that cross the y- axis, award ***A1A0*** if the y- axis intercepts are not approximately equidistant from O.

eg.



Note: Accept correct sketches that show the point(s) where the line(s) is/are normal. Candidates are not required to show dashed lines as shown above.

Award a maximum ***A1A0*** for a normal line sketched for $0 < m < 1$ and a normal line sketched for $m > 1$.

[2 marks]

continued…

Question 1 continued.

(e) (i) $\frac{k^2}{x} = t^2x - kt^3 + \frac{k}{t}$ ***(A1)***

attempts to form a quadratic equation in x^2 ***M1***

eg. $t^2x^2 - kt^3x + \frac{k}{t}x = k^2$ OR $x^2 - ktx + \frac{k}{t^3}x - \frac{k^2}{t^2} = 0$ ***A1***

OR $\frac{k^2}{t^2} = x^2 - \left(kt - \frac{k}{t^3}\right)x$ OR $t^2x^2 - k\left(t^3 - \frac{1}{t}\right)x - k^2 = 0$

$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0$ ***AG***

Note: Award ***(A0)M1A0*** for attempting to verify that kt is a root of the ***AG*** (or its equivalent quadratic equation).
Award ***(A0)M0A0*** for showing that A lies on F.

[3 marks]

continued…

Question 1 continued.

(ii) **METHOD 1**

let α be the x- coordinate of A and let β be the x- coordinate of B

EITHER

recognizes that the sum of roots is $k\left(t-\frac{1}{t^3}\right)$ ***(M1)***

Note: Award ***(M1)*** for $-k\left(t-\frac{1}{t^3}\right)$.

$kt+\beta=k\left(t-\frac{1}{t^3}\right)$

OR

recognizes that the product of roots is $-\frac{k^2}{t^2}$ ***(M1)***

Note: Award ***(M1)*** for $\frac{k^2}{t^2}$.

$kt\beta=-\frac{k^2}{t^2}$

THEN

$\left(\beta=\right)-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},-kt^3\right)$

continued…

Question 1 continued.

METHOD 2

recognizes that $(x-kt)$ is a factor of $x^2-k\left(t-\frac{1}{t^3}\right)x-\frac{k^2}{t^2}(=0)$ ***(M1)***

$(x-kt)\left(x+\frac{k}{t^3}\right)(=0)$

$\Rightarrow x=kt,-\frac{k}{t^3}$

$-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},-kt^3\right)$

METHOD 3

attempts to use the quadratic formula ***(M1)***

$$x=\frac{k\left(t-\frac{1}{t^3}\right)\pm\sqrt{\left(k\left(t-\frac{1}{t^3}\right)\right)^2+4\left(\frac{k^2}{t^2}\right)}}{2}$$

$\Rightarrow x=kt,-\frac{k}{t^3}$

$-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},-kt^3\right)$

[3 marks]

continued…

Question 1 continued.

(f) **METHOD 1**

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ ***(A1)***

attempts to find their m_{AC} or m_{BC} ***(M1)***

$m_{\mathrm{AC}} = \frac{1}{t^2}$ AND $m_{\mathrm{BC}} = \frac{-\frac{k}{t} - \left(-kt^3\right)}{-kt - \left(-\frac{k}{t^3}\right)} = -t^2 \left(= \frac{kt^3 - \frac{k}{t}}{\frac{k}{t^3} - kt} = -t^2 \right) \left(t \neq \pm 1\right)$ (or equivalent) ***A1***

attempts to find their $m_{\mathrm{AC}} \times m_{\mathrm{BC}}$ OR recognizes that $m_{\mathrm{AC}} = -\frac{1}{m_{\mathrm{BC}}}$ (or equivalent) ***(M1)***

$= \frac{1}{t^2} \times \left(-t^2\right)$ OR eg. $\frac{1}{t^2} \times \frac{kt^4 - k}{t} \times \frac{t^3}{k - kt^4} = \frac{t^3}{t^3} \times \frac{kt^4 - k}{-\left(kt^4 - k\right)}$ OR $-\frac{1}{\frac{1}{t^2}} = -t^2$ (or equivalent)

A1

$= -1$ and so $\left[\mathrm{BC}\right] \perp \left[\mathrm{CA}\right]$ ***R1***

hence $\mathrm{B\hat{C}A}$ is a right angle ***AG***

METHOD 2

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ ***(A1)***

$m_{\mathrm{AC}} = \frac{1}{t^2}$ ***A1***

attempts to find their equation of the perpendicular to $\left[\mathrm{AC}\right]$ at C ***(M1)***

$y - \left(-\frac{k}{t}\right) = -t^2\left(x - \left(-kt\right)\right)$

substitutes $x = -\frac{k}{t^3}$ into their equation of the perpendicular to $\left[\mathrm{AC}\right]$ at C ***(M1)***

$y = -kt^3$ ***A1***

B lies on the perpendicular to $\left[\mathrm{AC}\right]$ at C $(\mathrm{B} \neq \mathrm{C})$ and so $\left[\mathrm{BC}\right] \perp \left[\mathrm{CA}\right]$ ***R1***

hence $\mathrm{B\hat{C}A}$ is a right angle ***AG***

continued…

Question 1 continued.

METHOD 3

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ OR $\vec{\mathrm{OC}} = \begin{pmatrix} -kt \\ -\frac{k}{t} \end{pmatrix}$ ***(A1)***

attempts to find their $\vec{\mathrm{CA}}$ OR their $\vec{\mathrm{CB}}$ ***(M1)***

$\vec{\mathrm{CA}} = \begin{pmatrix} 2kt \\ \frac{2k}{t} \end{pmatrix}$ AND $\vec{\mathrm{CB}} = \begin{pmatrix} -\frac{k}{t^3} + kt \\ -kt^3 + \frac{k}{t} \end{pmatrix}$ ***A1***

attempts to find their $\vec{\mathrm{CA}} \cdot \vec{\mathrm{CB}}$ ***(M1)***

$\vec{\mathrm{CA}} \cdot \vec{\mathrm{CB}} = 2kt\left(-\frac{k}{t^3} + kt\right) + \frac{2k}{t}\left(-kt^3 + \frac{k}{t}\right)$

$= -\frac{2k^2}{t^2} + 2k^2t^2 - 2k^2t^2 + \frac{2k^2}{t^2}$ ***A1***

$= 0$ (and since $t \neq \pm 1$, $\left|\vec{\mathrm{CB}}\right| \neq 0$) ***R1***

then $\mathrm{B\hat{C}A}$ is a right angle ***AG***

Note: Award as above if $\vec{\mathrm{OA}} = \begin{pmatrix} kt \\ \frac{k}{t} \end{pmatrix}$ is used instead of $\vec{\mathrm{CA}} = \begin{pmatrix} 2kt \\ \frac{2k}{t} \end{pmatrix}$.

continued …

Question 1 continued.

METHOD 4

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ ***(A1)***

attempts to find one of their AB^2, AC^2 or BC^2 ***(M1)***

EITHER

$\mathrm{AB}^2 = \left(kt - \left(-\frac{k}{t^3}\right)\right)^2 + \left(\frac{k}{t} - \left(-kt^3\right)\right)^2$ ***A1***

OR

$\mathrm{AC}^2 = \left(kt - \left(-kt\right)\right)^2 + \left(\frac{k}{t} - \left(-\frac{k}{t}\right)\right)^2$ ***A1***

OR

$\mathrm{BC}^2 = \left(-kt - \left(-\frac{k}{t^3}\right)\right)^2 + \left(-\frac{k}{t} - \left(-kt^3\right)\right)^2$ ***A1***

THEN

attempts to show that their $\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{BC}^2$ ***(M1)***

$\mathrm{AB}^2 = \frac{k^2}{t^6}\left(t^4+1\right)^3$ (or equivalent) AND $\mathrm{AC}^2 + \mathrm{BC}^2 = \frac{k^2}{t^6}\left(t^4+1\right)^3$ (or equivalent) ***A1***

$\mathrm{AB}^2 = k^2t^6 + 3k^2t^2 + \frac{3k^2}{t^2} + \frac{k^2}{t^6}$ AND $\mathrm{AC}^2 + \mathrm{BC}^2 = k^2t^6 + 3k^2t^2 + \frac{3k^2}{t^2} + \frac{k^2}{t^6}$

$\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{BC}^2$ (and so, by the converse of Pythagoras' theorem) ***R1***

$\mathrm{B\hat{C}A}$ is a right angle ***AG***

[6 marks]

Total [26 marks]

2. (a) (i) that the stable/long term population will be the same as before the reduction. ***A1***

Note: Condone 'the maximum value / carrying capacity of the population will be 200'.

[1 mark]

(ii) attempt to substitute into logistic function ***(M1)***

$$40 = \frac{200}{1 + C\mathrm{e}^{0}}$$

$$40 + 40C = 200$$

$$C = 4$$ ***A1***

[2 marks]

(iii) correct substitution into their logistic function ***(A1)***

$$70 = \frac{200}{1 + 4\mathrm{e}^{-5k}}$$

$$k = 0.153\ (0.153451..., -\frac{1}{5}\ln\left(\frac{130}{280}\right))$$ ***A1***

[2 marks]

(iv) correct substitution into their logistic function ***(A1)***

$$x = \frac{200}{1 + 4\mathrm{e}^{-10\times 0.153451...}} \left(= 107.397....\right)$$

$$x = 107$$ ***A1***

[2 marks]

continued…

Question 2 continued

(b) (i) $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ ***A1***

[1 mark]

(ii) the population is steady/stable (at 200) **OR** this (200) is the equilibrium population ***A1***

Note: Accept phrases such as 'the rate of change of wolf population is zero'.
Do not accept 200 is the maximum. Do not follow through from an incorrect (b)(i).

[1 mark]

(iii) **EITHER**

$\frac{1}{\ln 200 - \ln x} \times$ an attempt to differentiate $\ln 200 - \ln x$ ***M1***

$f'(x) = \frac{1}{\ln 200 - \ln x} \times \left(-\frac{1}{x}\right)$ ***A1***

OR

$\frac{1}{\ln\left(\frac{200}{x}\right)} \times \frac{x}{200} \times -\frac{200}{x^2}$ ***M1A1***

Note: Award ***M1*** if there is a product of three expressions and two are correct.

$= \frac{-1}{x \ln\left(\frac{200}{x}\right)}$ ***AG***

[2 marks]

continued…

Question 2 continued

(iv) attempt to separate variables ***M1***

$$\int \frac{-1}{x\ln\left(\frac{200}{x}\right)}\mathrm{d}x = \int -a\mathrm{d}t$$ ***A1***

Note: Award ***A1*** also for $\int \frac{1}{x\left(\ln\left(\frac{200}{x}\right)\right)}\mathrm{d}x = \int a\mathrm{d}t$.

$$\ln\left(\ln 200 - \ln x\right) = -at\left(+c\right)$$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

$\ln 200 - \ln x = \mathrm{e}^{-at+c}$ **OR** $\ln 200 - \ln x = A\mathrm{e}^{-at}$, $\left(A = \mathrm{e}^{c}\right)$ ***A1***

$\ln x = \ln 200 - A\mathrm{e}^{-at}$ ***AG***

[5 marks]

(v) correct substitution $\ln 40 = \ln 200 - A\mathrm{e}^{-0}$ ***(A1)***

$A = \ln\left(5\right)$ ***A1***

$\left(p = 5\right)$

Note: Award ***A1A0*** for an unsupported 1.61 (1.60943…).

[2 marks]

continued…

Question 2 continued

(vi) correct substitution $\ln 70 = \ln 200 - \ln 5 \times e^{-5a}$ ***A1***

$a = 0.0854528...$ ***A1***

Note: Award ***A1*** if their unrounded answer has at least four significant figures and rounds to 0.0855 OR if an exact value equivalent to $0.2 \ln \left(\frac{\ln 5}{\ln \left(\frac{200}{70} \right)} \right)$ is seen.

$a = 0.0855$ to 3 significant figures ***AG***

[2 marks]

(vii) correct substitution $\ln x = \ln 200 - \ln 5 \times e^{-10 \times 0.0854528...}$ ***(A1)***

$100.839\ldots$ ***(A1)***

Note: Accept $100.871\ldots$ from use of 3sf 0.0855.

101 to the nearest whole number ***A1***

[3 marks]

(c) both are overestimates **OR** the Gompertz is closer to the true value ***A1***

[1 mark]

Note: Follow through on their answers for 2(a)(iv) and 2 (b)(vii) if both positive values and these values are compared with 85.

continued…

Question 2 continued

(d) (i) attempt to use Euler ***(M1)***

Note: Condone maximum of one error from missing $x_n +$ at start, or incorrect/missing subscripts, absence of 0.5.

$$x_{n+1} = x_n + 0.5\left(0.0855x_n \ln\left(\frac{200}{x_n}\right) - 0.03x_n\right)$$ ***(A1)***

evidence of correct use of Euler's method ***(A1)***

Note: Award ***(A1)*** for any correct value of x_{n+1} for $n = 2,3,...$ seen (including this intermediate value presented as a final answer).

after 10 years the population will be 89 wolves $(89.42803\ldots)$ ***A1***

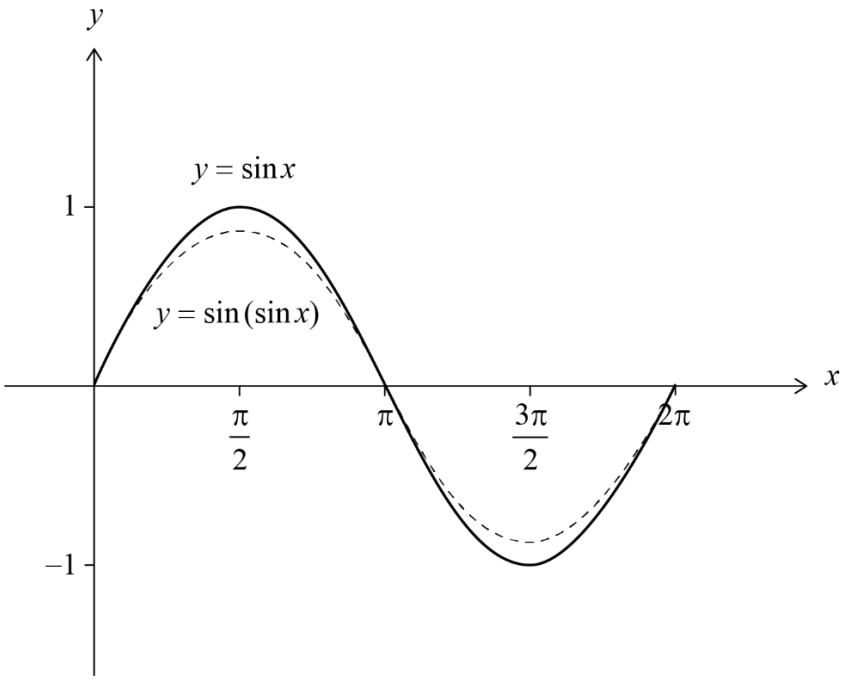
[4 marks]

(ii) correctly compare their result to the actual number of wolves or to the other models or both. Allow even if this is implicit, for example, it is more accurate / realistic than the other models ***A1***

[1 mark]

Total [29 marks]

1. (a)



correct shape $y = \sin x$ ***A1***

correct domain ***A1***

intercepts on x-axis at $0, \pi, 2\pi$ ***A1***

correct shape $y = \sin(\sin x)$ with a smaller maximum value and a larger minimum value than $y = \sin x$ and the same x-intercepts as $y = \sin x$ ***A1***

Note: For candidates who do not label their graphs with $\sin x$ and $\sin(\sin x)$ award at most ***A1A1A1A0.***

For candidates who sketch their graphs on separate axes award at most ***A1A1A1A0.***

These marks are independent of each other.

[4 marks]

continued…

Question 1 continued

(b) (i) 1 ***A1***

[1 mark]

(ii) $0.841(0.841470\ldots)$ ***A1***

[1 mark]

(iii) $0.746(0.745624\ldots)$ ***A1***

[1 mark]

(c) use of GDC eg tables (to vary the value of n) ***(M1)***

$\left(S_4\left(\frac{\pi}{2}\right)=0.678430...,S_5\left(\frac{\pi}{2}\right)=0.627571...,S_6\left(\frac{\pi}{2}\right)=0.587180...\right)$

$n=6$ ***A2***

[3 marks]

continued…

Question 1 continued

(d) attempt to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ using chain rule ***(M1)***

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} =\right) \cos(\sin x)\cos x$ ***A1***

$\cos(\sin x)\cos x = 0$

considers $\cos x = 0$ ***M1***

$x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ ***A1***

considers $\cos(\sin x) = 0$ ***M1***

$\sin x = \frac{\pi}{2} \Rightarrow$ no other solution for x ***R1***

Note: The ***M1R1*** above are independent of the previous ***M1A1***.

hence curve has exactly two points of zero gradient at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ ***AG***

[6 marks]

(e) $S_3'(x) = \cos(S_2(x))\cos(S_1(x))\cos x$ ***A1***

$S_3'(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$ ***AG***

[1 mark]

continued…

Question 1 continued

(f) consider case when $n=1$ $\text{LHS}=\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x=\text{RHS}$ ***A1***

Note: Subsequent marks after this ***A1*** are independent of this mark and can be awarded.

assume true for some k, ***M1***

i.e. $\frac{\mathrm{d}}{\mathrm{d}x}\left(S_k(x)\right)=\cos\left(S_{k-1}(x)\right)\cos\left(S_{k-2}(x)\right)\ldots\cos\left(S_1(x)\right)\cos x$

Note: Award ***M0*** for statements such as "let $n=k$" or "assume that $n=k$ is true". The assumption of truth of the statement must be clear. Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

EITHER

recognition that $S_{k+1}(x)=\sin\left(S_k(x)\right)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(S_{k+1}(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(S_k(x)\right)$

$=\cos\left(S_k(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(S_k(x)\right)$ ***A1***

$=\cos\left(S_k(x)\right)\cos\left(S_{k-1}(x)\right)\cos\left(S_{k-2}(x)\right)\ldots\cos x$ (by assumption) ***A1***

OR

recognition that $S_{k+1}(x)=S_k(\sin x)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(S_{k+1}(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(S_k(\sin x)\right)$

$=S_k'(\sin x)\cos x$ ***A1***

$=\cos\left(S_{k-1}(\sin x)\right)\cos\left(S_{k-2}(\sin x)\right)\ldots\cos x(\sin x)\cos x$ (by assumption)

$=\cos\left(S_k(x)\right)\cos\left(S_{k-1}(x)\right)\cos\left(S_{k-2}(x)\right)\ldots\cos x$ ***A1***

THEN

hence true for case $n=k+1$

true for the case $n=1$, when case $n=k$ assumed true, case $n=k+1$ proven to be true. Hence true for all positive integers n. ***R1***

Note: Only award **R1** if at least 3 of the previous 5 marks have been awarded.

[6 marks]

continued…

Question 1 continued

(g) attempt to differentiate numerator and denominator ***(M1)***

$$\lim_{x\to 0}\frac{S_n(x)}{x}=\lim_{x\to 0}\frac{\cos\left(S_{n-1}(x)\right)\cos\left(S_{n-2}(x)\right)\ldots\cos\left(S_1(x)\right)\cos x}{1}$$ ***A1***

$$=\frac{\cos(0)\cos(0)\ldots\cos(0)\cos(0)}{1}$$ ***A1***

$$=1$$ ***AG***

Note: For candidates who don't include the limit award a maximum of ***(M1)A1A0***

[3 marks]

Total [26 marks]

2. (a) attempt to use $S_\infty = \frac{u_1}{1-r}$ **(*M1*)**

$= \frac{1}{1+x^2}$ ***A1***

Note: Allow working with $u_1 = 1$ and $r = -x^2$ or $u_1 = 1 - x^2$ and $r = x^4$.

[2 marks]

(b) $\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x \ (+c)$ ***A1***

$\int (1 - x^2 + x^4 - x^6 + \ldots)\mathrm{d}x = (c+)\, x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots (= \arctan x)$ ***A1***

Note: The ***A1*** marks above are independent.

when $x = 0, \arctan 0 = c + 0 \Rightarrow c = 0$ ***R1***

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***AG***

[3 marks]

(c) $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}\left(\approx \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 \right)$ ***A1***

attempt to evaluate $\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5$ ***M1***

$\pi \approx 3.156$ ***A1***

[3 marks]

continued…

Question 2 continued

(d)

METHOD 1

recognition that in the Maclaurin expansion of $\arctan x$, the error term is $\frac{x^{2m-1}}{2m-1}$ ***(M1)***

attempt to solve $\frac{1}{2m-1}\times\left(\frac{1}{\sqrt{3}}\right)^{2m-1} < 0.0001$ OR $m = 6.60583...(=7)$ ***M1***

6 (non-zero) terms are needed ***A1***

Note: Accept the error term in an absolute value.

Give marks as above for candidates who use $2m+1$ instead of $2m-1$. This gives $m = 5.60583...(=6)$ and leads to needing 6 terms.

Give marks as above for candidates who use n as the exponent and divisor. This gives $n = 12.211675...$ and leads to needing 6 terms.

METHOD 2

recognition that in the Maclaurin expansion of $\arctan x$, the error term is $\frac{x^{2m-1}}{2m-1}$ ***(M1)***

attempt to find the value of the error term for at least two consecutive values of m ***M1***

error terms are

$$\frac{\left(\frac{1}{\sqrt{3}}\right)^7}{7} = 0.003054..., \frac{\left(\frac{1}{\sqrt{3}}\right)^9}{9} = 0.0007919..., \frac{\left(\frac{1}{\sqrt{3}}\right)^{11}}{11} = 0.0002159..., \frac{\left(\frac{1}{\sqrt{3}}\right)^{13}}{13} = 0.00006092...$$

6 (non-zero) terms are needed ***A1***

Note: Accept the error term in an absolute value.

Accept $2m+1$ or n in place of $2m-1$.

continued…

Question 2 continued

METHOD 3

attempt to find the actual error using $\arctan x -$ (first m non-zero terms) ***(M1)***

attempt to find the error for at least two consecutive values of m ***M1***

$(m = 4 \Rightarrow 0.000623, m = 5 \Rightarrow 0.000169, m = 6 \Rightarrow 0.0000473)$

6 (non-zero) terms are needed ***A1***

[3 marks]

(e) $u = \arctan x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$ (or equivalent) (***A1)***

$$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}x = \left[x \arctan x\right]_0^{\frac{1}{\sqrt{3}}} - \int_0^{\frac{1}{\sqrt{3}}} \frac{x}{1+x^2} \, \mathrm{d}x \quad \textbf{A1A1}$$

Note: Award ***A1*** for each term.

$$= \left[x \arctan x\right]_0^{\frac{1}{\sqrt{3}}} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^{\frac{1}{\sqrt{3}}} \left(= \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}\right) \quad \textbf{A1}$$

$$= \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3} \quad \textbf{AG}$$

[4 marks]

(f) $\left(\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) \mathrm{d}x = \right) 0.158422$ ***A2***

[2 marks]

(g) attempt to solve (for π) $\frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3} \approx 0.158422$ ***M1***

$\pi \approx 3.1412$ ***A1***

[2 marks]

continued…

Question 2 continued

(h) theorem error is given by $\left|\int_0^{\frac{1}{\sqrt{3}}} \frac{x^9}{9}\mathrm{d}x\right| \left(= \left[\frac{x^{10}}{90}\right]_0^{\frac{1}{\sqrt{3}}}\right)$ ***(M1)***

$= 4.57\times10^{-5}$ ***A1***

attempt to evaluate their error ***(M1)***

$\left(\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3}\right) - 0.158422$ OR $0.158458558\ldots - 0.158422$

$= 3.69\times10^{-5}$ ***A1***

Note: Accept $= 3.73\times10^{-5}$ from using the unrounded answer from part (f).
The (***M1)A1*** above are independent of the previous (***M1)A1***.

$3.69\times10^{-5} < 4.57\times10^{-5}$ ***R1***

hence theorem holds ***AG***

[5 marks]

continued…

Question 2 continued

(i) **METHOD 1**

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^n}{n}\right) \mathrm{d}x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\frac{\left(\frac{1}{\sqrt{3}}\right)^{n+1}}{n(n+1)} < 1\times10^{-6}$ ***A1***

attempt to solve using GDC ***(M1)***

$n = 14.3336\ldots$

$\Rightarrow n = 15$ ***A1***

(hence 8^{th} term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 2

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \frac{x^{2m-1}}{2m-1} dx$ OR $\int_0^{\frac{1}{\sqrt{3}}} \frac{x^{2m+1}}{2m+1} dx$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\frac{\left(\frac{1}{\sqrt{3}}\right)^{2m}}{2m(2m-1)} < 1\times10^{-6}$ OR $\frac{\left(\frac{1}{\sqrt{3}}\right)^{2m+2}}{(2m+1)(2m+2)} < 1\times10^{-6}$ ***A1***

attempt to solve using GDC ***(M1)***

$\Rightarrow m = 7.66681\ldots$ OR $\Rightarrow m = 6.66681\ldots$ ***A1***

(hence 8^{th} term is the error term so) we require 7 (non-zero) terms ***A1***

continued…

Question 2 continued

METHOD 3

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^n}{n}\right) \mathrm{d}x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{13}}{13}\right) \mathrm{d}x = 2.51234... \times 10^{-6}$ OR $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{15}}{15}\right) \mathrm{d}x = 6.35065... \times 10^{-7}$ ***A1***

attempt to find the value of the integral for at least two consecutive odd values of n, $n > 9$ ***(M1)***

$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{13}}{13}\right) \mathrm{d}x = 2.51234... \times 10^{-6}$, $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{15}}{15}\right) \mathrm{d}x = 6.35065... \times 10^{-7}$ ***A1***

Note: Award this ***A1*** for the second correct value.

(hence 8^{th} term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 4

attempt to find the actual error using $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx -$ (first m non-zero terms) ***(M1)***

error with 6 terms is $2.00830\ldots \times 10^{-6}$ **OR** error with 7 terms is $5.04041\ldots \times 10^{-7}$ ***A1***

attempt to find the error for at least two consecutive values of m ***(M1)***

error with 6 terms is $2.00830\ldots \times 10^{-6}$ and error with 7 terms is $5.04041\ldots \times 10^{-7}$ ***A1***

Note: Award this ***A1*** for the second correct value.

we require 7 (non-zero) terms ***A1***

[5 marks]

Total [29 marks]