

APOLLO

Calculus

HL

Mathematics: analysis and approaches

First assessment 2021

135 questions | questions

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!} (1+x)^{\frac{1}{2}-n}$
for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = e^{mx}$, $m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

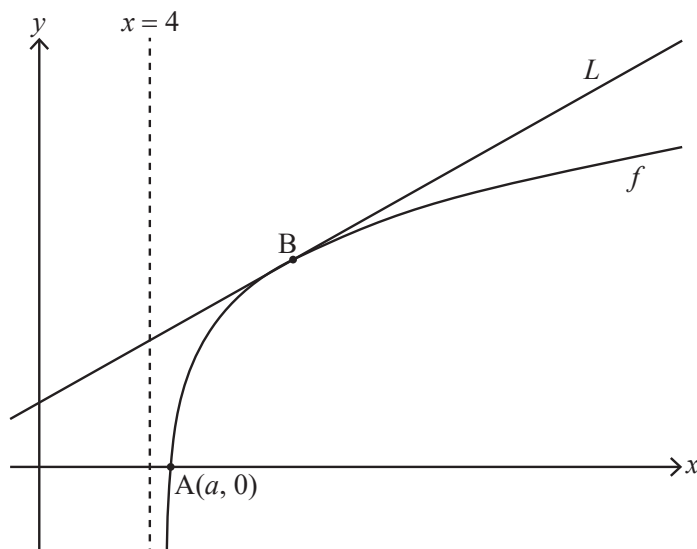
(c) Find the possible values of m . [8]

References:

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4. [Maximum mark: 9]



- (a) Find the exact value of α . [3]
- (b) Given that the gradient of L is $\frac{1}{3}$, find the x -coordinate of B. [6]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

The acceleration, $a \text{ ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v \text{ ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0 \text{ ms}^{-1}$.

- (a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)e^{-t} - 1$. [6]
- (b) Initially at O , the particle moves in the positive direction until it reaches its maximum displacement from O . The particle then returns to O .

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O .

- (i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $e^T = 1 + v_0$.
- (ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0 . [7]

Let $v(T - k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T - k) = (1 + v_0)e^{-(T-k)} - 1.$$

- (c) By using the result to part (b) (i), show that $v(T - k) = e^k - 1$. [2]

Similarly, let $v(T + k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

- (d) Deduce a similar expression for $v(T + k)$ in terms of k . [2]
- (e) Hence, show that $v(T - k) + v(T + k) \geq 0$. [3]



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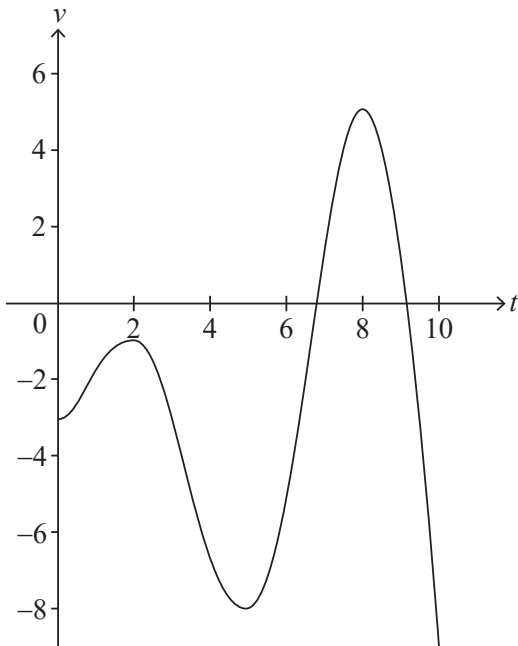
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\text{ ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t \sin t - 3$, for $0 \leq t \leq 10$.

The following diagram shows the graph of v .



- (a) Find the smallest value of t for which the particle is at rest. [2]
- (b) Find the total distance travelled by the particle. [2]
- (c) Find the acceleration of the particle when $t = 7$. [2]

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9. [Maximum mark: 7]

Initially, boat B is positioned 50 metres due east of boat A.

diagram not to scale



Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T . [6]



Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 90e^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

(a) Find the x -coordinate of P.

[2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

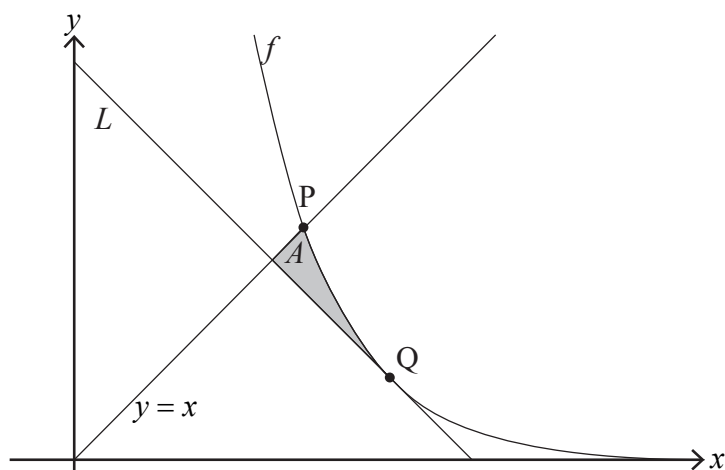
(b) Find the exact coordinates of Q.

[4]

(c) Show that the equation of L is $y = -x + 2 \ln 45 + 2$.

[2]

The shaded region A is enclosed by the graph of f and the lines $y = x$ and L .



(d) (i) Find the x -coordinate of the point where L intersects the line $y = x$.

(ii) Hence, find the area of A .

[5]

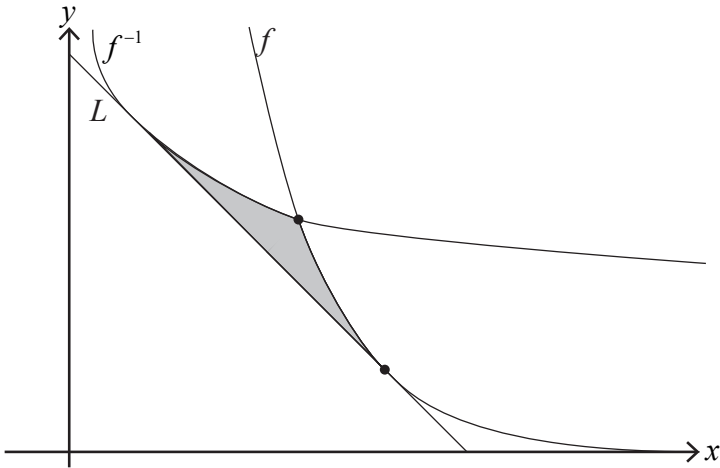
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(Question 10 continued)

The line L is tangent to the graphs of both f and the inverse function f^{-1} .



- (e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L . [2]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

The function f has a derivative given by $f'(x) = \frac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{x} + \frac{b}{k-x}$, where $a, b \in \mathbb{R}$.
Find a and b in terms of k . [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider P , the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\frac{dP}{dt} = \frac{P(k-P)}{5k}$,
where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

(c) By solving the differential equation, show that $P = \frac{1200k}{(k-1200)e^{\frac{t}{5}} + 1200}$. [8]

At $t = 10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of k , giving your answer correct to four significant figures. [3]

(e) Find the value of t when the rate of change of the population is at its maximum. [3]

References:

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

(a) Write down the first three terms of the binomial expansion of $(1 + t)^{-1}$ in ascending powers of t . [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1 + \frac{x^2}{2} + \frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x \rightarrow 0} \left(\frac{x \arctan 2x}{\sec x - 1} \right)$. [3]



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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

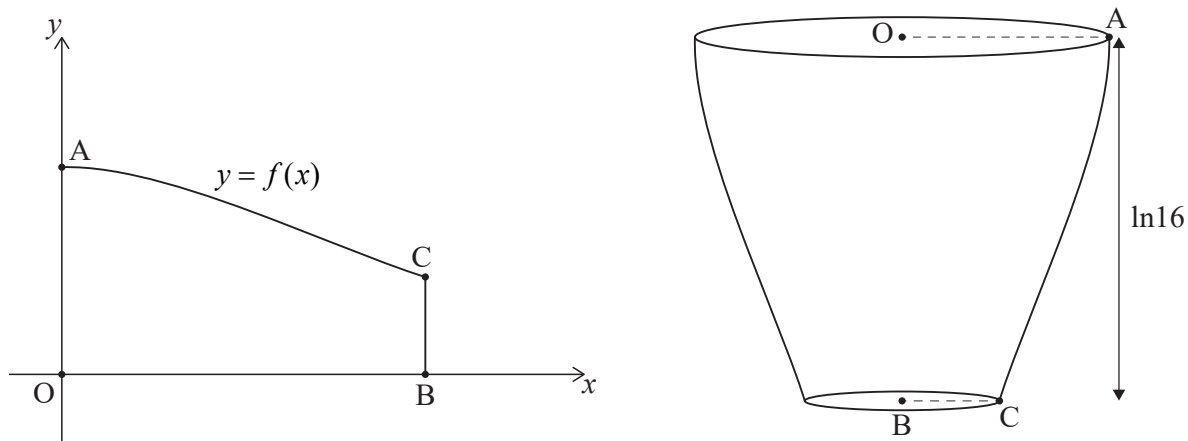
11. [Maximum mark: 18]

A function f is defined by $f(x) = \frac{ke^{\frac{x}{2}}}{1+e^x}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^+$.

The region enclosed by the graph of $y = f(x)$, the x -axis, the y -axis and the line $x = \ln 16$ is rotated 360° about the x -axis to form a solid of revolution.

- (a) Show that the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of 300 cm^3 based on the result from part (a). Pedro's design is shown in the following diagrams.



The vertical height of the bowl, BO , is measured along the x -axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC . All lengths are measured in cm.

- (b) Find the value of k that satisfies the requirements of Pedro's design. [2]
- (c) Find
- (i) OA ;
 - (ii) BC . [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

- (d) (i) By sketching the graph of a suitable derivative of f , find where the cross-sectional radius of the bowl is decreasing most rapidly. [6]
- (ii) State the cross-sectional radius of the bowl at this point.



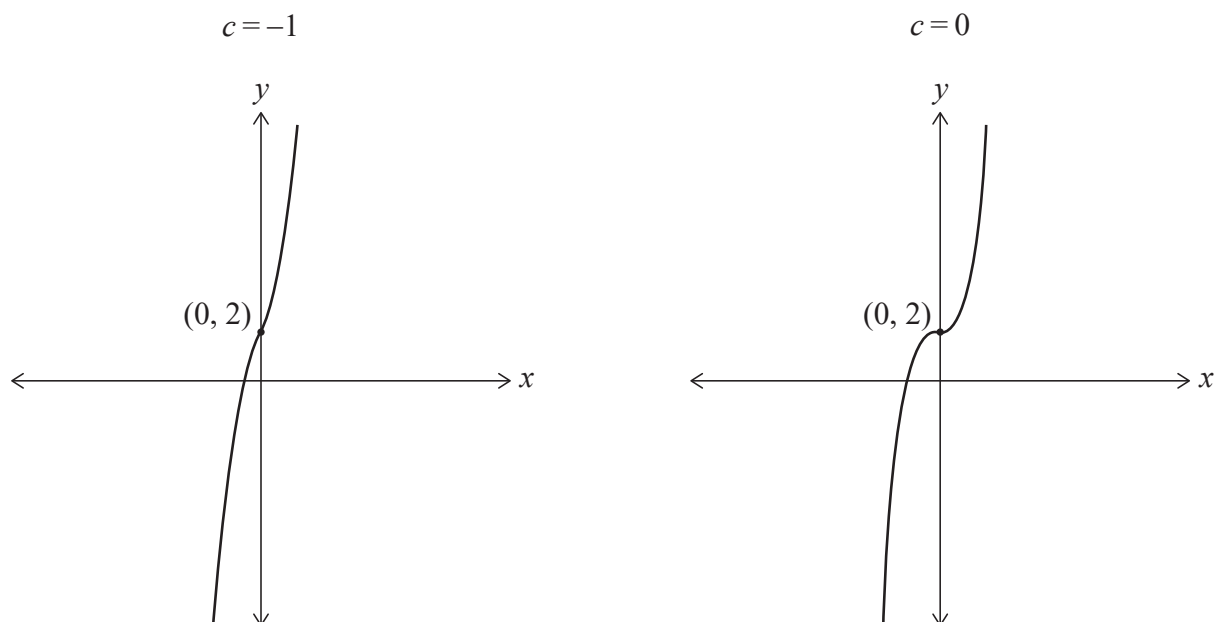
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1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.



- (a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y -intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

- (b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

- (c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has
- (i) a point of inflexion with zero gradient; [1]
 - (ii) one local maximum point and one local minimum point; [2]
 - (iii) no points where the gradient is equal to zero. [1]
- (d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that
- (i) the y -coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]
 - (ii) the y -coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]
- (e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has
- (i) exactly one x -axis intercept; [2]
 - (ii) exactly two x -axis intercepts; [2]
 - (iii) exactly three x -axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c, d \in \mathbb{R}$.

- (f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x -axis intercept, explaining your reasoning. [6]

Turn over

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a - x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2 - x)$.

- (a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2 - x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

- (b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for
- the odd values $n = 3$ and $n = 5$;
 - the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	Number of local minimum points	Number of points of inflexion with zero gradient
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a - x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

- (c) Show that $f'_n(x) = nx^{n-1}(a - 2x)(a - x)^{n-1}$. [5]
- (d) State the three solutions to the equation $f'_n(x) = 0$. [2]
- (e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(This question continues on the following page)

(Question 1 continued)

(f) Hence, or otherwise, show that $f'_n\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f'_n(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n , where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n , where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a - x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a - x)^n = k$ has four solutions for x . [5]

Turn over

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of h at $x = 4$. [3]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A particle P moves along the x -axis. The velocity of P is $v \text{ m s}^{-1}$ at time t seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \leq t \leq 3$. When $t = 0$, P is at the origin O .

- (a) (i) Find the value of t when P reaches its maximum velocity. [7]
- (ii) Show that the distance of P from O at this time is $\frac{88}{27}$ metres. [4]
- (b) Sketch a graph of v against t , clearly showing any points of intersection with the axes. [5]
- (c) Find the total distance travelled by P .

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{d^n}{dx^n}(x^2e^x) = [x^2 + 2nx + n(n-1)]e^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2e^x$ in ascending powers of x , up to and including the term in x^4 . [3]

(c) Hence or otherwise, determine the value of $\lim_{x \rightarrow 0} \left[\frac{(x^2e^x - x^2)^3}{x^9} \right]$. [4]



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Turn over

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2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{dx}{dt} = x - y \text{ and } \frac{dy}{dt} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

- (a) (i) By solving the differential equation $\frac{dy}{dt} = y$, show that $y = Ae^t$ where A is a constant. [3]
- (ii) Show that $\frac{dx}{dt} - x = -Ae^t$. [1]
- (iii) Solve the differential equation in part (a)(ii) to find x as a function of t . [4]

Now consider the case where $a = -1$.

- (b) (i) By differentiating $\frac{dy}{dt} = -x + y$ with respect to t , show that $\frac{d^2y}{dt^2} = 2\frac{dy}{dt}$. [3]
- (ii) By substituting $Y = \frac{dy}{dt}$, show that $Y = Be^{2t}$ where B is a constant. [3]
- (iii) Hence find y as a function of t . [2]
- (iv) Hence show that $x = -\frac{B}{2}e^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

- (c) (i) Show that $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = Fe^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

- (ii) Find the two values for λ that satisfy $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2 .

- (iii) Verify that $y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References:

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Let $f(x) = \frac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \frac{1 - \ln 5x}{kx^2}$. [3]

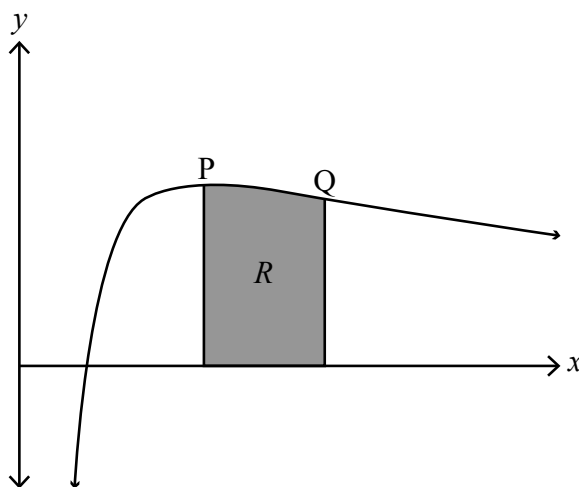
The graph of f has exactly one maximum point P.

(b) Find the x -coordinate of P. [3]

The second derivative of f is given by $f''(x) = \frac{2 \ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x -coordinate of Q is $\frac{1}{5}e^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f , the x -axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k . [7]



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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The function f is defined by $f(x) = e^{\sin x}$.

- (a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term.

[8]
 - (b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero.

[4]
 - (c) Using the Maclaurin series for $\arctan x$ and $e^{3x} - 1$, find the Maclaurin series for $\arctan(e^{3x} - 1)$ up to and including the x^3 term.

[6]
 - (d) Hence, or otherwise, find $\lim_{x \rightarrow 0} \frac{f(x) - 1}{\arctan(e^{3x} - 1)}$.

[3]
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The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t + 2t \sin t$, $0 \leq t \leq 5$.

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank. The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let x grams represent the amount of salt in the tank and let t minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{dx}{dt}$, is described by the differential equation $\frac{dx}{dt} = 10e^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t+1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40e^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of x versus t for $0 \leq t \leq 60$ and hence find the maximum amount of salt in the tank and the value of t at which this occurs. [5]

(d) Find the value of t at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first 60 minutes. [4]

Find the value of k .

[illegible]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The function f is defined by $f(x) = e^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 e^{x^2} \sin(x^2) dx$. [4]

The function g is defined by $g(x) = e^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2(g'(x) - g(x))$.

(ii) Hence, deduce that $g^{(4)}(x) = 2(g'''(x) - g''(x))$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim_{x \rightarrow 0} \frac{e^x \cos x - 1 - x}{x^3}$. [3]

References:

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 20]

Consider the function $f(x) = \sqrt{x^2 - 1}$, where $1 \leq x \leq 2$.

(a) Sketch the curve $y = f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x) = \sqrt{x^2 + 1}$.

(ii) State the domain and range of f^{-1} . [5]

The curve $y = f(x)$ is rotated 2π about the y -axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\text{m}^3$, of water in the container when it is filled to a height of h metres is given by $V = \pi\left(\frac{1}{3}h^3 + h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t = 0$, the container is empty. Water is then added to the container at a constant rate of $0.4\text{m}^3\text{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]



Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

Consider the differential equation $x^2 \frac{dy}{dx} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 1.5$. [4]

(b) Use the substitution $y = vx$ to show that $x \frac{dv}{dx} = v^2 - v - 2$. [3]

(c) (i) By solving the differential equation, show that $y = \frac{8x + x^4}{4 - x^3}$.

(ii) Find the actual value of y when $x = 1.5$.

(iii) Using the graph of $y = \frac{8x + x^4}{4 - x^3}$, suggest a reason why the approximation given

by Euler's method in part (a) is not a good estimate to the actual value of y at $x = 1.5$. [12]

References:

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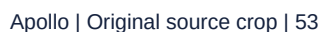


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The derivative of a function g is given by $g'(x) = 3x^2 + 5e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0, 4)$. Find $g(x)$.

[illegible]

5. [Maximum mark: 7]

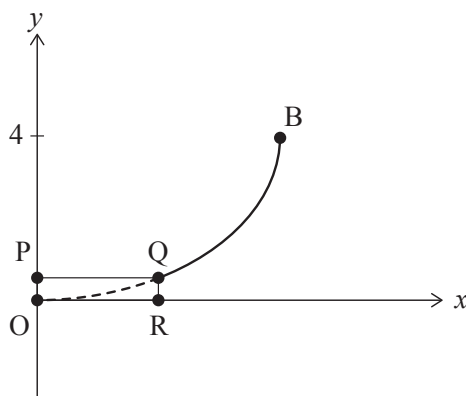
$$v = \frac{(t^2 + 1)\cos t}{4}, \quad 0 \leq t \leq 3.$$

- Determine when the particle changes its direction of motion. [2]
- Find the times when the particle's acceleration is -1.9 m s^{-2} . [3]
- Find the particle's acceleration when its speed is at its greatest. [2]

[illegible]

6. [Maximum mark: 5]

diagram not to scale



Given that the interior volume of the bowl is to be 285 cm^3 , determine the height of the base.

[illegible]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B , of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A , h_A cm, at time t weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \leq t \leq 9$.

The scientist found that the height of Plant B , h_B cm, at time t weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 \leq t \leq 9$.

- (a) Use the scientist's models to find the initial height of
 - (i) Plant B ;
 - (ii) Plant A correct to three significant figures. [3]
- (b) Find the values of t when $h_A(t) = h_B(t)$. [3]
- (c) For $t > 6$, prove that Plant A was always taller than Plant B . [3]
- (d) For $0 \leq t \leq 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A . [6]



Turn over

— 17 —

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The population, P , of a particular species of marsupial on a small remote island can be modelled by the logistic differential equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N} \right)$$

where t is the time measured in years and k, N are positive constants.

The constant N represents the maximum population of this species of marsupial that the island can sustain indefinitely.

- (a) In the context of the population model, interpret the meaning of $\frac{dP}{dt}$. [1]

- (b) Show that $\frac{d^2P}{dt^2} = k^2P \left(1 - \frac{P}{N} \right) \left(1 - \frac{2P}{N} \right)$. [4]

- (c) Hence show that the population of marsupials will increase at its maximum rate when $P = \frac{N}{2}$. Justify your answer. [5]

- (d) Hence determine the maximum value of $\frac{dP}{dt}$ in terms of k and N . [2]

Let P_0 be the initial population of marsupials.

- (e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln \frac{P}{P_0} \left(\frac{N - P_0}{N - P} \right). \quad [7]$$

After 10 years, the population of marsupials is $3P_0$. It is known that $N = 4P_0$.

- (f) Find the value of k for this population model. [2]

References:

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2. [Maximum marks: 28]

This question asks you to explore cubic polynomials of the form $(x - r)(x^2 - 2ax + a^2 + b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r = 1$, $a = 4$ and $b = 1$.

Consider the equation $(z - 1)(z^2 - 8z + 17) = 0$ for $z \in \mathbb{C}$.

(a) (i) Given that 1 and $4 + i$ are roots of the equation, write down the third root. [1]

(ii) Verify that the mean of the two complex roots is 4. [1]

Consider the function $f(x) = (x - 1)(x^2 - 8x + 17)$ for $x \in \mathbb{R}$.

(b) Show that the line $y = x - 1$ is tangent to the curve $y = f(x)$ at the point $A(4, 3)$. [4]

(c) Sketch the curve $y = f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x -axis. [2]

Consider the function $g(x) = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b > 0$.

(d) (i) Show that $g'(x) = 2(x - r)(x - a) + x^2 - 2ax + a^2 + b^2$. [2]

(ii) Hence, or otherwise, prove that the tangent to the curve $y = g(x)$ at the point $A(a, g(a))$ intersects the x -axis at the point $R(r, 0)$. [6]

The equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ for $z \in \mathbb{C}$ has roots r and $a \pm bi$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b > 0$.

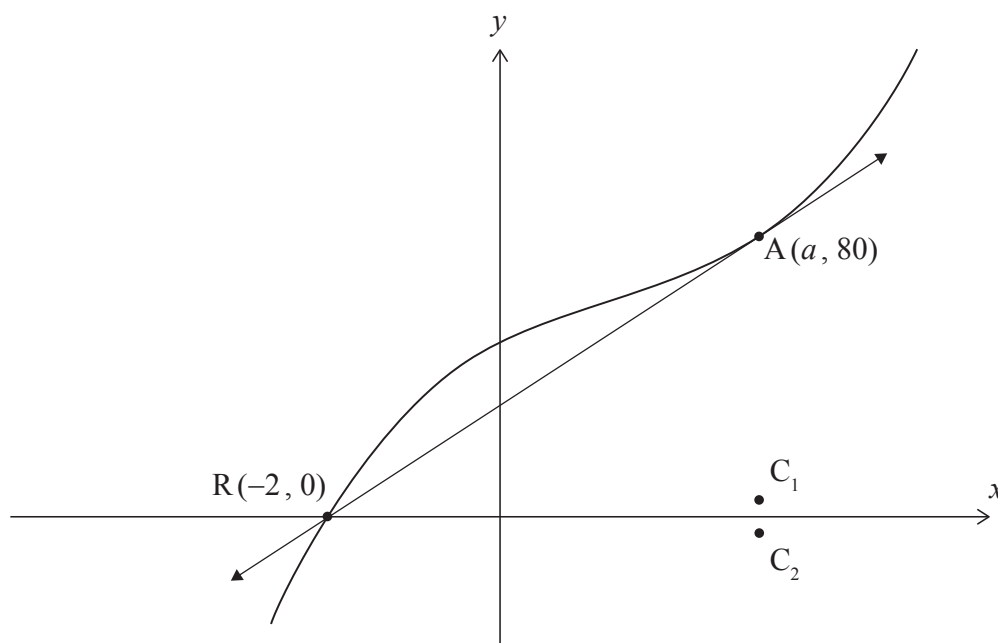
(e) Deduce from part (d)(i) that the complex roots of the equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$ can be expressed as $a \pm i\sqrt{g'(a)}$. [1]

(This question continues on the following page)

(Question 2 continued)

On the Cartesian plane, the points $C_1(a, \sqrt{g'(a)})$ and $C_2(a, -\sqrt{g'(a)})$ represent the real and imaginary parts of the complex roots of the equation $(z - r)(z^2 - 2az + a^2 + b^2) = 0$.

The following diagram shows a particular curve of the form $y = (x - r)(x^2 - 2ax + a^2 + 16)$ and the tangent to the curve at the point $A(a, 80)$. The curve and the tangent both intersect the x -axis at the point $R(-2, 0)$. The points C_1 and C_2 are also shown.



- (f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z - r)(z^2 - 2az + a^2 + 16) = 0$ for $z \in \mathbb{C}$. [4]
- (ii) State the coordinates of C_2 . [1]

Consider the curve $y = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $a \neq r$, $b > 0$. The points $A(a, g(a))$ and $R(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

- (g) (i) Show that the x -coordinate of P is $\frac{1}{3}(2a + r)$. [2]
- You are **not** required to demonstrate a change in concavity.
- (ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A. [1]

Consider the special case where $a = r$ and $b > 0$.

- (h) (i) Sketch the curve $y = (x - r)(x^2 - 2ax + a^2 + b^2)$ for $a = r = 1$ and $b = 2$. [2]
- (ii) For $a = r$ and $b > 0$, state in terms of r , the coordinates of points P and A. [1]

References:

1. [Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b , where $a, b \in \mathbb{N}$.

- (a) On the same set of axes, sketch the following curves for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3, x \geq 0$ [2]

(ii) $y^2 = x^3 + 1, x \geq -1$ [2]

- (b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

- (ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \geq -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

- (c) By varying the value of b , suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x, x \geq 0$.

- (d) (i) Show that $\frac{dy}{dx} = \pm \frac{3x^2 + 1}{2\sqrt{x^3 + x}}$, for $x > 0$. [3]

- (ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x -coordinate.

- (e) Find the value of this x -coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3} + q}{r}}$, where $p, q, r \in \mathbb{Z}$. [7]

(This question continues on the following page)

(Question 1 continued)

$P(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q .

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $P(-1, -1)$ lies on C .

- (f) (i) Find the equation of the tangent to C at P . [2]
- (ii) Hence, find the coordinates of the rational point Q where this tangent intersects C , expressing each coordinate as a fraction. [2]
- (g) The point $S(-1, 1)$ also lies on C . The line $[QS]$ intersects C at a further point. Determine the coordinates of this point. [5]

Turn over

9. [Maximum mark: 10]

It is given that $y = 2$ when $x = 1$.

- (a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

(b) Find the values of m . [2]

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2. [Maximum mark: 6]

(c) Find the total area enclosed by the graph of f , the x -axis and the line $x = 2$. [3]

[illegible]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 15]

The function f is defined by $f(x) = e^{2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x -axis and the y -axis is rotated through 2π radians about the x -axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^n i^q = 1^q + 2^q + 3^q + \dots + n^q \text{ where } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n , for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$.

[1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^n i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^n i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p .

[1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^n i^2 = a_1n + a_2n^2 + a_3n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1 , a_2 and a_3 .

[3]

(iii) Hence, find the values of a_1 , a_2 and a_3 .

[2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q .

Consider the function $f(x) = 1 + x + x^2 + \dots + x^n$, $n \in \mathbb{Z}^+$.

(c) Show that $xf'(x) = x + 2x^2 + 3x^3 + \dots + nx^n$. [1]

Let $f_1(x) = xf'(x)$ and consider the following family of functions:

$$f_2(x) = xf_1'(x)$$

$$f_3(x) = xf_2'(x)$$

$$f_4(x) = xf_3'(x)$$

...

$$f_q(x) = xf_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^n i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^n i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \dots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \frac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

(g) (i) Show that $\lim_{x \rightarrow 1} f_1(x)$ is in indeterminate form. [1]

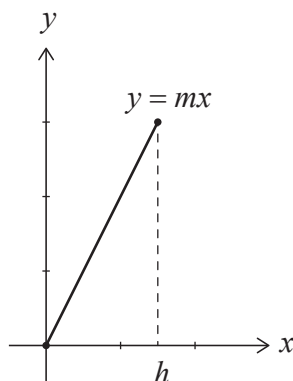
(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \rightarrow 1} f_1(x) = \frac{1}{2}n(n+1)$. [5]

Turn over

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \leq x \leq h$ and m, h are positive constants.



When this line is rotated through 360° about the x -axis, a cone is formed with a curved surface area A given by:

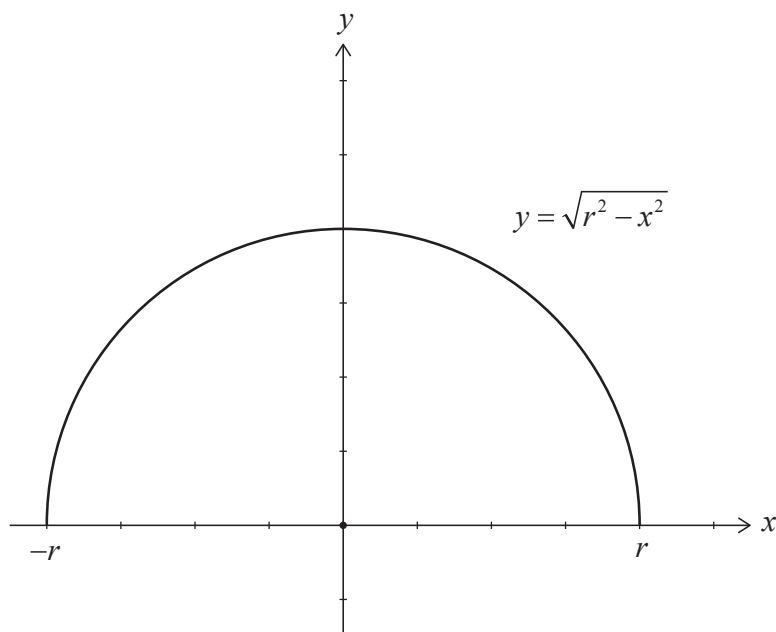
$$A = 2\pi \int_0^h y \sqrt{1 + m^2} \, dx.$$

- (a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\pi$. [2]
- (b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \leq x \leq h$ through 360° about the x -axis.
- (i) Deduce an expression for the radius of this cone r in terms of h and m . [1]
- (ii) Deduce an expression for the slant height l in terms of h and m . [2]
- (iii) Hence, by using the above integral, show that $A = \pi r l$. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the semi-circle, with radius r , defined by $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.



- (c) Find an expression for $\frac{dy}{dx}$. [2]

A differentiable curve $y = f(x)$ is defined for $x_1 \leq x \leq x_2$ and $y \geq 0$. When any such curve is rotated through 360° about the x -axis, the surface formed has an area A given by:

$$A = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

- (d) A sphere is formed by rotating the semi-circle $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$ through 360° about the x -axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(This question continues on the following page)

Turn over

(Question 2 continued)

- (e) Let $f(x) = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.

The graph of $y = f(x)$ is transformed to the graph of $y = f(kx)$, $k > 0$. This forms a different curve, called a semi-ellipse.

- (i) Describe this geometric transformation. [2]
- (ii) Write down the x -intercepts of the graph $y = f(kx)$ in terms of r and k . [1]
- (iii) For $y = f(kx)$, find an expression for $\frac{dy}{dx}$ in terms of x , r and k . [2]
- (iv) The semi-ellipse $y = f(kx)$ is rotated 360° about the x -axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A , of the ellipsoid.

Give your answer in the form $2\pi \int_{x_1}^{x_2} \sqrt{p(x)} \, dx$, where $p(x)$ is a polynomial. [4]

- (v) Planet Earth can be modelled as an ellipsoid. In this model:

- the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
- the distance from the North Pole to the South Pole is 12 714 km.
- the diameter of the equator is 12 756 km.

By choosing suitable values for r and k , find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a \times 10^q$ where $1 \leq a < 10$ and $q \in \mathbb{Z}^+$. [4]

References:

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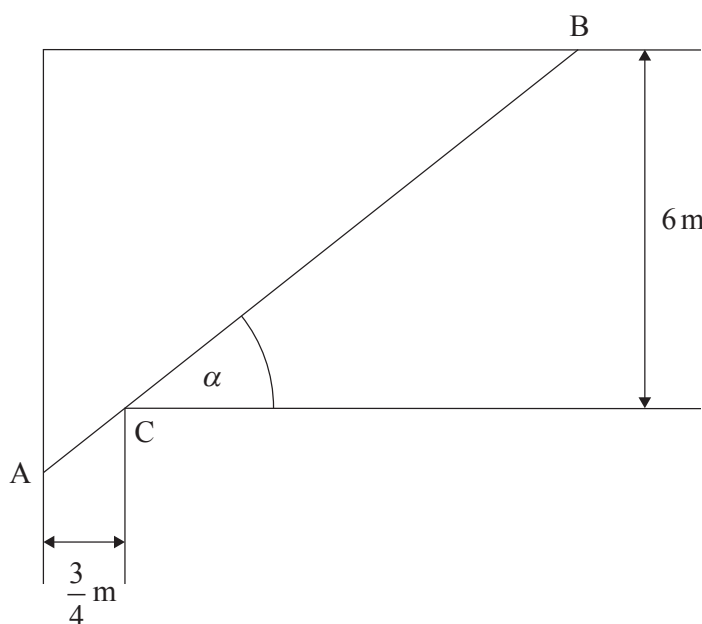
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with width $\frac{3}{4}$ m is perpendicular to a room of width 6 m. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that [AB] makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4} \sec \alpha + 6 \operatorname{cosec} \alpha$. [2]

(b) (i) Find $\frac{dL}{d\alpha}$.

(ii) When $\frac{dL}{d\alpha} = 0$, show that $\alpha = \arctan 2$. [5]

(This question continues on the following page)



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(Question 11 continued)

- (c) (i) Find $\frac{d^2L}{d\alpha^2}$.
- (ii) When $\alpha = \arctan 2$, show that $\frac{d^2L}{d\alpha^2} = \frac{45}{4}\sqrt{5}$. [7]
- (d) (i) Hence, justify that L is a minimum when $\alpha = \arctan 2$.
- (ii) Determine this minimum value of L . [3]

Two people need to carry a pole of length 11.25 m from the passageway into the room.
It must be carried horizontally.

- (e) Determine whether this is possible, giving a reason for your answer. [2]



Turn over

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4. [Maximum mark: 6]

A graph in the xy -plane showing a shaded region R . The region is bounded by the x -axis, a curve that starts at the origin, rises to a peak, and then decreases, and a vertical line segment at $x=c$. The curve continues to the right of $x=c$, but the shaded region R ends at $x=c$.

Find the value of c .

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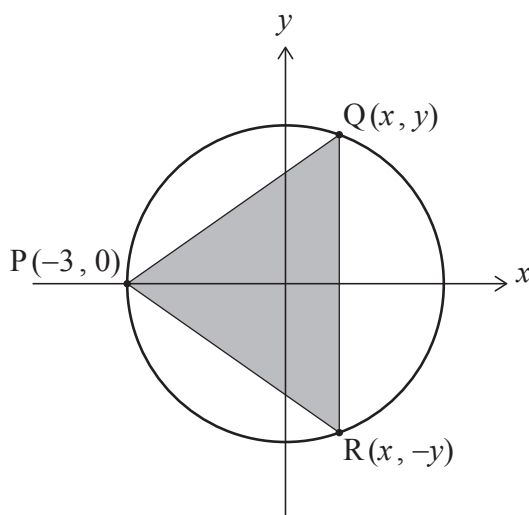
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0, 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $P(-3, 0)$, $Q(x, y)$ and $R(x, -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



- (a) For point Q , show that $y = \sqrt{9 - x^2}$. [1]
- (b) Hence, find an expression for A , the area of triangle PQR, in terms of x . [3]
- (c) Show that $\frac{dA}{dx} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$. [4]
- (d) Hence or otherwise, find the y -coordinate of R such that A is a maximum. [6]



Turn over

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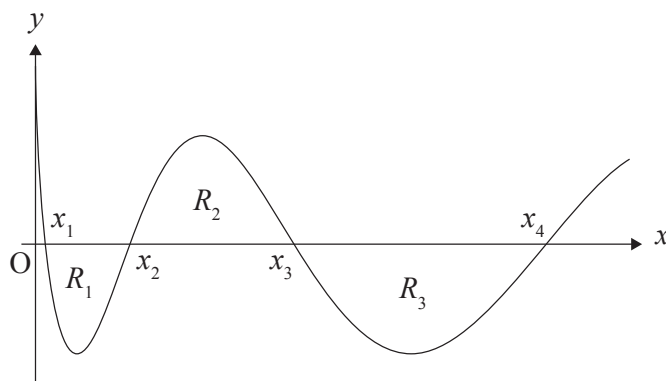
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 17]

- (a) By using an appropriate substitution, show that $\int \cos \sqrt{x} \, dx = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos \sqrt{x}$ for $x \geq 0$.



The curve intersects the x -axis at $x_1, x_2, x_3, x_4, \dots$

The n th x -intercept of the curve, x_n , is given by $x_n = \frac{(2n-1)^2 \pi^2}{4}$, where $n \in \mathbb{Z}^+$.

- (b) Write down a similar expression for x_{n+1} . [1]

The regions bounded by the curve and the x -axis are denoted by $R_1, R_2, R_3, \dots$, as shown on the above diagram.

- (c) Calculate the area of region R_n .

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

- (d) Hence, show that the areas of the regions bounded by the curve and the x -axis, $R_1, R_2, R_3, \dots$, form an arithmetic sequence. [3]

References:

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

- (a) Use Euler's method, with a step length of 0.03, to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [4]

- (b) (i) Write down the value of $\frac{dy}{dx}$ when $x = 0$.

- (ii) Show that $\frac{d^2y}{dx^2} = 3$ when $x = 0$. [5]

- (c) (i) Given that $\frac{d^3y}{dx^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for y .

- (ii) Use the Maclaurin series to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [3]

- (d) (i) It is given that $\frac{x^2 - 1}{x^2 + 1} \equiv 1 - \frac{2}{x^2 + 1}$.

Solve the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

- (ii) Hence, find the value of y when $x = 0.15$. Give your answer correct to six significant figures. [7]

- (e) For $0 \leq x < 1$, explain why the approximate value for y obtained using Euler's method will always be less than the actual value for y . [2]

References:

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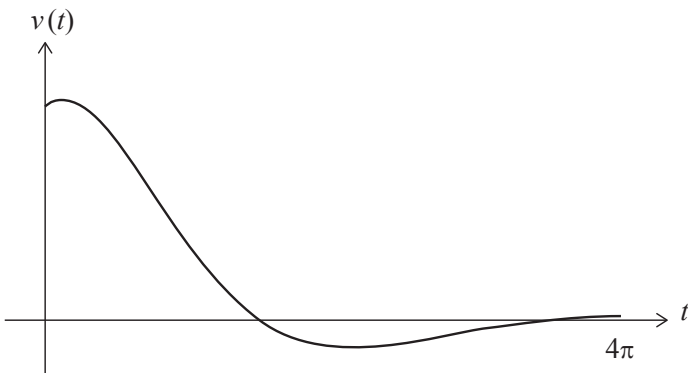


Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v \text{ m s}^{-1}$, at time t seconds is given by $v(t) = 4e^{-\frac{t}{3}} \cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \leq t \leq 4\pi$. The graph of v is shown in the following diagram.



Let t_1 be the first time when the particle's **acceleration** is zero.

(a) Find the value of t_1 . [2]

Let t_2 be the **second** time when the particle is instantaneously at rest.

(b) Find the value of t_2 . [2]

(c) Find the distance travelled by the particle between $t = t_1$ and $t = t_2$. [2]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$, where $x > 0, y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.1$. [2]

(b) By solving the differential equation, show that $y = x\sqrt{\frac{9x^4 - 1}{2}}$. [8]

(c) Find the value of y when $x = 1.1$. [1]

(d) With reference to the concavity of the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $1 \leq x \leq 1.1$, explain why the value of y found in part (c) is greater than the approximate value of y found in part (a). [2]

The graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $\frac{\sqrt{3}}{3} < x < 1$ has a point of inflexion at the point P.

(e) By sketching the graph of an appropriate derivative of y , determine the x -coordinate of P. [2]

It can be shown that $\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$, where $x > 0, y > 0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point P lies on the straight line $y = mx$ where the exact value of m is to be determined. [6]

References:

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1. [Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = x e^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x -axis and the line $x = b$, where $b > 0$, is given by $\frac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, A_n , of the region between the graph $y = f_n(x)$ and the x -axis can be written as $A_n = \int_0^\infty f_n(x) dx$ and is given by $\lim_{b \rightarrow \infty} \int_0^b f_n(x) dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b \rightarrow \infty} \frac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1 . [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4 ; [2]

(ii) A_5 . [1]

(e) Suggest an expression for A_n in terms of n , where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m , $\lim_{x \rightarrow \infty} x^m e^{-x} = 0$. [8]

Turn over

2. [Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers $x_1, x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

- (a) Find the product of x_1 and x_2 as a function, f , of x_1 only. [2]
- (b) (i) Find the value of x_1 for which the function is maximum. [1]
- (ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S , where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

- (c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, $x_1, x_2, \dots, x_n$.

The geometric mean is defined as $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1 + x_2 + \dots + x_n)}{n}$.

- (d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \dots = x_n$. [2]
- (ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]
- (e) Hence determine the value of
- (i) $M_3(12)$; [1]
- (ii) $M_4(12)$; [1]
- (iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n .

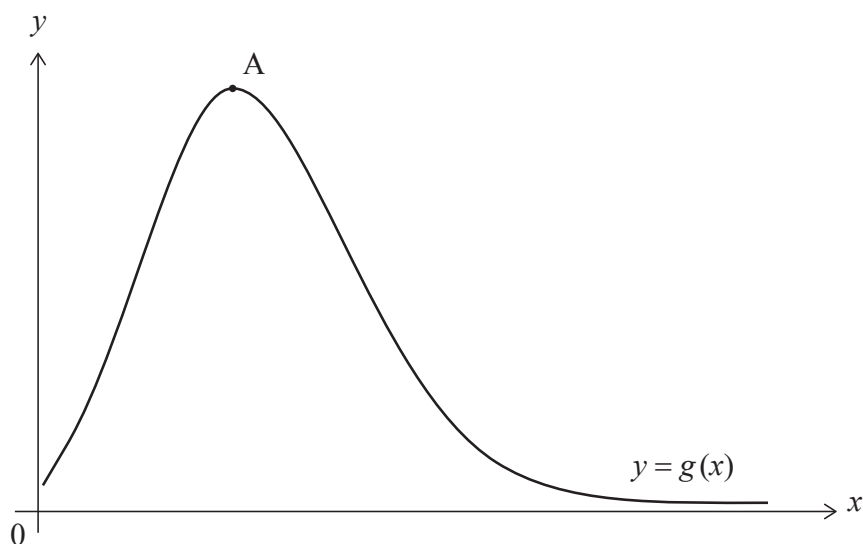
- (f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]
- (g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the function g , defined by $\ln(g(x)) = x \ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



- (h) Find, in terms of S , the x -coordinate of point A. [6]
- (i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]
- (j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}^+$. [3]

References:

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1. [Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a .

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator “logarithm to any base feature”.

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+, a \neq 1.$$

- (a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x -axis intercepts.

[4]

(This question continues on the following page)

(Question 1 continued)

In parts (b) and (c), consider the case where $a = e$. Note that $\ln x \equiv \log_e x$.

- (b)

Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a minimum.

[5]
- (c)

Hence deduce that $x > \ln x$.

[1]
- (d)

There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a .

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a , write down the values of p , q and r .

[4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

- (e)

Find the exact coordinates of P and the exact value of a .

[8]
- (f)

Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have

(i)

two intersection points;

[1]

(ii)

no intersection points.

[1]

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x \rightarrow 0} f(x) = 16$, find the value of k . [6]



Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Consider the function $f(x) = e^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

- (a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]
- (b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]
- (c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]
- (d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4 .
(ii) Hence, find the Maclaurin series for $e^{\cos 2x - 1}$, up to and including the term in x^4 .
(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4 . [6]
- (e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that

$$\int_0^{1/10} e^{\cos 2x} dx \approx \frac{149e}{1500}.$$
 [3]



Do **not** write solutions on this page.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x \rightarrow 0} f(x) = 16$, find the value of k . [6]



Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Consider the function $f(x) = e^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

- (a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]
- (b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]
- (c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]
- (d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4 .
(ii) Hence, find the Maclaurin series for $e^{\cos 2x - 1}$, up to and including the term in x^4 .
(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4 . [6]
- (e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that

$$\int_0^{1/10} e^{\cos 2x} dx \approx \frac{149e}{1500}.$$
 [3]



Do **not** write solutions on this page.

4. [Maximum mark: 5]

The particle first comes to rest after q seconds.

- (a) Find the value of q . [2]
- (b) Find the total distance that the particle travels in the first q seconds. [3]

[illegible]

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 9]

Consider the differential equation $\frac{dy}{dx} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

- (a) Use Euler’s method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]
- (b) By solving the differential equation, show that $y = 4 - 2e^{-\frac{x}{10}}$. [5]
- (c) Find the absolute value of the error in your approximation in part (a). [1]



Turn over

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4. [Maximum mark: 5]

The particle first comes to rest after q seconds.

- (a) Find the value of q . [2]
- (b) Find the total distance that the particle travels in the first q seconds. [3]

[illegible]

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 9]

Consider the differential equation $\frac{dy}{dx} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

- (a) Use Euler’s method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]
- (b) By solving the differential equation, show that $y = 4 - 2e^{-\frac{x}{10}}$. [5]
- (c) Find the absolute value of the error in your approximation in part (a). [1]



Turn over

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2. [Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α .

- (a) Consider a family of straight lines, L , with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C , at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

- (i) Write down an expression for the gradient of L in terms of x and y . [1]

- (ii) Hence show that the gradient of C is given by $\frac{dy}{dx} = -\frac{x}{y}$. [1]

- (iii) By solving the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, show that the family of curves, C , has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

- (b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x -intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

- (c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $M(a - b, 2\sqrt{ab})$ and $N(a - b, -2\sqrt{ab})$. [6]

- (d) At point M , show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

(This question continues on the following page)

(Question 2 continued)

Consider two families of curves, F and G .

The gradient of F is denoted by $f(x, y)$.

The gradient of G is denoted by $g(x, y)$.

Each member of F intersects every member of G at an acute angle, α .

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan \alpha}{1 - f(x, y) \tan \alpha}.$$

In part (e), consider the specific case where $f(x, y) = -\frac{x}{y}$, for $x \neq 0$, $y \neq 0$ and $\alpha = \frac{\pi}{4}$.

(e) (i) Show that $g(x, y) = \frac{y-x}{y+x}$. [2]

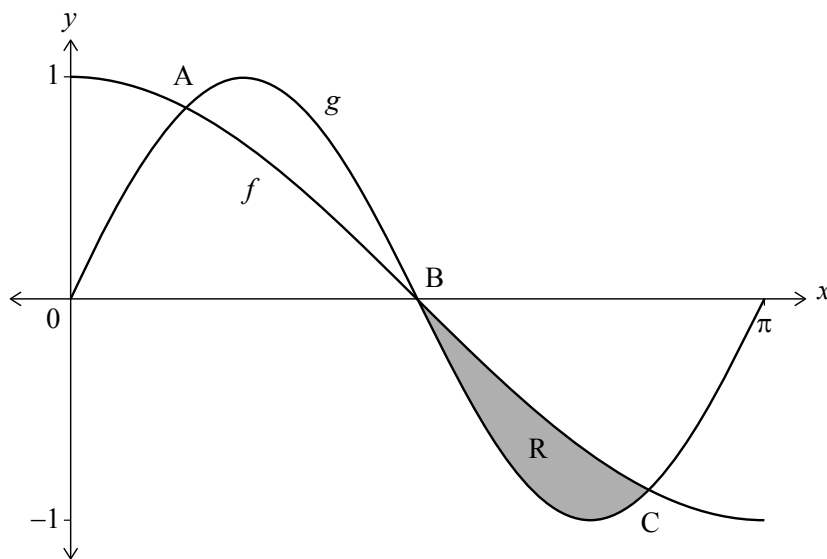
(ii) Hence, by solving the homogeneous differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, find a general equation that represents this family of curves, G . Give your answer in the form $h(x, y) = d$ where d is a parameter. [9]

(f) By considering $\lim_{\alpha \rightarrow \frac{\pi}{2}} \tan \alpha$, show that, for all finite $f(x, y)$,

$$\lim_{\alpha \rightarrow \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}. \quad [2]$$

4. [Maximum mark: 7]

The graph of f intersects the graph of g at the point A, the point B $\left(\frac{\pi}{2}, 0\right)$ and the point C as shown on the following diagram.



- (b) Find the area of R . [4]

[illegible]

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8. [Maximum mark: 7]

- (i) $\sin(x^2)$;

- (ii) $\sin^2(x^2)$. [5]

- (b) Hence, or otherwise, find the first two non-zero terms in the Maclaurin series of $4x \sin(x^2) \cos(x^2)$.

[illegible]

Turn over

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

Consider the polynomial $P(x) = 3x^3 + 5x^2 + x - 1$.

(a) Show that $(x + 1)$ is a factor of $P(x)$. [2]

(b) Hence, express $P(x)$ as a product of three linear factors. [3]

Now consider the polynomial $Q(x) = (x + 1)(2x + 1)$.

(c) Express $\frac{1}{Q(x)}$ in the form $\frac{A}{x+1} + \frac{B}{2x+1}$, where $A, B \in \mathbb{Z}$. [3]

(d) Hence, or otherwise, show that $\frac{1}{(x+1)Q(x)} = \frac{4}{2x+1} - \frac{2}{x+1} - \frac{1}{(x+1)^2}$. [2]

(e) Hence, find $\int \frac{1}{(x+1)^2(2x+1)} dx$. [4]

Consider the function defined by $f(x) = \frac{P(x)}{(x+1)Q(x)}$, where $x \neq -1$, $x \neq -\frac{1}{2}$.

(f) Find

(i) $\lim_{x \rightarrow -1} f(x)$;

(ii) $\lim_{x \rightarrow \infty} f(x)$. [3]



Turn over

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

- (a) Let $f(x) = (1 - ax)^{-\frac{1}{2}}$, where $ax < 1$, $a \neq 0$.

The n^{th} derivative of $f(x)$ is denoted by $f^{(n)}(x)$, $n \in \mathbb{Z}^+$.

Prove by induction that $f^{(n)}(x) = \frac{a^n (2n-1)! (1-ax)^{-\frac{2n+1}{2}}}{2^{2n-1} (n-1)!}$, $n \in \mathbb{Z}^+$. [8]

- (b) By using part (a) or otherwise, show that the Maclaurin series for $f(x) = (1 - ax)^{-\frac{1}{2}}$ up to and including the x^2 term is $1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2$. [2]

- (c) Hence, show that $(1 - 2x)^{-\frac{1}{2}}(1 - 4x)^{-\frac{1}{2}} \approx \frac{2 + 6x + 19x^2}{2}$. [4]

- (d) Given that the series expansion for $(1 - ax)^{-\frac{1}{2}}$ is convergent for $|ax| < 1$, state the restriction which must be placed on x for the approximation $(1 - 2x)^{-\frac{1}{2}}(1 - 4x)^{-\frac{1}{2}} \approx \frac{2 + 6x + 19x^2}{2}$ to be valid. [1]

- (e) Use $x = \frac{1}{10}$ to determine an approximate value for $\sqrt{3}$.

Give your answer in the form $\frac{c}{d}$, where $c, d \in \mathbb{Z}^+$. [5]



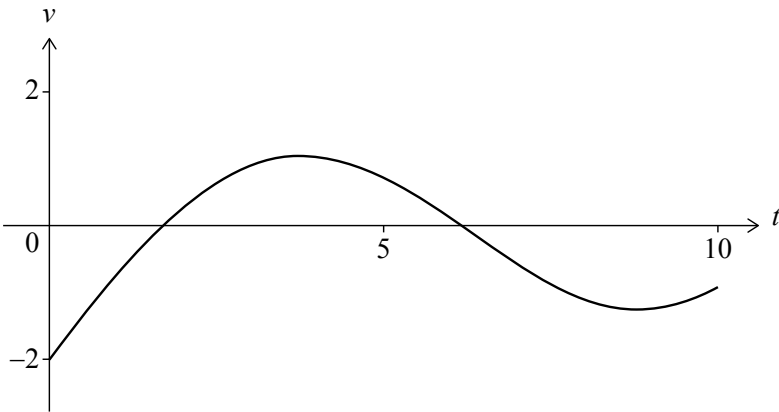
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

A particle moves in a straight line such that it passes through a fixed point O at time $t = 0$, where t represents time measured in seconds after passing O . For $0 \leq t \leq 10$ its velocity, v metres per second, is given by $v = 2 \sin(0.5t) + 0.3t - 2$.

The graph of v is shown in the following diagram.



(a) Find the smallest value of t when the particle changes direction. [2]

The displacement of the particle is measured in metres from O .

(b) Find the range of values of t for which the displacement of the particle is increasing. [2]

(c) Find the displacement of the particle relative to O when $t = 10$. [2]

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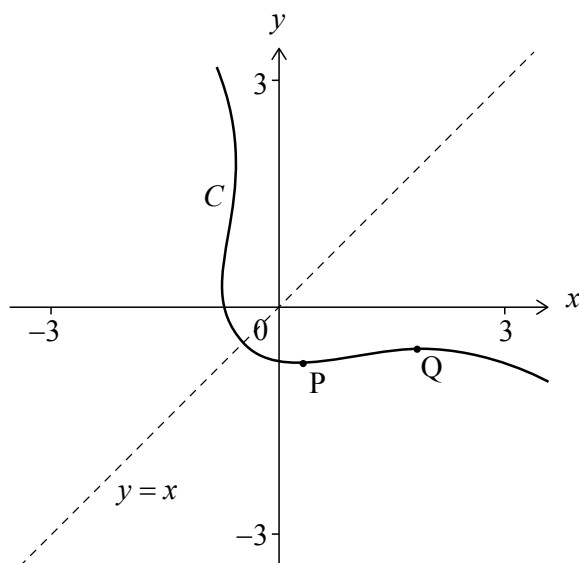
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the curve C defined by the equation $e^{x+y} = x^2 + y^2$, shown on the following diagram. The curve has a line of symmetry $y = x$.

There are two points on the curve C where the tangent is horizontal. These points are labelled P and Q.



- (a) Show that $\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - 2y}$. [5]
- (b) (i) Show that the x -coordinates of points P and Q satisfy the equation $2x^2 + (\ln(2x))^2 - 2x \ln(2x) - 2x = 0$. [9]
- (ii) Hence, find the coordinates of P and the coordinates of Q. [1]
- (c) Using the line of symmetry, write down the coordinates of the points on the curve C where the tangent is vertical. [4]
- (d) Find the coordinates of the point on the curve C where the tangent has a gradient of -1 . [4]



Do **not** write solutions on this page.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The functions f and g are both defined for $-1 \leq x \leq 0$ by

$$f(x) = 1 - x^2$$

$$g(x) = e^{2x}.$$

The graphs of f and g intersect at $x = a$ and $x = b$, where $a < b$.

- (a) Find the value of a and the value of b . [3]
- (b) Find the area of the region enclosed by the graphs of f and g . [3]

[illegible]

[illegible]

9. [Maximum mark: 8]

Find the values of a , b and c .

[illegible]

— 11 —

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} - y \operatorname{cosec} 2x = \sqrt{\tan x}$, where $0 < x < \frac{\pi}{2}$ and $y = \frac{\pi}{4}$ at $x = \frac{\pi}{4}$.

- (a) Use Euler's method with step length $\frac{\pi}{12}$ to find an approximate value of y when $x = \frac{5\pi}{12}$.

Give your answer correct to three significant figures. [3]

- (b) Show that $\frac{d}{dx} \left(\frac{1}{2} \ln(\cot x) \right) = -\operatorname{cosec} 2x$. [4]

- (c) Show that $\sqrt{\cot x}$ is an integrating factor for this differential equation. [4]

- (d) Hence, by solving the differential equation, show that $y = x\sqrt{\tan x}$. [5]

- (e) Consider the curve $y = x\sqrt{\tan x}$ for $0 < x < \frac{\pi}{2}$ and the Euler's method approximation calculated in part (a).

- (i) Find the y -coordinate at $x = \frac{5\pi}{12}$. Give your answer correct to three significant figures.

- (ii) By considering the gradient of the curve, suggest a reason why Euler's method does not give a good approximation for the y -coordinate at $x = \frac{5\pi}{12}$.

- (iii) State why this approximation is less than the y -coordinate at $x = \frac{5\pi}{12}$. [3]

- (f) By considering $\frac{dy}{dx} - y \operatorname{cosec} 2x = \sqrt{\tan x}$, deduce that the curve $y = x\sqrt{\tan x}$ has a positive gradient for $0 < x < \frac{\pi}{2}$. [2]



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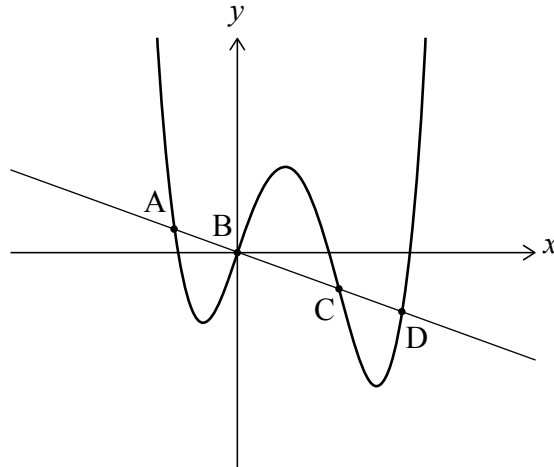
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2. [Maximum mark: 28]

This question investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.

The curve $y = x^4 - 3x^3 + 3x$ has points of inflexion at B and C. The line passing through B and C intersects the curve again at points A and D. This is shown in the following graph.



- (a) Find $\frac{d^2y}{dx^2}$. [3]
- (b) Find the coordinates of B and C. [4]
- (c) Show that the equation of the line through B and C is $y = -0.375x$. [2]
- (d) Find, correct to three decimal places, the x -coordinate of D. [2]

Now consider the general curve $y = x^4 - mx^3 + nx$, where $m, n \in \mathbb{R}$ and $m > 0$.

- (e) Find the x -coordinates of the two points of inflexion in terms of m . [3]

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x -coordinate of point A, and similarly for x_B , x_C and x_D . It is given that $x_A < x_B < x_C < x_D$.

- (f) (i) Write down the coordinates of B. [1]
- (ii) Find, in terms of m and n , the coordinates of C. [2]

(This question continues on the following page)

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(Question 2 continued)

(g) Show that the equation of the line through B and C is $y = \left(-\frac{m^3}{8} + n\right)x$. [2]

(h) Show that $x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$. [7]

(i) Hence, find the exact value of $\frac{x_B - x_A}{x_C - x_B}$. [2]

1. [Maximum mark: 24]

If two functions $f(x)$ and $g(x)$ are differentiable, then their product is differentiable and the two functions satisfy the product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$.

In this question, you will meet examples of pairs of differentiable functions, $f(x)$ and $g(x)$, that also satisfy $(f(x)g(x))' = f'(x)g'(x)$.

In part (a), consider $f(x) = \frac{1}{(2-x)^2}$, where $x \in \mathbb{R}$, $x \neq 2$, and $g(x) = x^2$, where $x \in \mathbb{R}$.

(a) (i) Find an expression for $f'(x)$. [2]

(ii) Show that $f'(x)g'(x) = \frac{4x}{(2-x)^3}$. [2]

(iii) Show that $f(x)g'(x) + g(x)f'(x) = \frac{4x}{(2-x)^3}$. [4]

In parts (b) and (c), consider two non-constant functions, $f(x)$ and $g(x)$, where $f(x) > 0$ and $g(x) \neq g'(x)$.

(b) By rearranging the equation $f(x)g'(x) + g(x)f'(x) = f'(x)g'(x)$, show that $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$. [2]

(This question continues on the following page)

(Question 1 continued)

- (c) Hence, by integrating both sides of $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$, show that $f(x) = Ae^{\left(\int \frac{g'(x)}{g'(x) - g(x)} dx\right)}$, where A is an arbitrary positive constant. [2]

The result from part (c) can be used to find pairs of functions, $f(x)$ and $g(x)$, which satisfy **both** of the following:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \text{ and } (f(x)g(x))' = f'(x)g'(x).$$

In parts (d) and (e), use the result in part (c) with $A = 1$.

- (d) Consider $g(x) = xe^x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations. [5]

- (e) Consider $g(x) = \sin x + \cos x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations over the domain $0 < x < \pi$.

Give your answer in the form $f(x) = \sqrt{e^x h(x)}$, where $h(x)$ is a function to be determined. [7]

Turn over

7. [Maximum mark: 7]

(a) Determine the range of f . [3]

(b) Find the volume of revolution generated. [4]

[illegible]

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Section B

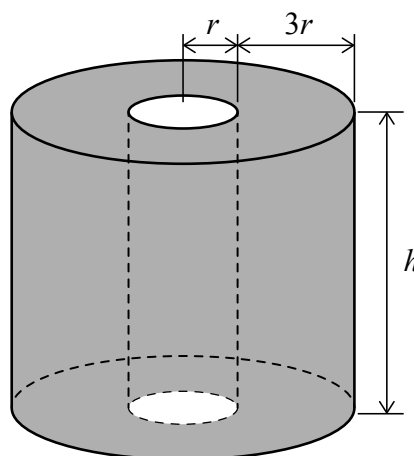
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

Consider a cylinder of radius $4r$ and height h . A smaller cylinder of radius r is removed from the centre to form a hollow cylinder. This is shown in the following diagram.

All lengths are measured in centimetres.

diagram not to scale



The total surface area of the hollow cylinder, in cm^2 , is given by S .

The volume of the hollow cylinder, in cm^3 , is given by V .

(a) Show that $S = 30\pi r^2 + 10\pi r h$. [3]

(b) The total surface area of the hollow cylinder is $240\pi \text{ cm}^2$.

Show that $V = 360\pi r - 45\pi r^3$. [6]

(c) Find an expression for $\frac{dV}{dr}$. [2]

The hollow cylinder has its maximum volume when $r = p\sqrt{\frac{2}{3}}$, where $p \in \mathbb{Z}^+$.

(d) Find the value of p . [3]

(e) Hence, find this maximum volume, giving your answer in the form $q\pi\sqrt{\frac{2}{3}}$, where $q \in \mathbb{Z}^+$. [3]



Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

A curve is given by the equation $y = \frac{e^{2x} - 1}{e^{2x} + 1}$, $x \in \mathbb{R}$.

(a) By applying l'Hôpital's rule or otherwise, show that $\lim_{x \rightarrow \infty} \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right) = 1$. [2]

(b) (i) Show that $\frac{dy}{dx} = \frac{4e^{2x}}{(e^{2x} + 1)^2}$.

(ii) Hence, show that $1 - y^2 = \frac{dy}{dx}$. [6]

(c) (i) By using implicit differentiation and the result in part (b)(ii), show that $\frac{d^2y}{dx^2} = 2y^3 - 2y$.

(ii) Hence, find an expression for $\frac{d^3y}{dx^3}$ in terms of y . [5]

(d) By using your results from parts (b) and (c), find the Maclaurin series for $\frac{e^{2x} - 1}{e^{2x} + 1}$ up to and including the term in x^3 . [4]



Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The following table shows the population of Canada t years after the year 2000.

t (years after 2000)	0	5	10	15	20
p (population in millions)	30.7	32.2	34.0	35.7	37.9

A student uses linear regression to model the population of Canada using these data. The student model is $p = at + b$.

- (a) (i) Write down the value of a and the value of b .
- (ii) Interpret, in context, the value of a . [3]

The student uses this model to predict the population of Canada in the year 2030, where $t = 30$, and calculates a population of approximately 41.3 million people.

- (b) Comment on the reliability of the student's prediction. [1]

A data scientist, Benoit, uses additional information to develop an exponential model for Canada's future population.

In this model, $B(t) = 33.5(1.005)^t$ represents the millions of people in Canada t years after the year 2000, where $25 \leq t \leq 100$.

- (c) (i) Use Benoit's model to predict the population of Canada in the year 2100.
- (ii) Interpret, in context, the value 1.005 in Benoit's model. [3]

(This question continues on the following page)



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(Question 10 continued)

Another data scientist, Cecilia, develops a third model for the Canadian population.

In this model, $C(t) = \frac{62}{1 + e^{-0.02t}}$ represents the millions of people in Canada t years after the year 2000, where $25 \leq t \leq 100$.

- (d) Use Cecilia's model to predict the population of Canada in the year 2100. [1]
- (e) Determine the year in which the difference between the predictions from Benoit's model and Cecilia's model is greatest. [3]
- (f) Find the value of
 - (i) $B'(75)$;
 - (ii) $C'(75)$. [2]
- (g) Compare and interpret, in context, the values of $B'(75)$ and $C'(75)$. [2]



Turn over

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1. [Maximum mark: 5]

The graph of $y = f(x)$ has a local minimum point at (p, q) where $p > 0$.

Find the value of p and the value of q .

[illegible]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

Consider the family of functions $f_n(x) = \cos^n x$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

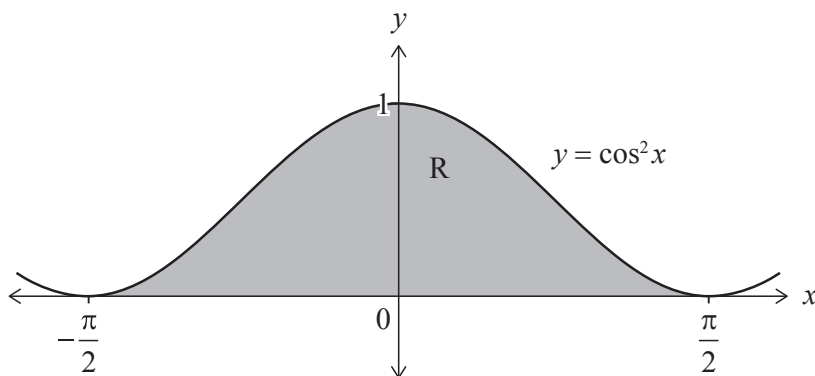
(a) By writing $\cos^n x$ as $\cos^{n-1} x \cos x$, show that

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \text{ for } n > 1. \quad [4]$$

(b) Hence, show that $\int f_n(x) \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int f_{n-2}(x) \, dx$ for $n > 1$. [2]

(c) Hence, find an expression for $\int \cos^4 x \, dx$, giving your answer in the form $p \cos^3 x \sin x + q \cos x \sin x + rx + c$ where $p, q, r \in \mathbb{Q}^+$. [4]

The region R is enclosed by the graph of $y = \cos^2 x$ and the x -axis where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in the following diagram.



The region R is rotated by 2π radians around the x -axis to form a solid of revolution.

(d) Find the volume of the solid. [4]

(e) (i) Find the Maclaurin series of $f_n(x)$ up to the term in x^2 .

(ii) Hence or otherwise, find $\lim_{x \rightarrow 0} \frac{f_n(x) - 1}{x^2}$ in terms of n . [5]



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6. [Maximum mark: 7]

(a) Show that the distance, l , between the origin and any point on the graph of f is given by $l = \sqrt{x^2 \ln x + 4}$. [1]

(b) Hence, find the x -coordinate of the point on the graph of f which is closest to the origin. [6]

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3. [Maximum mark: 6]

Based on this model, find the predicted population at the start of 2026.

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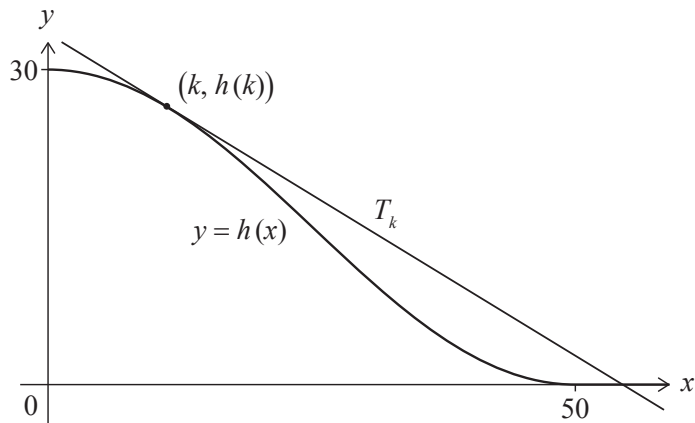
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

Consider the function $h(x) = 15 \cos\left(\frac{\pi x}{50}\right) + 15$, where $0 \leq x \leq 50$.

The tangent, T_k , to the curve $y = h(x)$ at the point $(k, h(k))$ is shown on the following diagram.



(a) Find the gradient of T_k in terms of k . [3]

Consider the case where the angle between T_k and the x -axis is $\frac{\pi}{8}$ radians.

(b) Find the possible values of k . [3]

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Turn over

9. [Maximum mark: 6]

The solution curve passes through the point $(1, 0)$.

- (a) Use Euler's method with a step value of 0.25 to estimate the value of y when $x = 2$. [3]

- (b) (i) Determine whether your answer to part (a) is an overestimate or an underestimate, justifying your answer.
- (ii) Justify why the use of Euler's method starting at $(1, 0)$ does not lead to an estimate of the negative value of y when $x = 2$.
- [3]

[illegible]

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3. [Maximum mark: 6]

(a) Find the values of x for which f is decreasing. [3]

(b) Find the values of x for which the graph of f is concave-up. [3]

[illegible]

Section B

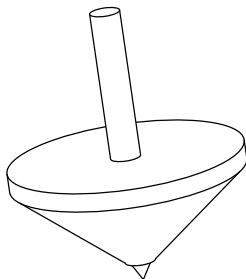
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 14]

A mathematics class of 15 students plays a game which requires three equal size teams.

- (a) Find the total number of ways that the three teams can be chosen. [3]

The game involves the spinning of a top.



The time, T , in minutes that the spinning top is in motion can be modelled by the probability density function f where

$$f(t) = \begin{cases} kte^{-3t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $k \in \mathbb{Z}^+$.

- (b) Show that $\int_0^a f(t) dt = \frac{k}{9} [1 - (3a+1)e^{-3a}]$, where $a \in \mathbb{R}^+$. [4]

- (c) (i) Use l'Hôpital's rule to find $\lim_{x \rightarrow \infty} (3x+1)e^{-3x}$.

- (ii) Hence, by considering $\lim_{a \rightarrow \infty} \int_0^a f(t) dt$, find the value of k . [5]

- (d) Find the median length of time that a spinning top is in motion. [2]



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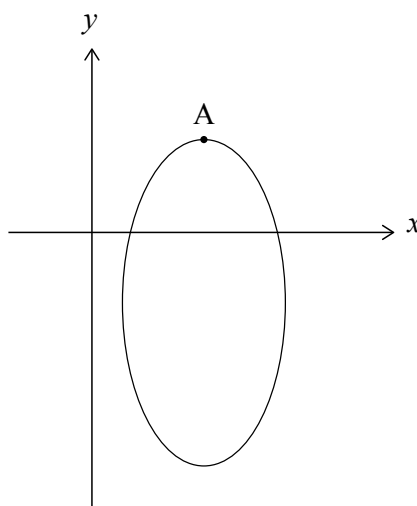
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

The curve C has equation $4x^2 + y^2 - 24x + 4y + 20 = 0$.

The following diagram shows C with a maximum point at A .



(a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{4(3-x)}{y+2}$. [4]

(b) Hence, determine the domain of C . Give your answer in the form $3 - \sqrt{a} \leq x \leq 3 + \sqrt{a}$, where $a \in \mathbb{Z}^+$. [4]

(c) Find (x_A, y_A) , the coordinates of A . [3]

A line $y = mx$ is a tangent to C , where $m \in \mathbb{Z}$.

(d) Find the possible values of m . [4]

The line $y = -4x$ touches C at point B .

(e) Find y_B , the y -coordinate of B . [3]

The region bounded by the curve C , the y -axis and the lines $y = y_A$ and $y = y_B$, is rotated 360° about the y -axis to form a solid of revolution.

(f) Find the volume of the solid formed. [4]



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5. [Maximum mark: 7]

(a) Find an expression for $f'(x)$, in terms of a . [3]

(b) Find the two possible values of a . [4]

[illegible]

9. [Maximum mark: 9]

This is shown in the following diagram.

The diagram shows an airplane at point P , which is 6 km above the ground. The ground is represented by a horizontal line with points G , A , and B marked. A vertical dashed line connects P to G , with a right-angle symbol at G . The distance PG is labeled as 6 . The distance between A and B is labeled as 2 . The distance from G to A is labeled as x . The angle of elevation from A to P is labeled as θ .

- (b) Find the speed of the airplane. [7]

[illegible]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

Two athletes, Fiona and Lucy, compete in a 200 metres race along a straight track.

Fiona's velocity, in m s^{-1} , during the race can be modelled by $v(t) = \frac{8.14t}{\sqrt{t^2 + 0.2}}$, where $t \geq 0$. Time, t , is measured in seconds from when the race starts.

- (a) (i) Write down the value of $v(1)$.
- (ii) Find the time when Fiona's velocity is 5 m s^{-1} . [3]
- (b) Find the time when Fiona's acceleration is 4 m s^{-2} . [2]
- (c) (i) Write down the limit of $v(t)$ as t approaches infinity.
- (ii) State a reason why the value in part (c)(i) is not valid in the context of this question. [3]

Lucy's velocity, in m s^{-1} , during the race can be modelled by $w(t) = \frac{8t}{\sqrt{t^2 + 0.3}}$, where $t \geq 0$.

Fiona completes the race and crosses the finishing line in front of Lucy.

- (d) Find the distance Lucy is from the finishing line when Fiona completes the 200 metres. [6]



Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

- (a) Find $\int (x^2 - 5)e^x dx$. [6]

Consider the differential equation $\frac{dy}{dx} = x^2 - y - 5$.

- (b) By solving the differential equation, show that its solution can be expressed in the form $y = x^2 - 2x - 3 + Ce^{-x}$, where C is a constant. [4]

- (c) Sketch the curve of the particular solution which passes through the point $(-3, 2)$, for $-4 \leq x \leq 4$, clearly labelling the coordinates of any local maximum and minimum points. [5]

Consider the family of curves that are solutions of the differential equation.

The tangent at $x = -3$ is drawn for each of these curves.

- (d) By considering the curve which passes through the point $(-3, p)$ and the curve which passes through the point $(-3, q)$, where $p, q \in \mathbb{R}$, $p \neq q$, show that all these tangents intersect at a common point, and state its coordinates. [6]



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2. [Maximum mark: 32]

Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.

Consider any function f that can be differentiated twice.

The curvature, k , of any function f is defined by $k(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}}$.

Consider the family of linear functions $g(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

- (a) Show that $k(x) = 0$ for this family of linear functions. [2]

Consider the family of quadratic functions $h(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b, c \in \mathbb{R}$.

For this family of quadratic functions, it is given that

$$k(x) = \frac{2|a|}{\left(1 + (2ax + b)^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = -\frac{12a|a|(2ax + b)}{\left(1 + (2ax + b)^2\right)^{\frac{5}{2}}}.$$

The maximum value of $k(x)$ is denoted by $k_{\max}$.

- (b) (i) By solving $k'(x) = 0$, find the value of x where $k_{\max}$ occurs. [1]
You are **not** required to justify that this value of x leads to a maximum.
- (ii) Determine an expression for $k_{\max}$, in terms of a only. [2]
- (iii) State the value of $\lim_{x \rightarrow \infty} k(x)$ and explain briefly the significance of this result. [2]
- (iv) Consider the quadratic functions

$$p(x) = -2x^2 + 2x - 10 \text{ where } x \in \mathbb{R} \text{ and}$$

$$q(x) = 2x^2 + 5x + 25 \text{ where } x \in \mathbb{R}.$$

State which one of the following statements is true and justify your answer.

A. $k_{\max}$ of $p > k_{\max}$ of q

B. $k_{\max}$ of $p < k_{\max}$ of q

C. $k_{\max}$ of $p = k_{\max}$ of q

[2]

(This question continues on the following page)

(Question 2 continued)

Consider the function $v(x) = \ln x$, where $x \in \mathbb{R}$, $x > 0$.

For v , it is given that

$$k(x) = \frac{x}{(1+x^2)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}.$$

- (c) (i) Determine the exact value of x where $k_{\max}$ occurs.

You are **not** required to justify that this value of x leads to a maximum. [2]

- (ii) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [4]

Consider the function $w(x) = e^x$, where $x \in \mathbb{R}$.

For w , it is given that

$$k(x) = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{e^x(1-2e^{2x})}{(1+e^{2x})^{\frac{5}{2}}}.$$

- (d) (i) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [5]

- (ii) Suggest a reason why v and w have the same $k_{\max}$. [1]

Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r$, $y > 0$ and r is a positive constant.

- (e) (i) Show that $\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}$. [6]

- (ii) Hence, show that the curvature, k , is constant for this family of curves. [5]

1. [Maximum mark: 28]

The following question explores features of a family of curves. The family is then linked to a homogeneous differential equation.

Consider the curve given by $y = \frac{x(x^2 - 16)}{x^2 + 16}$.

- (a) (i) Sketch the curve of y for $-10 \leq x \leq 10$. [1]
- (ii) State the coordinates of the points where the curve crosses the x -axis. [1]
- (iii) State the coordinates of the local maximum point and the coordinates of the local minimum point. [2]
- (b) State whether the function $f(x) = \frac{x(x^2 - 16)}{x^2 + 16}$ is odd, even or neither. Justify your answer. [2]

Now consider the general curve given by $y = \frac{x(x^2 - A)}{x^2 + A}$, where A is a positive constant and $x \in \mathbb{R}$.

- (c) Given $f(x) = \frac{x(x^2 - A)}{x^2 + A}$, prove that $f'(\sqrt{A})$ is independent of A . [4]
- (d) (i) Show that $x - \frac{2Ax}{x^2 + A} \equiv \frac{x(x^2 - A)}{x^2 + A}$. [2]
- (ii) Hence, determine the equation of the oblique asymptote to the curve. [1]
- (iii) Write down the coordinates of a point on the curve where the oblique asymptote is parallel to the tangent to the curve at that point. [1]

(This question continues on the following page)

(Question 1 continued)

Now consider the differential equation $x^2 \frac{dy}{dx} = x(x+y) - y^2$, where $x \neq 0$, $y \neq \pm x$.

Using the substitution $y = vx$, the differential equation can be written as $x \frac{dv}{dx} = 1 - v^2$.

(e) Using partial fractions, show that $\int \frac{1}{1-v^2} dv = \frac{1}{2} \ln \left| \frac{A(1+v)}{1-v} \right|$, where A is a positive constant. [5]

(f) Hence, show that a solution to the original differential equation may be expressed in the form $x^2 = \left| \frac{A(x+y)}{x-y} \right|$, where A is a positive constant. [5]

Now consider only the case where $\frac{A(x+y)}{x-y} > 0$.

(g) Show that a solution to the original differential equation is $y = \frac{x(x^2 - A)}{x^2 + A}$. [4]

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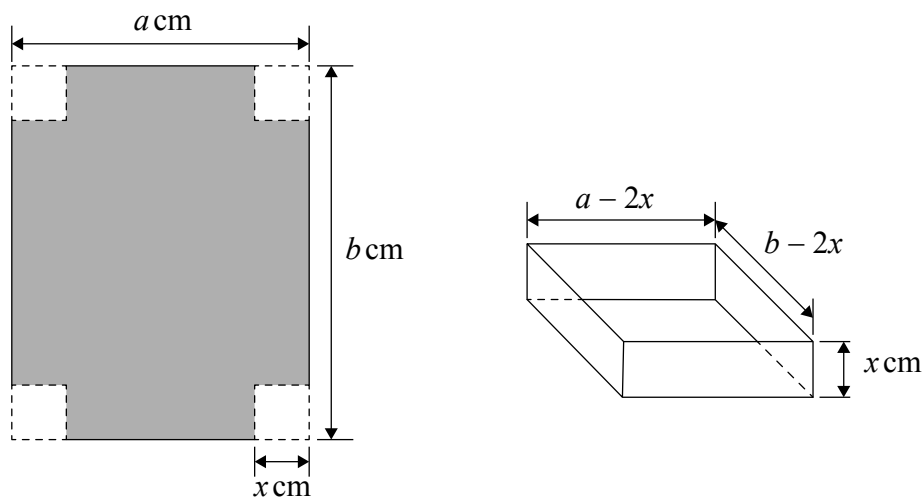
2. [Maximum mark: 28]

This question asks you to consider open-topped boxes formed by cutting square corners from an $a \times b$ sheet of cardboard and folding the sides. In particular, you are asked to investigate conditions for which the maximum volume of a box is an integer.

A rectangular sheet of cardboard measures a cm by b cm where $a < b$.

A square of side length x cm is cut from each corner and the resulting sheet of cardboard is then folded to form an open-topped box with a rectangular base.

This is shown in the following diagrams.



The volume, $V \text{ cm}^3$, of a box is given by $V = x(a - 2x)(b - 2x)$ where $0 \leq x \leq \frac{a}{2}$.

(a) Write down the value of V when

(i) $x = 0$; [1]

(ii) $x = \frac{a}{2}$. [1]

(b) Show that $V = 4x^3 - 2(a + b)x^2 + abx$. [2]

(c) Show that the only solutions of $\frac{dV}{dx} = 0$ are $x = \frac{(a + b) \pm \sqrt{a^2 - ab + b^2}}{6}$. [6]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), the following three results are given.

$$x = \frac{(a+b) + \sqrt{a^2 - ab + b^2}}{6} \text{ lies outside the interval } 0 \leq x \leq \frac{a}{2}.$$

$$x = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6} \text{ lies inside the interval } 0 \leq x \leq \frac{a}{2}.$$

$$\sqrt{a^2 - ab + b^2} > 0.$$

Let $x_m = \frac{(a+b) - \sqrt{a^2 - ab + b^2}}{6}.$

(d) Use $\frac{d^2V}{dx^2}$ to show that there is a local maximum of V at $x = x_m$. [5]

(e) Use your answers to parts (a) and (d) to explain why x_m will give the maximum volume, V_m , of the box. [1]

In parts (f) to (h), consider a method for generating sets of positive integer values for x_m and hence V_m .

Consider the case where $a = 2st - s^2$, $b = t^2 - s^2$, and $s, t \in \mathbb{Z}^+$.

(f) Given that $a < b$, show that $t > 2s$. [2]

It is given that $\sqrt{a^2 - ab + b^2} = s^2 - st + t^2$. You are **not** required to show this result.

(g) By substituting for each of a , b and $\sqrt{a^2 - ab + b^2}$ in terms of s and t , show that $x_m = \frac{s(t-s)}{2}$. [3]

(h) (i) Given that $a, b \in \mathbb{Z}^+$, use $V_m = x_m(a - 2x_m)(b - 2x_m)$ to explain why s being even implies that V_m is an integer. [2]

(ii) State another condition on s and a corresponding condition on t , that implies that both x_m and V_m are integers. [1]

(iii) Hence, or otherwise, find an integer value of x_m and the corresponding integer value of V_m . [4]

Section A

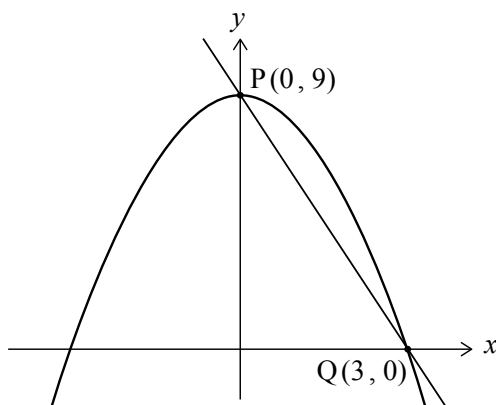
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

A line with equation $y = -3x + 9$ intersects the axes at the points $P(0, 9)$ and $Q(3, 0)$.

A parabola of the form $y = ax^2 + c$, where $a, c \in \mathbb{Z}$, also passes through the points P and Q .

This is shown in the following diagram.

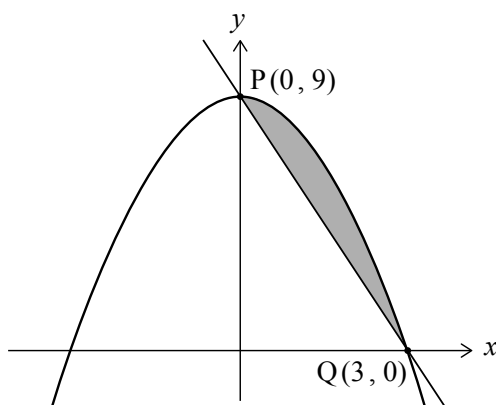


(a) (i) Write down the value of c .

(ii) Find the value of a .

[3]

The region enclosed by the line and the parabola is shaded in the following diagram.



(b) Find the area of the shaded region.

[4]

(This question continues on the following page)



(Question 3 continued)

[illegible]

Turn over

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The first four terms of the Maclaurin series expansion of $(1 - x)^{-4}$ are

$$1 + ax + bx^2 + 20x^3, \quad a, b \in \mathbb{Z}^+.$$

- (a) (i) Show that $a = 4$.
- (ii) Find the value of b . [5]

A car was purchased four years ago. The car depreciated in value by 10% each year. The value of the car today is \$ 1000.

- (b) Using the results of part (a), estimate the value of the car four years ago. [2]

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Turn over

7. [Maximum mark: 8]

$$\frac{dy}{dx} = y^2 \cos^2 x, y \neq 0$$
$$y = \frac{a}{2x + \sin 2x + b},$$

(a) By solving the differential equation, find the value of a . [6]

(b) Given that $y = 1$ when $x = \frac{\pi}{2}$, find the value of b . [2]

[illegible]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

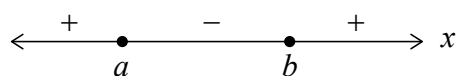
9. [Maximum mark: 14]

The function f has a derivative given by $f'(x) = 3x^2 + 12x - 15$.

The graph of $y = f(x)$ has horizontal tangents at the points where $x = a$ and $x = b$, $a < b$.

(a) Find the value of a and the value of b . [3]

The following diagram shows three intervals along the x -axis defined by a and b . The sign of the **first derivative** of f is shown in each interval.

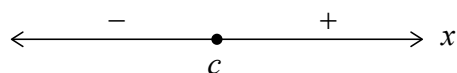


(b) State, with a reason, whether there is a local maximum point or a local minimum point on the graph of $y = f(x)$ at $x = a$. [2]

The second derivative $f''(x)$ is zero at $x = c$.

(c) Find the value of c . [3]

The following diagram shows two intervals along the x -axis defined by c . The sign of the **second derivative** is shown in each interval.



(d) State, with a reason, whether there is a point of inflexion on the graph of $y = f(x)$ at $x = c$. [2]

(e) Given that $f(-2) = 36$, find $f(x)$. [4]



Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the expression $\frac{1}{1+x^2}$, where $|x| < 1$.

(a) (i) Show that the first four terms in the binomial expansion of $\frac{1}{1+x^2}$ are $1 - x^2 + x^4 - x^6$. [3]

(ii) Hence, find an approximation for $\int \frac{1}{1+x^2} dx$ up to and including the term in x^7 . [2]

Now consider the expression $\frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.

(b) (i) Expand $\frac{1}{\sqrt{1-x^2}}$ up to and including the term in x^4 .

(ii) Hence, find a polynomial expression for $\int \frac{1}{\sqrt{1-x^2}} dx$ up to and including the term in x^5 . [5]

The expression $\frac{25}{48} + \frac{k}{1280}$, where $k \in \mathbb{Z}^+$, can be used to approximate $\arcsin \frac{1}{2}$.

(c) Use the result from part (b)(ii) to find the value of k . [4]

(d) Hence, find an approximation for π in the form $\frac{p}{640}$ where $p \in \mathbb{Z}^+$. [4]



Do **not** write solutions on this page.

6. [Maximum mark: 7]

Express your answer in the form $y = f(x)$.

[illegible]

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

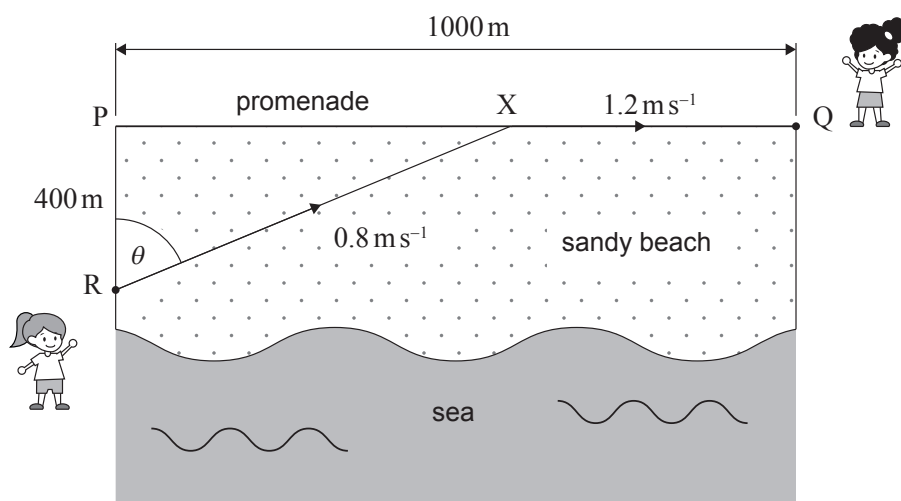
8. [Maximum mark: 9]

Astrid and Bronwyn are on vacation at Blackpool beach.

Astrid is standing on the beach at point R and sees Bronwyn standing on the promenade at point Q. $PR = 400\text{ m}$, $PQ = 1000\text{ m}$, $\hat{P}RX = \theta$ where $0 < \tan \theta \leq \frac{5}{2}$.

Astrid walks in a straight line from R with speed 0.8 m s^{-1} until reaching point X on the promenade. She then jogs along the promenade with speed 1.2 m s^{-1} .

This is shown in the following diagram.



It is given that $RX = 400 \sec \theta$ and $XQ = 1000 - 400 \tan \theta$.

The time, in seconds, for Astrid to reach Bronwyn is given by T .

(a) Show that $T = 500 \sec \theta + \frac{2500 - 1000 \tan \theta}{3}$. [2]

(b) Find $\frac{dT}{d\theta}$. [3]

Astrid chooses the angle θ of her walk across the beach to point X, in order to reach Bronwyn in the shortest time possible. You may assume that T has exactly one minimum value.

(c) Show that in this case $PX = 160\sqrt{5}\text{ m}$. [4]

(This question continues on the following page)



[illegible]

Turn over

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 15]

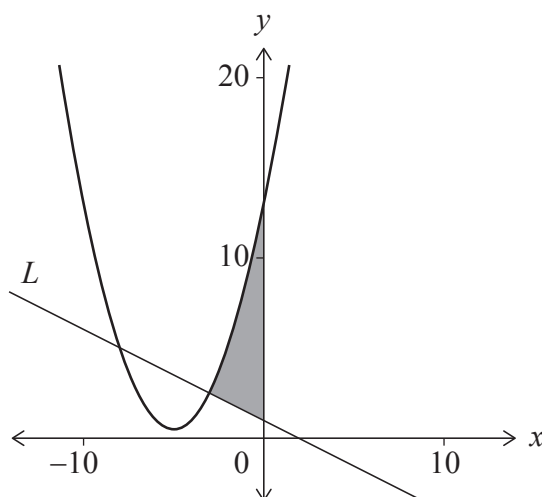
Consider the function defined by $f(x) = \frac{1}{2}x^2 + kx + 13$, where $x \in \mathbb{R}$ and $k \in \mathbb{Z}^+$.

- (a) Given that the equation $f(x) = 0$ has no real roots, show that the greatest possible value of k is 5. [2]

For the remainder of this question, consider the case $k = 5$.

- (b) (i) Write down the equation of the axis of symmetry of the graph of f .
 (ii) Hence, or otherwise, determine the coordinates of the minimum point on the graph of f . [3]

The following diagram shows the graph of f and a line L which is normal to the curve at $x = -3$. The shaded area shown is bounded by the curve, the line L and the y -axis.



- (c) Show that the equation of L is given by $y = -\frac{1}{2}x + 1$. [5]
 (d) Hence, find the shaded area. [5]



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Section A

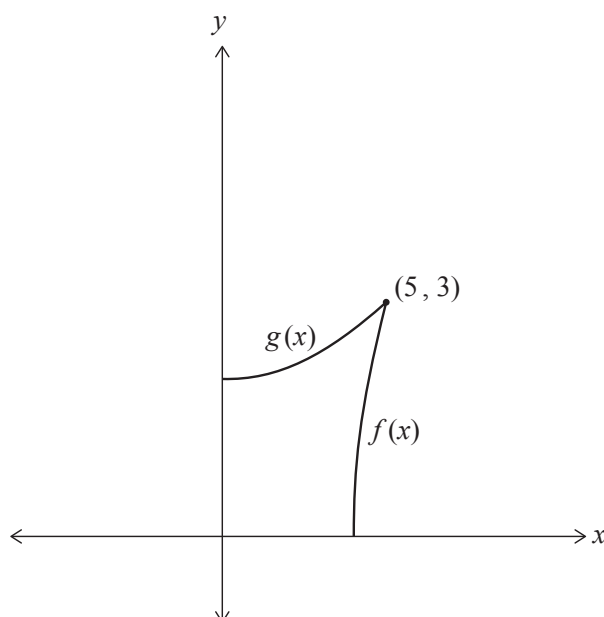
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

The following diagram shows the graphs of two functions, f and g , where

$$f(x) = \sqrt{x^2 - 16} \text{ for } 4 \leq x \leq 5, \text{ and}$$

$$g(x) = 2 + \frac{x^2}{25} \text{ for } 0 \leq x \leq 5.$$



The graphs of f and g intersect at the point $(5, 3)$.

The region bounded by the graphs of f , g and both axes is rotated 360° about the y -axis to form a solid of revolution.

Find the volume of the solid formed.

(This question continues on the following page)



(Question 8 continued)

[illegible]

Turn over

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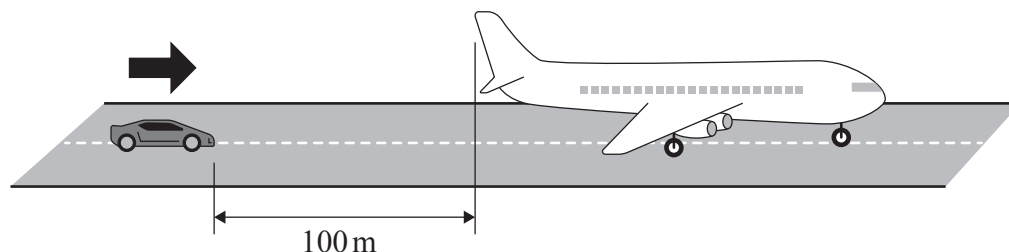
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

An airplane lands on a runway 100 metres in front of a stationary car. At the instant the airplane lands, the car begins to travel in the same direction towards the airplane.

diagram not to scale



Let t represent the number of seconds after the airplane lands. For $t \geq 0$, the velocities of the airplane and the car in m s^{-1} , can be modelled by the following equations:

$$v_{\text{air}} = 60e^{-0.1t}$$

$$v_{\text{car}} = 5t$$

(a) When the airplane lands, write down the speed of

(i) the airplane;

(ii) the car.

[2]

(b) Find

(i) the value of t when the airplane and the car have the same speed;

(ii) the speed at this time.

[3]

Let $d(t)$ represent the distance, in metres, between the car and the back of the airplane after t seconds.

(c) If $d(0) = 100$, find $d(t)$.

[7]

(d) Hence, find how long it takes for the car to reach the back of the airplane.

[2]

(e) Find the distance travelled by the car when it reaches the back of the airplane.

[3]



Turn over

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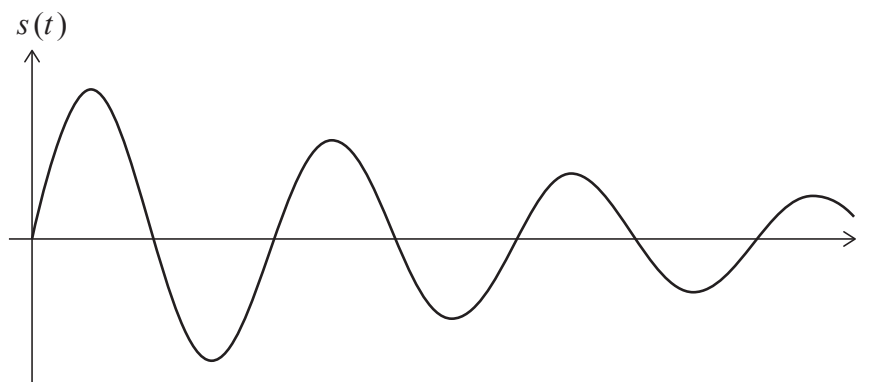
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

A particle P moves in a straight line so that its displacement, s cm, from a fixed point O at time t seconds is given by $s(t) = 2^{\left(1 - \frac{t}{5}\right)} \sin\left(\frac{2\pi t}{3}\right)$, where $t \geq 0$.

The following diagram shows part of the graph of $y = s(t)$.



(a) Find

- (i) the maximum displacement of P from O ;
- (ii) the maximum velocity of P .

[5]

(b) Find

- (i) the minimum value of the displacement function $s(t)$;
- (ii) the displacement of P from O when $t = 3.5$.

[3]

(c) Hence, determine the **total distance** travelled by P in the first 3.5 seconds.

[3]

The first time that P returns and passes through O is when $t = T$.

(d) Write down the value of T .

[1]

(This question continues on the following page)



Do **not** write solutions on this page.

(Question 10 continued)

The particle passes through O every T seconds.

A sequence $u_1, u_2, u_3 \dots$ is formed where $u_1, u_2, u_3 \dots$ are the largest **distances** from O in each of the intervals $0 < t < T$, $T < t < 2T$, $2T < t < 3T \dots$ respectively.

It is known that $u_1, u_2, u_3 \dots$ form a geometric sequence.

- (e) (i) Determine the value of the common ratio r of this geometric sequence.
- (ii) Calculate the **total distance** travelled by the particle if it were to continue to move in this way indefinitely.

[5]



Turn over

— 17 —

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

Consider the following homogenous differential equation

$$(x^2 + xy) \frac{dy}{dx} = x^2 + xy - 3y^2 \text{ where } x > 0 \text{ and } y > \frac{x}{2}.$$

It is given that $y = \frac{3}{2}$ when $x = 1$.

- (a) (i) Find the value of $\frac{dy}{dx}$ when $x = 1$.
- (ii) Use Euler's method with two equal steps to estimate the value of y when $x = 1.4$.
- (iii) Hence, state the concavity of the solution curve for $1 \leq x \leq 1.4$. You may assume that the concavity does not change in this interval. Give a reason for your answer. [6]

- (b) (i) Show that

$$(x^2 + xy) \frac{d^2y}{dx^2} = 2x + y - x \left(\frac{dy}{dx} \right)^2 - (x + 7y) \frac{dy}{dx}.$$

- (ii) Find the value of $\frac{d^2y}{dx^2}$ when $x = 1$. [6]

- (c) Determine constants $A, B \in \mathbb{R}$ such that $\frac{1+v}{1-4v^2} \equiv \frac{A}{1-2v} + \frac{B}{1+2v}$. [2]

- (d) By solving the differential equation $(x^2 + xy) \frac{dy}{dx} = x^2 + xy - 3y^2$ where $x > 0$, $y > \frac{x}{2}$ and $y = \frac{3}{2}$ when $x = 1$, show that $x^6(2y - x)^3 = 2(x + 2y)$. [8]



Please **do not** write on this page.

Answers written on this page
will not be marked.



1. [Maximum mark: 26]

This question asks you to investigate lines normal to curves of the form $y = \frac{k^2}{x}$.

The curve H has equation $y = \frac{1}{x}$ where $x \in \mathbb{R}, x \neq 0$.

(a) A line N is normal to H at $x = t$.

(i) Show that the gradient of N is t^2 . [2]

(ii) Hence, show that the equation of N is $y = t^2x + \frac{1}{t} - t^3$. [1]

(b) The equation for N given in part (a)(ii) is of the form $y = mx + c$.

(i) Show that either $c = \frac{1}{\sqrt{m}}(1 - m^2)$ or $c = \frac{1}{\sqrt{m}}(m^2 - 1)$. [4]

(ii) Determine the set of values of m for which there exists at least one line normal to H . [1]

(c) Hence, or otherwise, determine the set of values of m for which there exists exactly

(i) one line normal to H ; [1]

(ii) two lines normal to H . [2]

(d) On the same set of axes, sketch

(i) the curve H ; [1]

(ii) for an appropriate value of m , two lines that satisfy the result found in part (c)(ii). Clearly indicate the point at which each line is normal to H . [2]

You are **not** required to state the equations of these lines nor determine where they intersect H or the coordinate axes.

(This question continues on the following page)

Turn over

(Question 1 continued)

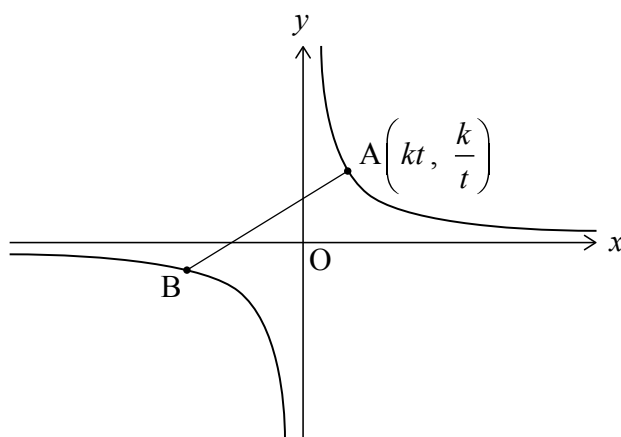
The curve F has equation $y = \frac{k^2}{x}$ where $x \in \mathbb{R}, x \neq 0$ and $k \in \mathbb{R}, k \neq 0$.

The point $A\left(kt, \frac{k}{t}\right)$, where $t \in \mathbb{R}, t \neq \pm 1$, lies on F .

The line normal to F at A intersects F again at point B .

The line segment $[AB]$ is shown in the following diagram.

diagram not to scale



(e) The equation of the line normal to F at A is given by $y = t^2x - kt^3 + \frac{k}{t}$.

(i) Show that the x -coordinates of A and B satisfy the quadratic equation

$$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0. \quad [3]$$

(ii) Hence, by considering either the sum or product of the roots of this quadratic equation, or otherwise, determine the coordinates of B . [3]

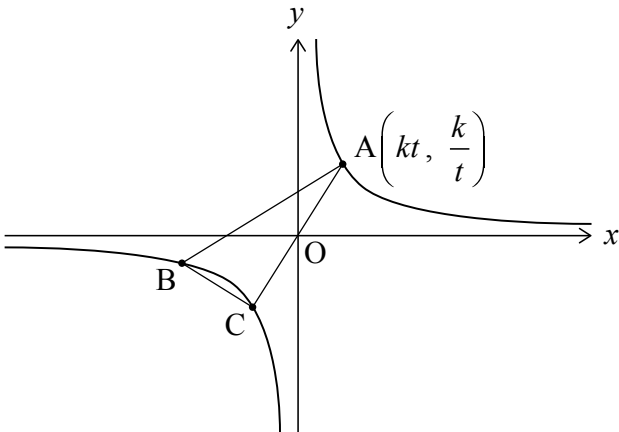
(This question continues on the following page)

(Question 1 continued)

From A , the line passing through the origin O intersects F again at point C .

Points A , B and C form triangle ABC as shown in the following diagram.

diagram not to scale



(f) Prove that $\widehat{BCA}$ is a right angle.

[6]

2. [Maximum mark: 29]

In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population, x , as a function of t , where t is the time, in years, since the area became protected.

(a) Initially, the researchers consider using the logistic model

$$x = \frac{L}{1 + Ce^{-kt}}, \text{ where } L, C, k \in \mathbb{R}^+.$$

The researchers decide to let $L = 200$.

(i) State the assumption being made in assuming $L = 200$. [1]

At $t = 0$, the population of wolves is 40.

(ii) Find the value of C . [2]

At $t = 5$, the population of wolves is found to have increased to 70.

(iii) Find the value of k . [2]

(iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

(b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$$\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right), \quad a \in \mathbb{R}^+.$$

(i) Write down the value of $\frac{dx}{dt}$ when $x = 200$. [1]

(ii) Interpret your answer to part (b)(i) in context. [1]

(This question continues on the following page)

(Question 2 continued)

Consider the function $f(x) = \ln(\ln 200 - \ln x)$, where $0 < x < 200$.

(iii) Show that $f'(x) = \frac{-1}{x \ln\left(\frac{200}{x}\right)}$. [2]

(iv) Hence, use separation of variables to show that the general solution of

$$\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right), \text{ where } 0 < x < 200,$$

can be written as

$$\ln x = \ln 200 - Ae^{-at},$$

where A is an arbitrary positive constant. [5]

(v) Use the size of the wolf population at $t = 0$ to find the value of A .
Give your answer in the form $A = \ln p$, where $p \in \mathbb{Z}^+$. [2]

(vi) Use the size of the wolf population at $t = 5$, given in part (a), to show that $a = 0.0855$, correct to three significant figures. [2]

(vii) Use the Gompertz model to predict the size of the wolf population at $t = 10$.
Give your answer correct to the nearest whole number. [3]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$$\frac{dx}{dt} = 0.0855x \ln\left(\frac{200}{x}\right) - 0.03x.$$

(d) (i) Use Euler's method, with a step size of 0.5 years and an initial value of $x_0 = 70$ when $t = 5$, to find an estimate for the size of the wolf population when $t = 10$.
Give your answer correct to the nearest whole number. [4]

(ii) Comment on your answer. [1]

1. [Maximum mark: 26]

The following question explores features of composed trigonometric functions, such as $\sin(\sin x)$, $\sin(\sin(\sin x))$.

Suppose $S_n(x)$ denotes the function $\sin x$ composed within itself $n - 1$ times, defined for $n \geq 1$, $n \in \mathbb{Z}^+$, where $0 \leq x \leq 2\pi$.

For example, $S_1(x) = \sin x$ and $S_2(x) = \sin(\sin x)$ where $0 \leq x \leq 2\pi$.

- (a) On the same axes, sketch and label the graphs of $y = S_1(x)$ and $y = S_2(x)$. On your sketch, show the values of the intercepts with the axes. [4]
- (b) Determine the maximum value of
- (i) $S_1(x)$; [1]
 - (ii) $S_2(x)$; [1]
 - (iii) $S_3(x)$. [1]
- (c) Find the least value of n for which the maximum value of $S_n(x)$ is less than 0.6. [3]

Consider the graph of $y = S_2(x)$.

- (d) By considering the equation $\frac{dy}{dx} = 0$, show that there are exactly two points of zero gradient, one at $x = \frac{\pi}{2}$ and one at $x = \frac{3\pi}{2}$. [6]

(This question continues on the following page)

(Question 1 continued)

The derivative $S'_n(x) = \frac{d}{dx}(S_n(x))$ can be expressed as a product of cosine functions, as follows:

$$S'_n(x) = \cos(S_{n-1}(x)) \cos(S_{n-2}(x)) \dots \cos(S_1(x)) \cos x.$$

(e) Hence, show that $S'_3(x) = \cos(\sin(\sin x)) \cos(\sin x) \cos x$. [1]

(f) Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$

$$S'_n(x) = \cos(S_{n-1}(x)) \cos(S_{n-2}(x)) \dots \cos(S_1(x)) \cos x. \quad [6]$$

(g) Use l'Hôpital's rule to show that $\lim_{x \rightarrow 0} \frac{S_n(x)}{x} = 1$, for $n \in \mathbb{Z}^+$. [3]

Turn over

2. [Maximum mark: 29]

The following question uses Maclaurin series to investigate approximations of mathematical constants and the accuracies of such approximations.

(a) Given $|x| < 1$, find the sum to infinity of the geometric series $1 - x^2 + x^4 - x^6 + \dots$. [2]

(b) Hence, use integration to show that the Maclaurin series of $\arctan x$ may be expressed as $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$. [3]

(c) Using $x = \frac{1}{\sqrt{3}}$ and the first **three** (non-zero) terms of the Maclaurin series of $\arctan x$, find an approximation for π to three decimal places. [3]

The Maclaurin series of $\arctan x$ is an example of an alternating series, ie a series where consecutive terms are positive and negative. Consider the following theorem.

Theorem: For alternating series with terms of decreasing magnitude, the error obtained in using a finite number of terms is less than or equal to the absolute value of the next term in the sequence.

Using the theorem, the maximum error in using the first three (non-zero) terms as an approximation to $\arctan x$ is given by $\left| -\frac{x^7}{7} \right|$. In other words, $\left| \arctan x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} \right) \right| \leq \left| -\frac{x^7}{7} \right|$.

(d) Determine how many (non-zero) terms of the series would need to be used, such that the error in approximating $\arctan\left(\frac{1}{\sqrt{3}}\right)$ is less than 0.0001. [3]

(e) By using integration by parts, show that $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx = \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}$. [4]

(f) Determine the value of $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \right) dx$, giving your answer to six decimal places. [2]

(g) Use the results in parts (e) and (f) to find an approximation for π . Give your answer to four decimal places. [2]

(This question continues on the following page)

(Question 2 continued)

$\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) dx$ may be considered as the sum of alternating terms.

Hence, $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) dx = \int_0^{\frac{1}{\sqrt{3}}} x \, dx + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^3}{3} \right) dx + \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^5}{5} \right) dx + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^7}{7} \right) dx + \dots$

$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx$ is approximated using the sum of the first four definite integrals.

(h) Verify that the theorem used in part (d) holds in this case. [5]

Suppose that the maximum error in approximating $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx$ is required to be at most 1×10^{-6} .

(i) Determine the smallest number of (non-zero) terms of the Maclaurin series for $\arctan x$ that should be used. [5]