APOLLO

Mathematics: analysis and approaches HL Workbook

First assessment 2021
Whole questions arranged by syllabus topic.
Original question and scheme crops preserve source diagrams.

APOLLO

Number and algebra

102 questions

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider an arithmetic sequence where $u_8 = S_8 = 8$. Find the value of the first term, u_1, and the value of the common difference, d.

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, $z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where a, $b \in \mathbb{Z}$.

Find the possible values of a.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

A farmer has six sheep pens, arranged in a grid with three rows and two columns as shown in the following diagram.



Five sheep called Amber, Brownie, Curly, Daisy and Eden are to be placed in the pens. Each pen is large enough to hold all of the sheep. Amber and Brownie are known to fight.

Find the number of ways of placing the sheep in the pens in each of the following cases:

(a) Each pen is large enough to contain five sheep. Amber and Brownie must not be placed in the same pen. [4]

(b) Each pen may only contain one sheep. Amber and Brownie must not be placed in pens which share a boundary. [4]

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Consider two consecutive positive integers, n and $n+1$.

Show that the difference of their squares is equal to the sum of the two integers.

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

In the expansion of $(x+k)^7$, where $k \in \mathbb{R}$, the coefficient of the term in x^5 is 63.

Find the possible values of k.

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

Consider the expansion of $(3+x^2)^{n+1}$, where $n \in \mathbb{Z}^+$.

Given that the coefficient of x^4 is 20412, find the value of n.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$ and $w = 8\left(\cos\frac{2k\pi}{5} - \mathrm{i}\sin\frac{2k\pi}{5}\right)$, where $k \in \mathbb{Z}^+$.

(a) Find the modulus of zw. [1]

(b) Find the argument of zw in terms of k. [2]

Suppose that $zw \in \mathbb{Z}$.

(c) (i) Find the minimum value of k.

(ii) For the value of k found in part (i), find the value of zw. [4]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

An arithmetic sequence has first term 60 and common difference -2.5.

(a) Given that the kth term of the sequence is zero, find the value of k. [2]

Let S_n denote the sum of the first n terms of the sequence.

(b) Find the maximum value of S_n. [3]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

Eight runners compete in a race where there are no tied finishes. Andrea and Jack are two of the eight competitors in this race.

Find the total number of possible ways in which the eight runners can finish if Jack finishes

(a) in the position immediately after Andrea; [2]

(b) in any position after Andrea. [3]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 5]

Consider $z = \cos\theta + \mathrm{i}\sin\theta$ where $z \in \mathbb{C}$, $z \neq 1$.

Show that $\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$.

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Question 8 - working space

2. [Maximum mark: 24]

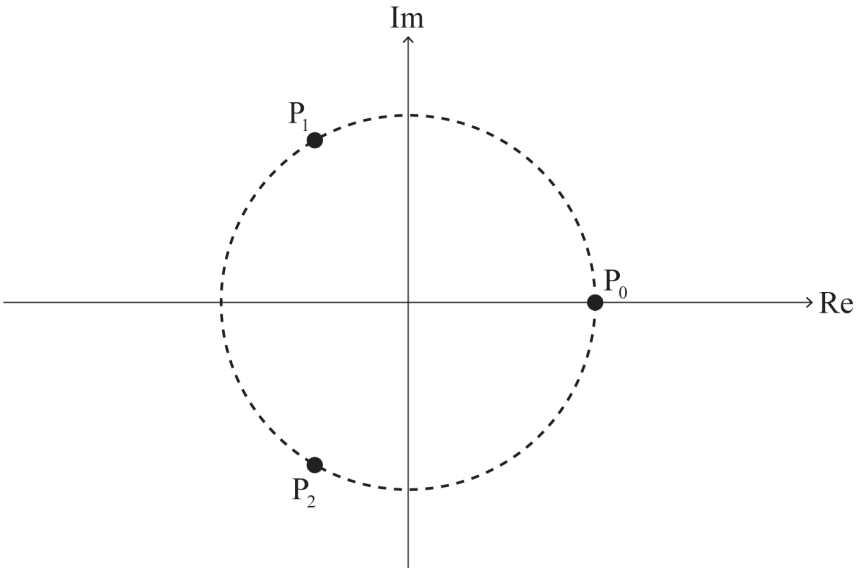
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2, ..., ω^{n-1}, where $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$. Each root can be represented by a point P_0, P_1, P_2, ..., P_{n-1}, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω. On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1.

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1, ω and ω^2. On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $\mathrm{O}(0, 0)$.



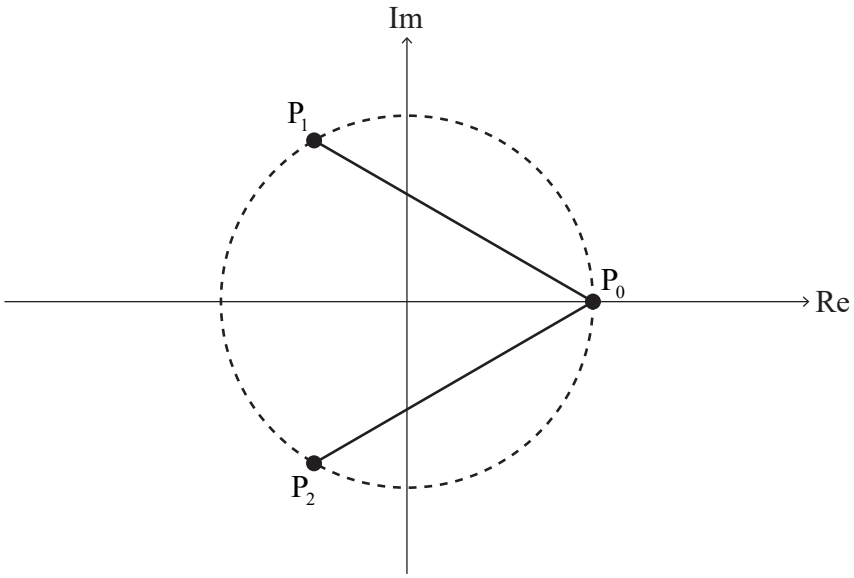
(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

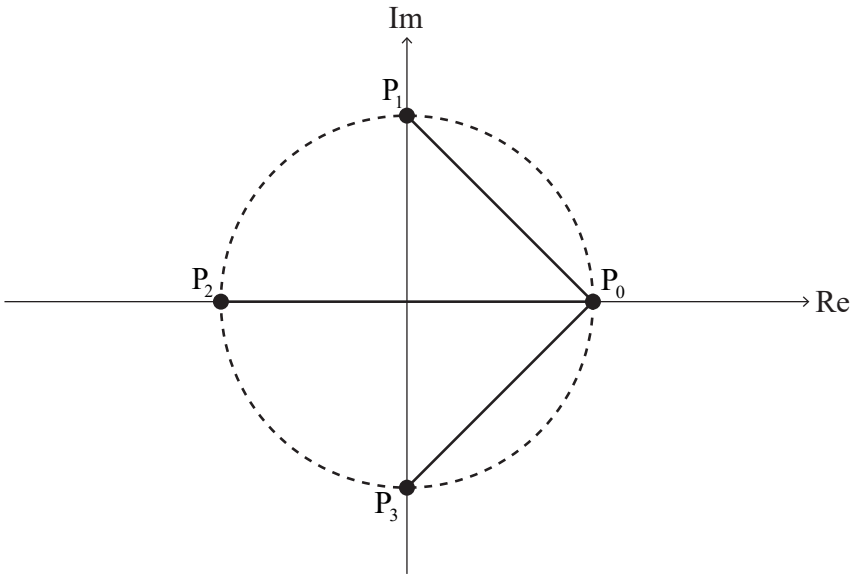
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2 and ω^3.

(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $O(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



(d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1, ω, ω^2, ω^3 and ω^4.

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2, ..., ω^{n-1}. On an Argand diagram, these roots can be represented by the points P_0, P_1, P_2, ..., P_{n-1} respectively where $[P_0P_1]$, $[P_0P_2]$, ..., $[P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $O(0, 0)$.

(e) Suggest a value for $P_0P_1 \times P_0P_2 \times ... \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω. [2]

(ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω. [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + ... + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + ... + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

References:

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q}$, $a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of a and the value of b. [6]

(b) State the restriction which must be placed on x for this expansion to be valid. [1]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are ω_1, ω_2 and ω_3, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find ω_2 and ω_3, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots ω_1, ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1. [2]

(b) Find S_∞. [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

(a) Prove the identity $(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$. [2]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β.

Consider the equation $x^2 + mx + n = 0$, where m, $n \in \mathbb{Z}$ and which has roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$.

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n. [6]

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Question 6 - working space

1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z)=\frac{e^{z}+e^{-z}}{2}$ and $g(z)=\frac{e^{z}-e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u, such that $t,\ u \in \mathbb{R}$.

(a) Verify that $u=f(t)$ satisfies the differential equation $\frac{d^2u}{dt^2}=u$. [2]

(b) Show that $(f(t))^2+(g(t))^2=f(2t)$. [3]

(c) Using $e^{iu}=\cos u+i\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

(i) $f(iu)$; [3]

(ii) $g(iu)$. [2]

(d) Hence find, and simplify, an expression for $(f(iu))^2+(g(iu))^2$. [2]

(e) Show that $(f(t))^2-(g(t))^2=(f(iu))^2-(g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta, \sin\theta)$ defines points on the unit circle with equation $x^2+y^2=1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2-y^2=1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2-y^2=1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2-y^2=1$ can be rotated to coincide with the curve defined by $xy=k,\ k \in \mathbb{R}$.

(g) Find the possible values of k. [5]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n \in \mathbb{Z}$. [2]

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

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Question 2 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 18]

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that
$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0.$ [4]

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

On 1st January 2020, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of 5.5%, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, r.

(a) Find the value of r, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

The complex numbers w and z satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 6]

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = (1 - b^2) - 2b\mathrm{i}$, where $b \in \mathbb{R}$, $b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the series $\ln x + p\ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R}, x > 1$ and $p \in \mathbb{R}, p \neq 0$.

(a) Consider the case where the series is geometric.

(i) Show that $p = \pm\frac{1}{\sqrt{3}}$.

(ii) Hence or otherwise, show that the series is convergent.

(iii) Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [6]

(b) Now consider the case where the series is arithmetic with common difference d.

(i) Show that $p = \frac{2}{3}$.

(ii) Write down d in the form $k\ln x$, where $k \in \mathbb{Q}$.

(iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n. [12]

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The nth term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, u_1. [1]

(b) Given that the nth term of this sequence is -33, find the value of n. [2]

(c) Find the common difference, d. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

Consider any three consecutive integers, $n-1$, n and $n+1$.

(a) Prove that the sum of these three integers is always divisible by 3. [2]

(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + ... + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

(b) Find the possible values of x. [5]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

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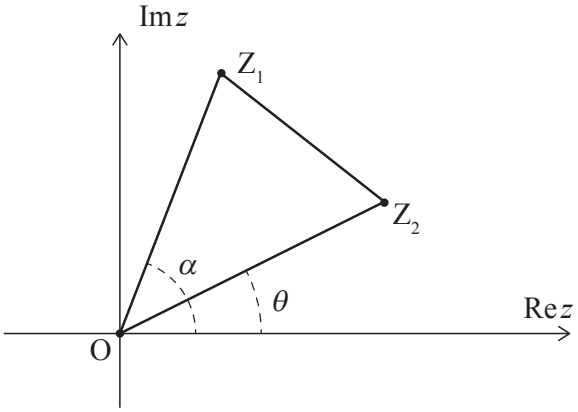
Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 18]

In the following Argand diagram, the points Z_1, O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles α, θ are measured anticlockwise from the positive direction of the real axis such that $0 \leq \alpha$, $\theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2^* = r_1 r_2 e^{i(\alpha - \theta)}$ where z_2^* is the complex conjugate of z_2. [2]

(b) Given that $\text{Re}(z_1 z_2^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2.

(ii) Hence show that $z_1^2 + z_2^2 = z_1 z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

Mary, three female friends, and her brother, Peter, attend the theatre. In the theatre there is a row of 10 empty seats. For the first half of the show, they decide to sit next to each other in this row.

(a) Find the number of ways these five people can be seated in this row. [3]

For the second half of the show, they return to the same row of 10 empty seats. The four girls decide to sit at least one seat apart from Peter. The four girls do not have to sit next to each other.

(b) Find the number of ways these five people can now be seated in this row. [4]

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 4]

Consider the set of six-digit positive integers that can be formed from the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

Find the total number of six-digit positive integers that can be formed such that

(a) the digits are distinct; [2]

(b) the digits are distinct and are in increasing order. [2]

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Question 9 - working space

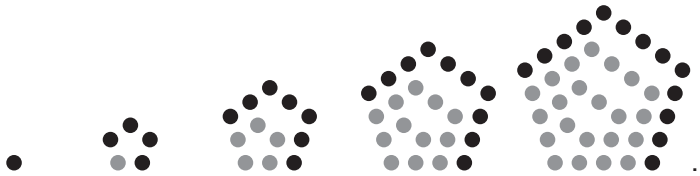
1. [Maximum marks: 27]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are $1, 3, 6, 10$ and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		1, 3, 6, 10, 15, ...
Square numbers		1, 4, 9, 16, 25, ...
Pentagonal numbers		1, 5, 12, 22, 35, ...

For an r-sided regular polygon, where $r \in \mathbb{Z}^+$, $r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

(This question continues on the following page)

(Question 1 continued)

Hence, for square numbers, $P_4(n) = \frac{(4-2)n^2 - (4-4)n}{2} = n^2$.

(a) (i) For triangular numbers, verify that $P_3(n) = \frac{n(n+1)}{2}$. [2]

(ii) The number 351 is a triangular number. Determine which one it is. [2]

(b) (i) Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. [2]

(ii) State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. [1]

(iii) For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). [1]

(c) Show that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3]

The nth pentagonal number can be represented by the arithmetic series

$$P_5(n) = 1 + 4 + 7 + ... + (3n - 2).$$

(d) Hence show that $P_5(n) = \frac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. [3]

(e) By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. [5]

A polygonal number, $P_r(n)$, can be represented by the series

$$\sum_{m=1}^{n} \left(1 + (m-1)(r-2)\right) \text{ where } r \in \mathbb{Z}^+,\ r \geq 3.$$

(f) Use mathematical induction to prove that $P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}$ where $n \in \mathbb{Z}^+$. [8]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

Let a be a constant, where $a > 1$.

(a) Show that $a^2 + \left(\frac{a^2-1}{2}\right)^2 = \left(\frac{a^2+1}{2}\right)^2$. [3]

Consider a right-angled triangle with sides of length a, $\left(\frac{a^2-1}{2}\right)$ and $\left(\frac{a^2+1}{2}\right)$.

(b) Find an expression for the area of the triangle in terms of a. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

(b) Find the value of p. [3]

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 18]

Let z_n be the complex number defined as $z_n = (n^2 + n + 1) + \mathrm{i}$ for $n \in \mathbb{N}$.

(a) (i) Find $\arg(z_0)$.

(ii) Write down an expression for $\arg(z_n)$ in terms of n. [3]

Let $w_n = z_0 z_1 z_2 z_3 \ldots z_{n-1} z_n$ for $n \in \mathbb{N}$.

(b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.

(ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]

(c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

A geometric sequence has a first term of 50 and a fourth term of 86.4.

The sum of the first n terms of the sequence is S_n.

Find the smallest value of n such that $S_n > 33\,500$.

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

Consider the expansion of $\frac{(ax+1)^9}{21x^2}$, where $a \neq 0$. The coefficient of the term in x^4 is $\frac{8}{7}a^5$.

Find the value of a.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a) (i) Find the sum of the first five terms.

(ii) Given that $S_6 = 60$, find u_6. [4]

(b) Find u_1. [2]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

(e) Given that $v_{99} < 0$, find v_5. [2]

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Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}\,\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The coefficient of x^6 in the expansion of $(ax^3 + b)^8$ is 448.

The coefficient of x^6 in the expansion of $(ax^3 + b)^{10}$ is 2880.

Find the value of a and the value of b, where $a, b > 0$.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Consider $z = \cos\frac{11\pi}{18} + \mathrm{i}\sin\frac{11\pi}{18}$.

(a) Find the smallest value of n that satisfies $z^n = -\mathrm{i}$, where $n \in \mathbb{Z}^+$. [4]

(b) Hence or otherwise, describe a single geometric transformation applied to z on the Argand diagram that results in z^{10}. [3]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 6]

Prove by contradiction that $p^2 - 8q - 11 \neq 0$, for any $p, q \in \mathbb{Z}$.

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

The expansion of $(x+h)^8$, where $h \in \mathbb{Q}^+$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

Given that the coefficients, a, b and d, are the first three terms of a geometric sequence, find the value of h.

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 5]

Let S be the set of 30 positive integers $\{1, 2, 3, ..., 28, 29, 30\}$.

Raghu randomly selects three positive integers from S without replacement. He then adds them together and determines whether the sum is divisible by 3.

Determine the total number of selections Raghu can make to obtain a sum that is divisible by 3.

You may assume that order is not important, for example, $\{1, 2, 3\}, \{1, 3, 2\}, \{2, 3, 1\}, \{2, 1, 3\}, \{3, 1, 2\}, \{3, 2, 1\}$ are all considered to be the same selection. [5]

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Question 9 - working space

2. [Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where $m,\ c \in \mathbb{R}$ and $m,\ c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c, in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

(a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

(b) (i) Show that $r = -\frac{c}{m}$. [1]

(ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \frac{m^2}{m+2}$. [4]

(iii) State any further restrictions on the value of m. [1]

There are only three **integer** sets of values of m, r and c, that form an AS-linear function. One of these is $L(x) = -x - 1$.

(c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c. [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R},\ a \neq 0$ and $b,\ c \in \mathbb{R}$.

Let $r_1,\ r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

(d) Write down an expression for

(i) the sum of roots, $r_1 + r_2$, in terms of a and b. [1]

(ii) the product of roots, $r_1 r_2$, in terms of a and c. [1]

(This question continues on the following page)

(Question 2 continued)

If a, r_1, b, r_2 and c, in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(i) write down an expression for $r_2 - r_1$ in terms of a and b; [1]

(ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \frac{a^2 - ab - b}{2a}$; [2]

(iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\frac{1}{2}$. [3]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where $a = -\frac{1}{2}$.

(g) (i) Find an expression for r_1 in terms of b. [2]

(ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\frac{-p \pm q\sqrt{s}}{2}$ where p, q, $s \in \mathbb{Z}^+$. [5]

References:

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

(a) Find the value of p and the value of q. [5]

(b) Find the value of u_5. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + ... + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by 17 for all $n \in \mathbb{Z}^+$.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

It is given that $z = 5 + q\mathrm{i}$ satisfies the equation $z^2 + \mathrm{i}z = -p + 25\mathrm{i}$, where $p, q \in \mathbb{R}$.

Find the value of p and the value of q.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

The sum of the first n terms of an arithmetic sequence is given by $S_n = pn^2 - qn$, where p and q are positive constants.

It is given that $S_4 = 40$ and $S_5 = 65$.

(a) Find the value of p and the value of q. [5]

(b) Find the value of u_5. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

The binomial expansion of $(1 + kx)^n$ is given by $1 + 12x + 28k^2x^2 + ... + k^nx^n$ where $n \in \mathbb{Z}^+$ and $k \in \mathbb{Q}$.

Find the value of n and the value of k.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Prove by mathematical induction that $5^{2n} - 2^{3n}$ is divisible by 17 for all $n \in \mathbb{Z}^+$.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

It is given that $z = 5 + q\mathrm{i}$ satisfies the equation $z^2 + \mathrm{i}z = -p + 25\mathrm{i}$, where $p, q \in \mathbb{R}$.

Find the value of p and the value of q.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

A junior baseball team consists of six boys and three girls.

The team members are to be placed in a line to have their photograph taken.

(a) In how many ways can the team members be placed if

(i) there are no restrictions;

(ii) the girls must be placed next to each other. [3]

(b) Five members of the team are selected to attend a baseball summer camp. Find the number of possible selections that contain at least two girls. [3]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

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The team members are to be placed in a line to have their photograph taken.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n - 1}{b}$, where $a, b \in \mathbb{Z}^+$. [1]

(b) Hence, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10\left(10^n - 1\right) - 9n}{81}$. [4]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

Using mathematical induction and the definition ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$, prove that $\sum_{r=1}^{n} {}^{r}C_{1} = {}^{n+1}C_{2}$ for all $n \in \mathbb{Z}^{+}$.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider $\phi = (a + b\mathrm{i})^3$, where $a, b \in \mathbb{R}$.

(a) In terms of a and b, find

(i) the real part of ϕ;

(ii) the imaginary part of ϕ. [3]

(b) Hence, or otherwise, show that $\left(1+\sqrt{3}\,\mathrm{i}\right)^3 = -8$. [2]

The roots of the equation $z^3 = -8$ are u, v and w, where $u = 1+\sqrt{3}\,\mathrm{i}$ and $v \in \mathbb{R}$.

(c) Write down v and w, giving your answers in Cartesian form. [2]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(d) Find the area of the triangle UVW. [3]

Each of the points U, V and W is rotated counter-clockwise (anticlockwise) about 0 through an angle of $\frac{\pi}{4}$ to form three new points U', V' and W'. These points represent the complex numbers u', v' and w' respectively.

(e) Find u', v' and w', giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$. [4]

(f) Given that u', v' and w' are the solutions of $z^3 = c + d\,\mathrm{i}$, where $c, d \in \mathbb{R}$, find the value of c and the value of d. [3]

It is given that u, v, w, u', v' and w' are all solutions of $z^n = \alpha$ for some $\alpha \in \mathbb{C}$, where $n \in \mathbb{N}$.

(g) Find the smallest positive value of n. [3]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

A function $g(x)$ is defined by $g(x) = 2x^3 - 7x^2 + dx - e$, where $d, e \in \mathbb{R}$.

α, β and γ are the three roots of the equation $g(x) = 0$ where $\alpha, \beta, \gamma \in \mathbb{R}$.

(a) Write down the value of $\alpha + \beta + \gamma$. [1]

A function $h(z)$ is defined by $h(z) = 2z^5 - 11z^4 + rz^3 + sz^2 + tz - 20$, where $r, s, t \in \mathbb{R}$.

α, β and γ are also roots of the equation $h(z) = 0$.

It is given that $h(z) = 0$ is satisfied by the complex number $z = p + 3\mathrm{i}$.

(b) Show that $p = 1$. [3]

It is now given that $h\left(\frac{1}{2}\right) = 0$, and $\alpha, \beta \in \mathbb{Z}^+$, $\alpha < \beta$ and $\gamma \in \mathbb{Q}$.

(c) (i) Find the value of the product $\alpha\beta$.

(ii) Write down the value of α and the value of β. [3]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

A teacher takes n students on a field trip. The students are assigned randomly into two groups.

For safety reasons there must be exactly three students in the first group and at least three students in the second group.

The teacher will randomly assign three students to the first group and the other students to the second group.

(a) Write down an expression for the number of ways that the students could be assigned. [1]

Two of the students ask the teacher not to work in the same group.

The teacher agrees and now finds that the number of ways to assign the students is halved.

(b) Determine the value of n. [6]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Consider the arithmetic sequence $a, p, q \ldots$, where $a, p, q \neq 0$.

(a) Show that $2p - q = a$. [2]

Consider the geometric sequence $a, s, t \ldots$, where $a, s, t \neq 0$.

(b) Show that $s^2 = at$. [2]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Consider the case where $a = 9$, $s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(i) arithmetic sequence;

(ii) geometric sequence. [4]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9$, $u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e) (i) Find the common difference of the new sequence in terms of $\ln 3$.

(ii) Show that $\sum_{i=1}^{10} u_i = -90 - 25\ln 3$. [6]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

Darren buys a car for $\$35\,000$. The value of the car decreases by 15% in the first year.

(a) Find the value of the car at the end of the first year. [2]

After the first year, the value of the car decreases by 11% in each subsequent year.

(b) Find the value of Darren's car 10 years after he buys it, giving your answer to the nearest dollar. [2]

When Darren has owned the car for n complete years, the value of the car is less than 10% of its original value.

(c) Find the least value of n. [3]

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Question 1 - working space

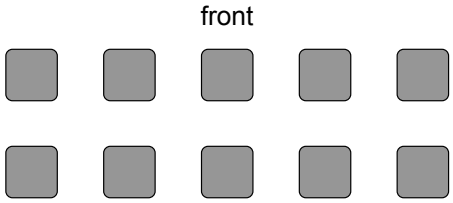
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

A group of 10 children includes one pair of brothers, Alvin and Bobby, and one pair of sisters, Catalina and Daniela.

The children are to be seated at 10 desks which are arranged in two rows of five as shown in the following diagram.



Alvin and Bobby must be seated next to each other in the same row.

(a) Find the total number of ways the children can be seated. [3]

After an argument, Catalina and Daniela must not be seated next to each other. Alvin and Bobby must still be seated next to each other.

(b) Find the total number of ways the children can be seated. [4]

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 10]

Let $z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Show that

(i) $\arg z = \theta$;

(ii) $|z| = 2\cos\theta$. [7]

(b) Hence or otherwise, find the value of θ such that $\arg(z^2) = |z^3|$. [3]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 4]

Prove that $(3n+2)^2-(3n-2)^2$ is a multiple of 12 for all $n \in \mathbb{Z}^+$.

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

For a particular arithmetic sequence, $u_{10} = 16$ and $S_{25} = 100$.

Find the value of k such that $u_k = 0$.

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider the equation $z^4 = 16\mathrm{i}$, where $z \in \mathbb{C}$.

The equation has four roots z_1, z_2, z_3, z_4, where $z_i = r(\cos\theta_i + \mathrm{i}\sin\theta_i)$, $r > 0$ and $0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$.

(a) Find z_1, z_2, z_3 and z_4. [6]

The roots z_1, z_2, z_3 and z_4 form a geometric sequence.

(b) Find the common ratio of the sequence, expressing your answer in Cartesian form. [3]

The roots z_1, z_2, z_3 and z_4 are represented by the points A, B, C and D respectively on an Argand diagram.

(c) Plot the points A, B, C and D on an Argand diagram. [3]

The equation $v^4 = a + b\mathrm{i}$, where $v \in \mathbb{C}$ and $a, b \in \mathbb{R}$ has roots z_1^*, z_2^*, z_3^* and z_4^*.

(d) Determine the value of a and the value of b. [3]

The midpoint of $[\mathrm{AB}]$ is A', the midpoint of $[\mathrm{BC}]$ is B', the midpoint of $[\mathrm{CD}]$ is C' and the midpoint of $[\mathrm{DA}]$ is D'.

Consider the equation $w^p = 2^q$, where $w \in \mathbb{C}$ and $p, q \in \mathbb{Z}^+$.

Four of the roots of $w^p = 2^q$ are represented by the points A', B', C' and D'.

(e) Find the least possible value of p and the corresponding value of q. [5]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 4]

Find the coefficient of x^6 in the expansion of $(2x - 5)^9$.

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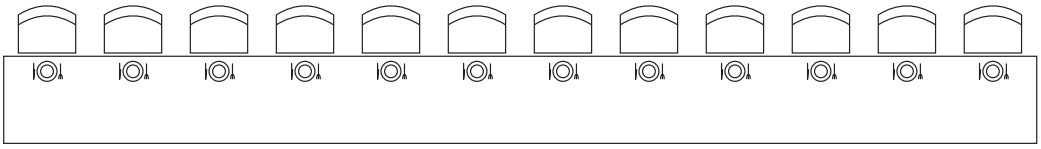
Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

A self-service sushi restaurant has a row of 12 available seats, as shown in the following diagram.



Anvi, Vanya and Parita decide to go to the restaurant for lunch.

(a) Find the number of possible ways that they can be seated in this row, if they decide **not** to sit together as a group of 3. [3]

The next day, Anvi, Vanya and Parita are joined by 3 additional people in the same restaurant, and they sit in the same row of 12 available seats. Anvi, Vanya and Parita now decide to sit next to each other as a group of 3.

(b) Find the number of possible ways that these 6 people can be seated. [3]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

(a) Given that $w \in \mathbb{C}$, prove that $ww^* = |w|^2$. [2]

Two complex numbers z and w satisfy the following equations:

$$5w^* = (1 - 2\mathrm{i})z^2$$

$$zw = 10 + 10\mathrm{i}.$$

(b) Given that $|w| = 2\sqrt{5}$, find z expressing your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$. [4]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=0}^{n-1} 5\left(\log_2 c\right)^r$.

(a) Given that S_n converges, find the range of possible values of c. [3]

(b) In the case where $c = 1.5$, find the least value of n such that $|S_\infty - S_n| < 0.1$. [4]

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Question 9 - working space

1. [Maximum mark: 27]

This question asks you to investigate models for the population of trout in a lake.

Trout is a type of fish. At the start of a year, a lake is estimated to contain 6000 trout.

The owner of the lake estimates that the number of trout will increase by $10\,\%$ per year.

At the end of each year, the owner proposes to remove 500 trout from the lake to prevent overpopulation.

Therefore, the relationship between T_n, the predicted number of trout at the start of year n, and T_{n+1}, the predicted number of trout at the start of year $n+1$, is given by

$$T_{n+1} = 1.1T_n - 500 \text{ and } T_1 = 6000.$$

For example, the predicted number of trout at the start of the second year is given by

$$T_2 = 1.1T_1 - 500.$$

(a) Use this formula to verify that $T_2 = 6100$. [2]

(b) (i) Verify that $T_3 = 6210$. [1]

(ii) Find T_4. [2]

It is also known that $T_n = 6000(1.1)^{n-1} - \dfrac{500\left((1.1)^{n-1} - 1\right)}{1.1 - 1}$.

(c) (i) Show that $T_n = 1000(1.1)^{n-1} + 5000$. [2]

(ii) Hence, or otherwise, find T_6. Give your answer to the nearest whole number. [2]

(This question continues on the following page)

(Question 1 continued)

After deciding that the trout population would increase too quickly, the lake owner proposes instead to remove 750 trout at the end of each year.

The relationship between D_n, the predicted number of trout at the start of year n, and D_{n+1}, the predicted number of trout at the start of year $n+1$, is now given by

$$D_{n+1} = 1.1D_n - 750 \text{ and } D_1 = 6000.$$

It is also known that $D_n = -1500(1.1)^{n-1} + 7500$.

(d) (i) Show that $D_{n+1} - D_n = -150(1.1)^{n-1}$. [3]

(ii) Use the result in part (d)(i) to deduce that the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year. [1]

(e) Determine the first year during which there will be no trout remaining in the lake. [4]

The lake owner now considers a more general approach where d trout are removed at the end of each year.

Let C_n denote the predicted number of trout in the lake at the start of the nth year where

$$C_n = 6000(1.1)^{n-1} - 10d\left((1.1)^{n-1} - 1\right).$$

(f) Find the value of d such that the predicted number of trout at the start of each year is constant. [3]

To model predicted numbers of trout, the lake owner has been using sequences generated by

$$u_{n+1} = ru_n - d, \text{ where } d, r \in \mathbb{R}^+ \text{ and } r \neq 1.$$

(g) Use mathematical induction to prove that $u_n = u_1 r^{n-1} - \dfrac{d\left(r^{n-1} - 1\right)}{r-1}$, for $n \in \mathbb{Z}^+$. [7]

Turn over

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 7]

Bob invests 1000 dinar in an account which pays a nominal annual interest rate of 4% compounded **quarterly**.

The amount of money in the account after one complete year can be written as $1000(1+k)^4$ where $k \in \mathbb{Q}$.

(a) Write down the value of k. [1]

(b) Expand and simplify $(1+x)^4$. [2]

(c) Hence or otherwise, find the amount of money in the account after one complete year, giving your answer correct to the nearest dinar. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 8]

Consider a sequence of ten rectangular picture frames $F_1, F_2, \ldots, F_9, F_{10}$.

Picture frame F_1 has width $4\,\text{cm}$ and height $5\,\text{cm}$.

The width and height of picture frame F_n, are each increased by $50\,\%$ to generate the width and height of the next picture frame F_{n+1}, for $n \in \mathbb{Z}^+,\ 1 \leq n \leq 9$.

(a) (i) Show that the area of picture frame F_n is $20\left(\frac{9}{4}\right)^{n-1}\text{cm}^2$.

(ii) Hence, find the mean area of the ten picture frames, giving your answer in the form $p\left(\left(\frac{9}{4}\right)^a - 1\right)\text{cm}^2$, where $p \in \mathbb{Q}^+,\ a \in \mathbb{Z}^+$. [5]

(b) Find the median area of the ten picture frames, giving your answer in the form $q\left(\frac{9}{4}\right)^4\text{cm}^2$, where $q \in \mathbb{Q}^+$. [3]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

Consider the complex number $z = 3^{\mathrm{i}-1}$.

(a) Write the integer 3 in the form e^{a} where $a \in \mathbb{R}$. [1]

(b) Hence, giving your answers in the form $p\cos(\ln q)$ where $p, q \in \mathbb{Q}^{+}$, find

(i) $\mathrm{Re}(z)$;

(ii) $\mathrm{Re}\left(\frac{1}{z}\right)$. [4]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Seema claims that $n > \log_2 n$ for $n \in \mathbb{Z}^+$.

(a) Show that $1 + \log_2 n \geq \log_2 (n+1)$ for $n \in \mathbb{Z}^+$. [2]

(b) Use mathematical induction and the result from part (a) to prove that Seema's claim is valid. [5]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 6]

Prove by contradiction that $\frac{1}{x(1-x)} \geq 4$ for $x \in \mathbb{R}$, $0 < x < 1$.

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

(a) Find the first four terms in the binomial expansion of $\sqrt{1+5x}$ in ascending powers of x. [4]

Consider the expression $(1+px)(1+qx)^{-1}$, where $p, q \in \mathbb{Q}$.

(b) Find the expansion of $(1+px)(1+qx)^{-1}$ in ascending powers of x, up to and including the term in x^2. [3]

The expansions found in parts (a) and (b) are identical up to the first three terms, for a value of p and a value of q.

(c) Show that $q = \frac{5}{4}$. [4]

(d) The expression $\frac{1+px}{1+qx}$, with $p = \frac{15}{4}$ and $q = \frac{5}{4}$, can be used as an approximation for $\sqrt{1+5x}$ where $|x| < \frac{1}{5}$.

(i) Hence, by finding a suitable value for x, find the approximation for $\sqrt{1.2}$ in the form $\frac{m}{n}$, where $m, n \in \mathbb{Z}$.

(ii) Now consider the approximation for $\frac{\sqrt{5}}{2}$. Explain why the approximation for $\frac{\sqrt{5}}{2}$ is not as accurate as the approximation for $\sqrt{1.2}$. [6]

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Do **not** write solutions on this page.

Question 11 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

(a) Solve $z^2 = -1 - \sqrt{3}\,\mathrm{i}$, giving your answers in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$. [4]

Let z_1 and z_2 be the square roots of $-1 - \sqrt{3}\,\mathrm{i}$, where $\mathrm{Re}(z_1) > 0$.

Let z_3 and z_4 be the square roots of $-1 + \sqrt{3}\,\mathrm{i}$, where $\mathrm{Re}(z_3) > 0$.

(b) Expressing your answers in the form $z = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$,

(i) find z_1 and z_2;

(ii) deduce z_3 and z_4. [4]

The four roots z_1, z_2, z_3 and z_4 are represented by the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D respectively on an Argand diagram.

(c) (i) Plot the points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D on an Argand diagram.

(ii) Find the area of the polygon formed by these four points. [4]

The four roots z_1, z_2, z_3 and z_4 satisfy the equation $z^4 + 2z^2 + 4 = 0$.

The four roots $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ and $\frac{1}{z_4}$ satisfy the equation $pw^4 + qw^2 + r = 0$ where $p, q, r \in \mathbb{Z}$.

(d) Find the value of p, q and r. [3]

The four roots $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ and $\frac{1}{z_4}$ are represented by the points $\mathrm{E}, \mathrm{F}, \mathrm{G}$ and H respectively on an Argand diagram.

(e) (i) Find $\frac{1}{z_1}$ in the form $z = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

(ii) Hence, deduce the area of the polygon formed by these four points. [4]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let $\log_{10} 2 = p$ and $\log_{10} 3 = q$.

(a) Find an expression for $\log_{10} 24$ in terms of p and q. [3]

(b) Find an expression for $\log_3 8$ in terms of p and q. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 8]

Consider the complex number $z_1 = \sqrt{3} - 3\mathrm{i}$.

(a) Express z_1 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

Consider the complex number $z_2 = 2\sqrt{3}e^{i\frac{5\pi}{6}}$.

The cube roots of $\frac{z_2}{z_1}$ are denoted by u, v and w.

(b) Find u, v and w. [5]

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Consider the sequence $\{u_n\}$, with nth term given by u_n. The first three terms are

$$u_1 = k - 5,\ u_2 = 3 - 2k \text{ and } u_3 = 5k + 3, \text{ where } k \in \mathbb{R}.$$

(a) Consider the case when $\{u_n\}$ is arithmetic.

(i) Find the value of k.

(ii) Hence, or otherwise, find u_3. [5]

(b) Consider the case where $k = 12$.

(i) Show that the first three terms of $\{u_n\}$ form a geometric sequence.

(ii) Given that $\{u_n\}$ is geometric, state a reason why the sum of an infinite number of terms of this sequence does not exist. [4]

(c) The sequence, $\{u_n\}$, is geometric for a second value of k.

(i) Show that $k^2 - 10k - 24 = 0$.

(ii) Find the first three terms of $\{u_n\}$ for this second value of k.

(iii) Hence, write down the value of S_{2m}, the sum of the first $2m$ terms, for this second value of k. [7]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the complex number $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$, such that $|z - (2 + \mathrm{i})| = 3$.

(a) Show that $x^2 + y^2 - 4x - 2y - 4 = 0$. [3]

The argument of $\frac{z+p}{z-1}$ is $\frac{\pi}{4}$, where $p \in \mathbb{R}$.

(b) Show that $x^2 + y^2 + (p-1)x + (p+1)y - p = 0$. [7]

Two roots of the equation $z^4 + az^3 + bz^2 + cz + d = 0$ are z_1 and z_2, where $z \in \mathbb{C}$ and $a, b, c, d \in \mathbb{R}$.

Both z_1 and z_2 satisfy the conditions $|z - (2 + \mathrm{i})| = 3$ and $\mathrm{Re}\left(\frac{z+4}{z-1}\right) = \mathrm{Im}\left(\frac{z+4}{z-1}\right)$.

(c) Use the results from parts (a) and (b) to find z_1 and z_2. [7]

(d) Find the value of a. [4]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Consider the functions f, g and h defined as follows for $t \in \mathbb{R}$.

$$f(t) = \sin(2t + 1)$$

$$g(t) = \sin(2t + 3)$$

$$h(t) = f(t) + g(t)$$

(a) Show that $h(t) = \text{Im}\left(e^{2ti}(e^{i} + e^{3i})\right)$. [2]

(b) Write $e^{i} + e^{3i}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(c) Hence or otherwise, write $h(t)$ in the form $p\sin(2t + q)$, where $p > 0$ and $0 < q < 2\pi$. [3]

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Question 8 - working space

Section B

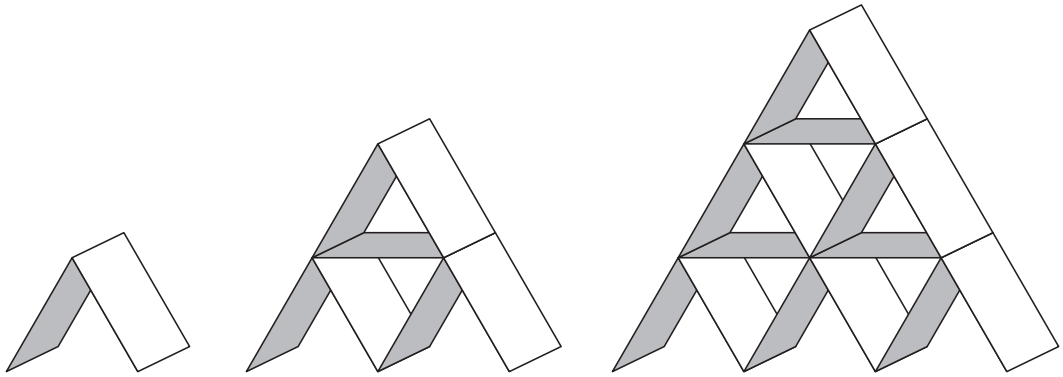
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Rectangular playing cards are stacked in the shape of a pyramid with n rows, where $n \geq 1$.

Some cards are placed horizontally and some cards are stacked at an angle of $60°$ to the horizontal.

The following diagrams represent pyramid stacks for $n = 1$, $n = 2$ and $n = 3$.



$n = 1$ $n = 2$ $n = 3$

Let t_n represent the number of cards used to create a pyramid stack with n rows.

(a) Write down t_3. [1]

(b) Find t_4. [2]

(c) Show that $t_n = \frac{n(3n+1)}{2}$. [3]

There are 52 cards in a full pack of playing cards.

(d) A complete pyramid stack is created using playing cards taken from 14 full packs. Find the maximum number of rows in this stack. [3]

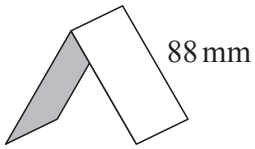
(e) A complete pyramid stack is created using playing cards taken from full packs with no cards left over. Find the minimum number of rows in this stack. [2]

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(Question 10 continued)

The long edge of each playing card measures $88\,\text{mm}$ as illustrated in the following diagram.



(f) Find the minimum number of cards needed to create a complete pyramid stack with a vertical height of more than 2 metres. The thickness of the cards may be ignored. [5]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider the family of functions f_n defined by $f_n(x) = \sum_{r=0}^{n} \left(-2x^2\right)^r$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Show that f_n is an even function for all values of n. [3]

(b) (i) Show that $f_3(x) = 1 - 2x^2 + 4x^4 - 8x^6$.

(ii) Write down a similar expression for $f_4(x)$ in ascending powers of x. [2]

Consider the function $f(x) = \lim_{n \to \infty} f_n(x)$ defined over the domain $-k < x < k$ where $k > 0$.

The largest possible value of k is K.

(c) (i) Find the value of K, giving your answer in exact form.

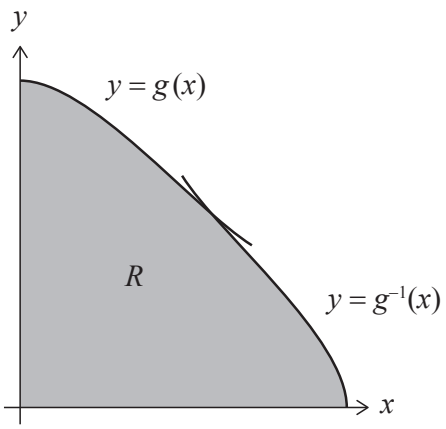
(ii) Express $f(x)$ as a rational function in the form $\frac{1}{a + bx^2}$, where a and b are constants to be determined. [5]

The function g is defined as $g(x) = f(x)$ for $0 \leq x < K$.

(d) (i) Justify that g^{-1} exists.

(ii) Find $g^{-1}(x)$, giving its domain. [6]

The region R is completely enclosed by the curves $y = g(x)$, $y = g^{-1}(x)$ and the x- and y-axes, as shown on the following diagram.



(e) Find the area of R. [4]

Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

Alex purchases a car for $€30\,000$. The value of the car depreciates at 15% per annum.

(a) Find the value of the car after ten years. Give your answer to two decimal places. [2]

Alex invests $€50\,000$ in a bank account that pays a compound interest rate of 1.5% per month.

Inflation over the same time period was 0.8% per month.

(b) Find the number of months required for the real value of the investment to first exceed $€55\,000$. [4]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

A geometric sequence has first term 80 and fourth term 0.74088.

(a) Find the second term. [3]

The first two terms of this geometric sequence are also the first term and eleventh term respectively, of an arithmetic sequence.

Let S_n denote the sum of the first n terms of the arithmetic sequence.

(b) Find the greatest value of S_n, giving your answer to two decimal places. [4]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

In the expansion of $\frac{1}{\sqrt{q-x^2}}$, where $q \in \mathbb{Q}^+$, the coefficient of x^6 is 5120.

Find the value of q.

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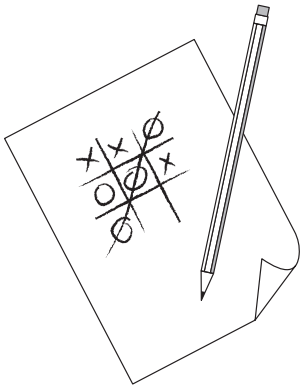
Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 5]

A class of students plays a tic-tac-toe competition among themselves. Each individual game in the competition involves only two students.



Every student in the class is to play every other student twice. However, Stephen left the class after he had played only seven games. All other games, not involving Stephen, were played.

By the end of the competition a total of 513 games had been played.

Determine the number of students that were originally in the class.

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The 1st and 5th terms of an arithmetic sequence are 36 and 12 respectively.

(a) Find the 13th term of this arithmetic sequence. [4]

The sum of the first n terms of this arithmetic sequence is zero.

(b) Find the value of n. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

A linear system of equations is given by

$$x - z = 4$$

$$2x + 2y + z = 14$$

$$x + 2y + \alpha z = \beta$$

Find the value of α and the value of β, where $\alpha, \beta \in \mathbb{Z}$, for which the system has an infinite number of solutions.

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 22]

Consider the complex number $w = \dfrac{4\mathrm{i}}{1+\mathrm{i}\sqrt{3}}$.

(a) Show that $w = \sqrt{3} + \mathrm{i}$. [2]

(b) Find

(i) $\left|w\right|$;

(ii) $\arg w$. [3]

(c) Use De Moivre's theorem to find the two square roots of w. Give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

The complex number w can be expressed in the form $(a + b\mathrm{i})^2$, where $a, b \in \mathbb{R}^+$.

(d) (i) Show that $a^2 = \dfrac{\sqrt{3}}{2} + 1$.

(ii) Find the value of b^2. [8]

(e) Hence, express $\tan\dfrac{\pi}{12}$ in the form $p + q\sqrt{3}$, where $p, q \in \mathbb{Z}$. [5]

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Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The 7th term of an arithmetic sequence is 6.

The sum of the 6th term and the 12th term is 24.

Find the first term and the common difference.

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Question 1 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

(a) It is given that $w = 1 + \mathrm{i}\sqrt{3}$.

(i) Express w in the form $w = r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$.

(ii) Show that $\left(1 + \mathrm{i}\sqrt{3}\right)^8 + \left(1 - \mathrm{i}\sqrt{3}\right)^8 = -256$. [6]

It is given that $\left(1 + \mathrm{i}\tan\theta\right)^n \equiv \dfrac{\cos n\theta + \mathrm{i}\sin n\theta}{\cos^n \theta}$ for $\cos\theta \neq 0$, $n \in \mathbb{Z}^+$.

(b) Show that $\left(1 + \mathrm{i}\tan\theta\right)^4 + \left(1 - \mathrm{i}\tan\theta\right)^4 \equiv \dfrac{2\cos 4\theta}{\cos^4\theta}$. [3]

(c) Hence, determine the four roots of the equation $(1+z)^4 + (1-z)^4 = 0$, giving your answers in the form $z = \mathrm{i}\tan\left(\dfrac{k\pi}{8}\right)$, where $k \in \mathbb{Z}^+$. [4]

(d) By considering the binomial expansion of $(1+z)^4 + (1-z)^4$, show that $\tan\dfrac{\pi}{8} = \sqrt{3 - 2\sqrt{2}}$. [7]

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Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 4]

A particular service group at a school has ten members.

The group intends to select a committee of seven by first choosing a chairperson, a vice-chairperson and a treasurer, then selecting four additional members to complete the committee.

Determine the number of ways in which this committee can be chosen.

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

Consider the infinite series

$$\sum_{k=0}^{\infty}\left(\frac{53}{1000}\right)\left(\frac{1}{100}\right)^{k}.$$

(a) Find the exact value of S_{∞}. [3]

Let $0.2\overline{53}$ represent the repeating decimal $0.2535353...$.

(b) Use your answer from part (a) to express $0.2\overline{53}$ as a fraction such that the numerator and denominator have no common factors. [2]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

Consider the following two equations.

$$2x + 6y - 8z = 13$$

$$3x - y + 3z = 12$$

(a) Determine the general solution giving the answer in parametric form. [2]

Consider a third equation $ax + 12y - 16z = k$, where $a, k \in \mathbb{R}$.

(b) In the case where $a = 3$ and $k = -3$, determine the unique solution to the system of three equations. [2]

(c) (i) Write down the value of a for which there is no unique solution to the system of three equations.

(ii) Hence, write down the corresponding value of k for which there is an infinite number of solutions to the system of three equations. [2]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Malik owns 15 recipe books which he bought while travelling.

(a) Find the number of different ways of arranging the books in a line on a shelf. [1]

Malik's books were bought in three continents.

Six books are from Asia, five from Europe and four from Africa.

Malik decides to arrange the books on the shelf so that the books from each of the continents are grouped together.

(b) Determine the number of different ways that Malik can do this. [3]

Malik chooses four books, all from the same continent.

(c) Determine the number of different choices Malik could make. [3]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 5]

Prove by contradiction that $\sqrt{2xy} \leq \frac{1}{2}(2x + y)$ for all $x, y \in \mathbb{R}^+$.

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Question 9 - working space

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Functions

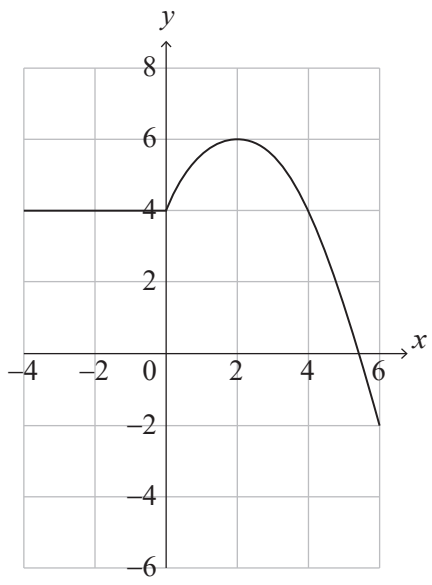
68 questions

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \le x \le 6$ is shown in the following diagram.



(a) Write down the value of

(i) $f(2)$;

(ii) $(f \circ f)(2)$. [2]

(b) Let $g(x) = \frac{1}{2} f(x) + 1$ for $-4 \le x \le 6$. On the axes above, sketch the graph of g. [3]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

The function f is defined by $f(x) = \frac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q. [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x-coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \frac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\frac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 6x^2 - 12x + 1$ and $g(x) = -x + c$, where $c \in \mathbb{R}$.

(a) Find the range of f. [2]

(b) Given that $(g \circ f)(x) \leq 0$ for all $x \in \mathbb{R}$, determine the set of possible values for c. [4]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and A_0, k are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 9]

The function f is defined by $f(x)=\frac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of f;

(ii) the horizontal asymptote of the graph of f. [2]

(b) Find the coordinates where the graph of f crosses

(i) the x-axis;

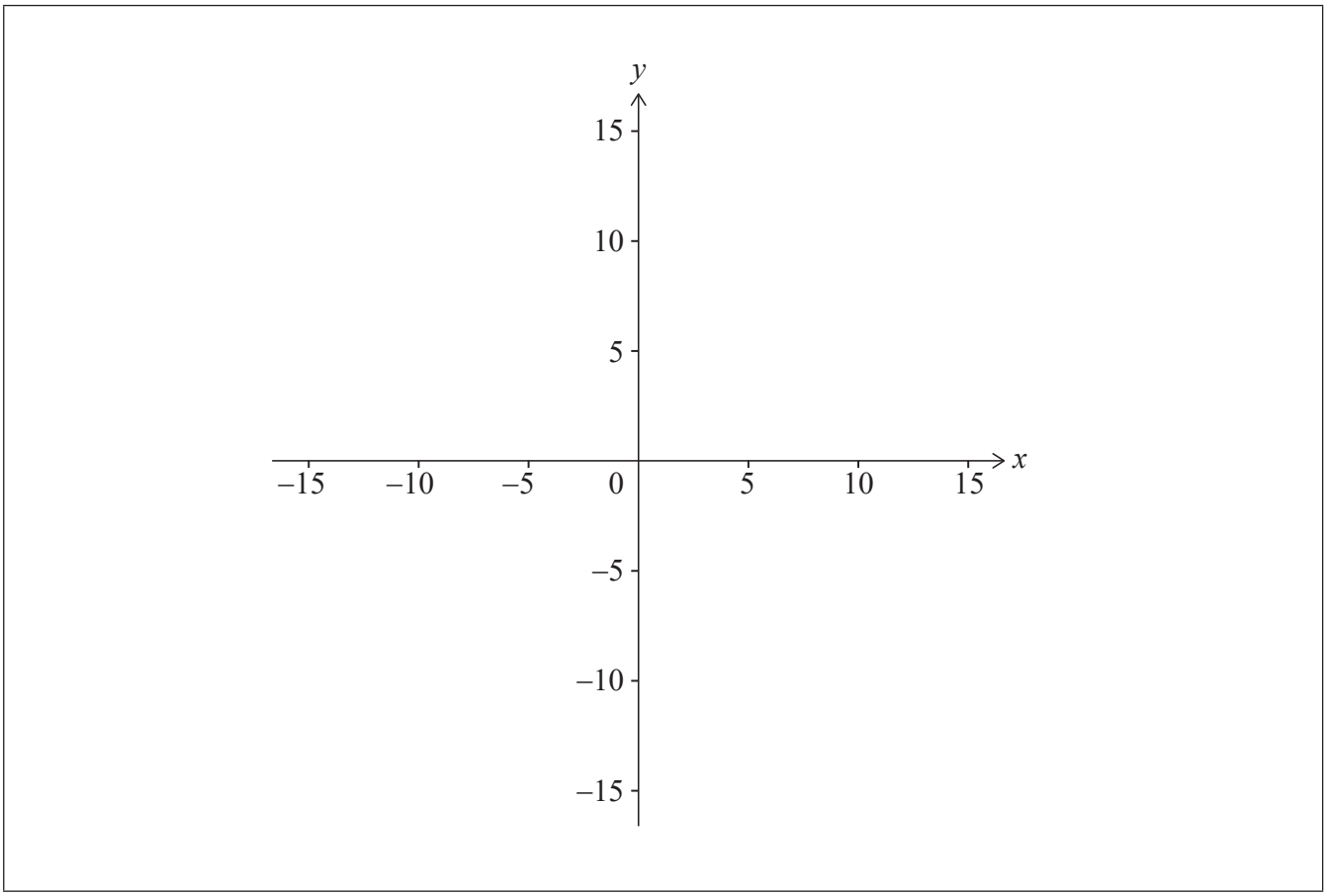
(ii) the y-axis. [2]

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(Question 2 continued)

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x) = \frac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of a. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the function $f(x) = \frac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \frac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x-axis;

(ii) y-axis. [3]

(b) Write down the equation of the vertical asymptote of the graph of f. [1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Q}$.

Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes. [3]

(e) (i) Express $\frac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\int_0^3 \frac{1}{f(x)}\,\mathrm{d}x$, expressing your answer as a single logarithm. [7]

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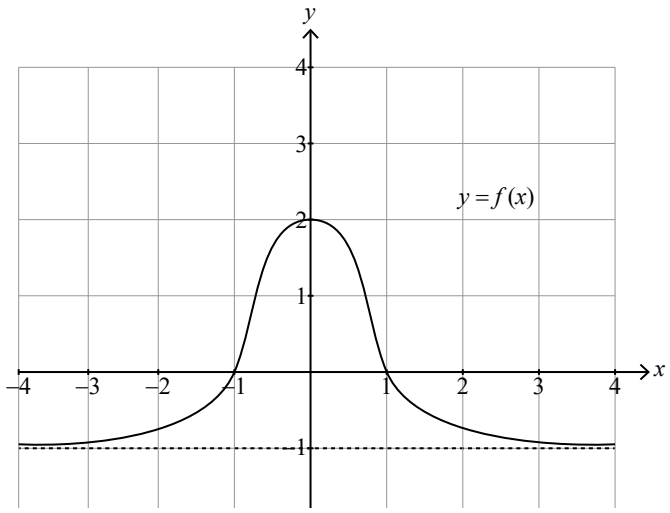
Question 10 - working space

Section A

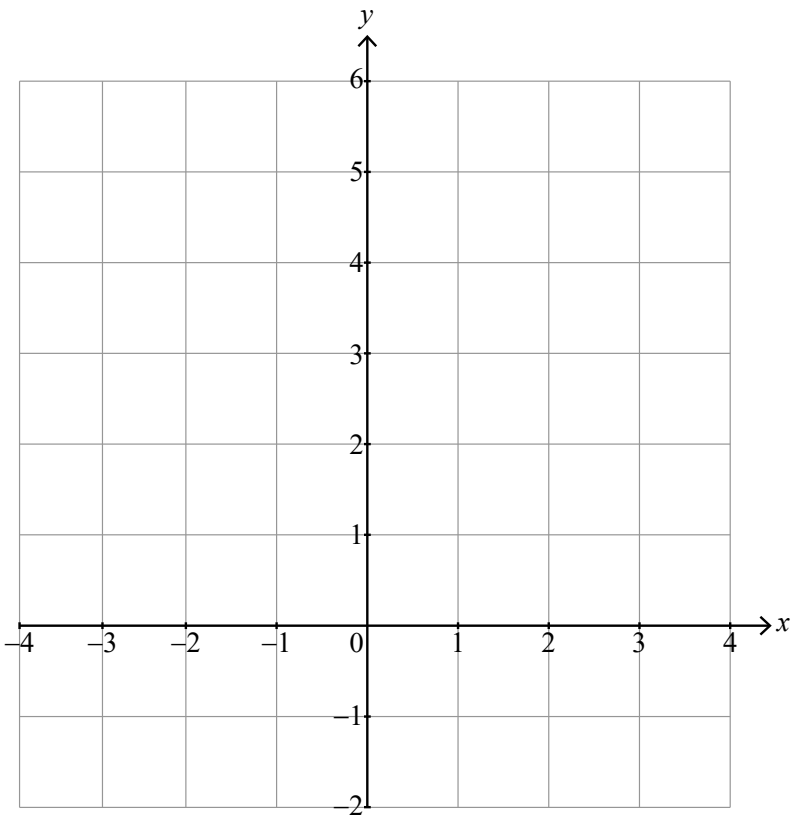
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

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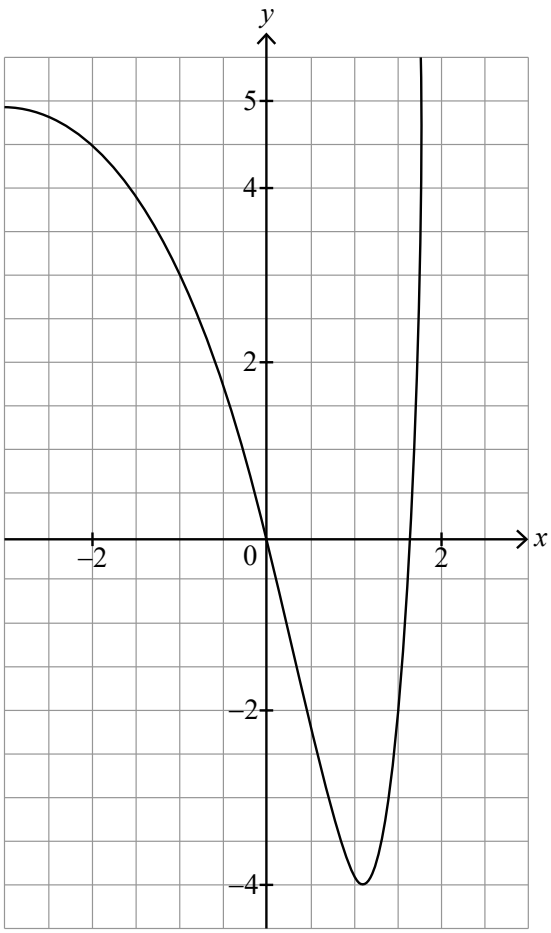
Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

The function f is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

(This question continues on the following page)

(Question 9 continued)

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 5]

Consider the graphs of $y = \frac{x^2}{x-3}$ and $y = m(x+3)$, $m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points.

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Question 9 - working space

2. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \leq x \leq 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \leq x \leq 1$. [2]

(b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \leq x \leq 1$. [2]

(d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

(g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$. [2]

(h) Hence

(i) show that $f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x)$, $n \in \mathbb{Z}^+$;

(ii) express $f_3(x)$ as a cubic polynomial. [5]

Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

Consider $f(x) = 4\sin x + 2.5$ and $g(x) = 4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q$, where $x \in \mathbb{R}$ and $q > 0$.

The graph of g is obtained by two transformations of the graph of f.

(a) Describe these two transformations. [2]

The y-intercept of the graph of g is at $(0, r)$.

(b) Given that $g(x) \geq 7$, find the smallest value of r. [5]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 8]

A function f is defined by $f(x)=\frac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y=f(x)$ has a vertical asymptote and a horizontal asymptote.

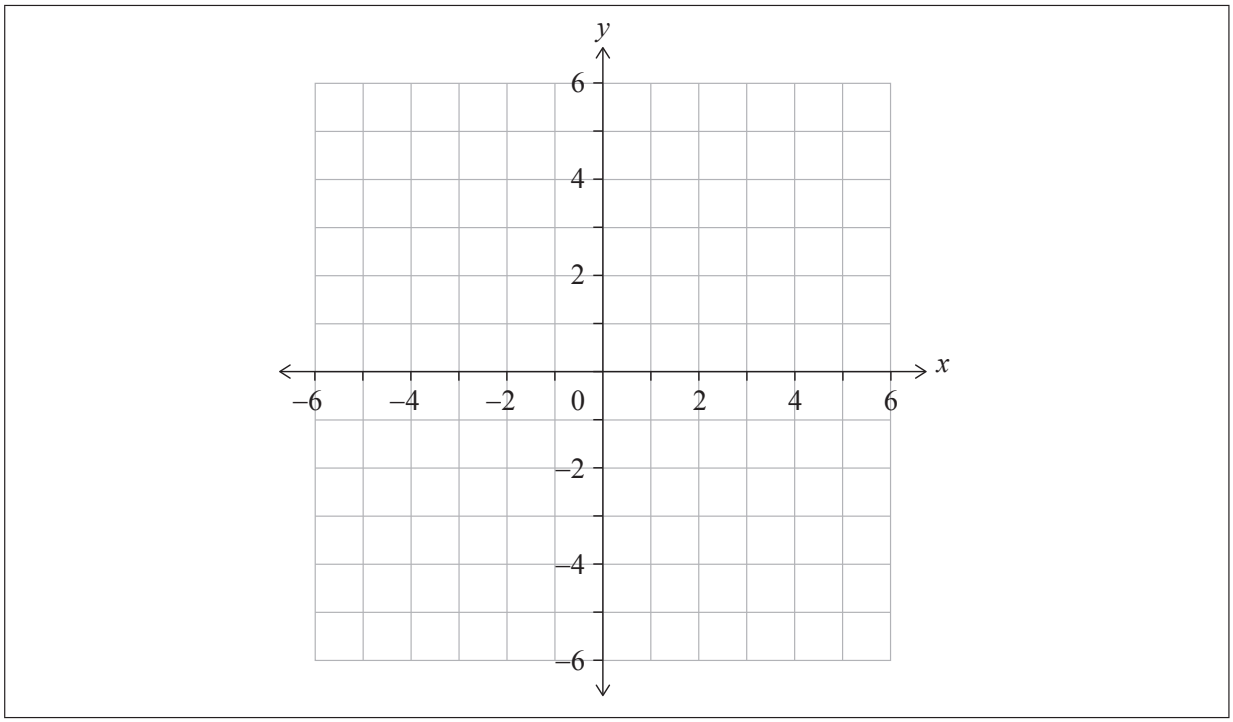
Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) On the set of axes below, sketch the graph of $y=f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes. [3]



(c) Hence, solve the inequality $0<\frac{2x-1}{x+1}<2$. [1]

(d) Solve the inequality $0<\frac{2|x|-1}{|x|+1}<2$. [2]

(This question continues on the following page)

(Question 3 continued)

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

A function f is defined by $f(x)=\frac{1}{x^2-2x-3}$, where $x \in \mathbb{R}, x \neq -1, x \neq 3$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x)=\frac{1}{x^2-2x-3}$, where $x \in \mathbb{R}, x > 3$.

(b) The inverse of g is g^{-1}.

(i) Show that $g^{-1}(x)=1+\frac{\sqrt{4x^2+x}}{x}$.

(ii) State the domain of g^{-1}. [7]

A function h is defined by $h(x)=\arctan\frac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a)=\frac{\pi}{4}$, find the value of a.

Give your answer in the form $p+\frac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]

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Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

Consider the function $f(x) = 2^x - \frac{1}{2^x}$, $x \in \mathbb{R}$.

(a) Show that f is an odd function. [2]

The function g is given by $g(x) = \frac{x-1}{x^2-2x-3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

(b) Solve the inequality $f(x) \geq g(x)$. [4]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 4]

Consider the equation $kx^2 - (k+3)x + 2k + 9 = 0$, where $k \in \mathbb{R}$.

(a) Write down an expression for the product of the roots, in terms of k. [1]

(b) Hence or otherwise, determine the values of k such that the equation has one positive and one negative real root. [3]

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Question 8 - working space

2. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p, q, r \in \mathbb{R}$, has roots α, β and γ.

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma.$$ [3]

(b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

(e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

(ii) Explain why the equation will have at least one real root for all values of q. [1]

(This question continues on the following page)

(Question 2 continued)

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p, q, r, s \in \mathbb{R}$, has roots α, β, γ and δ.

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

(f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

(ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

(h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

(ii) Write down the integer root of this equation. [1]

(iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

References:

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 5]

Let $f(x) = \cos(x - k)$, where $0 \leq x \leq a$ and $a, k \in \mathbb{R}^+$.

(a) Consider the case where $k = \frac{\pi}{2}$.

By sketching a suitable graph, or otherwise, find the largest value of a for which the inverse function f^{-1} exists. [2]

(b) Find the largest value of a for which the inverse function f^{-1} exists in the case where $k = \pi$. [1]

(c) Find the largest value of a for which the inverse function f^{-1} exists in the case where $\pi < k < 2\pi$. Give your answer in terms of k. [2]

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 16]

Consider a three-digit code abc, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated. [4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a, b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a.

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes. [12]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

The population of a town t years after 1 January 2014 can be modelled by the function

$$P(t) = 15\,000\mathrm{e}^{kt}, \text{ where } k < 0 \text{ and } t \geq 0.$$

It is known that between 1 January 2014 and 1 January 2022 the population decreased by $11\,\%$.

Use this model to estimate the population of this town on 1 January 2041.

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

The function f is defined by $f(x)=\frac{7x+7}{2x-4}$ for $x \in \mathbb{R}$, $x \neq 2$.

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y=f(x)$, write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

Find the range of possible values of k such that $e^{2x} + \ln k = 3e^{x}$ has at least one real solution.

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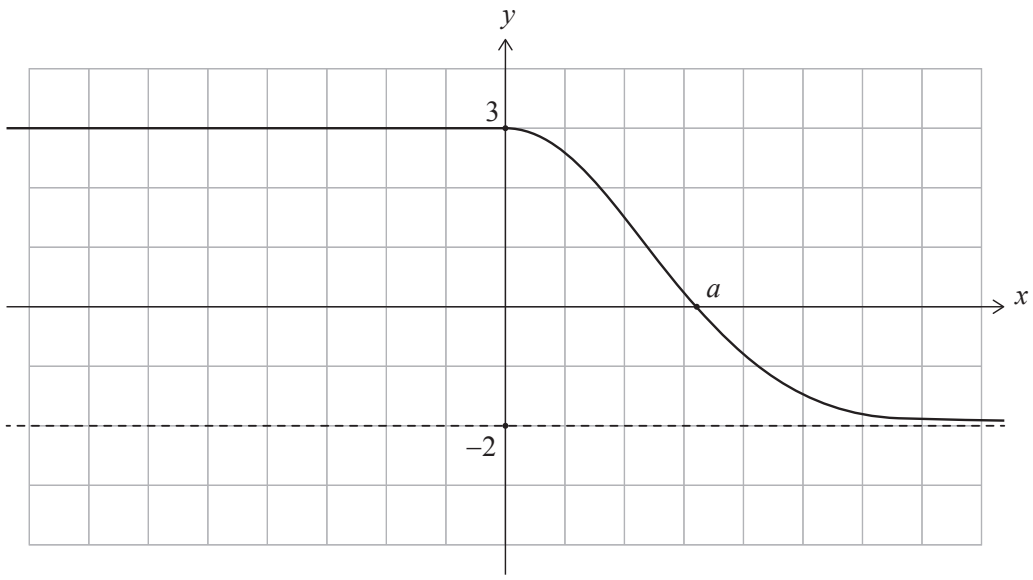
Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0, 3)$, an x-intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.

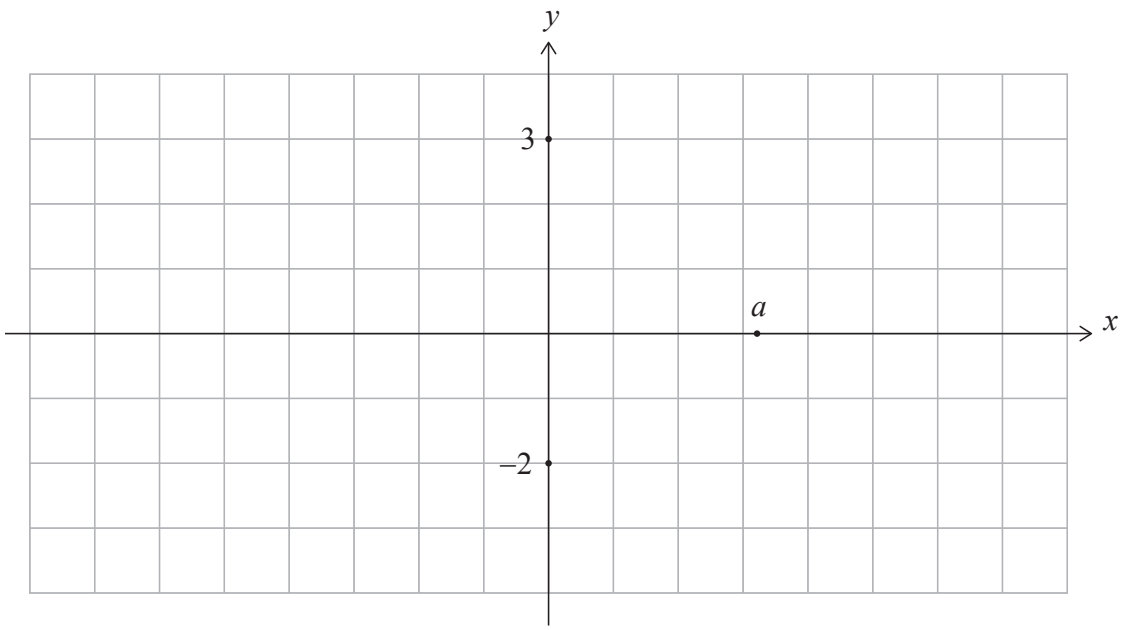


(This question continues on the following page)

(Question 8 continued)

Consider the function $g(x) = \left| f\left(|x|\right)\right|$.

(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote. [4]



(b) Find the possible values of k such that $\left(g(x)\right)^2 = k$ has exactly two solutions. [3]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}$, $x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(i) the y-axis;

(ii) the x-axis. [2]

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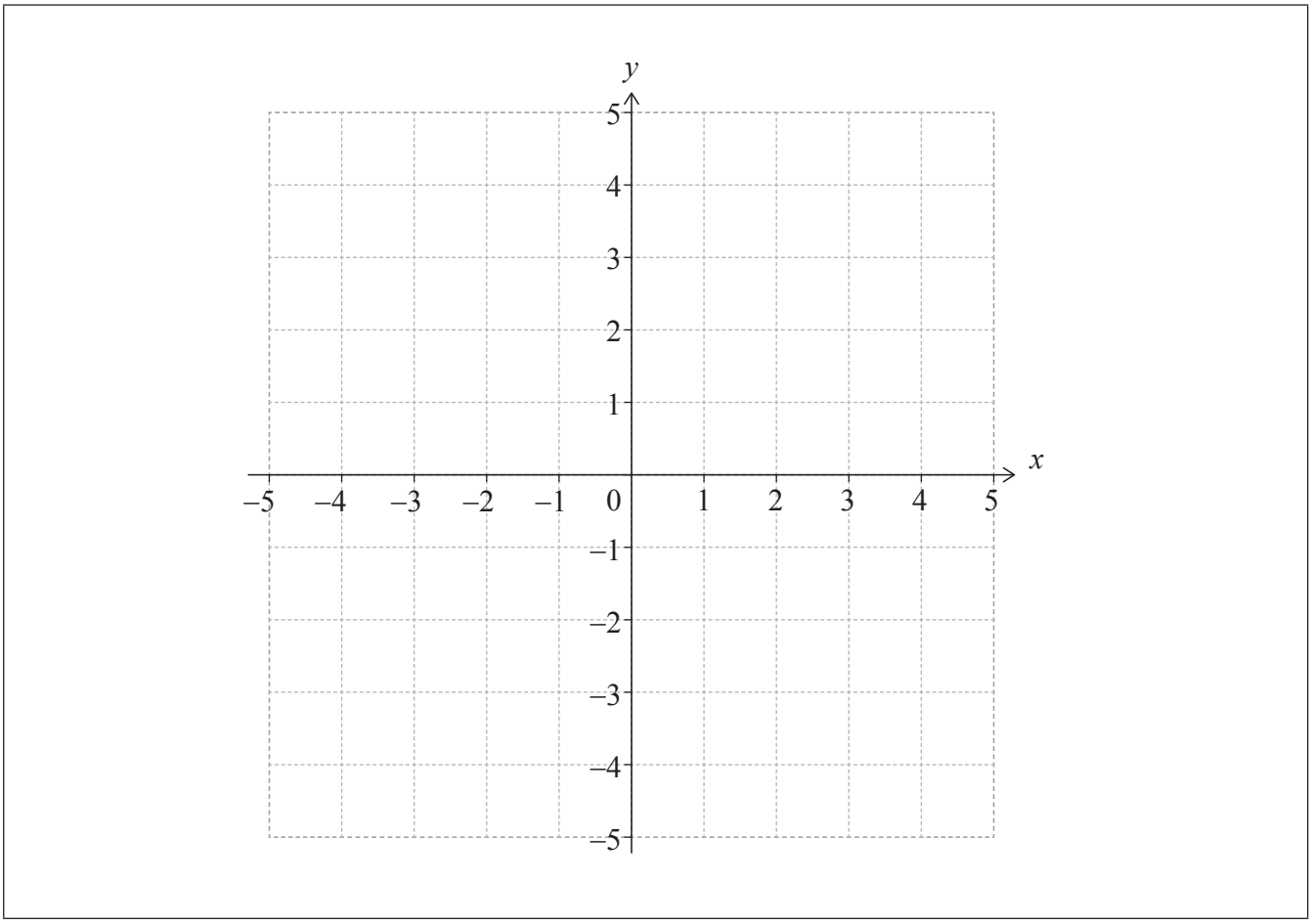
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(Question 2 continued)

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b). [1]



Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b, \text{ where } a,\ b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3.$$

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

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Question 5 - working space

Section A

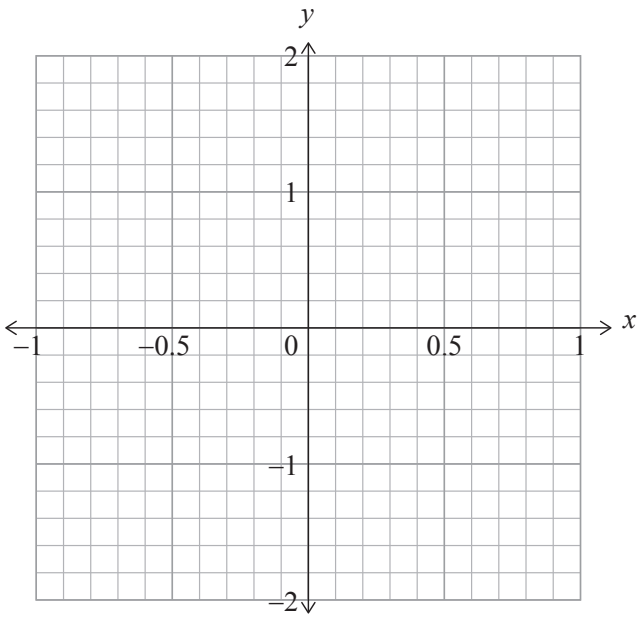
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

The functions f and g are defined by $f(x) = 2x - x^3$ and $g(x) = \tan x$.

(a) Find $(f \circ g)(x)$. [2]

(b) On the following grid, sketch the graph of $y = (f \circ g)(x)$ for $-1 \leq x \leq 1$. Write down and clearly label the coordinates of any local maximum or minimum points. [3]



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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

A function g is defined by $g(x)=\frac{2x-5}{x^2-3}$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$.

(a) Determine the range of g. [4]

A function h is defined by $h(x)=g(|x|)\cos t$, where $x \in \mathbb{R}$, $x \neq \pm\sqrt{3}$ and t is a constant where $\frac{\pi}{2} < t \leq \pi$.

(b) Find the set of values of x such that $h(x) \leq 0$. [3]

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will not be marked.

Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

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Question 1 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 9)$, where $x > \frac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

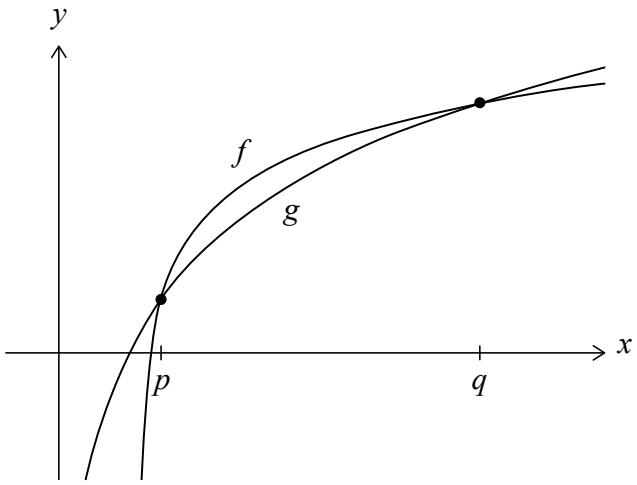
(iii) Find the range of possible values of d. [9]

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Do **not** write solutions on this page.

(Question 10 continued)

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where a, $b \in \mathbb{Z}^+$. [5]

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Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Consider the functions $f(x) = x - 3$ and $g(x) = x^2 + k^2$, where k is a real constant.

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k. [3]

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Question 1 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The functions f and g are defined by

$f(x) = \ln(2x - 9)$, where $x > \frac{9}{2}$

$g(x) = 2\ln x - \ln d$, where $x > 0$, $d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

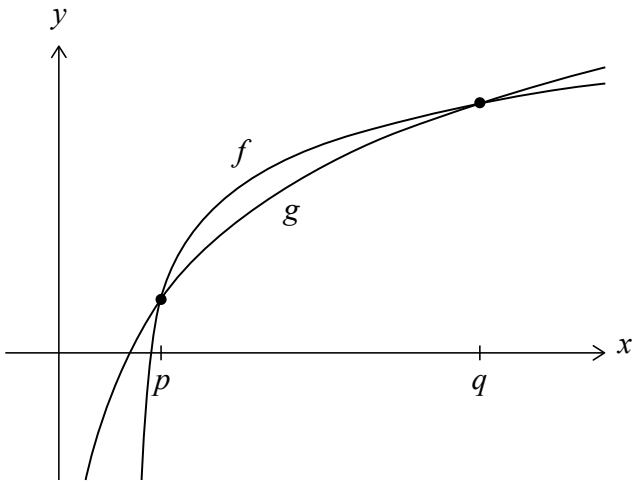
(iii) Find the range of possible values of d. [9]

(This question continues on the following page)

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(Question 10 continued)

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where a, $b \in \mathbb{Z}^+$. [5]

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Do **not** write solutions on this page.

Question 10 - working space

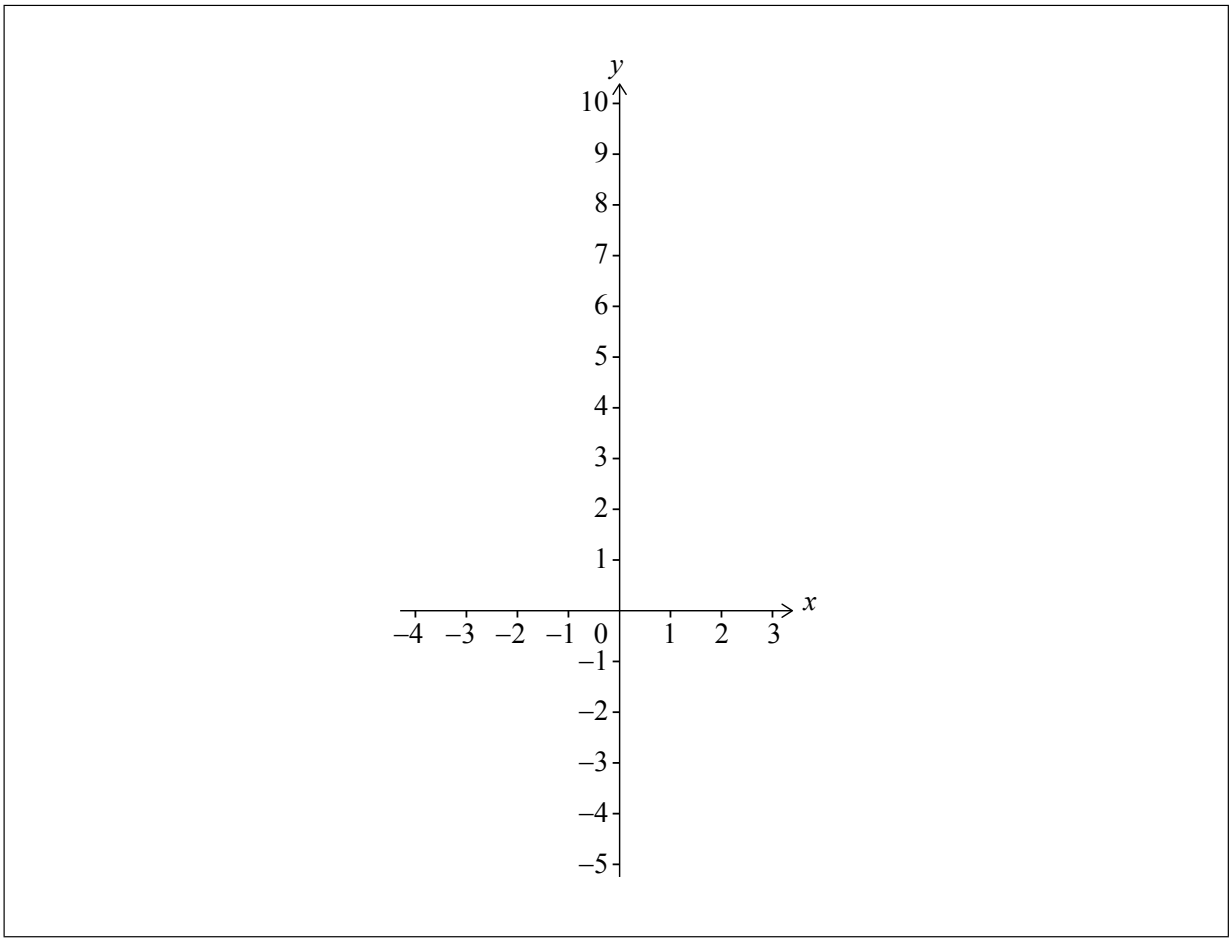
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of f for $-4 \leq x \leq 3$. [3]



The function g is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of g is obtained from the graph of f by a horizontal stretch with scale factor k, followed by a vertical translation of c units.

Find the value of k and the value of c. [2]

(This question continues on the following page)

(Question 2 continued)

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Question 2 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the function defined by $f(x) = \frac{x^2 - 14x + 24}{2x + 6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f. [1]

(b) Find the coordinates of the points where the graph of f crosses the x-axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f. [4]

(f) Solve the inequality $f(x) > x$. [4]

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Question 11 - working space

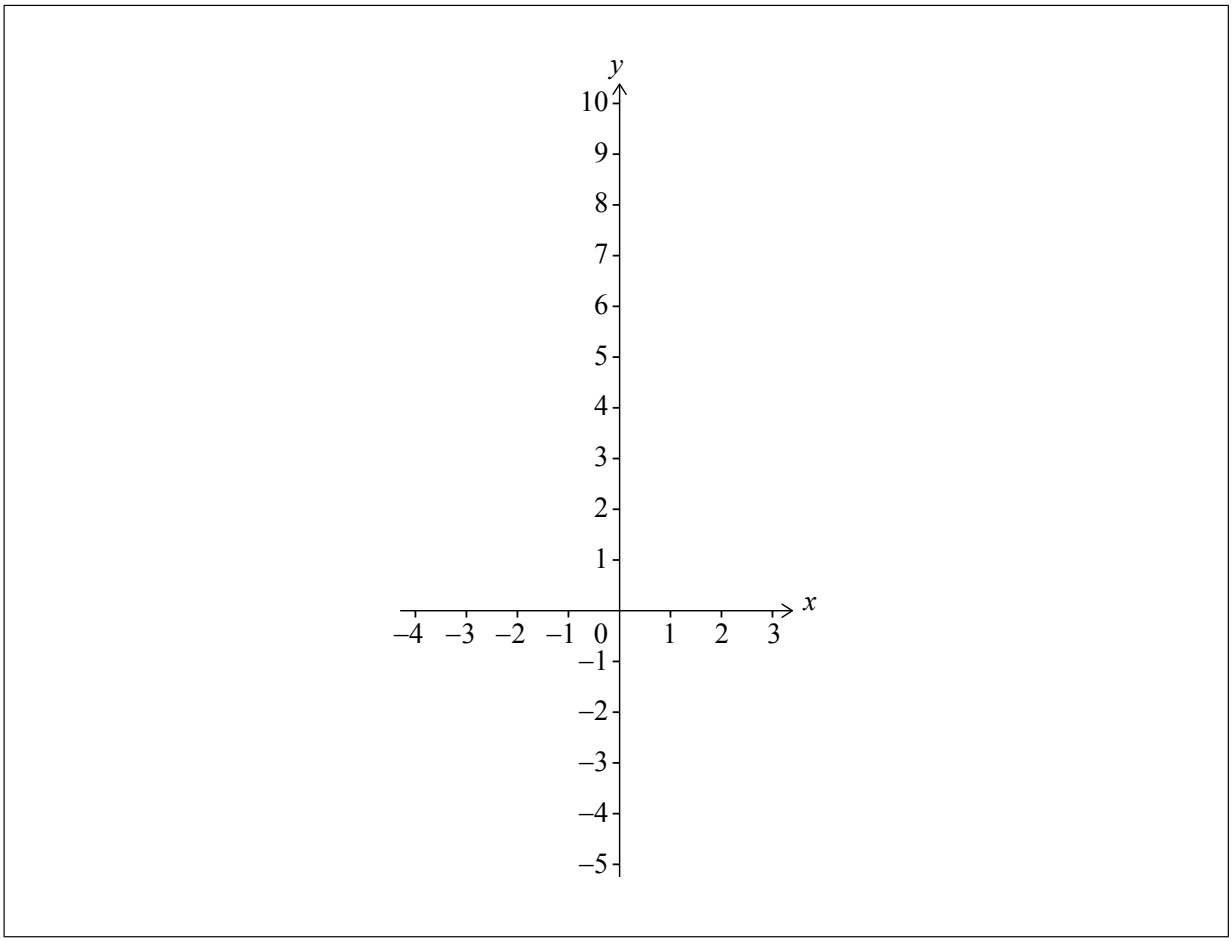
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of f for $-4 \leq x \leq 3$. [3]



The function g is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of g is obtained from the graph of f by a horizontal stretch with scale factor k, followed by a vertical translation of c units.

Find the value of k and the value of c. [2]

(This question continues on the following page)

(Question 2 continued)

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Question 2 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the function defined by $f(x) = \frac{x^2 - 14x + 24}{2x + 6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f. [1]

(b) Find the coordinates of the points where the graph of f crosses the x-axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of a and the value of b. [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f. [4]

(f) Solve the inequality $f(x) > x$. [4]

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Do **not** write solutions on this page.

Question 11 - working space

1. [Maximum mark: 24]

This question asks you to explore some properties of the family of curves $y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

(a) By systematically varying the value of b, or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x-axis intercepts. [2]

(b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly

(i) one x-axis intercept; [1]

(ii) three x-axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

(c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly

(i) two x-axis intercepts; [1]

(ii) one x-axis intercept; [1]

(iii) three x-axis intercepts. [1]

(This question continues on the following page)

(Question 1 continued)

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

(d) Consider the case where the curve has exactly three x-axis intercepts. State whether each point of zero gradient is located above or below the x-axis. [1]

(e) Show that the curve has a point of zero gradient at $\mathrm{P}(0, b)$ and a point of zero gradient at $\mathrm{Q}\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$. [5]

(f) Consider the points P and Q for $a > 0$ and $b > 0$.

(i) Find an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]

(ii) Determine whether each point is located above or below the x-axis. [1]

(g) Consider the points P and Q for $a < 0$ and $b > 0$.

(i) State whether P is a local maximum or a local minimum and whether it is above or below the x-axis. [1]

(ii) State the conditions on a and b that determine when Q is below the x-axis. [1]

(h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x-axis intercepts. [5]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10}\left(\frac{1}{a}\right)$; [2]

(b) $\log_{1000} a$. [3]

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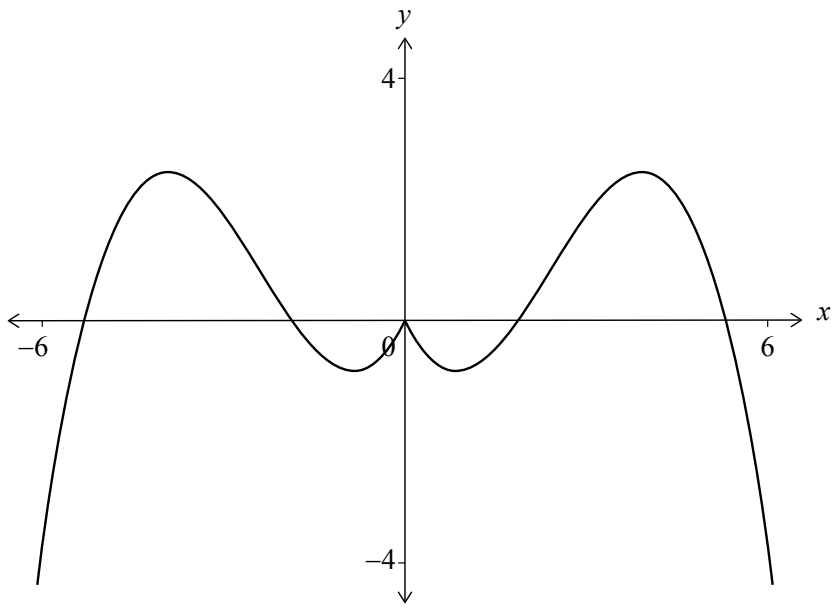
Question 2 - working space

Section A

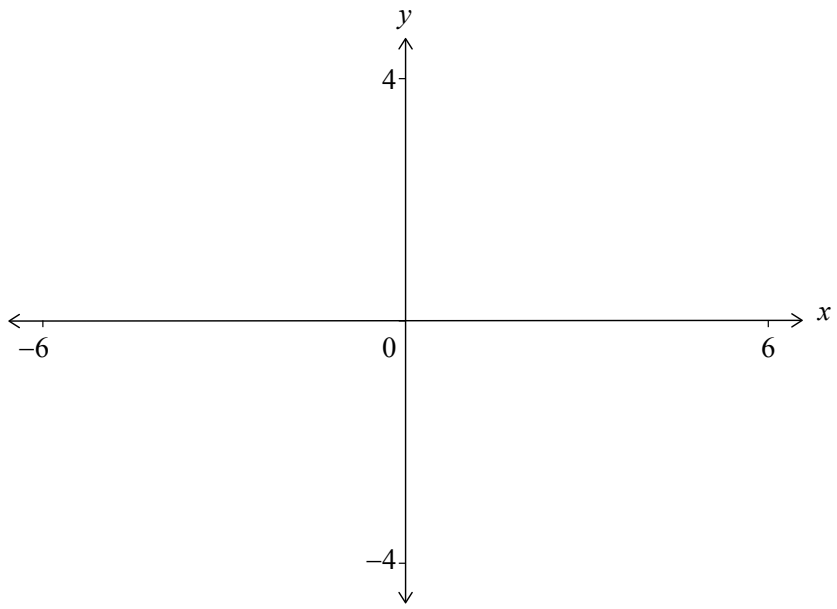
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 6]

The graph of $y = f(|x|)$ for $-6 \leq x \leq 6$ is shown in the following diagram.



(a) On the following axes, sketch the graph of $y = |f(|x|)|$ for $-6 \leq x \leq 6$. [2]

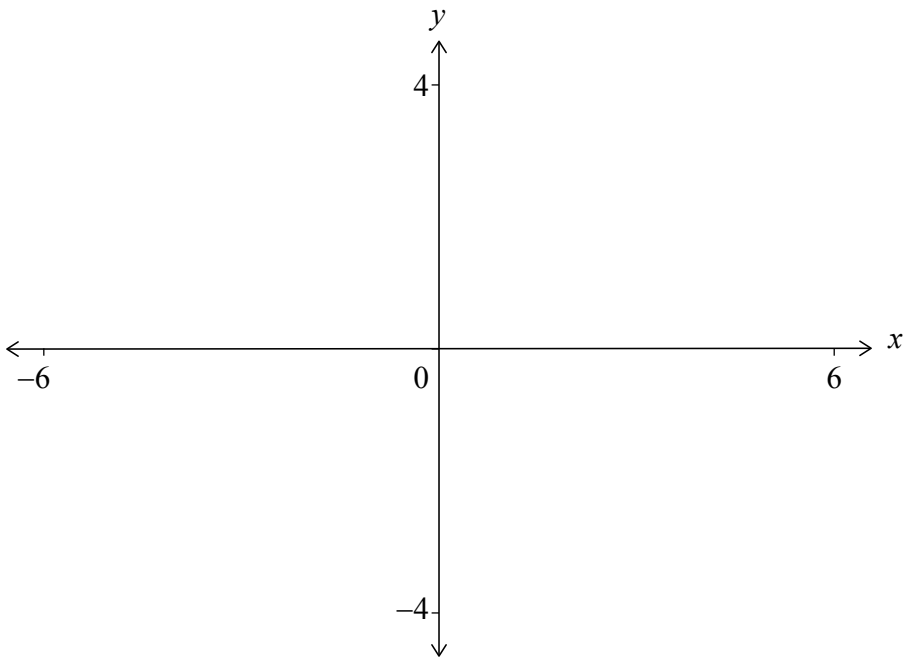


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(Question 9 continued)

It is given that f is an odd function.

(b) On the following axes, sketch the graph of $y = f(x)$ for $-6 \leq x \leq 6$. [2]



It is also given that $\int_0^4 f\left(|x|\right)\mathrm{d}x = 1.6$.

(c) Write down the value of

(i) $\int_{-4}^{0} f\left(x\right)\mathrm{d}x$;

(ii) $\int_{-4}^{4}\left(f\left(|x|\right) + f\left(x\right)\right)\mathrm{d}x$. [2]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Consider the function $f(x) = \frac{4x+2}{x-2}$, $x \neq 2$.

(a) Sketch the graph of $y = f(x)$. On your sketch, indicate the values of any axis intercepts and label any asymptotes with their equations. [5]

(b) Write down the range of f. [1]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p, where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$. [1]

(d) Find the value of b and the value of c. [3]

(e) Find the y-coordinate of the vertex of the graph of $y = g(x)$. [2]

(f) Find the product of the solutions of the equation $f(x) = g(x)$. [4]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$.

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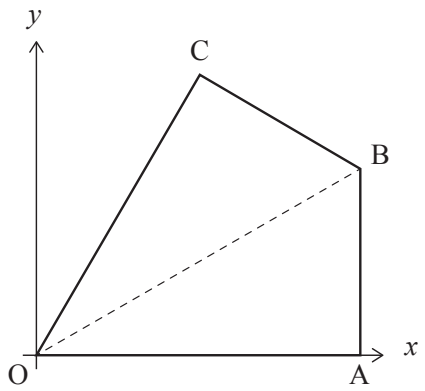
Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

Quadrilateral OABC is shown on the following set of axes.



OABC is symmetrical about $[\mathrm{OB}]$.

A has coordinates $(6, 0)$ and C has coordinates $\left(3, 3\sqrt{3}\right)$.

(a) (i) Write down the coordinates of the midpoint of $[\mathrm{AC}]$.

(ii) Hence or otherwise, find the equation of the line passing through the points O and B. [4]

(b) Given that $[\mathrm{OA}]$ is perpendicular to $[\mathrm{AB}]$, find the area of the quadrilateral OABC. [3]

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Question 3 - working space

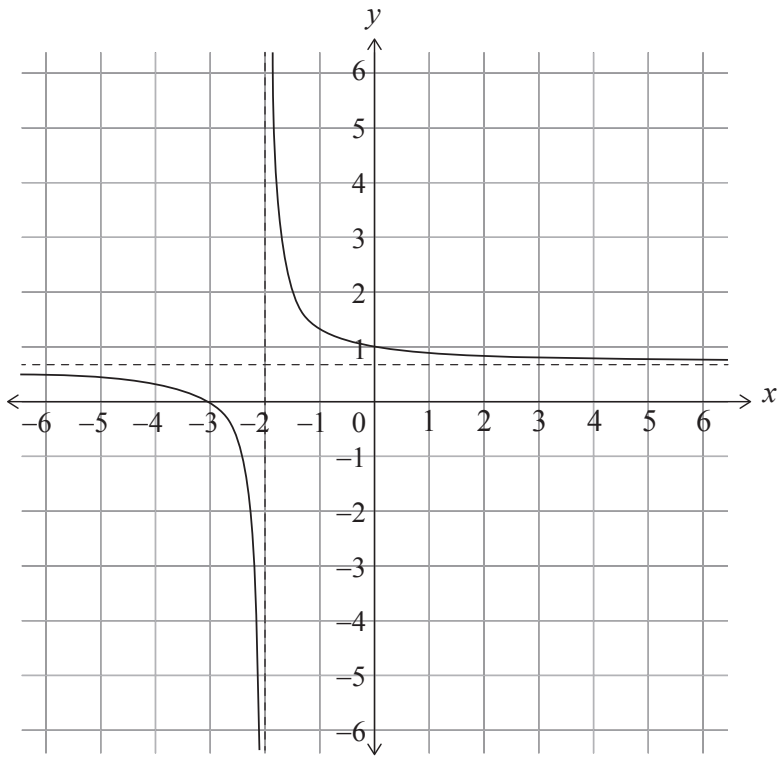
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 8]

A function f is defined by $f(x)=\frac{2(x+3)}{3(x+2)}$, where $x \in \mathbb{R}, x \neq -2$.

The graph $y=f(x)$ is shown below.



(a) Write down the equation of the horizontal asymptote. [1]

Consider $g(x)=mx+1$, where $m \in \mathbb{R}, m \neq 0$.

(b) (i) Write down the number of solutions to $f(x)=g(x)$ for $m>0$.

(ii) Determine the value of m such that $f(x)=g(x)$ has only one solution for x.

(iii) Determine the range of values for m, where $f(x)=g(x)$ has two solutions for $x \geq 0$. [7]

(This question continues on the following page)

(Question 5 continued)

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

The loudness of a sound, L, measured in decibels, is related to its intensity, I units, by $L = 10 \log_{10}(I \times 10^{12})$.

Consider two sounds, S_1 and S_2.

S_1 has an intensity of 10^{-6} units and a loudness of 60 decibels.

S_2 has an intensity that is twice that of S_1.

(a) State the intensity of S_2. [1]

(b) Determine the loudness of S_2. [2]

The maximum loudness of thunder in a thunderstorm was measured to be 115 decibels.

(c) Find the corresponding intensity, I, of the thunder. [3]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

(a) Solve $2x^2 - 15x + 18 < 0$. [3]

(b) The function f is defined by $f(x) = \sqrt{2x^2 - 15x + 18}$, where $x \in \mathbb{R}$, $x \leq k$.

Find the greatest value of k for which f^{-1} exists, justifying your answer. [2]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

Consider the function $f(x) = 11\sqrt{x} - 2x - 11$, where $0 \le x \le 20$.

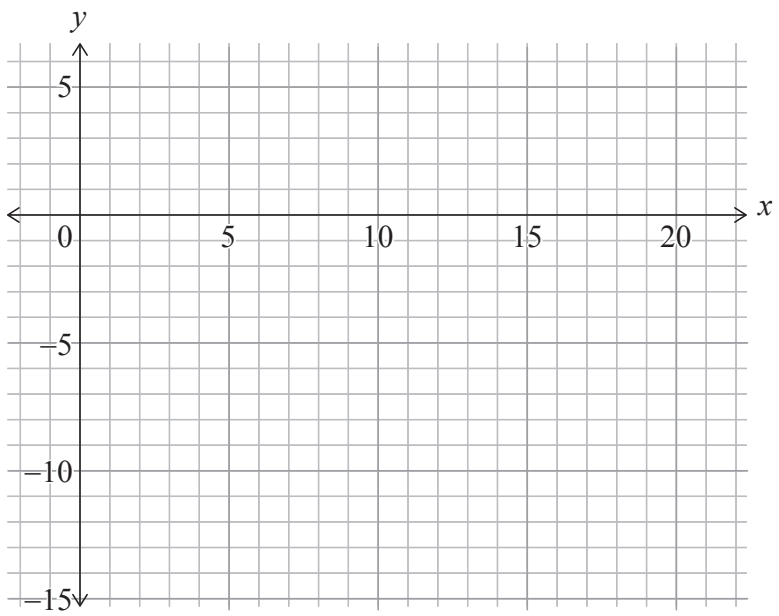
(a) Find the value of

(i) $f(0)$;

(ii) $f(20)$. [2]

(b) Find the two roots of $f(x) = 0$. [2]

(c) Sketch the graph of $y = f(x)$ on the following grid. [3]



(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

Consider the function $h(x) = \log_{10}(4x^2 - rx + r - 1)$, where $x \in \mathbb{R}$.

Find the possible values of r. [5]

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

A curve C has equation $y = \frac{2x^2 + 6x - 3}{x + k}$, $x \in \mathbb{R}$, $x \neq -k$, where k is a real positive constant.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + 4kx + 6k + 3}{(x + k)^2}$. [4]

(b) Find the range of values of k for which a local minimum or maximum point exists. [4]

Consider the curve C, when $k = 2$.

(c) Write down the equation of the vertical asymptote. [1]

(d) Find the equation of the oblique asymptote. [4]

(e) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} > 2$, for $x \in \mathbb{R}$, $x \neq -2$. [4]

(f) Sketch the curve C, showing clearly both asymptotes and the general behaviour of C as it approaches each asymptote. [You are not required to find any axes intercepts.] [4]

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Question 12 - working space

2. [Maximum mark: 28]

A polynomial is said to be palindromic if the sequence of its coefficients remains the same in reverse. This question asks you to investigate some properties and solutions of palindromic polynomial equations.

In parts (a) and (b), consider quadratic equations of the form $ax^2 + bx + a = 0$, where $a \neq 0$.

The sequence of coefficients, $\{a, b, a\}$, remains the same in reverse.

The following table shows three palindromic quadratic equations and their sequence of coefficients.

Palindromic quadratic equation	Sequence of coefficients
$2x^2 - 5x + 2 = 0$	$\{2, -5, 2\}$
$x^2 + 4x + 1 = 0$	$\{1, 4, 1\}$
$x^2 + 1 = 0$	$\{1, 0, 1\}$

The quadratic equation $2x^2 - 5x + 2 = 0$ has roots 2 and $\frac{1}{2}$.

These roots form a “reciprocal pair”, since one root is the reciprocal of the other.

(a) (i) Determine the roots of $x^2 + 4x + 1 = 0$.

Give these roots in the form $s \pm \sqrt{t}$, where $s \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. [3]

(ii) Hence, or otherwise, show that these roots form a reciprocal pair. [2]

(b) Show that the complex roots of $x^2 + 1 = 0$ form a reciprocal pair. [2]

Let $p(x) = ax^2 + bx + a$, where $a \neq 0$.

(c) Verify that $p(x) = x^2 p\left(\frac{1}{x}\right)$ where $x \neq 0$. [2]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), you may assume the result that a polynomial, $p(x)$, of degree n is palindromic if and only if $p\left(x\right) = x^n p\left(\frac{1}{x}\right)$.

(d) Use $p\left(x\right) = x^n p\left(\frac{1}{x}\right)$ to show that if $\alpha \neq 0$ is a root of $p(x) = 0$, then $\frac{1}{\alpha}$ is also a root. [2]

Let $f(x) = p(x)q(x)$, where p and q are palindromic polynomials of degree n and m respectively.

(e) Show that f is a palindromic polynomial. [4]

Consider the palindromic polynomial $f(x) = x^4 + 2x^3 - x^2 + 2x + 1$.

This polynomial can be expressed in the form

$f(x) = (x^2 + ux + 1)(x^2 + vx + 1)$, where $u, v \in \mathbb{Z}$ and $u < v$.

(f) By forming and solving an appropriate system of equations in u and v, determine the value of u and the value of v. [6]

(g) Hence, find all the exact complex and purely real roots of $x^4 + 2x^3 - x^2 + 2x + 1 = 0$. [3]

Consider the palindromic polynomial equation

$x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_2x^2 + a_1x + 1 = 0$, where n is odd.

(h) Show that -1 is always a root of this equation. [4]

Question 2 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

The function f is defined by $f(x) = 5(x+1)(x+3)$, where $x \in \mathbb{R}$.

(a) Write $f(x)$ in the form $a(x-h)^2 + k$, where $a, h, k \in \mathbb{Z}$. [4]

(b) Sketch the graph of $y = f(x)$, showing the values of any intercepts with the axes and the coordinates of the vertex. [4]

(c) Solve the inequality $f(x) \leq 40$. [4]

The function g is defined by $g(x) = \ln x$, where $x \in \mathbb{R}$, $x > 0$.

(d) (i) Write down an expression for $(f \circ g)(x)$.

(ii) Solve the inequality $(f \circ g)(x) \leq 40$. [3]

(e) Find the domain of $g \circ f$. [3]

Do **not** write solutions on this page.

Question 10 - working space

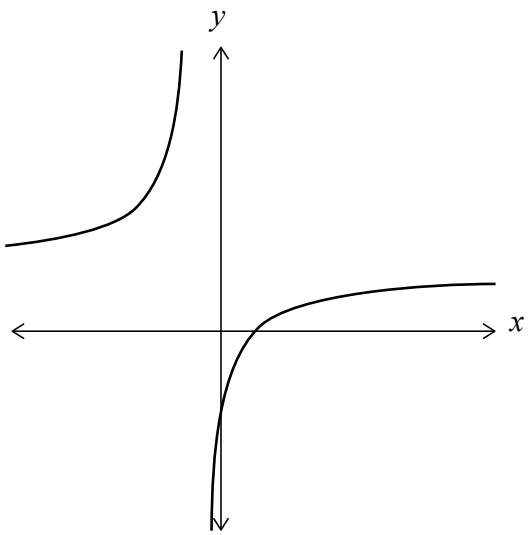
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The function f is defined by $f(x) = \frac{3x-2}{2x+1}$ for $x \in \mathbb{R}$, $x \neq -\frac{1}{2}$.

The following diagram shows part of the graph of $y = f(x)$.



(a) Write down the value of $f(0)$. [1]

(b) Write down the equation of the horizontal asymptote. [1]

The function g is defined by $g(x) = -f(x)$ for $x \geq 0$.

(c) Find the range of g. [3]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The quadratic equation $x^2 + kx + 15 - k = 0$ has two distinct real roots.

(a) Find the possible values of k. [5]

(b) Find the possible values of k in the case where the two distinct real roots are both positive or both negative. [2]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

It is given that $x^4 + px^3 - 2x^2 + qx - 3$ is exactly divisible by $(x+1)^2$.

Find the value of p and the value of q, where $p, q \in \mathbb{R}$.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

The function f is defined by $f(x) = 4^x$, where $x \in \mathbb{R}$.

(a) Find $f^{-1}(8)$. Express your answer in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. [3]

The function g is defined by $g(x) = 1 + \log_2 x$, where $x \in \mathbb{R}^+$.

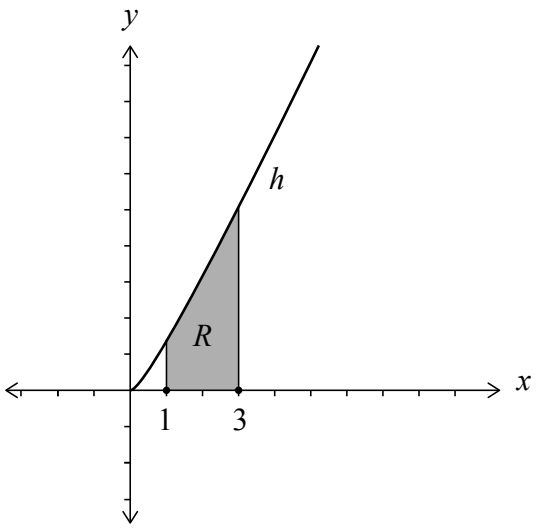
(b) (i) Find an expression for $g^{-1}(x)$.

(ii) Describe a sequence of transformations that transforms the graph of $y = g^{-1}(x)$ to the graph of $y = f(x)$. [4]

(c) Show that $(f \circ g)(x) = 4x^2$. [3]

The function h is defined by $h(x) = \frac{4x^2}{2x+1}$, $x \neq -\frac{1}{2}$.

The following diagram shows part of the graph of h. Let R be the region enclosed by the graph of h and the x-axis, between the lines $x = 1$ and $x = 3$.



(d) (i) Show that $2x - 1 + \frac{1}{2x+1} = \frac{4x^2}{2x+1}$.

(ii) Hence or otherwise, find the area of R, giving your answer in the form $p + q\ln r$, where $p, q, r \in \mathbb{Q}^+$. [7]

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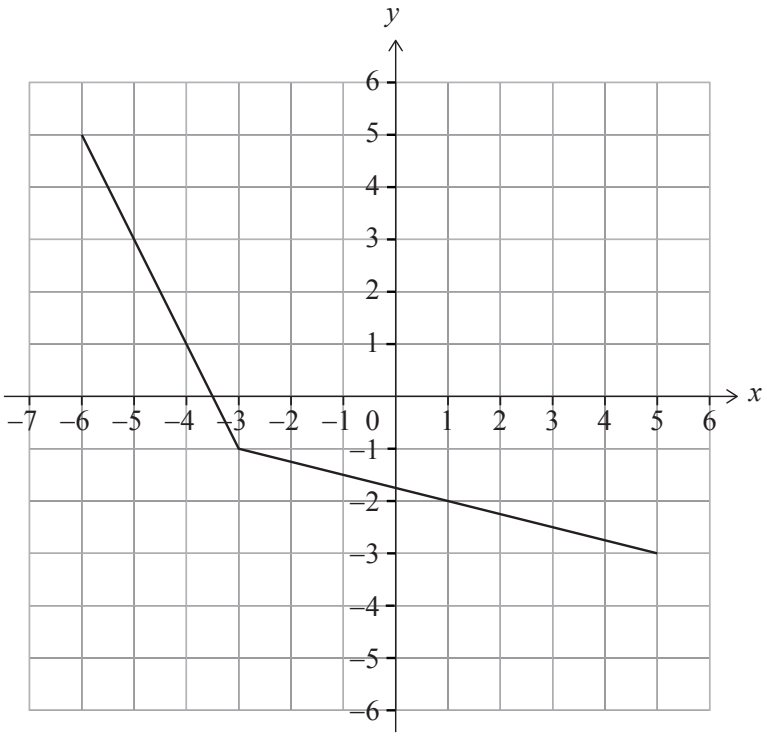
Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$, for $-6 \leq x \leq 5$.



(a) Write down the value of $f(-3)$. [1]

(b) State the domain of f^{-1}, the inverse function of f. [1]

(c) Find the value of x that satisfies $f^{-1}(2x - 7) = -3$. [3]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

Consider the curve $y = x^2 - x - 1$ and the line $y = mx - 3$, where $m \in \mathbb{R}$.

(a) Show that the curve and the line meet when $x^2 - (m+1)x + 2 = 0$. [2]

(b) Hence, find the values of m when the line is tangent to the curve. [5]

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Question 4 - working space

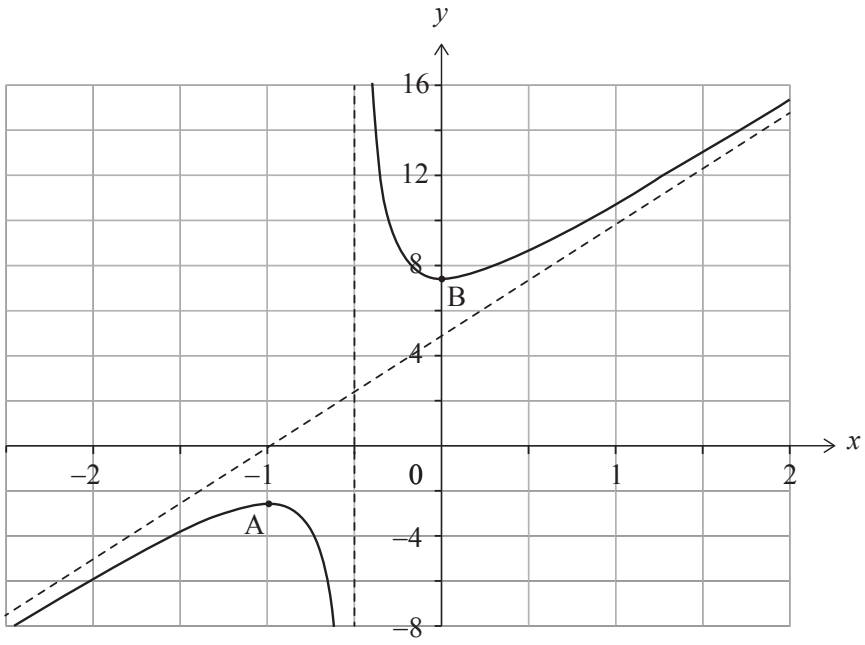
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Consider the function f. The graph of f has a local maximum at $\mathrm{A}\left(-1, -\frac{5}{2}\right)$, a local minimum at $\mathrm{B}\left(0, \frac{15}{2}\right)$, a vertical asymptote at $x = -\frac{1}{2}$ and an oblique asymptote $y = 5x + 5$.

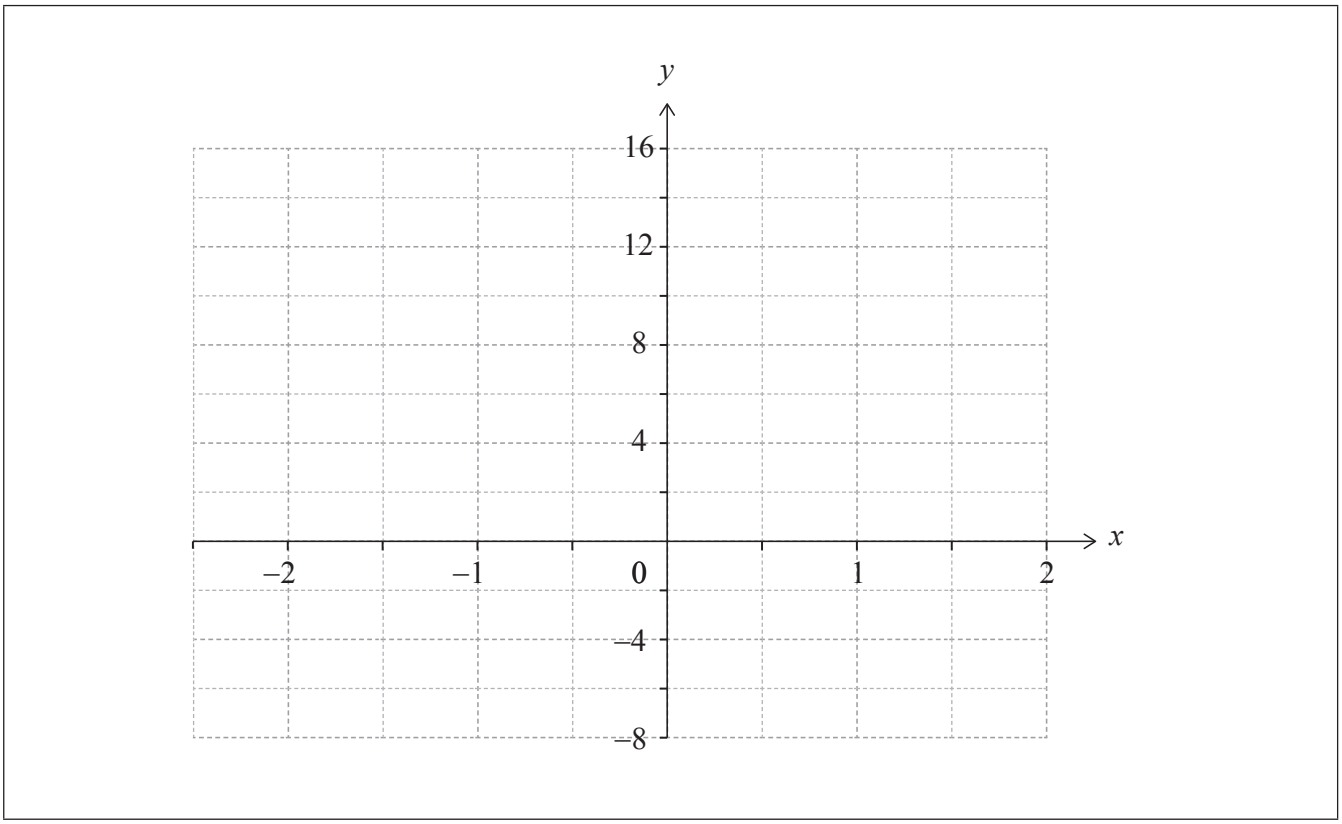
This information and part of the graph of f is shown in the following diagram.



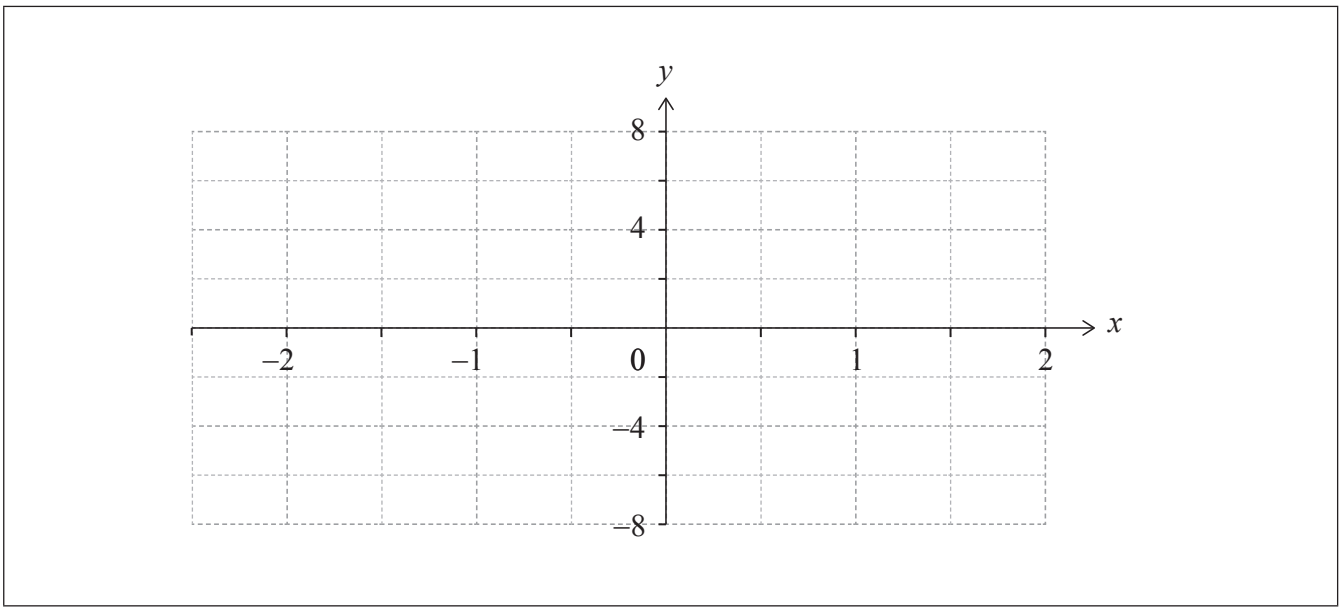
(This question continues on the following page)

(Question 6 continued)

(a) On the following grid, sketch the graph of $y = |f(x)|$, clearly indicating any asymptotes. [4]



(b) On the following grid, sketch the graph of $y = \frac{15}{f(x)}$, clearly indicating any asymptotes and intercepts with the axes. [3]



Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider the function $f(x) = 2x^4 - 6x^3 + px^2 + qx - 2$, where $p, q \in \mathbb{R}$.
A factor of $f(x)$ is $(x - 1)$, and when $f(x)$ is divided by $(x - 3)$ the remainder is -2.

Find the value of p and of q.

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Question 2 - working space

1. [Maximum mark: 27]

This question asks you to explore self-composite linear functions, of the form $f(x) = mx + c$, for varying values of m.

A function composed with itself is called a self-composite function.

For a function f, the function composition with itself is given by $(f \circ f)(x) = f\big(f(x)\big)$.

Let f^n denote the nth composition of f with itself where $f^n(x) = \underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}}(x)$.

Hence, for example, $f^2(x) = (f \circ f)(x)$ and $f^3(x) = (f \circ f \circ f)(x) = f\big(f\big(f(x)\big)\big)$.

Consider the linear function $f(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

(a) Show that

(i) $f^2(x) = m^2x + c(1 + m)$; [3]

(ii) $f^3(x) = m^3x + c(1 + m + m^2)$. [2]

(b) (i) Write down an expression for $f^4(x)$. [1]

(ii) Suggest a similar expression for $f^n(x)$, $n \in \mathbb{Z}^+$. [2]

(iii) By using your expression from part (b)(ii), or otherwise, find an expression in terms of n for $f^n(x)$ when $m = 1$. [3]

(c) For $m \neq 1$, use mathematical induction to prove that

$$f^n(x) = m^n x + c\left(\frac{1 - m^n}{1 - m}\right) \text{ for } n \in \mathbb{Z}^+. \quad [8]$$

(This question continues on the following page)

(Question 1 continued)

Consider $f^n(x) = m^n x + c\left(\frac{1-m^n}{1-m}\right)$ where $-1 < m < 1$.

(d) As $n \to \infty$, the family of graphs $y = f^n(x)$ approaches the graph of a straight line, L. Determine the equation of L, giving your answer in terms of c and m. [4]

Consider $f^n(x) = m^n x + c\left(\frac{1-m^n}{1-m}\right)$ where $m = -1$.

(e) (i) Show that $f^n(x) = -x + c$ when n is odd. [2]

(ii) Find an expression for $f^n(x)$ when n is even. [2]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

(a) Given that $x > 7$, show that $\frac{x}{x^2-8x+7} \times \frac{x^2-1}{x+1} \equiv \frac{x}{x-7}$. [2]

(b) Hence, or otherwise, solve $\log_2[x(x^2-1)] - 1 = \log_2[(x^2-8x+7)(x+1)]$. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 8]

For $x \in \mathbb{R}$, the functions f and g are defined by

$$f(x) = -x^2 + 4x + p \text{ and}$$

$$g(x) = x^2 + qx - 1,$$

where p and q are non-zero constants.

The graph of $y = f(x)$ has a vertex at A and the graph of $y = g(x)$ has a vertex at B.

Points A and B both lie on the line $y = 2x - 1$.

(a) Show that $p = -1$. [3]

(b) Find the value of q. [5]

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Question 4 - working space

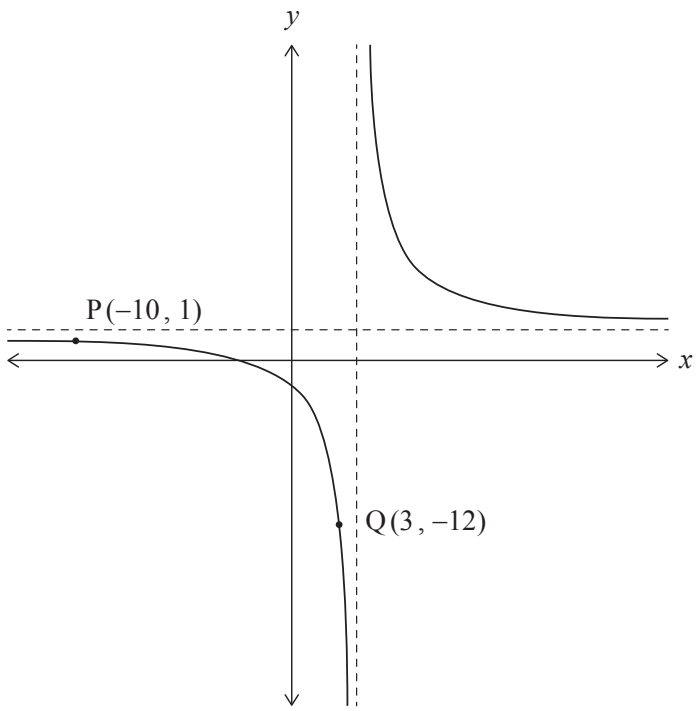
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 8]

The following diagram shows the graph of $y = \frac{Ax + B}{x - 4}$, where $x \in \mathbb{R}, x \neq 4$ and $A, B \in \mathbb{Z}$.

The graph passes through the points $\mathrm{P}(-10, 1)$ and $\mathrm{Q}(3, -12)$.



(a) Determine the value of A and the value of B. [3]

(b) Describe a sequence of transformations that would map the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{Ax + B}{x - 4}$. [5]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

Solve the equation $3\log_8 10x - \log_4 x = 1$ for $x > 0$.

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Question 4 - working space

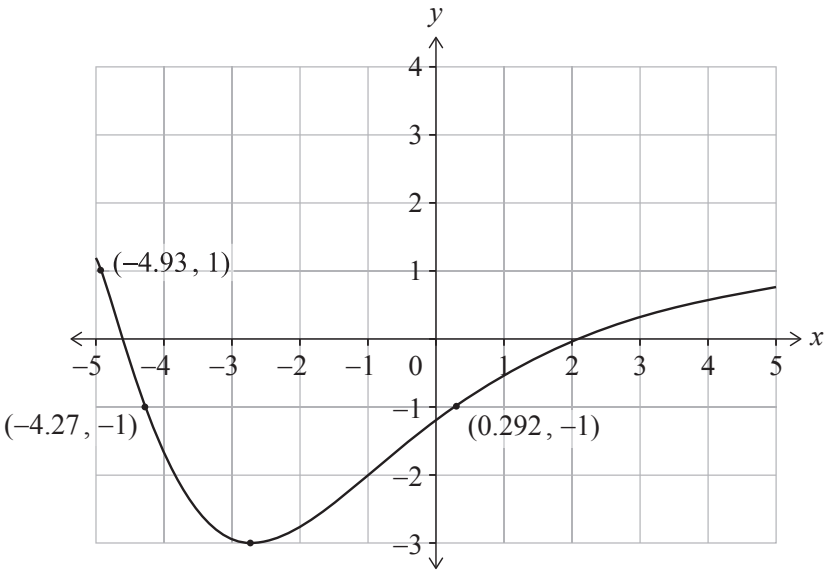
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 8]

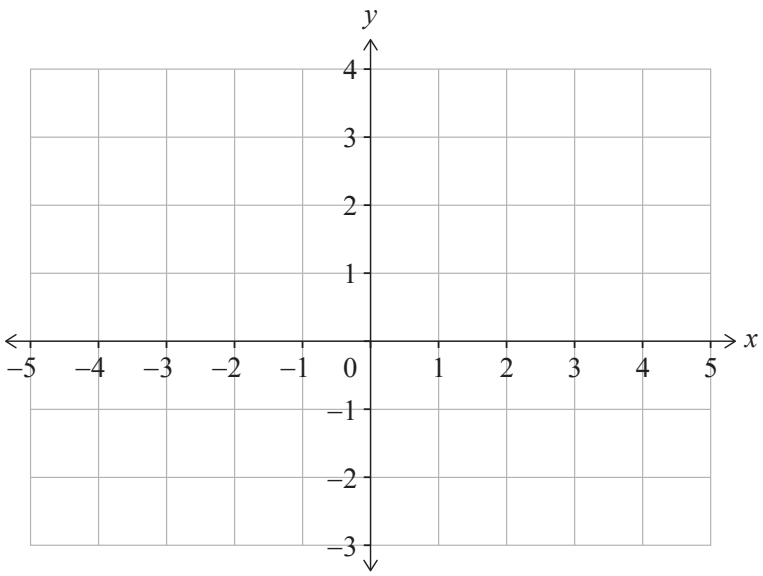
The graph of $y = f(x)$ for $-5 \leq x \leq 5$ is shown in the following diagram.

The curve passes through the points $(-4.93, 1)$, $(-4.27, -1)$ and $(0.292, -1)$.



(a) On the following set of axes,

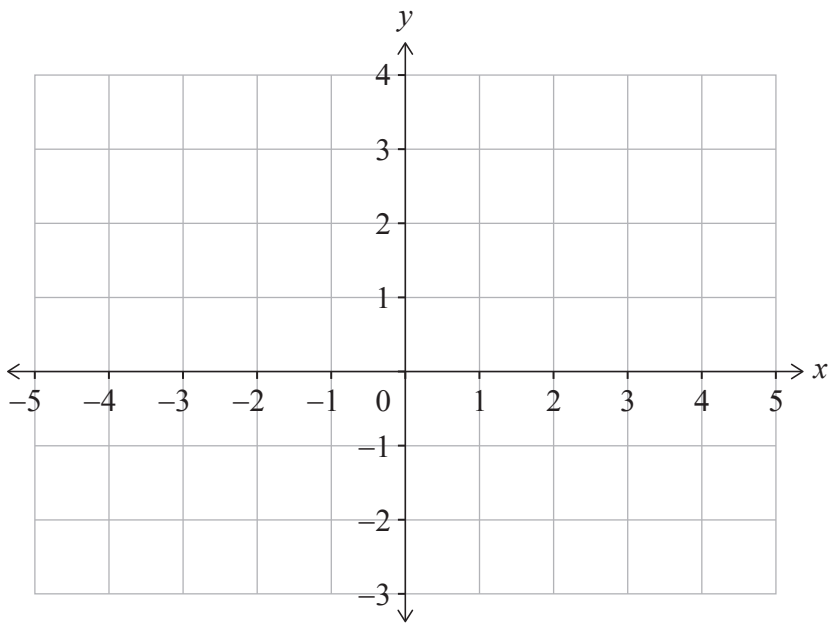
(i) sketch the graph $y = f(|x|)$ showing the approximate intercepts with the axes;



(This question continues on the following page)

(Question 5 continued)

(ii) sketch the graph $y = |f(x)|$ showing the approximate intercepts with the axes. [4]



(b) Solve the inequality $|f(x)| \geq \left(f(x)\right)^2$. [4]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Consider the quadratic polynomial given by $P(x) = 2x^2 + qx + r$ where $q, r \in \mathbb{R}$.

The equation $P(x) = 0$ has roots α and β, where $\alpha, \beta \in \mathbb{R}$.

When $P(x)$ is divided by $(x + 1)$, the remainder is 12.

Given that $\alpha^2 + \beta^2 = \frac{37}{4}$, find the value of q and the value of r.

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Question 6 - working space

APOLLO

Geometry and trigonometry

97 questions

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 8]

(a) Show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$. [2]

(b) Hence or otherwise, solve $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for $0 < x < 2\pi$. [6]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 4]

It is given that $\text{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\ \theta$.

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

(a) (i) Show that the point $(-1, 0, 3)$ lies on L_1.

(ii) Find a vector equation of L_1. [4]

Consider a second line L_2 defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$,

where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

(b) Find the possible values of a when the acute angle between L_1 and L_2 is $45°$. [8]

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

(c) Find the value of k, and find the coordinates of the point A in terms of a. [7]

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4,\ 0 \le x \le 2\pi$.

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 4]

Given any two non-zero vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

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Question 5 - working space

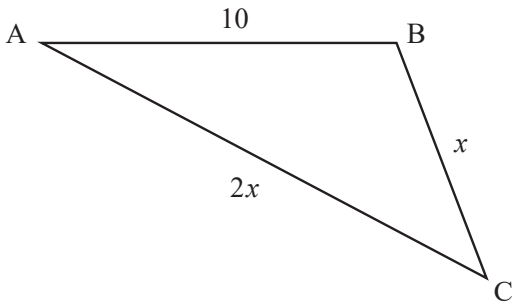
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

The following diagram shows triangle ABC, with $\mathrm{AB} = 10$, $\mathrm{BC} = x$ and $\mathrm{AC} = 2x$.

diagram not to scale



Given that $\cos \hat{\mathrm{C}} = \frac{3}{4}$, find the area of the triangle.

Give your answer in the form $\frac{p\sqrt{q}}{2}$ where $p, q \in \mathbb{Z}^+$.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 8]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$l_2 : \boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(a) Show that l_1 and l_2 do not intersect. [3]

(b) Find the minimum distance between l_1 and l_2. [5]

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Question 8 - working space

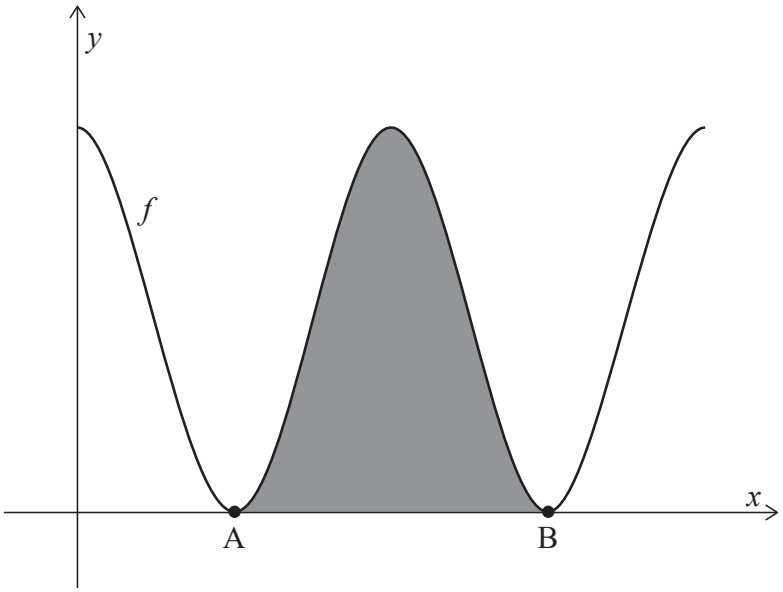
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \le x \le 4\pi$.

The following diagram shows the graph of $y = f(x)$.



The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

(a) Find the x-coordinates of A and B. [3]

(b) Show that the area of the shaded region is 12π. [5]

(This question continues on the following page)

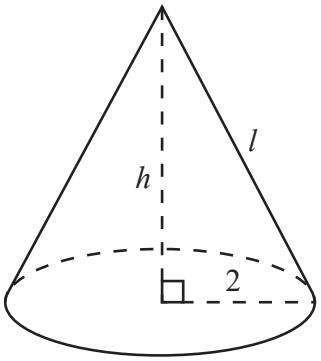
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(Question 10 continued)

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

diagram not to scale



(c) Find the value of l. [3]

(d) Hence, find the volume of the cone. [4]

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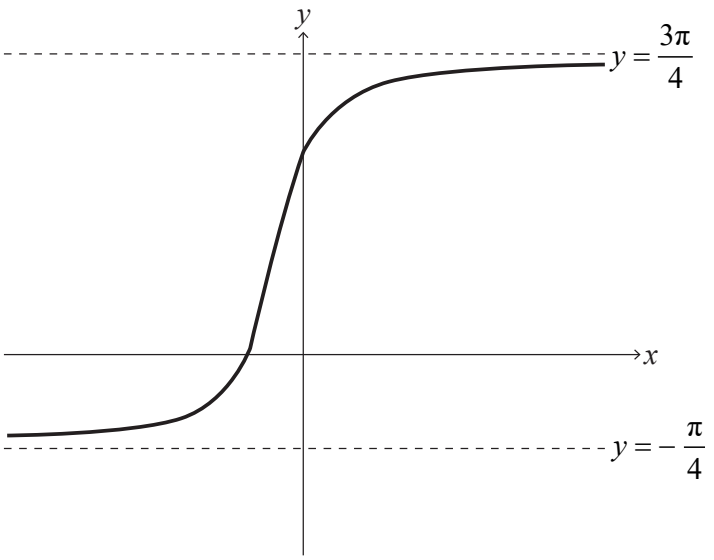
Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

The following diagram shows the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, with asymptotes at $y = -\frac{\pi}{4}$ and $y = \frac{3\pi}{4}$.



(a) Describe a sequence of transformations that transforms the graph of $y = \arctan x$ to the graph of $y = \arctan(2x+1) + \frac{\pi}{4}$ for $x \in \mathbb{R}$. [3]

(b) Show that $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where p, $q > 0$ and $pq < 1$. [4]

(c) Verify that $\arctan(2x+1) = \arctan\left(\frac{x}{x+1}\right) + \frac{\pi}{4}$ for $x \in \mathbb{R}$, $x > 0$. [3]

(d) Using mathematical induction and the result from part (b), prove that $\sum_{r=1}^{n} \arctan\left(\frac{1}{2r^2}\right) = \arctan\left(\frac{n}{n+1}\right)$ for $n \in \mathbb{Z}^+$. [9]

References:

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Question 12 - working space

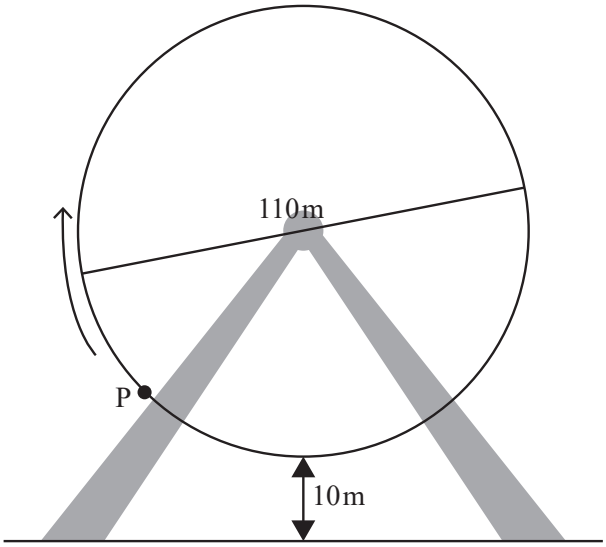
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.

diagram not to scale



The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where $a, b, c \in \mathbb{R}$.

Find the values of a, b and c.

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Turn over

Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

Consider the planes Π_1 and Π_2 with the following equations.

$\Pi_1 : 3x + 2y + z = 6$

$\Pi_2 : x - 2y + z = 4$

(a) Find a Cartesian equation of the plane Π_3 which is perpendicular to Π_1 and Π_2 and passes through the origin $(0, 0, 0)$. [3]

(b) Find the coordinates of the point where Π_1, Π_2 and Π_3 intersect. [2]

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Question 6 - working space

2. [Maximum mark: 28]

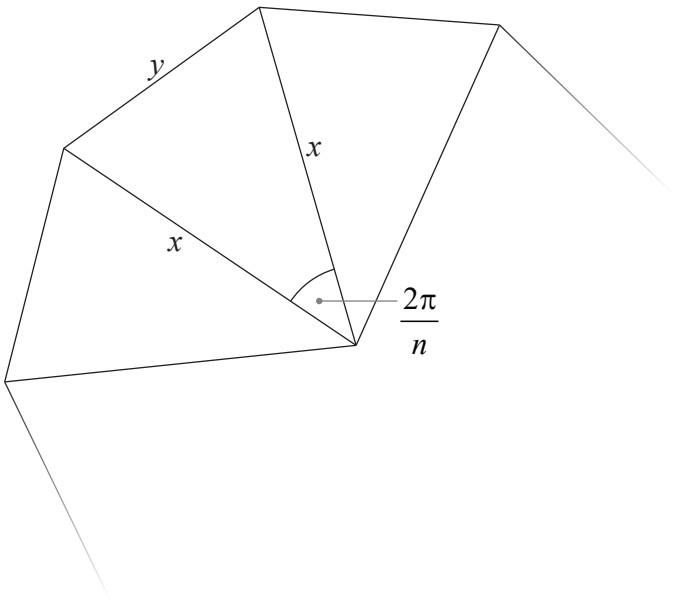
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P.

(a) Find the side length, s, where $s > 0$, of a square such that $A = P$. [3]

An n-sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n-sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n, an expression for the area, A_T, of one of these isosceles triangles. [1]

(c) Show that $y = 2x\sin\frac{\pi}{n}$. [2]

Consider a n-sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n\tan\frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x+\frac{x^3}{3}+\frac{2x^5}{15}+...$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2+b^2}$, where $a \geq b$, such that $A = P$.

(f) Show that $a=\frac{8}{b-4}+4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a, b, A, P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References:

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

(a) Show that $2x-3-\frac{6}{x-1}=\frac{2x^2-5x-3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta - 3 - \frac{6}{\sin 2\theta - 1} = 0$ for $0 \leq \theta \leq \pi$, $\theta \neq \frac{\pi}{4}$. [5]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider a triangle ABC, where $AC = 12$, $CB = 7$ and $B\hat{A}C = 25°$.

Find the smallest possible perimeter of triangle ABC.

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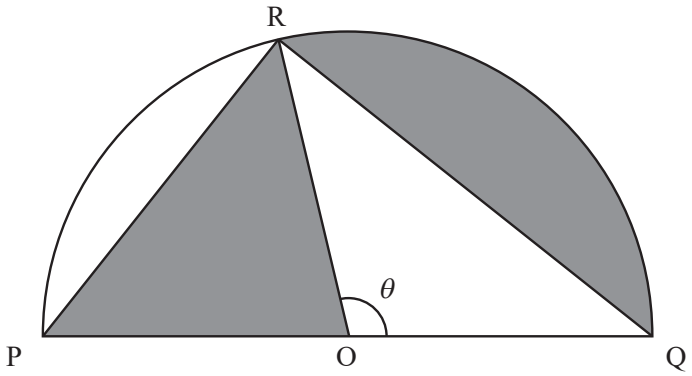
Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. [1]

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Question 4 - working space

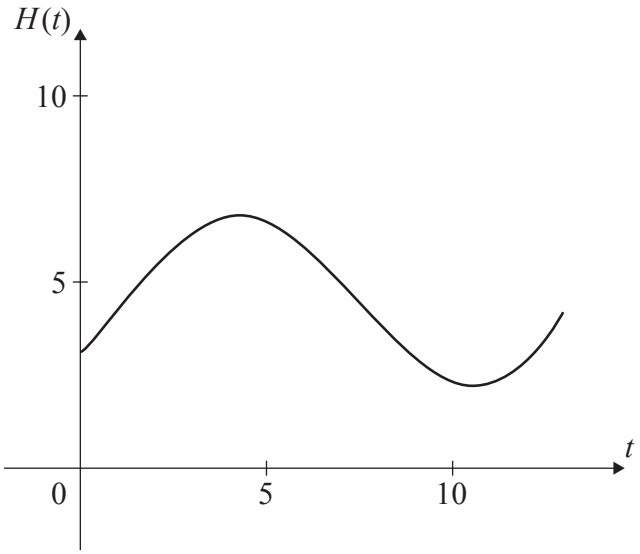
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin(b(t-c)) + d$, where t is the number of hours after midnight, and a, b, c and d are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\text{m}$ and $6.8\,\text{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \frac{\pi}{6}$. [1]

(b) Find the value of a. [2]

(c) Find the value of d. [2]

(d) Find the smallest possible value of c. [3]

(e) Find the height of the water at 12:00. [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 9 continued)

A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

(g) Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Three points $\mathrm{A}(3, 0, 0)$, $\mathrm{B}(0, -2, 0)$ and $\mathrm{C}(1, 1, -7)$ lie on the plane Π_1.

(a) (i) Find the vector $\vec{\mathrm{AB}}$ and the vector $\vec{\mathrm{AC}}$.

(ii) Hence find the equation of Π_1, expressing your answer in the form $ax + by + cz = d$, where $a, b, c, d \in \mathbb{Z}$. [7]

Plane Π_2 has equation $3x - y + 2z = 2$.

(b) The line L is the intersection of Π_1 and Π_2. Verify that the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. [2]

(c) The plane Π_3 is given by $2x - 2z = 3$. The line L and the plane Π_3 intersect at the point P.

(i) Show that at the point P, $\lambda = \frac{3}{4}$.

(ii) Hence find the coordinates of P. [3]

(d) The point $\mathrm{B}(0, -2, 0)$ lies on L.

(i) Find the reflection of the point B in the plane Π_3.

(ii) Hence find the vector equation of the line formed when L is reflected in the plane Π_3. [9]

References:

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Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3 \sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

The plane Π has the Cartesian equation $2x + y + 2z = 3$.

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

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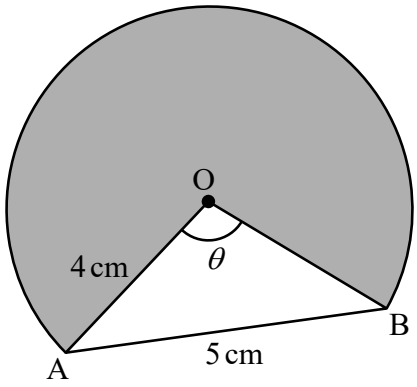
Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius $4\,\mathrm{cm}$.



Chord AB has a length of $5\,\mathrm{cm}$ and $\mathrm{A\hat{O}B} = \theta$.

(a) Find the value of θ, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$
$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1 - \tan^2 \theta}{2 \tan \theta}$. [1]

(b) Verify that $x = \tan \theta$ and $x = -\cot \theta$ satisfy the equation $x^2 + (2 \cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan \frac{\pi}{12} = 2 - \sqrt{3}$. [5]

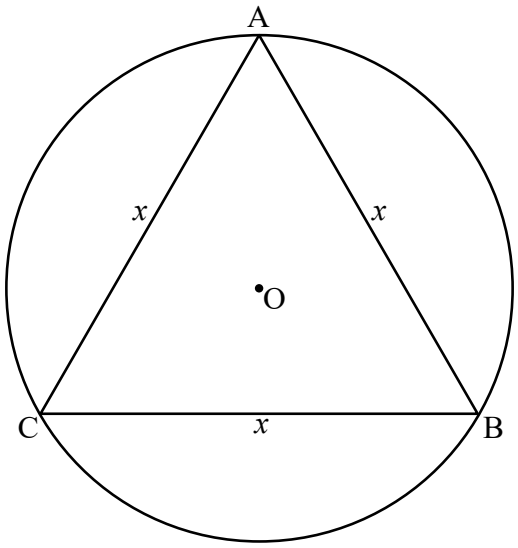
(d) Using the results from parts (b) and (c) find the exact value of $\tan \frac{\pi}{24} - \cot \frac{\pi}{24}$.
Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Question 12 - working space

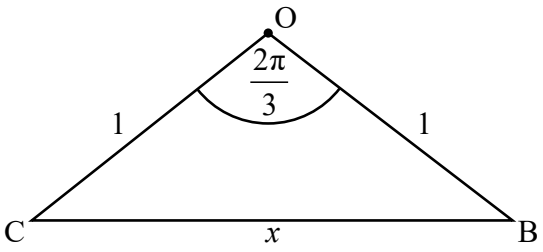
1. [Maximum mark: 30]

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

(a) Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram.



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units. [3]

(b) Consider a square of side length, x units, inscribed in a circle of radius 1 unit.
By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. [3]

(This question continues on the following page)

(Question 1 continued)

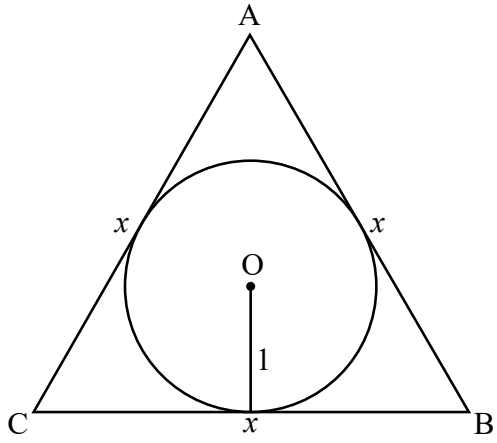
(c) Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. [2]

Let $P_i(n)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

(d) Show that $P_i(n) = 2n\sin\left(\frac{\pi}{n}\right)$. [3]

(e) Use an appropriate Maclaurin series expansion to find $\lim_{n\to\infty} P_i(n)$ and interpret this result geometrically. [5]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

(f) Show that $P_c(n) = 2n\tan\left(\frac{\pi}{n}\right)$. [4]

(g) By writing $P_c(n)$ in the form $\dfrac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}$, find $\lim_{n\to\infty} P_c(n)$. [5]

(h) Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. [2]

The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π.

(i) Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π. [3]

Turn over

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

Consider the functions $f(x) = \sqrt{3}\sin x + \cos x$ where $0 \le x \le \pi$ and $g(x) = 2x$ where $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve the equation $(f \circ g)(x) = 2\cos 2x$ where $0 \le x \le \pi$. [5]

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 15]

Consider the three planes

$$\Pi_1 : 2x - y + z = 4$$
$$\Pi_2 : x - 2y + 3z = 5$$
$$\Pi_3 : -9x + 3y - 2z = 32$$

(a) Show that the three planes do not intersect. [4]

(b) (i) Verify that the point $\mathrm{P}(1, -2, 0)$ lies on both Π_1 and Π_2.

(ii) Find a vector equation of L, the line of intersection of Π_1 and Π_2. [5]

(c) Find the distance between L and Π_3. [6]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$.

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Question 4 - working space

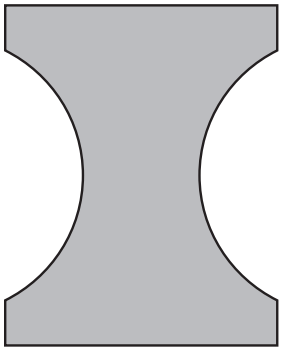
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

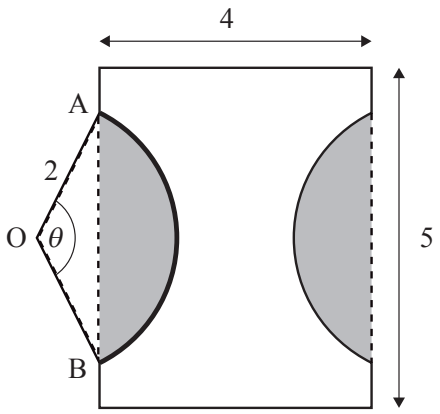
A company is designing a new logo. The logo is created by removing two equal segments from a rectangle, as shown in the following diagram.

diagram not to scale



The rectangle measures $5\,\text{cm}$ by $4\,\text{cm}$. The points A and B lie on a circle, with centre O and radius $2\,\text{cm}$, such that $\text{A}\hat{\text{O}}\text{B} = \theta$, where $0 < \theta < \pi$. This information is shown in the following diagram.

diagram not to scale



(a) Find the area of one of the shaded segments in terms of θ. [3]

(b) Given that the area of the logo is $13.4\,\text{cm}^2$, find the value of θ. [3]

(This question continues on the following page)

(Question 2 continued)

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 9]

Consider the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $|\boldsymbol{b}| = 15$.

(a) Find the possible range of values for $|\boldsymbol{a} + \boldsymbol{b}|$. [2]

Consider the vector $\boldsymbol{p}$ such that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$.

(b) Given that $|\boldsymbol{a} + \boldsymbol{b}|$ is a minimum, find $\boldsymbol{p}$. [2]

Consider the vector $\boldsymbol{q}$ such that $\boldsymbol{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x, y \in \mathbb{R}^+$.

(c) Find $\boldsymbol{q}$ such that $|\boldsymbol{q}| = |\boldsymbol{b}|$ and $\boldsymbol{q}$ is perpendicular to $\boldsymbol{a}$. [5]

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Question 7 - working space

Section A

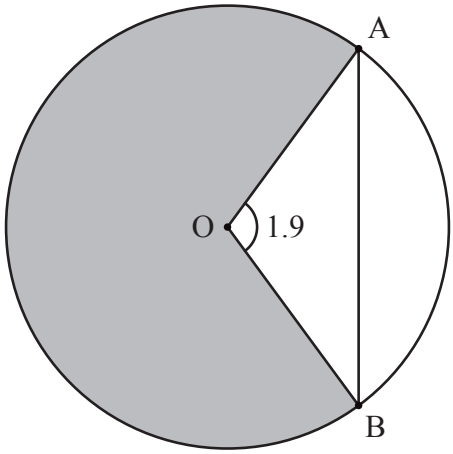
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius 5 metres.

Points A and B lie on the circle and $A\hat{O}B = 1.9$ radians.

diagram not to scale



(a) Find the length of the chord $[AB]$. [3]

(b) Find the area of the shaded sector. [3]

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Question 1 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

Two airplanes, A and B, have position vectors with respect to an origin O given respectively by

$$\boldsymbol{r}_A = \begin{pmatrix} 19 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -6 \\ 2 \\ 4 \end{pmatrix}$$

$$\boldsymbol{r}_B = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

where t represents the time in minutes and $0 \leq t \leq 2.5$.

Entries in each column vector give the displacement east of O, the displacement north of O and the distance above sea level, all measured in kilometres.

(a) Find the three-figure bearing on which airplane B is travelling. [2]

(b) Show that airplane A travels at a greater speed than airplane B. [2]

(c) Find the acute angle between the two airplanes' lines of flight. Give your answer in degrees. [4]

The two airplanes' lines of flight cross at point P.

(d) (i) Find the coordinates of P.

(ii) Determine the length of time between the first airplane arriving at P and the second airplane arriving at P. [7]

Let $D(t)$ represent the distance between airplane A and airplane B for $0 \leq t \leq 2.5$.

(e) Find the minimum value of $D(t)$. [5]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 7]

Consider a circle with a diameter AB, where A has coordinates $(1, 4, 0)$ and B has coordinates $(-3, 2, -4)$.

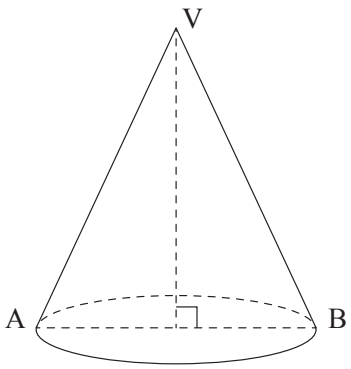
(a) Find

(i) the coordinates of the centre of the circle;

(ii) the radius of the circle. [4]

The circle forms the base of a right cone whose vertex V has coordinates $(-1, -1, 0)$.

diagram not to scale



(b) Find the exact volume of the cone. [3]

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Question 2 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 20]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x,\ 0 \le x \le \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f'(x) = 0$. [7]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f'(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

Consider the vectors $\boldsymbol{u} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = \left(\cos\frac{1}{n}\right)\boldsymbol{i} + \left(\sin\frac{1}{n}\right)\boldsymbol{j}$, where $n \in \mathbb{Z}^+$.

Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Find an expression for $\cos\theta$ in terms of n. [3]

(b) Find the exact value of the limit approached by θ as $n \to \infty$. [3]

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Question 7 - working space

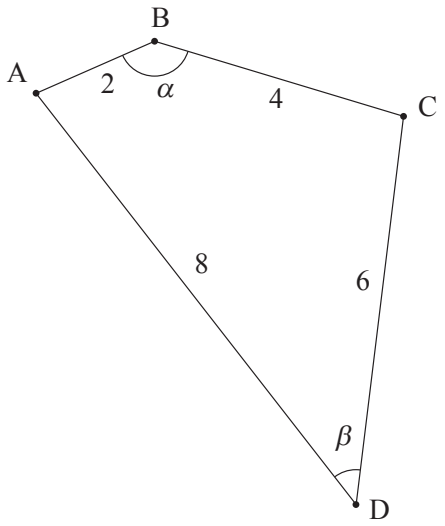
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider a quadrilateral $ABCD$ such that $AB = 2$, $BC = 4$, $CD = 6$ and $DA = 8$, as shown in the following diagram. Let $\alpha = A\hat{B}C$ and $\beta = A\hat{D}C$.

diagram not to scale



(a) (i) Find AC in terms of α.

(ii) Find AC in terms of β.

(iii) Hence or otherwise, find an expression for α in terms of β. [4]

(b) Find the maximum area of the quadrilateral $ABCD$. [4]

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Do **not** write solutions on this page.

Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

Consider the points $\mathrm{A}(1, 2, 3)$, $\mathrm{B}(k, -2, 1)$ and $\mathrm{C}(5, 0, 2)$, where $k \in \mathbb{R}$.

(a) Write down $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$. [2]

(b) Given that the points A, B and C lie on a straight line, show that $k = 9$. [1]

(c) For $k = 9$, let L_1 be the line passing through A, B and C.

(i) Find a vector equation of the line L_1.

(ii) Line L_2 has the equation $\frac{x-1}{2} = \frac{y}{3} = 1 - z$. Show that the lines L_1 and L_2 are skew. [10]

(d) For $k \neq 9$, let Π be the plane containing A, B and C.

(i) Find the Cartesian equation of the plane Π.

(ii) Find the coordinates of the point on the plane Π which is closest to the origin $(0, 0, 0)$. [9]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

Solve $\cos 2x = \sin x$, where $-\pi \leq x \leq \pi$.

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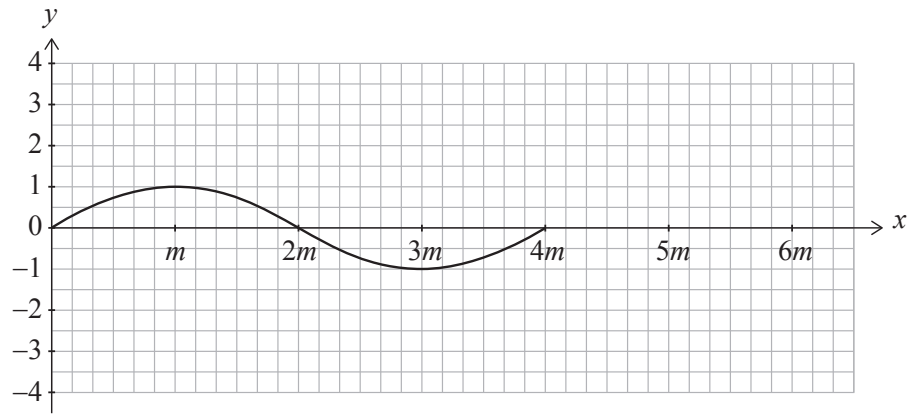
Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

The function f is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of f for $0 \le x \le 4m$, where x is in radians. There are x-intercepts at $x = 0$, $2m$ and $4m$.



(a) Find an expression for m in terms of q. [2]

The function g is defined by $g(x) = 3\sin\frac{2qx}{3}$, for $0 \le x \le 6m$.

(b) On the axes above, sketch the graph of g. [4]

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Question 5 - working space

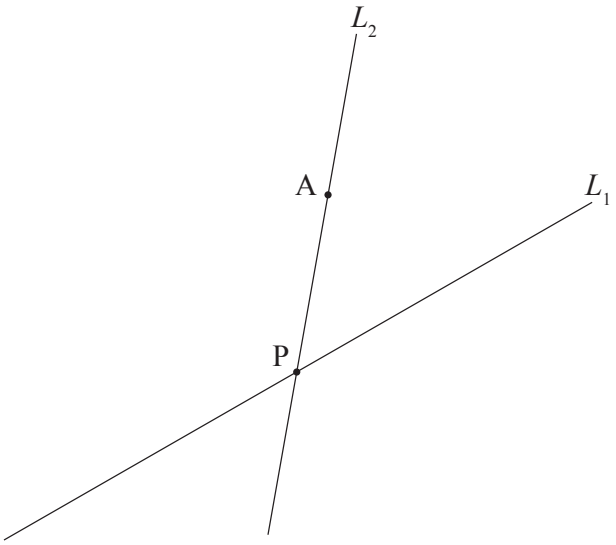
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Two lines, L_1 and L_2, intersect at point P. Point $\mathrm{A}(2t, 8, 3)$, where $t > 0$, lies on L_2. This is shown in the following diagram.

diagram not to scale



The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of L_1 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\overrightarrow{\mathrm{PA}} = \begin{pmatrix} 2t \\ 0 \\ 3+t \end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$. [4]

(b) Find the value of t. [4]

(c) Hence or otherwise, find the shortest distance from A to L_1. [4]

A plane, Π, contains L_1 and L_2.

(d) Find a normal vector to Π. [2]

The base of a right cone lies in Π, centred at A such that L_1 is a tangent to its base. The volume of the cone is $90\pi\sqrt{3}$ cubic units.

(e) Find the two possible positions of the vertex of the cone. [7]

References:

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Question 12 - working space

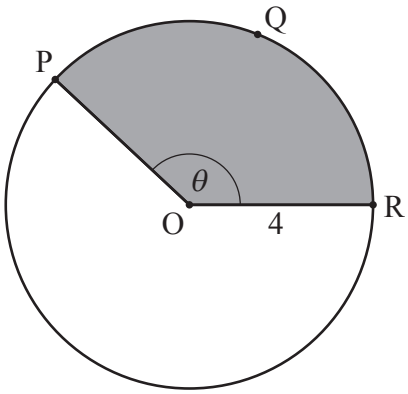
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius $4\,\mathrm{cm}$.

diagram not to scale



The points P, Q and R lie on the circumference of the circle and $\mathrm{P\hat{O}R} = \theta$, where θ is measured in radians.

The length of arc PQR is $10\,\mathrm{cm}$.

(a) Find the perimeter of the shaded sector. [2]

(b) Find θ. [2]

(c) Find the area of the shaded sector. [2]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

The functions f and g are defined by

$$f(x) = \cos x,\ 0 \leq x \leq \frac{\pi}{2}$$

$$g(x) = \tan x,\ 0 \leq x < \frac{\pi}{2}.$$

The curves $y = f(x)$ and $y = g(x)$ intersect at a point P whose x-coordinate is k, where $0 < k < \frac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

(b) Hence, show that the tangent to the curve $y = f(x)$ at P and the tangent to the curve $y = g(x)$ at P intersect at right angles. [3]

(c) Find the value of $\sin k$. Give your answer in the form $\frac{a + \sqrt{b}}{c}$, where $a,\ c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

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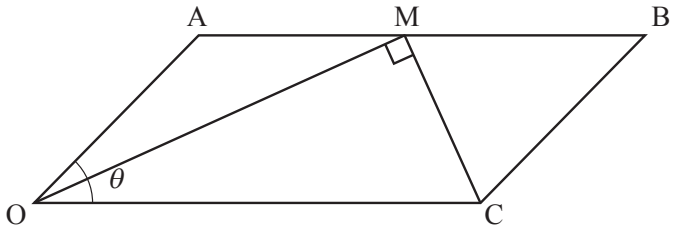
Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 9]

The following diagram shows parallelogram OABC with $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OC}} = \boldsymbol{c}$ and $|\boldsymbol{c}| = 2|\boldsymbol{a}|$, where $|\boldsymbol{a}| \neq 0$.



The angle between $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OC}}$ is θ, where $0 < \theta < \pi$.

Point M is on $[\mathrm{AB}]$ such that $\overrightarrow{\mathrm{AM}} = k\overrightarrow{\mathrm{AB}}$, where $0 \leq k \leq 1$ and $\overrightarrow{\mathrm{OM}} \bullet \overrightarrow{\mathrm{MC}} = 0$.

(a) Express $\overrightarrow{\mathrm{OM}}$ and $\overrightarrow{\mathrm{MC}}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$. [2]

(b) Hence, use a vector method to show that $|\boldsymbol{a}|^2 (1-2k)\big(2\cos\theta - (1-2k)\big) = 0$. [3]

(c) Find the range of values for θ such that there are two possible positions for M. [4]

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Question 9 - working space

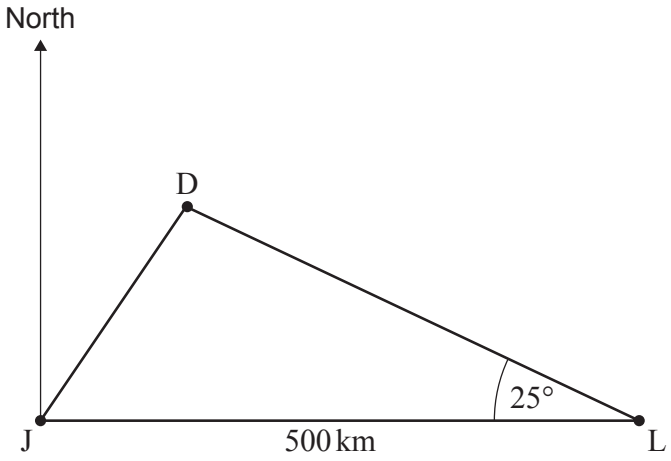
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The cities Lucknow (L), Jaipur (J) and Delhi (D) are represented in the following diagram. Lucknow lies $500\,\mathrm{km}$ directly east of Jaipur, and $\mathrm{JLD} = 25°$.

diagram not to scale



The bearing of D from J is $034°$.

(a) Find $\mathrm{J\hat{D}L}$. [2]

(b) Find the distance between Lucknow and Delhi. [3]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

The angle between a line and a plane is α, where $\alpha \in \mathbb{R},\ 0 < \alpha < \frac{\pi}{2}$.

The equation of the line is $\frac{x-1}{3} = \frac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

Find the value of α.

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Question 8 - working space

Section B

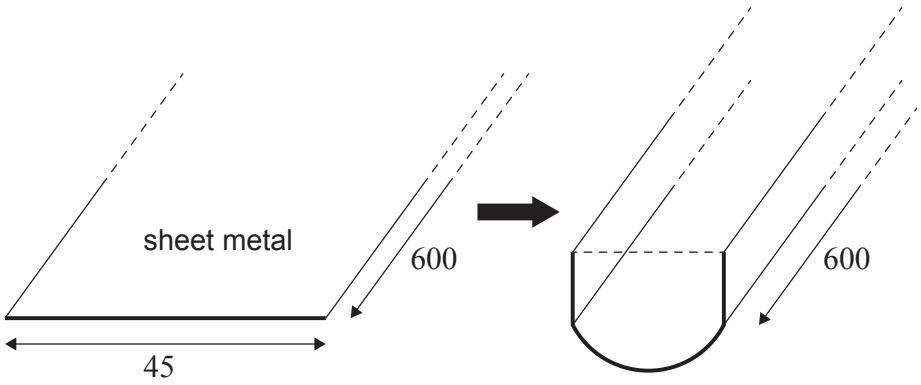
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

An engineer is designing a gutter to catch rainwater from the roof of a house.

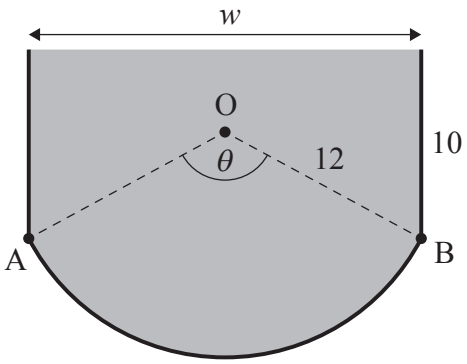
The gutter will be open at the top and is made by folding a piece of sheet metal $45\,\text{cm}$ wide and $600\,\text{cm}$ long.

diagram not to scale



The cross-section of the gutter is shaded in the following diagram.

diagram not to scale



The height of both vertical sides is $10\,\text{cm}$. The width of the gutter is $w\,\text{cm}$.

Arc AB lies on the circumference of a circle with centre O and radius $12\,\text{cm}$.

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

Let $\mathrm{A\hat{O}B} = \theta$ radians, where $0 < \theta < \pi$.

(a) Show that $\theta = 2.08$, correct to three significant figures. [3]

(b) Find the area of the cross-section of the gutter. [7]

In a storm, the total volume, in cm^3, of rainwater that enters the gutter can be modelled by a function $R(t)$, where t is the time, in seconds, since the start of the storm.

It was determined that the **rate** at which rainwater entered the gutter could be modelled by

$$R'(t) = 50\cos\left(\frac{2\pi t}{5}\right) + 3000\,,\ t \geq 0\,.$$

During any 60-second period, if the volume of rainwater entering the gutter is greater than the volume of the gutter, it will overflow.

(c) Determine whether the gutter overflowed in this storm. Justify your answer. [5]

Do **not** write solutions on this page.

Question 10 - working space

Section A

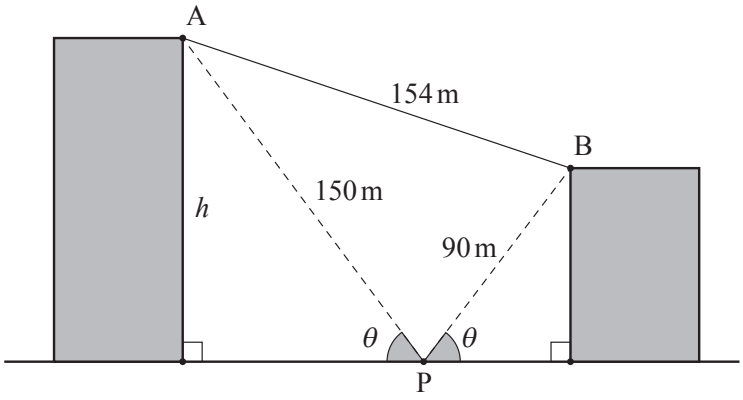
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

The following diagram shows two buildings situated on level ground.

From point P on the ground directly between the two buildings, the angle of elevation to the top of each building is θ.

diagram not to scale



The distance from point P to point A at the top of the taller building is 150 metres.

The distance from point P to point B at the top of the shorter building is 90 metres.

The distance between A and B is 154 metres.

Find the height, h, of the taller building.

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

Consider the two planes

$$\Pi_1 : 2x - y + 2z = 6$$

$$\Pi_2 : 4x + 3y - z = 2$$

Let L be the line of intersection of Π_1 and Π_2.

(a) Verify that a vector equation of L is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, where $\lambda \in \mathbb{R}$. [3]

(b) Find the coordinates of the point P on L that is nearest to the origin. [5]

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Question 6 - working space

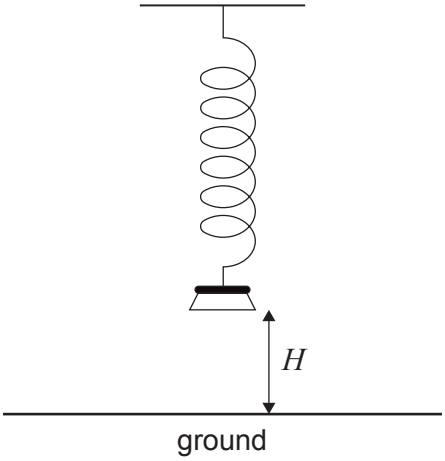
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 13]

A weight suspended on a spring is pulled down and released, so that it moves up and down vertically.

The height, H metres, of the base of the weight above the ground can be modelled by the function $H(t) = a\cos(7.8t) + b$, for $a, b \in \mathbb{R}$ and $0 \leq t \leq 10$, where t is the time in seconds after the weight is released.



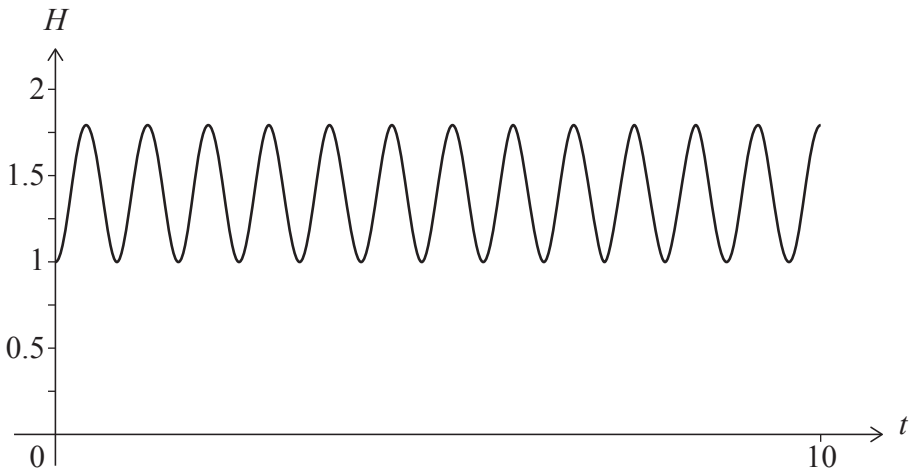
(a) Find the period of the function. [2]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The weight is released when its base is at a minimum height of 1 metre above the ground, and it reaches a maximum height of 1.8 metres above the ground. The graph of H is shown in the following diagram.



(b) Find the value of

(i) a;

(ii) b. [3]

(c) Find the number of times that the weight reaches its maximum height in the first five seconds of its motion. [2]

(d) Find the first time that the base of the weight reaches a height of 1.5 metres. [2]

A camera is set to take a picture of the weight at a random time during the first five seconds of its motion.

(e) Find the probability that the height of the base of the weight is greater than 1.5 metres at the time the picture is taken. [4]

Do **not** write solutions on this page.

Question 10 - working space

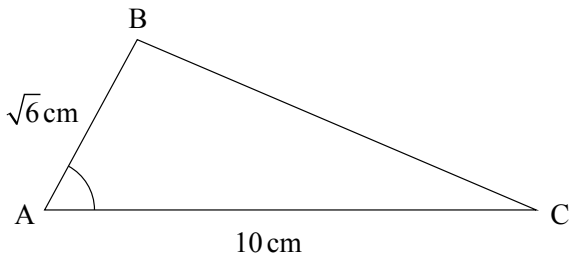
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

In the following triangle ABC, $\mathrm{AB} = \sqrt{6}$ cm, $\mathrm{AC} = 10\,\mathrm{cm}$ and $\cos \mathrm{B\hat{A}C} = \frac{1}{5}$.

diagram not to scale



Find the area of triangle ABC.

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 17]

(a) Find the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. [7]

Please **do not** write on this page.

Answers written on this page
will not be marked.

Question 12 - working space

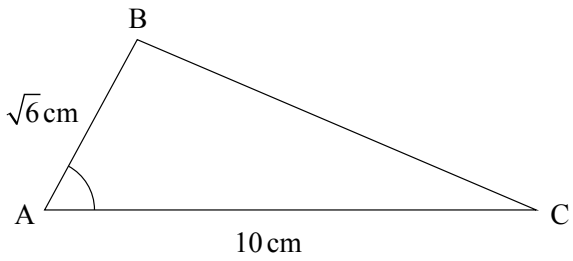
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

In the following triangle ABC, $\mathrm{AB} = \sqrt{6}$ cm, $\mathrm{AC} = 10\,\mathrm{cm}$ and $\cos \mathrm{B\hat{A}C} = \frac{1}{5}$.

diagram not to scale



Find the area of triangle ABC.

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 17]

(a) Find the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

(c) (i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\sin^4\theta - 20\sin^2\theta + 5 = 0$.

(ii) Hence, show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$. [7]

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Question 12 - working space

Section A

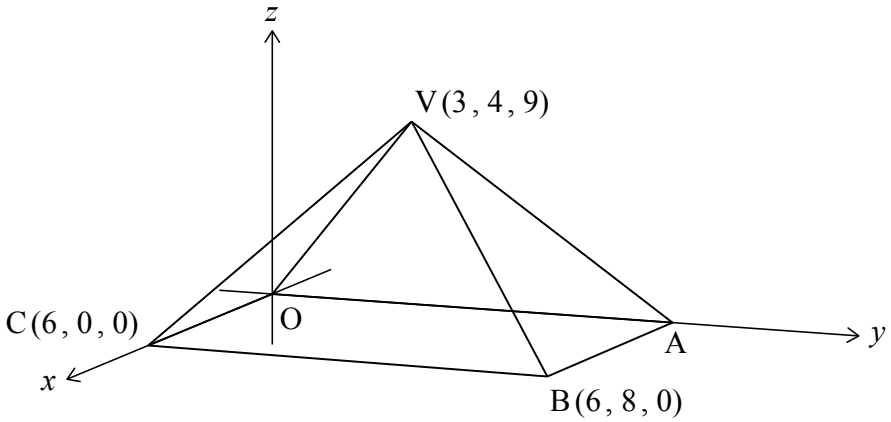
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a pyramid with vertex V and rectangular base OABC.

Point B has coordinates $(6, 8, 0)$, point C has coordinates $(6, 0, 0)$ and point V has coordinates $(3, 4, 9)$.

diagram not to scale



(a) Find BV. [2]

(b) Find the size of $\mathrm{B\hat{V}C}$. [4]

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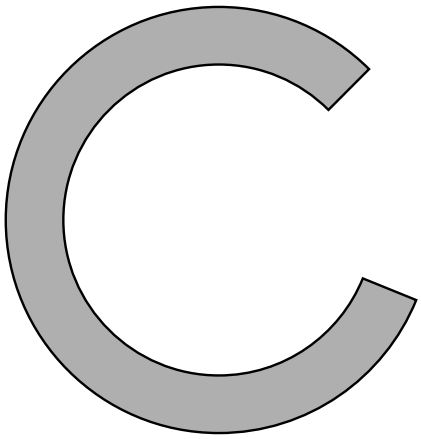
Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

A company is designing a new logo in the shape of a letter “C”.



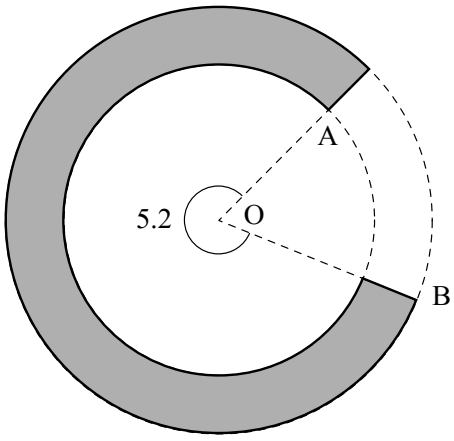
The letter “C” is formed between two circles with centre O.

The point A lies on the circumference of the inner circle with radius $r\,\mathrm{cm}$, where $r < 10$.

The point B lies on the circumference of the outer circle with radius $10\,\mathrm{cm}$.

The reflex angle $\mathrm{A\hat{O}B}$ is 5.2 radians. The letter “C” is shown by the shaded area in the following diagram.

diagram not to scale



(This question continues on the following page)

(Question 3 continued)

(a) Show that the area of the "C" is given by $260 - 2.6r^2$. [2]

The area of the "C" is $64\,\text{cm}^2$.

(b) (i) Find the value of r.

(ii) Find the perimeter of the "C". [5]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 9]

Three points are given by $\mathrm{A}(0, p, 2)$, $\mathrm{B}(1, 1, 1)$ and $\mathrm{C}(p, 0, 4)$, where p is a positive constant.

(a) Show that $\vec{\mathrm{AB}} \times \vec{\mathrm{AC}} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$. [4]

(b) Hence, find the smallest possible value of $\left|\vec{\mathrm{AB}} \times \vec{\mathrm{AC}}\right|^2$. [3]

(c) Hence, find the smallest possible area of triangle ABC. [2]

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 18]

Line L is given by the vector equation $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $s \in \mathbb{R}$.

Line M is given by the vector equation $\boldsymbol{r}_2 = \begin{pmatrix} 9 \\ 9 \\ 11 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ where $t \in \mathbb{R}$.

(a) Show that lines L and M intersect at a point A and find the position vector of A. [5]

(b) Verify that the lines L and M both lie in the plane Π given by $\boldsymbol{r} \bullet \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$. [3]

Point B has position vector $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$. A line through B perpendicular to Π intersects Π at point C.

(c) (i) Find the position vector of C.

(ii) Hence, find $\left|\vec{\mathrm{BC}}\right|$. [7]

(d) Find the reflection of the point B in the plane Π. [3]

Question 12 - working space

Section A

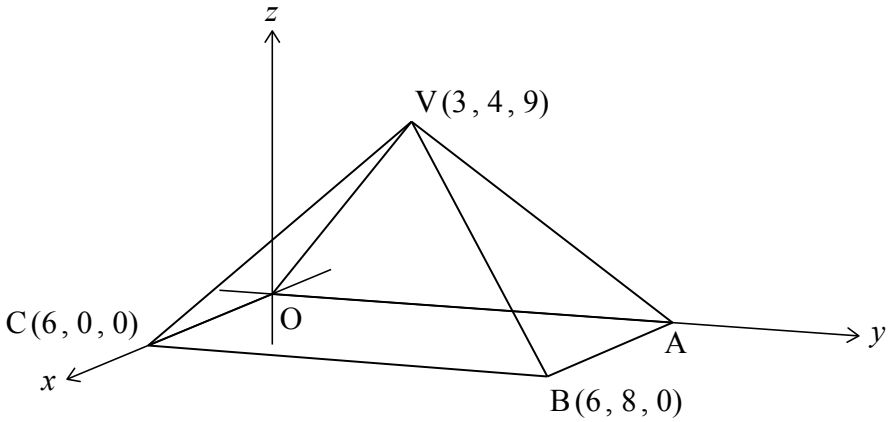
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1. [Maximum mark: 6]

The following diagram shows a pyramid with vertex V and rectangular base OABC.

Point B has coordinates $(6, 8, 0)$, point C has coordinates $(6, 0, 0)$ and point V has coordinates $(3, 4, 9)$.

diagram not to scale



(a) Find BV. [2]

(b) Find the size of $\mathrm{B\hat{V}C}$. [4]

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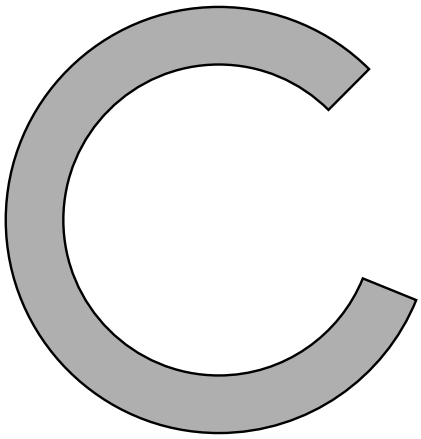
Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

A company is designing a new logo in the shape of a letter “C”.



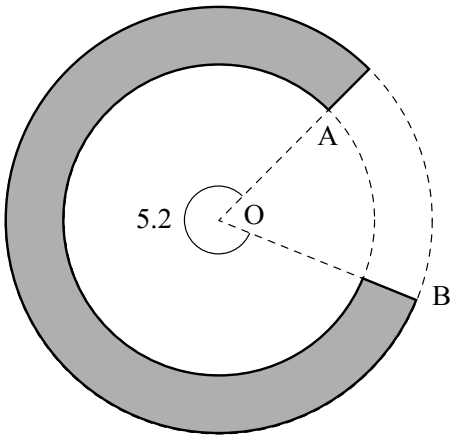
The letter “C” is formed between two circles with centre O.

The point A lies on the circumference of the inner circle with radius $r\,\mathrm{cm}$, where $r < 10$.

The point B lies on the circumference of the outer circle with radius $10\,\mathrm{cm}$.

The reflex angle $\mathrm{A\hat{O}B}$ is 5.2 radians. The letter “C” is shown by the shaded area in the following diagram.

diagram not to scale



(This question continues on the following page)

(Question 3 continued)

(a) Show that the area of the "C" is given by $260 - 2.6r^2$. [2]

The area of the "C" is $64\,\text{cm}^2$.

(b) (i) Find the value of r.

(ii) Find the perimeter of the "C". [5]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 9]

Three points are given by $\mathrm{A}(0, p, 2)$, $\mathrm{B}(1, 1, 1)$ and $\mathrm{C}(p, 0, 4)$, where p is a positive constant.

(a) Show that $\vec{\mathrm{AB}} \times \vec{\mathrm{AC}} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$. [4]

(b) Hence, find the smallest possible value of $\left|\vec{\mathrm{AB}} \times \vec{\mathrm{AC}}\right|^2$. [3]

(c) Hence, find the smallest possible area of triangle ABC. [2]

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 18]

Line L is given by the vector equation $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ where $s \in \mathbb{R}$.

Line M is given by the vector equation $\boldsymbol{r}_2 = \begin{pmatrix} 9 \\ 9 \\ 11 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ where $t \in \mathbb{R}$.

(a) Show that lines L and M intersect at a point A and find the position vector of A. [5]

(b) Verify that the lines L and M both lie in the plane Π given by $\boldsymbol{r} \bullet \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$. [3]

Point B has position vector $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$. A line through B perpendicular to Π intersects Π at point C.

(c) (i) Find the position vector of C.

(ii) Hence, find $\left|\vec{\mathrm{BC}}\right|$. [7]

(d) Find the reflection of the point B in the plane Π. [3]

Question 12 - working space

Section A

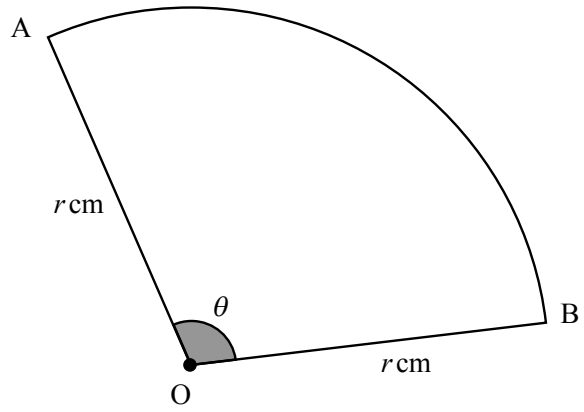
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 8]

Points A and B lie on the circumference of a circle of radius $r\,\mathrm{cm}$ with centre at O.

The sector OAB is shown on the following diagram. The angle $\mathrm{A\hat{O}B}$ is denoted as θ and is measured in radians.

diagram not to scale



The perimeter of the sector is $10\,\mathrm{cm}$ and the area of the sector is $6.25\,\mathrm{cm}^2$.

(a) Show that $4r^2 - 20r + 25 = 0$. [4]

(b) Hence, or otherwise, find the value of r and the value of θ. [4]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Solve $\tan(2x - 5°) = 1$ for $0° \leq x \leq 180°$.

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Question 1 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

The plane Π_1 has equation $2x + 6y - 2z = 5$.

(a) Verify that the point $\mathrm{A}\left(2, \frac{1}{2}, 1\right)$ lies on the plane Π_1. [1]

The plane Π_2 is given by $(k^2 - 6)x + (2k + 3)y + pz = q$, where p, q, $k \in \mathbb{R}$ and $p \neq 0$.

(b) In the case where $p = -6$, Π_2 is perpendicular to Π_1 and A lies on Π_2.
Find the value of k and the value of q. [5]

For parts (c), (d) and (e) it is now given that Π_2 is parallel to Π_1 with $k = 3$.

(c) Determine the value of p. [2]

It is also given that $q = -\frac{51}{2}$.

The line through A that is perpendicular to Π_1 meets Π_2 at the point B.

(d) (i) Find the coordinates of B.

(ii) Hence, show that the perpendicular distance between Π_1 and Π_2 is $\sqrt{11}$. [7]

(e) Find the equation of a third parallel plane Π_3 which is also a perpendicular distance of $\sqrt{11}$ from Π_1. [4]

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Do **not** write solutions on this page.

Question 11 - working space

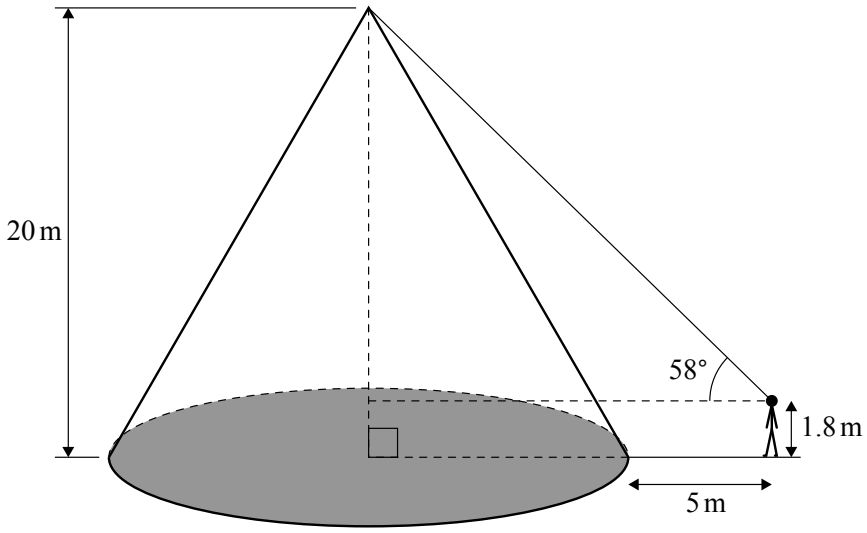
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

A monument is in the shape of a right cone with a vertical height of 20 metres. Oliver stands 5 metres from the base of the monument. His eye level is 1.8 metres above the ground and the angle of elevation from Oliver's eye level to the vertex of the cone is $58°$, as shown on the following diagram.

diagram not to scale



(a) Find the radius of the base of the cone. [3]

(b) Find the volume of the monument. [2]

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Question 2 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

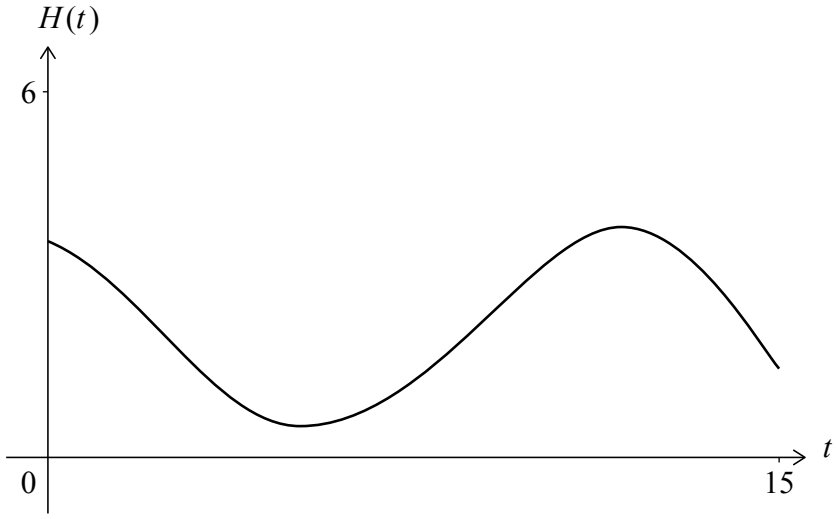
Sule Skerry and Rockall are small islands in the Atlantic Ocean, in the same time zone.

On a given day, the height of water in metres at Sule Skerry is modelled by the function $H(t) = 1.63\sin(0.513(t - 8.20)) + 2.13$, where t is the number of hours after midnight.

The following graph shows the height of the water for 15 hours, starting at midnight.

At low tide the height of the water is $0.50\,\text{m}$. At high tide the height of the water is $3.76\,\text{m}$.

All heights are given correct to two decimal places.



(a) The length of time between the first low tide and the first high tide is 6 hours and m minutes. Find the value of m to the nearest integer. [3]

(b) Between two consecutive high tides, determine the length of time, in hours, for which the height of the water is less than 1 metre. [2]

(c) Find the rate of change of the height of the water when $t = 13$, giving your answer in metres per hour. [2]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

On the same day, the height of water at the second island, Rockall, is modelled by the function $h(t) = a\sin\big(b(t-c)\big) + d$, where t is the number of hours after midnight, and $a, b, c, d > 0$.

The first low tide occurs at 02:41 when the height of the water is $0.40\,\text{m}$.

The first high tide occurs at 09:02 when the height of the water is $2.74\,\text{m}$.

(d) Find the values of a, b, c and d. [7]

When $t = T$, the height of the water at Sule Skerry is the same as the height of the water at Rockall for the first time.

(e) Find the value of T. [2]



Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider the non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) Using the definitions of $\boldsymbol{u} \bullet \boldsymbol{v}$ and $\boldsymbol{u} \times \boldsymbol{v}$ in terms of $|\boldsymbol{u}|$, $|\boldsymbol{v}|$ and θ, show that $(\boldsymbol{u} \bullet \boldsymbol{v})^2 + |\boldsymbol{u} \times \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 |\boldsymbol{v}|^2$. [2]

A triangle ABC has vertices $\mathrm{A}(0, 1, 2)$, $\mathrm{B}(p, q, 3)$ and $\mathrm{C}(3, 2, 1)$, $p, q \in \mathbb{Q}$.

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined as $\boldsymbol{u} = \overrightarrow{\mathrm{AB}}$ and $\boldsymbol{v} = \overrightarrow{\mathrm{AC}}$.

It is given that $\boldsymbol{u} \bullet \boldsymbol{v} = 3$ and the area of triangle ABC is $\sqrt{6}$.

(b) (i) Find the value of $|\boldsymbol{u} \times \boldsymbol{v}|$.

(ii) Hence, or otherwise, find the value of $|\boldsymbol{u}|$.

(iii) Hence, or otherwise, find the possible values of p and the corresponding values of q. [13]

Consider a new point D, the vector $\boldsymbol{w}$ is defined as $\boldsymbol{w} = \overrightarrow{\mathrm{CD}}$.

It is given that $\boldsymbol{u} \bullet \boldsymbol{w} = \boldsymbol{v} \bullet \boldsymbol{w} = 0$ and the area of triangle ACD is 5 square units.

(c) Assuming that $p = 1$, find the possible vectors for $\boldsymbol{w}$. [5]

Please **do not** write on this page.

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will not be marked.

Question 12 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

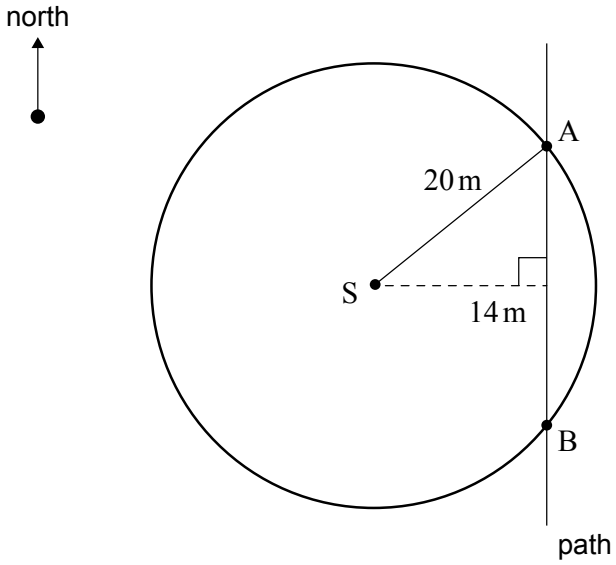
A rotating sprinkler is at a fixed point S.

It waters all points inside and on a circle of radius 20 metres.

Point S is 14 metres from the edge of a path which runs in a north-south direction.

The edge of the path intersects the circle at points A and B.

This information is shown in the following diagram.



(a) Show that $\mathrm{AB} = 28.57$, correct to four significant figures. [3]

The sprinkler rotates at a constant rate of one revolution every 16 seconds.

(b) Show that the sprinkler rotates through an angle of $\frac{\pi}{8}$ radians in one second. [1]

Let T seconds be the time that $[\mathrm{AB}]$ is watered in each revolution.

(c) Find the value of T. [4]

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(Question 11 continued)

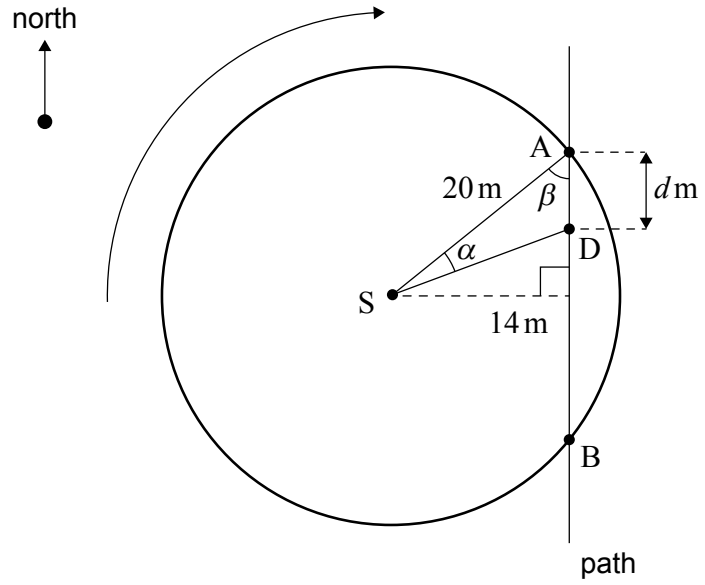
Consider one clockwise revolution of the sprinkler.

At $t = 0$, the water crosses the edge of the path at A.

At time t seconds, the water crosses the edge of the path at a movable point D which is a distance d metres south of point A.

Let $\alpha = \mathrm{A\hat{S}D}$ and $\beta = \mathrm{S\hat{A}B}$, where α, β are measured in radians.

This information is shown in the following diagram.



(d) Write down an expression for α in terms of t. [1]

It is known that $\beta = 0.7754$ radians, correct to four significant figures.

(e) By using the sine rule in $\Delta\mathrm{ASD}$, show that the distance, d, at time t, can be modelled by

$$d(t) = \frac{20\sin\left(\frac{\pi t}{8}\right)}{\sin\left(2.37 - \frac{\pi t}{8}\right)}.$$ [3]

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(Question 11 continued)

A turtle walks south along the edge of the path.

At time t seconds, the turtle's distance, g metres south of A, can be modelled by

$$g(t) = 0.05t^2 + 1.1t + 18, \text{ where } t \geq 0.$$

(f) At $t = 0$, state how far south the turtle is from A. [1]

Let w represent the distance between the turtle and point D at time t seconds.

(g) (i) Use the expressions for $g(t)$ and $d(t)$ to write down an expression for w in terms of t.

(ii) Hence find when and where on the path the water first reaches the turtle. [4]

Do **not** write solutions on this page.

Question 11 - working space

Section A

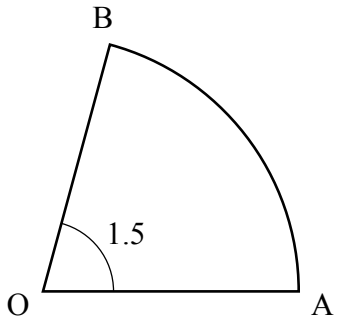
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Points A and B lie on a circle with centre O and radius $r\,\text{cm}$, where $A\hat{O}B = 1.5$ radians.

This is shown on the following diagram.

diagram not to scale



The area of sector OAB is $48\,\text{cm}^2$.

(a) Find the value of r. [3]

(b) Hence, find the perimeter of sector OAB. [2]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

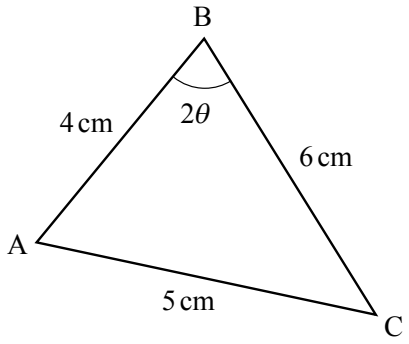
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

The following diagram shows triangle ABC, where $AB = 4\,\text{cm}$, $BC = 6\,\text{cm}$, $AC = 5\,\text{cm}$ and $A\hat{B}C = 2\theta$.

diagram not to scale



Find the exact value of $\cos\theta$, giving your answer in the form $\frac{p\sqrt{2}}{q}$, where $p, q \in \mathbb{Z}^+$.

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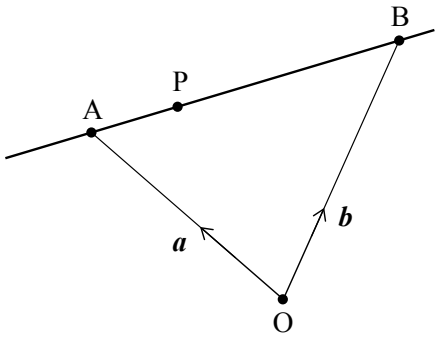
Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 8]

The following diagram shows two points A and B such that $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OB}} = \boldsymbol{b}$.



The point P lies on (AB) so that $\overrightarrow{\mathrm{AP}} = \lambda \overrightarrow{\mathrm{AB}}$ where $0 < \lambda < 1$.

(a) Show that $\overrightarrow{\mathrm{OP}} = (1-\lambda)\boldsymbol{a} + \lambda \boldsymbol{b}$. [1]

It is given that $|\boldsymbol{a}| = 1$, $|\boldsymbol{b}| = 2$ and $\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{4}$.

(b) In the case that $\overrightarrow{\mathrm{OP}}$ is perpendicular to $\overrightarrow{\mathrm{AB}}$, find the value of λ. [7]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 9]

(a) Prove that $\tan\left(\theta - \frac{\pi}{4}\right) \equiv \frac{\sin 2\theta - 1}{\cos 2\theta}$, where $\theta \neq \frac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$. [6]

(b) Hence, or otherwise, solve $\frac{\sin x - 1}{\cos x} = \sqrt{3}$ for $0 \leq x \leq 2\pi$. [3]

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Question 9 - working space

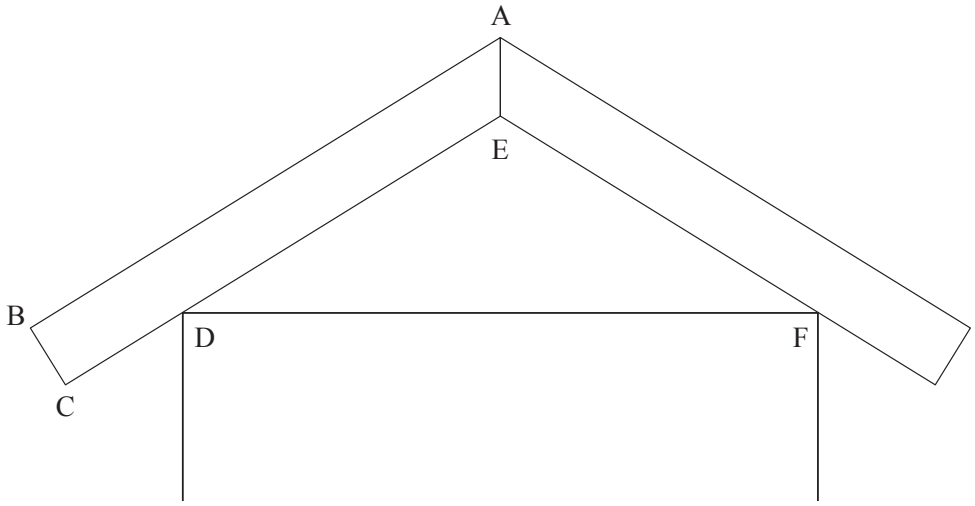
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 8]

The following diagram shows the cross section of the roof of a house. The cross section is symmetrical about the vertical line through points A and E.

diagram not to scale



The gradient of $[\mathrm{BA}]$ is $\frac{7}{12}$.

(a) Find the size of $\mathrm{B\hat{A}E}$, expressing your answer in degrees. [3]

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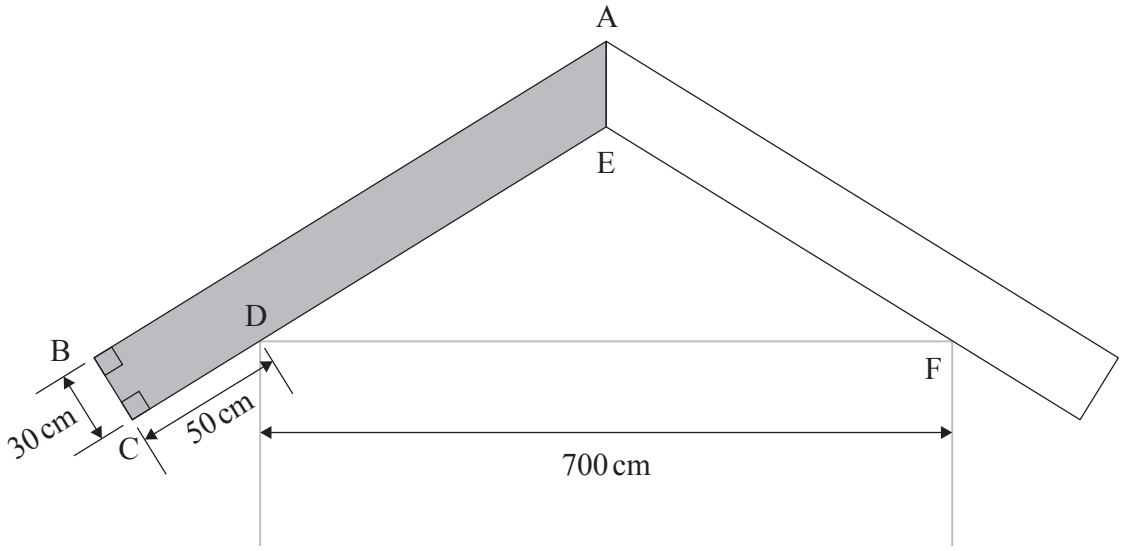
(Question 4 continued)

A builder requires the lengths of the sides $[BA]$ and $[CE]$.

The builder has the following measurements:

$A\hat{B}C = B\hat{C}E = 90°$, $DC = 50\,\text{cm}$, $BC = 30\,\text{cm}$, and $DF = 700\,\text{cm}$.

diagram not to scale



(b) Find

(i) CE;

(ii) BA. [5]

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 18]

A line L is defined by $L: \ -\frac{x}{2}+1=y+4=\frac{z}{3}$.

(a) Find the equation of L, expressing your answer in the form $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$, where $\lambda \in \mathbb{R}$. [3]

(b) Determine the minimum distance from the origin O to the line L. [5]

A plane Π is defined by $\Pi: \ 6x-3y+5z=24$.

(c) Verify that Π contains L. [3]

A second line M is parallel to Π.

The line M passes through the point $(4, 1, 2)$ and intersects the z-axis.

(d) Find the equation of M, expressing your answer in the form $\boldsymbol{s}=\boldsymbol{c}+\mu\boldsymbol{d}$, where $\mu \in \mathbb{R}$. [7]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

The line L_1 has vector equation $\boldsymbol{r} = 4\boldsymbol{i} - \boldsymbol{k} + \lambda(a\boldsymbol{j} + \boldsymbol{k})$, where $a, \lambda \in \mathbb{R}$.

The line L_2 has vector equation $\boldsymbol{r} = \boldsymbol{i} - b\boldsymbol{k} + \mu(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$, where $b, \mu \in \mathbb{R}$.

The lines L_1 and L_2 are perpendicular and intersect at a unique point.

Find the value of a and the value of b.

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

The plane Π_1 has equation $x + 2y + z = 0$ and the plane Π_2 has equation $x - y - 2z = 0$.

The acute angle between the planes Π_1 and Π_2 is θ.

(a) Show that $\theta = 60°$. [6]

A third plane Π_3 is perpendicular to both Π_1 and Π_2.

The unique point of intersection of all three planes is the point $\mathrm{R}(5, -5, 5)$.

(b) Find the Cartesian equation of Π_3. [4]

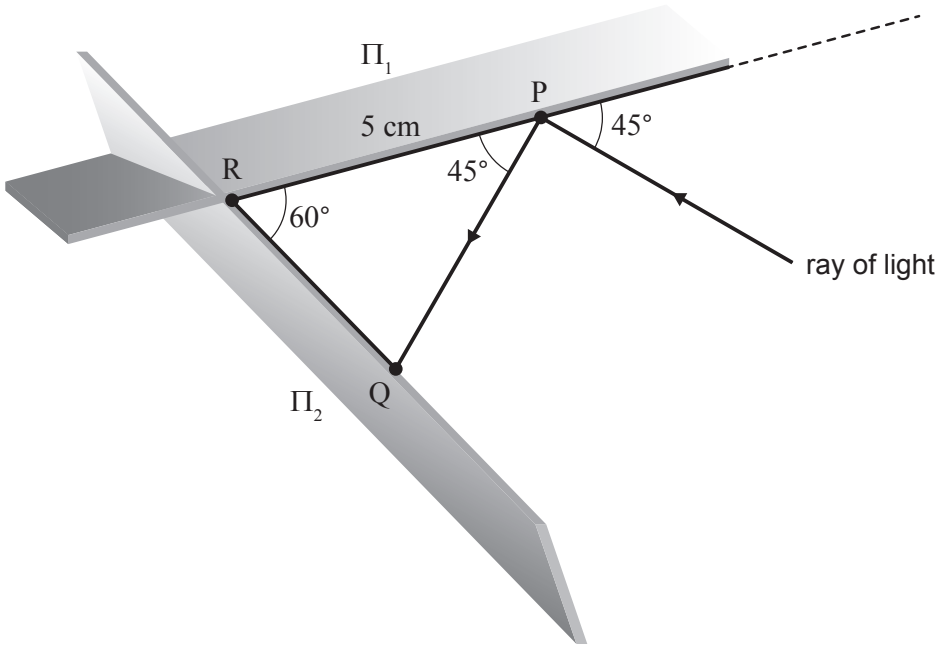
Each of the planes Π_1 and Π_2 contains a mirror.

A ray of light is directed towards the mirror in Π_1. The ray of light forms an angle of $45°$ with Π_1 and meets it at the point P.

The ray of light is then reflected towards the mirror in Π_2, and meets Π_2 at the point Q. The points P and Q are contained in Π_3.

It is given that $\mathrm{PR} = 5\,\mathrm{cm}$.

This information is shown on the following diagram.



(This question continues on the following page)

Do **not** write solutions on this page.

(Question 11 continued)

(c) (i) Using an appropriate compound angle identity, show that $\sin 75^\circ = \frac{\sqrt{2}+\sqrt{6}}{4}$.

(ii) Find QR, giving your answer in the form $p\left(\sqrt{q}-1\right)\text{cm}$ where $p, q \in \mathbb{Z}$. [7]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

The line L_1 is defined by the Cartesian equation $\frac{x-1}{2} = \frac{y+2}{3} = z$.

(a) Find a vector equation of L_1. [2]

A second line L_2 is defined by the vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 4 \\ -8 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, where $t \in \mathbb{R}$.

(b) Find the coordinates of the point where L_1 and L_2 intersect. [3]

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Question 2 - working space

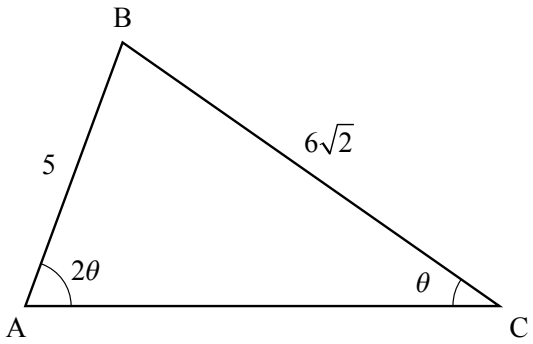
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

The following diagram shows a non-right angled triangle ABC.

diagram not to scale



$\mathrm{AB} = 5$, $\mathrm{BC} = 6\sqrt{2}$, $\mathrm{A\hat{C}B} = \theta$ and $\mathrm{B\hat{A}C} = 2\theta$, where $0 < \theta < \frac{\pi}{2}$.

(a) Using the sine rule, show that $\cos\theta = \frac{3\sqrt{2}}{5}$. [3]

(b) Hence, find $\sin\theta$. [2]

Point D is located on $[\mathrm{AC}]$ such that the area of triangle BCD is $2\sqrt{14}$.

(c) Find DC. [2]

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Question 3 - working space

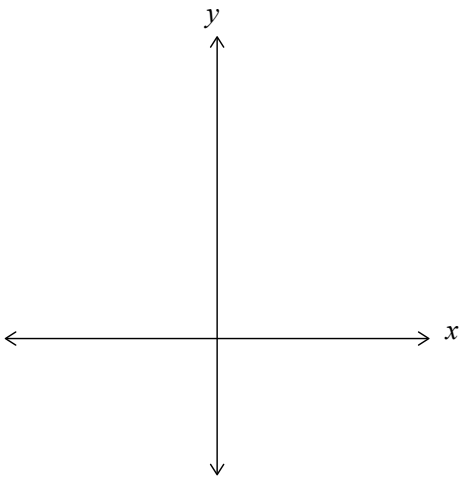
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 8]

Consider the function $f(x) = \arccos x$ for $-1 \leq x \leq 1$.

(a) On the set of axes below sketch the graph of $y = f(x)$.
On your sketch clearly indicate the y-intercept and coordinates of the end points. [2]



(b) Solve $\arccos(x) + \arccos\left(x\sqrt{3}\right) = \frac{3\pi}{2}$, for $-\frac{1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$. [6]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

Solve the equation $2\cos 2\theta - 5\cos\theta + 2 = 0$, where $\pi \leq \theta \leq 2\pi$.

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 18]

The points $\mathrm{A}(1, -4, 0)$, $\mathrm{B}(-3, -6, 2)$, $\mathrm{C}(-1, -2, 4)$ and D form a parallelogram, ABCD, where D is diagonally opposite B.

(a) Find the coordinates of D. [2]

The diagonals of the parallelogram, $[\mathrm{AC}]$ and $[\mathrm{BD}]$, intersect at point E.

(b) Find the coordinates of E. [2]

(c) (i) Given that $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AD}} = m\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, where $m \in \mathbb{Z}^+$, find the value of m.

(ii) Hence, find the area of parallelogram ABCD. [4]

The plane, Π_1, contains the parallelogram ABCD.

(d) Find the Cartesian equation of Π_1. [2]

A second plane, Π_2, has Cartesian equation $5x + y - 7z = 1$.

The acute angle between Π_1 and Π_2 is θ.

(e) Show that $\cos\theta = \frac{1}{5}$. [3]

The line L passes through E and is perpendicular to Π_1.

The line L intersects the plane Π_2 at point F.

(f) Find the coordinates of F. [5]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

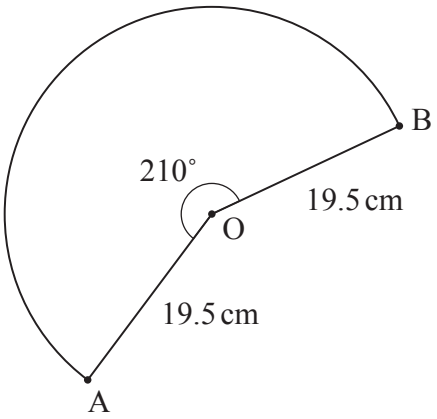
1. [Maximum mark: 6]

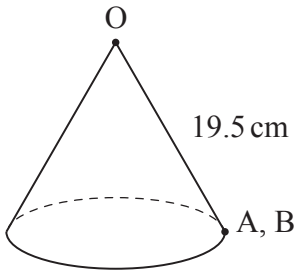
The points A and B lie on a circle, with centre O and radius $19.5\,\text{cm}$, such that $B\widehat{O}A = 210^\circ$.

A piece of paper is cut into the shape of the sector BOA.

A hollow cone with no base is constructed from the sector by joining the points A and B. The sector forms the curved surface of the cone.

This is shown in the following diagrams.





Find

(a) the area of the sector BOA; [3]

(b) the radius of the cone. [3]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 4]

Consider the function $f(x) = a\tan(2x) + b$, where $x \neq \frac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$ and $a, b \in \mathbb{R}$.

(a) Write down the period of f. [1]

The graph of $y = f(x)$ passes through the points $\left(\frac{\pi}{12}, 5\right)$ and $\left(\frac{\pi}{3}, 7\right)$.

(b) Find the value of a and the value of b. [3]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Consider the function $f(x) = 4\cot x + \sin x$, where $0 < x < \pi$.

(a) (i) Write $f(x)$ in terms of $\sin x$ and $\cos x$.

(ii) Hence or otherwise, sketch the graph of $y = f(x)$, showing the value of the x-intercept. [3]

(b) Find the value of $f^{-1}(2)$. [1]

It is given that $\sec\alpha = 1.5$, where $0 < \alpha < \pi$.

(c) Find the value of $f(\alpha)$. [3]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

At $09{:}00$ a helicopter is located at a point $(10, 3, 0.5)$ relative to a point O on horizontal ground. The x-direction is due east, the y-direction is due north and the z-direction is vertically upwards.

All distances are measured in kilometres.

The helicopter is flying at a constant height.

The helicopter's position relative to the point O is given by $\boldsymbol{r} = \begin{pmatrix} 10 \\ 3 \\ 0.5 \end{pmatrix} + 4t \begin{pmatrix} 10 \\ -25 \\ 0 \end{pmatrix}$, where t represents the time in hours since $09{:}00$.

(a) Find the speed of the helicopter. [2]

At $10{:}00$ the helicopter begins to descend.

During descent the helicopter's vertical height decreases at a constant rate of $16\,\mathrm{km\,h^{-1}}$ and its horizontal velocity remains unchanged.

The angle of descent, β, is defined as the angle between the helicopter's direction of travel and the horizontal.

(b) Find β, giving your answer in degrees. [3]

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Question 7 - working space

Section A

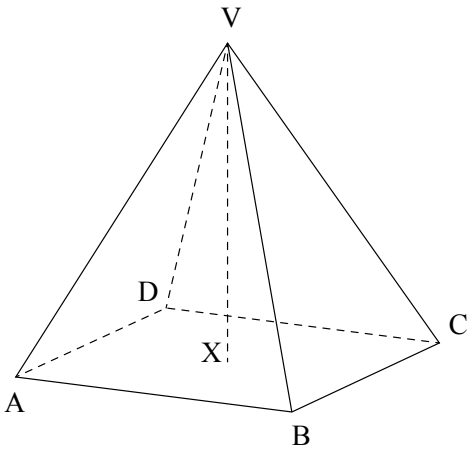
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

The following diagram shows a square-based right-pyramid with vertex $\mathrm{V}(1, 7, 0)$.

Point $\mathrm{X}(-3, 4, 2)$ is the centre of the base ABCD.

diagram not to scale



(a) Find VX. [2]

The square base has side length $5\,\mathrm{cm}$.

(b) Find AC. [2]

(c) Find the size of the angle between the edge $[\mathrm{VC}]$ and the base of the pyramid. [2]

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Question 2 - working space

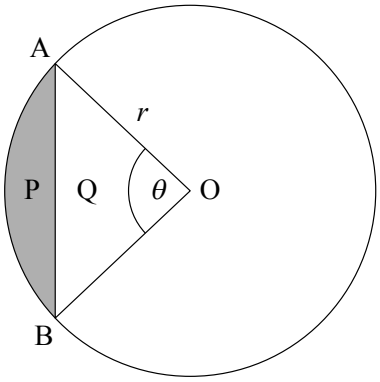
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius $r\,\mathrm{cm}$. Points A and B lie on the circle and $\mathrm{A\hat{O}B} = \theta$ radians.

Sector OAB is divided into two regions, a shaded segment P and a triangle Q.



The area of the shaded segment P is $12.8\,\mathrm{cm}^2$.

The areas of P and Q are in the ratio $3:5$.

Find the value of r.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

A line L_1 has vector equation $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ where $t \in \mathbb{R}$.

The plane Π_1 contains the line L_1 and passes through the point $(2, 1, 5)$.

(a) Show that the Cartesian equation of the plane Π_1 is $x + y - z = -2$. [4]

Consider the three planes

$$\Pi_1 : x + y - z = -2$$
$$\Pi_2 : 2x + by - z = 3$$
$$\Pi_3 : x - y + 2z = d$$

where $b, d \in \mathbb{Q}^+$.

The three planes intersect in a line.

(b) Find the value of b and the value of d. [4]

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Question 9 - working space

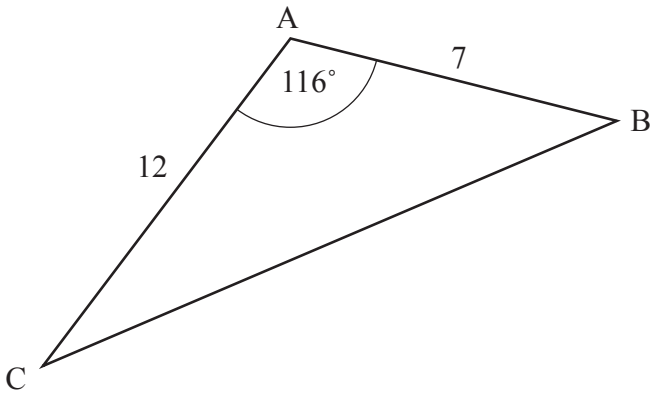
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a triangle ABC, with $AB = 7$, $AC = 12$ and $B\hat{A}C = 116°$.

diagram not to scale



(a) Find BC. [3]

(b) Find $A\hat{C}B$. [3]

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Question 1 - working space

Section A

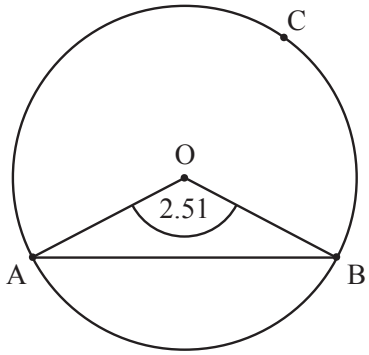
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

The following diagram shows a circle with centre O.

Points A, B and C lie on the circle.

diagram not to scale



The area of triangle AOB is $26\,\text{cm}^2$ and $A\hat{O}B = 2.51$ radians.

Find the length of arc ACB.

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

Consider the vectors $\boldsymbol{a} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -8 \\ 9 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} p \\ -6 \end{pmatrix}$, where $p \in \mathbb{R}$.

(a) Find an expression, in terms of p, for

(i) $\boldsymbol{a} \bullet \boldsymbol{c}$;

(ii) $\boldsymbol{b} \bullet \boldsymbol{c}$. [3]

The angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ is equal to the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$.

(b) Find the value of p. [5]

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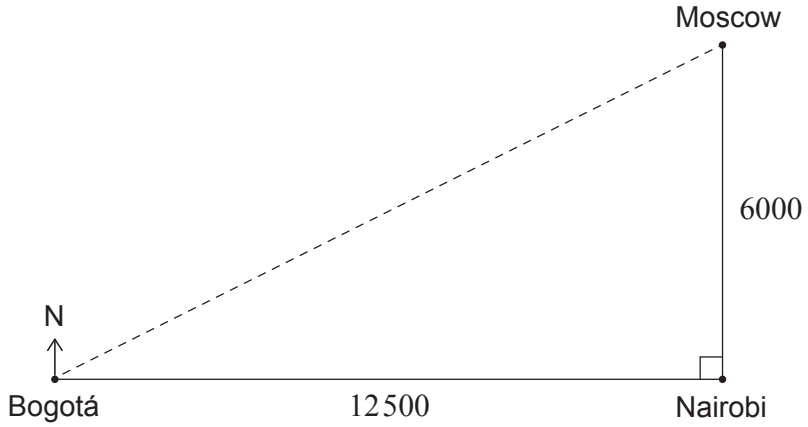
Question 6 - working space

2. [Maximum mark: 27]

The following question compares the distance and direction between cities on a flat surface to the distance and direction between cities on a sphere.

Consider a model where the cities of Bogotá, Moscow, and Nairobi lie on a flat surface. In this model, Nairobi is $6000\,\text{km}$ due south of Moscow and Bogotá is $12\,500\,\text{km}$ due west of Nairobi, as shown in the following diagram.

diagram not to scale



(a) (i) Find the distance from Bogotá to Moscow. [2]

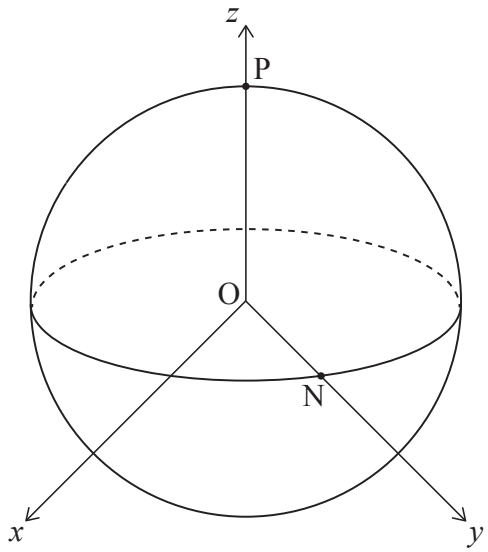
(ii) Find the bearing of Moscow from Bogotá. Give your answer in degrees. [3]

(This question continues on the following page)

(Question 2 continued)

In reality, these three cities lie on the curved surface of the Earth which will change the distances and directions found in part (a).

Now consider a curved model using a coordinate system (x, y, z) with its origin, O, at the centre of the Earth. The units of this system are thousands of kilometres and the Earth is modelled as a sphere with radius $6000\,\mathrm{km}$. The North Pole, P, lies on the z-axis, and Nairobi, N, is modelled as being on the equator and lying on the y-axis.



P has position vector $\overrightarrow{\mathrm{OP}} = \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ and N has position vector $\overrightarrow{\mathrm{ON}} = \boldsymbol{n} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$.

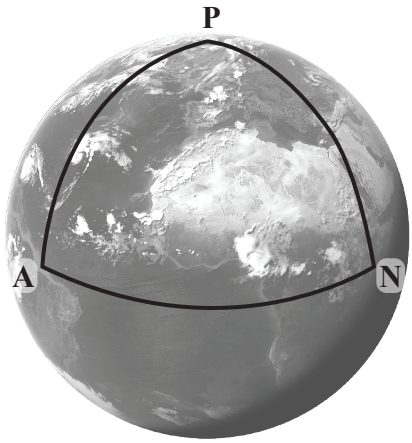
(b) (i) Use the scalar product to find the angle between $\boldsymbol{p}$ and $\boldsymbol{n}$. [2]

(ii) Show that the distance between P and N along the arc from P to N is $3000\pi\,\mathrm{km}$. [2]

(This question continues on the following page)

(Question 2 continued)

Point A, which is also on the equator, has position vector $\boldsymbol{a} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ as shown in the following diagram.



P, N and A, and the arcs connecting them, form a spherical triangle.

The angle at vertex A is defined as the angle between the vectors $\boldsymbol{a} \times \boldsymbol{p}$ and $\boldsymbol{a} \times \boldsymbol{n}$.

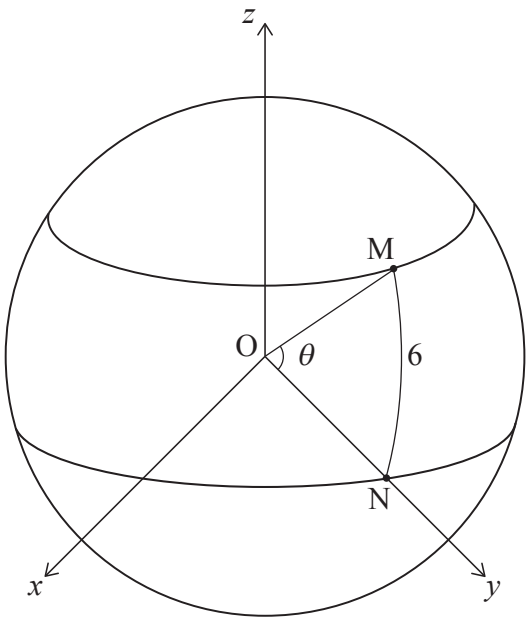
(c) (i) Find the vector $\boldsymbol{a} \times \boldsymbol{p}$. [2]

(ii) Show that the angle at vertex A in the spherical triangle is $90°$. [3]

(This question continues on the following page)

(Question 2 continued)

Moscow, M, has position vector $\overrightarrow{\mathrm{OM}} = \boldsymbol{m} = \begin{pmatrix} 0 \\ 6\cos\theta \\ 6\sin\theta \end{pmatrix}$, as shown in the following diagram.



The shortest distance between two points on the sphere lies along an arc of a circle on the sphere with centre O. In this model the shortest distance from Moscow to Nairobi is $6000\,\mathrm{km}$.

(d) Show that $\theta = 57.3°$, correct to three significant figures. [2]

Bogotá, B, is west of Nairobi and has position vector $\overrightarrow{\mathrm{OB}} = \boldsymbol{b} = \begin{pmatrix} 6\sin 120^\circ \\ 6\cos 120^\circ \\ 0 \end{pmatrix}$.

(e) Find the shortest distance from Bogotá to Moscow on the sphere. [5]

The bearing from B to M is defined as the angle at vertex B in the spherical triangle containing B, M and P. It is given that $\boldsymbol{b} \times \boldsymbol{p} = \begin{pmatrix} 36\cos 120^\circ \\ -36\sin 120^\circ \\ 0 \end{pmatrix}$.

(f) Using the method from part (c), find the bearing from Bogotá to Moscow. [6]

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References:

2(c) Google Maps/Google Earth. Data from Data SIO, NOAA, U.S. Navy, NGA, GEBCOLandsat / Copernicus, IBCAO, U.S. Geological Survey, PGC/NASA. Imagery from 14/12/2015. Image available at: https://earth.google.com/web/@12.01529518,-18.56070747,-158.39383184a,23597813.93249989d,30.00008083y,359.99981502h,0t,0r/data=CgRCAggBOgMKATBCAggASg0I_____ARAA.

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 8]

(a) Show that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$. [4]

(b) Hence, describe a sequence of transformations that maps the graph of $y = \sin\theta$ onto the graph of $y = 6\sin\left(\theta + \frac{\pi}{6}\right) - 8\sin^3\left(\theta + \frac{\pi}{6}\right)$. [4]

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Do **not** write solutions on this page.

Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 19]

The point $\mathrm{P}(-1, 1, -13)$ lies on the line L_1. The line L_1 has a direction vector $\begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$.

(a) Write down a vector equation for L_1 in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$. [1]

(b) Find a vector equation for line L_2 in the form $\boldsymbol{s} = \boldsymbol{c} + \mu \boldsymbol{d}$, given that L_2 passes through the points $\mathrm{A}(2, -4, 2)$ and $\mathrm{B}(7, -6, 1)$. [2]

(c) Show that L_1 and L_2 are skew. [5]

The point N lies on L_2.

(d) Find $\overrightarrow{\mathrm{PN}} \cdot \overrightarrow{\mathrm{AB}}$ in terms of μ. [4]

(e) Given that N is the point on L_2 that lies closest to P, find the coordinates of N. [3]

(f) Point O denotes the origin $(0, 0, 0)$.
Find the equation of the plane containing points O, P and N, giving your answer in the form $\alpha x + \beta y + \gamma z = \delta$, where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. [4]

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Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

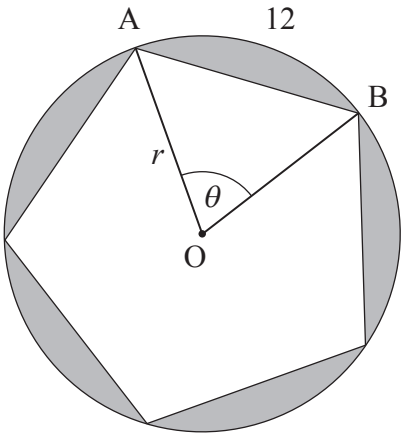
1. [Maximum mark: 6]

The following diagram shows a regular pentagon inscribed in a circle with centre O and radius $r\,\mathrm{cm}$.

The angle $\mathrm{A\hat{O}B}$ is θ, where θ is measured in radians.

The arc AB is $12\,\mathrm{cm}$.

diagram not to scale



(a) Find

(i) θ;

(ii) r. [3]

(b) Find the area of the shaded region. [3]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

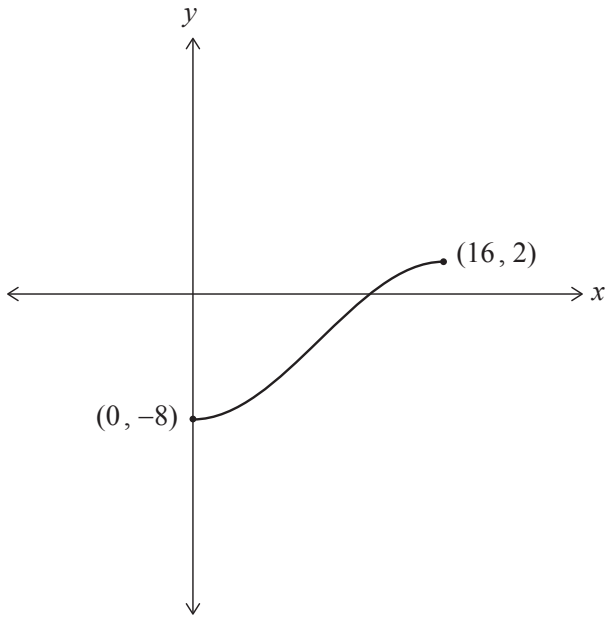
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

Consider the function $g(x) = a\cos(bx) - 3$, where $x \in \mathbb{R}$ and $a, b \in \mathbb{R}$.

The following diagram shows part of the graph of g.



The graph of g has a local minimum at $(0, -8)$ and a local maximum at $(16, 2)$.

(a) Find the value of

(i) a;

(ii) b, where $b > 0$. [3]

(b) Write down the smallest positive value of the constant k such that $g(x + k) = g(x)$ for all x. [1]

(c) The function $g(x)$ can be written in the form $f(x) = p\sin b(x - q) - 3$ where $p, q \in \mathbb{Z}^+$.

(i) Find the smallest positive value of q.

(ii) For this value of q, write down the value of p. [3]

(This question continues on the following page)

(Question 4 continued)

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The equations of two lines, L_1 and L_2, are given by:

$$L_1 : \boldsymbol{r}_1 = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ where } s \in \mathbb{R}$$

$$L_2 : \frac{x+1}{2} = y - 7 = \frac{z+5}{3}$$

(a) Show that the position vector of the point of intersection, L_1 and L_2 is $\begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$. [3]

The plane Π contains the lines L_1 and L_2.

(b) (i) Write down the equation of Π, giving your answer in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ where $\lambda, \mu \in \mathbb{R}$.

(ii) Given that $\boldsymbol{b} \times \boldsymbol{c} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}$, show that the Cartesian equation of Π is $4x + y - 3z = 18$. [3]

The plane intersects the coordinate axes at $\mathrm{P}(4.5, 0, 0)$, $\mathrm{Q}(0, q, 0)$, and $\mathrm{R}(0, 0, r)$.

(c) Write down the value of

(i) q;

(ii) r. [2]

(d) Use a vector method to find the area of the triangle PQR. [5]

Another line, L_3, is normal to Π and passes through the point of intersection of L_1 and L_2.

(e) Write down an equation for L_3 in the form $\boldsymbol{r}_3 = \boldsymbol{m} + \gamma\boldsymbol{n}$. [2]

(f) Given that the point $\mathrm{S}(-11, 5, 7)$ lies on the line L_3, find γ. [2]

(g) Hence, find the volume of pyramid PQRS. [4]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

(a) Show that $3\cos 2x + 11\sin x = 3 + 11\sin x - 6\sin^2 x$. [2]

(b) Hence, or otherwise, solve the equation $3\cos 2x + 11\sin x - 6 = 0$ for $0° \leq x \leq 180°$. [3]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

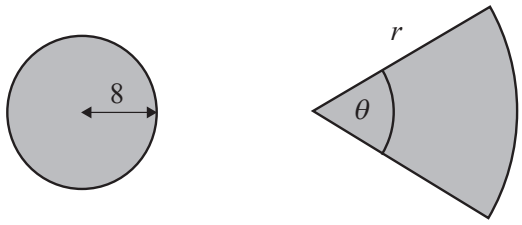
Consider a circle and a sector.

The radius of the circle is $8\,\text{mm}$.

The radius of the sector is $r\,\text{mm}$ and the acute angle at the centre is θ radians.

This is shown in the following diagram.

diagram not to scale



The perimeter of the sector is 1.5 times the circumference of the circle.

(a) Show that $r = \dfrac{24\pi}{2+\theta}$. [3]

It is given that the area of the circle is the same as the area of the sector.

(b) Determine the value of θ. [3]

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Question 3 - working space

APOLLO

Statistics and probability

87 questions

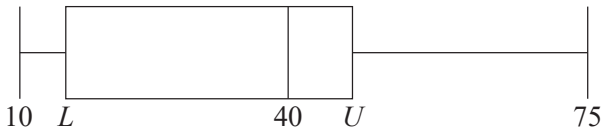
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

A research student weighed lizard eggs in grams and recorded the results. The following box and whisker diagram shows a summary of the results where L and U are the lower and upper quartiles respectively.

diagram not to scale



The interquartile range is 20 grams and there are no outliers in the results.

(a) Find the minimum possible value of U. [3]

(b) Hence, find the minimum possible value of L. [2]

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die, A, is rolled. Let X be the score obtained when die A is rolled. The probability distribution for X is given in the following table.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	p	p	$\frac{1}{2}p$

(a) Find the value of p. [2]

(b) Hence, find the value of $\mathrm{E}(X)$. [2]

A second biased four-sided die, B, is rolled. Let Y be the score obtained when die B is rolled. The probability distribution for Y is given in the following table.

y	1	2	3	4
$\mathrm{P}(Y=y)$	q	q	q	r

(c) (i) State the range of possible values of r.

(ii) Hence, find the range of possible values of q. [3]

(d) Hence, find the range of possible values for $\mathrm{E}(Y)$. [3]

Agnes and Barbara play a game using these dice. Agnes rolls die A once and Barbara rolls die B once. The probability that Agnes' score is less than Barbara's score is $\frac{1}{2}$.

(e) Find the value of $\mathrm{E}(Y)$. [6]

Do **not** write solutions on this page.

Question 10 - working space

Section A

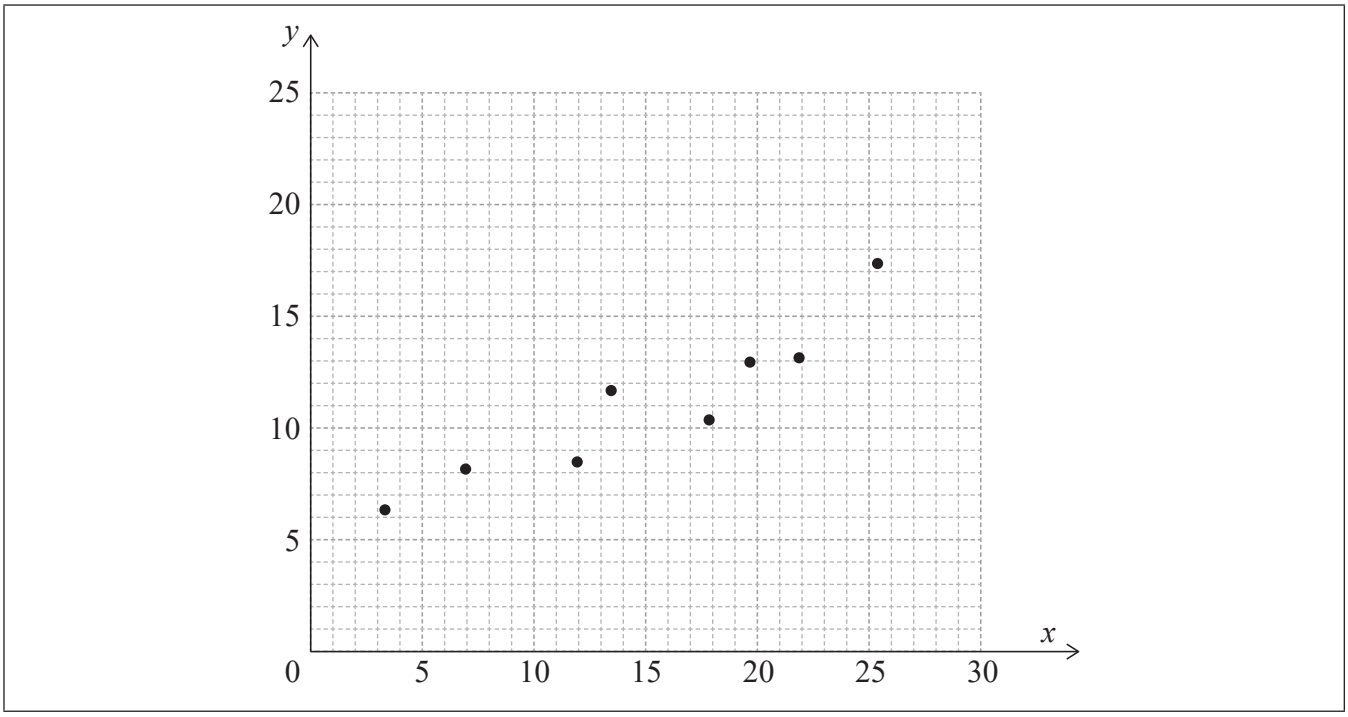
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

The following table shows the data collected from an experiment.

x	3.3	6.9	11.9	13.4	17.8	19.6	21.8	25.3
y	6.3	8.1	8.4	11.6	10.3	12.9	13.1	17.3

The data is also represented on the following scatter diagram.



(This question continues on the following page)

(Question 1 continued)

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$, where $a, b \in \mathbb{R}$.

(a) Write down the value of a and the value of b. [2]

(b) Use this model to predict the value of y when $x = 18$. [2]

(c) Write down the value of $\overline{x}$ and the value of $\overline{y}$. [1]

(d) Draw the line of best fit on the scatter diagram. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

A company produces bags of sugar whose masses, in grams, can be modelled by a normal distribution with mean 1000 and standard deviation 3.5. A bag of sugar is rejected for sale if its mass is less than 995 grams.

(a) Find the probability that a bag selected at random is rejected. [2]

(b) Estimate the number of bags which will be rejected from a random sample of 100 bags. [1]

(c) Given that a bag is not rejected, find the probability that it has a mass greater than 1005 grams. [3]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{x}{\sqrt{\left(x^2+k\right)^3}} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}^+$.

(a) Show that $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$. [5]

(b) Find the value of k. [2]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

At a café, the waiting time between ordering and receiving a cup of coffee is dependent upon the number of customers who have already ordered their coffee and are waiting to receive it.

Sarah, a regular customer, visited the café on five consecutive days. The following table shows the number of customers, x, ahead of Sarah who have already ordered and are waiting to receive their coffee and Sarah's waiting time, y minutes.

Number of customers (x)	3	9	11	10	5
Sarah's waiting time (y)	6	10	12	11	6

The relationship between x and y can be modelled by the regression line of y on x with equation $y = ax + b$.

(a) (i) Find the value of a and the value of b.

(ii) Write down the value of Pearson's product-moment correlation coefficient, r. [3]

(b) Interpret, in context, the value of a found in part (a)(i). [1]

On another day, Sarah visits the café to order a coffee. Seven customers have already ordered their coffee and are waiting to receive it.

(c) Use the result from part (a)(i) to estimate Sarah's waiting time to receive her coffee. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 8]

At a school, 70% of the students play a sport and 20% of the students are involved in theatre. 18% of the students do neither activity.

A student is selected at random.

(a) Find the probability that the student plays a sport and is involved in theatre. [2]

(b) Find the probability that the student is involved in theatre, but does not play a sport. [2]

At the school 48% of the students are girls, and 25% of the girls are involved in theatre.

A student is selected at random. Let G be the event "the student is a girl" and let T be the event "the student is involved in theatre".

(c) Find $\mathrm{P}(G \cap T)$. [2]

(d) Determine if the events G and T are independent. Justify your answer. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

A continuous random variable X has the probability density function f_n given by

$$f_n(x) = \begin{cases} (n+1)x^n, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $n \in \mathbb{R}$, $n \ge 0$.

(a) Show that $\mathrm{E}(X) = \frac{n+1}{n+2}$. [2]

(b) Show that $\mathrm{Var}(X) = \frac{n+1}{(n+2)^2(n+3)}$. [4]

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The flight times, T minutes, between two cities can be modelled by a normal distribution with a mean of 75 minutes and a standard deviation of σ minutes.

(a) Given that 2% of the flight times are longer than 82 minutes, find the value of σ. [3]

(b) Find the probability that a randomly selected flight will have a flight time of more than 80 minutes. [2]

(c) Given that a flight between the two cities takes longer than 80 minutes, find the probability that it takes less than 82 minutes. [4]

On a particular day, there are 64 flights scheduled between these two cities.

(d) Find the expected number of flights that will have a flight time of more than 80 minutes. [3]

(e) Find the probability that more than 6 of the flights on this particular day will have a flight time of more than 80 minutes. [3]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that "box 1 is chosen" and let R be the event that "a red ball is drawn".

(b) Determine whether events A and R are independent. [2]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

In Lucy's music academy, eight students took their piano diploma examination and achieved scores out of 150. For her records, Lucy decided to record the average number of hours per week each student reported practising in the weeks prior to their examination. These results are summarized in the table below.

Average weekly practice time (h)	28	13	45	33	17	29	39	36
Diploma score (D)	115	82	120	116	79	101	110	121

(a) Find Pearson's product-moment correlation coefficient, r, for these data. [2]

(b) The relationship between the variables can be modelled by the regression equation $D = ah + b$. Write down the value of a and the value of b. [1]

(c) One of these eight students was disappointed with her result and wished she had practised more. Based on the given data, determine how her score could have been expected to alter had she practised an extra five hours per week. [2]

(d) Lucy asserts that the number of hours a student practises has a direct effect on their final diploma result. Comment on the validity of Lucy's assertion. [1]

Lucy suspected that each student had not been practising as much as they reported. In order to compensate for this, Lucy deducted a fixed number of hours per week from each of the students' recorded hours.

(e) State how, if at all, the value of r would be affected. [1]

(This question continues on the following page)

(Question 1 continued)

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will not be marked.



Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

A factory manufactures lamps. It is known that the probability that a lamp is found to be defective is 0.05. A random sample of 30 lamps is tested.

(a) Find the probability that there is at least one defective lamp in the sample. [3]

(b) Given that there is at least one defective lamp in the sample, find the probability that there are at most two defective lamps. [4]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \arccos x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The median of this distribution is m.

(a) Determine the value of m. [2]

(b) Given that $P(|X - m| \leq a) = 0.3$, determine the value of a. [4]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{\pi x}{36}\sin\left(\frac{\pi x}{6}\right), & 0 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \leq X \leq 3)$.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

A six-sided biased die is weighted in such a way that the probability of obtaining a "six" is $\frac{7}{10}$.

The die is tossed five times. Find the probability of obtaining

(a) at most three "sixes". [3]

(b) the third "six" on the fifth toss. [3]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

The following table below shows the marks scored by seven students on two different mathematics tests.

Test 1 (x)	15	23	25	30	34	34	40
Test 2 (y)	20	26	27	32	35	37	35

Let L_1 be the regression line of x on y. The equation of the line L_1 can be written in the form $x = ay + b$.

(a) Find the value of a and the value of b. [2]

Let L_2 be the regression line of y on x. The lines L_1 and L_2 pass through the same point with coordinates (p, q).

(b) Find the value of p and the value of q. [3]

(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's mark on the first test. [2]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

In a city, the number of passengers, X, who ride in a taxi has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.60	0.30	0.03	0.05	0.02

After the opening of a new highway that charges a toll, a taxi company introduces a charge for passengers who use the highway. The charge is $2.40 per taxi plus $1.20 per passenger. Let T represent the amount, in dollars, that is charged by the taxi company per ride.

(a) Find $\mathrm{E}(T)$. [4]

(b) Given that $\mathrm{Var}(X) = 0.8419$, find $\mathrm{Var}(T)$. [2]

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The length, $X\,\text{mm}$, of a certain species of seashell is normally distributed with mean 25 and variance, σ^2.

The probability that X is less than 24.15 is 0.1446.

(a) Find $\text{P}(24.15 < X < 25)$. [2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a seashell selected at random has a length greater than $26\,\text{mm}$. [5]

A random sample of 10 seashells is collected on a beach. Let Y represent the number of seashells with lengths greater than $26\,\text{mm}$.

(c) Find $\text{E}(Y)$. [3]

(d) Find the probability that exactly three of these seashells have a length greater than $26\,\text{mm}$. [2]

A seashell selected at random has a length less than $26\,\text{mm}$.

(e) Find the probability that its length is between $24.15\,\text{mm}$ and $25\,\text{mm}$. [3]

Do **not** write solutions on this page.

Question 10 - working space

Section A

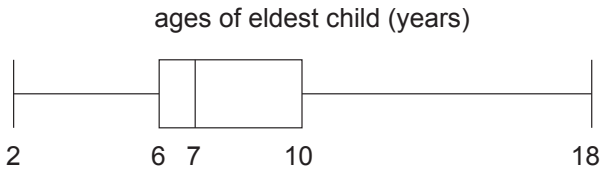
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 7]

A survey at a swimming pool is given to one adult in each family. The age of the adult, a years old, and of their eldest child, c years old, are recorded.

The ages of the eldest child are summarized in the following box and whisker diagram.

diagram not to scale



(a) Find the largest value of c that would not be considered an outlier. [3]

The regression line of a on c is $a = \frac{7}{4}c + 20$. The regression line of c on a is $c = \frac{1}{2}a - 9$.

(b) (i) One of the adults surveyed is 42 years old. Estimate the age of their eldest child.

(ii) Find the mean age of all the adults surveyed. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 8]

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{4-3x^2}}, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. [4]

(b) Find $\mathrm{E}(X)$. [4]

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Question 7 - working space

Section A

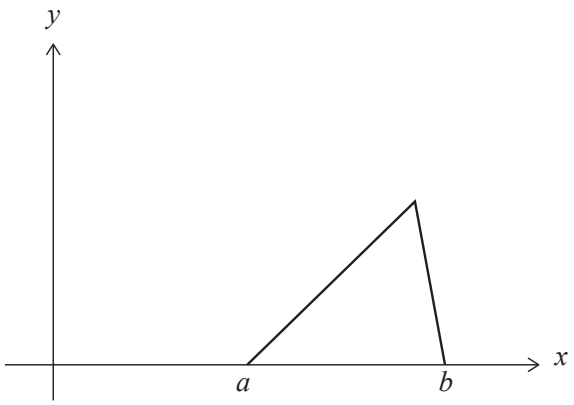
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

A continuous random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{2}{(b-a)(c-a)}(x-a), & a\le x\le c\\ \dfrac{2}{(b-a)(b-c)}(b-x), & c<x\le b\\ 0, & \text{otherwise}\end{cases}.$$

The following diagram shows the graph of $y=f(x)$ for $a\le x\le b$.



Given that $c\ge\dfrac{a+b}{2}$, find an expression for the median of X in terms of a, b and c.

Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A biased four-sided die with faces labelled 1, 2, 3 and 4 is rolled and the result recorded. Let X be the result obtained when the die is rolled. The probability distribution for X is given in the following table where p and q are constants.

x	1	2	3	4
$\mathrm{P}(X=x)$	p	0.3	q	0.1

For this probability distribution, it is known that $\mathrm{E}(X)=2$.

(a) Show that $p=0.4$ and $q=0.2$. [5]

(b) Find $\mathrm{P}(X>2)$. [2]

Nicky plays a game with this four-sided die. In this game she is allowed a maximum of five rolls. Her score is calculated by adding the results of each roll. Nicky wins the game if her score is at least ten.

After three rolls of the die, Nicky has a score of four.

(c) Assuming that rolls of the die are independent, find the probability that Nicky wins the game. [5]

David has two pairs of unbiased four-sided dice, a yellow pair and a red pair.
Both yellow dice have faces labelled 1, 2, 3 and 4. Let S represent the sum obtained by rolling the two yellow dice. The probability distribution for S is shown below.

s	2	3	4	5	6	7	8
$\mathrm{P}(S=s)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

The first red die has faces labelled 1, 2, 2 and 3. The second red die has faces labelled 1, a, a and b, where $a<b$ and $a, b \in \mathbb{Z}^+$. The probability distribution for the sum obtained by rolling the red pair is the same as the distribution for the sum obtained by rolling the yellow pair.

(d) Determine the value of b. [2]

(e) Find the value of a, providing evidence for your answer. [2]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The number of hours spent exercising each week by a group of students is shown in the following table.

Exercising time (in hours)	**Number of students**
2	5
3	1
4	4
5	3
6	x

The median is 4.5 hours.

(a) Find the value of x. [2]

(b) Find the standard deviation. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

A discrete random variable, X, has the following probability distribution:

x	0	1	2	3
$\mathrm{P}(X=x)$	0.41	$k-0.28$	0.46	$0.29-2k^2$

(a) Show that $2k^2-k+0.12=0$. [1]

(b) Find the value of k, giving a reason for your answer. [3]

(c) Hence, find $\mathrm{E}(X)$. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

Let A and B be two independent events such that $\mathrm{P}(A \cap B') = 0.16$ and $\mathrm{P}(A' \cap B) = 0.36$.

(a) Given that $\mathrm{P}(A \cap B) = x$, find the value of x. [4]

(b) Find $\mathrm{P}(A' \mid B')$. [2]

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 16]

A bakery makes two types of muffins: chocolate muffins and banana muffins.

The weights, C grams, of the chocolate muffins are normally distributed with a mean of $62\,\text{g}$ and standard deviation of $2.9\,\text{g}$.

(a) Find the probability that a randomly selected chocolate muffin weighs less than $61\,\text{g}$. [2]

(b) In a random selection of 12 chocolate muffins, find the probability that exactly 5 weigh less than $61\,\text{g}$. [2]

The weights, B grams, of the banana muffins are normally distributed with a mean of $68\,\text{g}$ and standard deviation of $3.4\,\text{g}$.

Each day $60\,\%$ of the muffins made are chocolate.

On a particular day, a muffin is randomly selected from all those made at the bakery.

(c) (i) Find the probability that the randomly selected muffin weighs less than $61\,\text{g}$.

(ii) Given that a randomly selected muffin weighs less than $61\,\text{g}$, find the probability that it is chocolate. [7]

The machine that makes the chocolate muffins is adjusted so that the mean weight of the chocolate muffins remains the same but their standard deviation changes to $\sigma\,\text{g}$. The machine that makes the banana muffins is not adjusted. The probability that the weight of a randomly selected muffin from these machines is less than $61\,\text{g}$ is now 0.157.

(d) Find the value of σ. [5]

Do **not** write solutions on this page

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

Events A and B are independent and $\mathrm{P}(A) = 3\mathrm{P}(B)$.

Given that $\mathrm{P}(A \cup B) = 0.68$, find $\mathrm{P}(B)$.

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Question 3 - working space

Section A

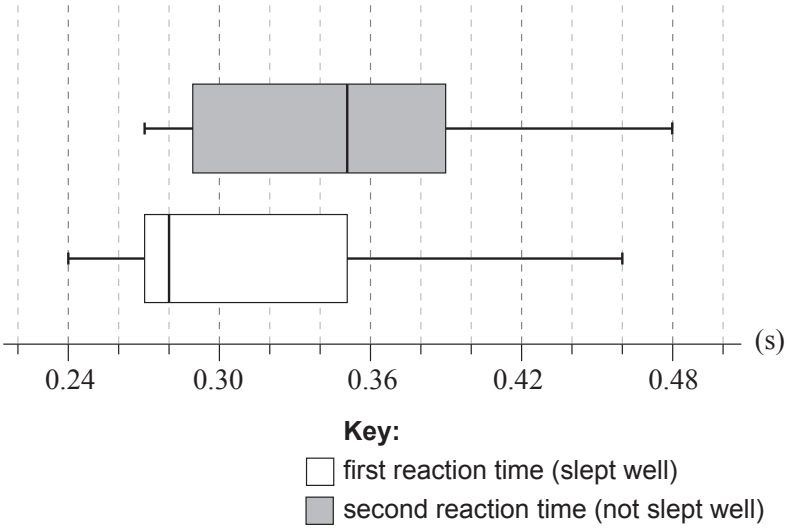
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

A random sample of nine adults were selected to see whether sleeping well affected their reaction times to a visual stimulus. Each adult's reaction time was measured twice.

The first measurement for reaction time was taken on a morning after the adult had slept well. The second measurement was taken on a morning after the same adult had not slept well.

The box and whisker diagrams for the reaction times, measured in seconds, are shown below.



Consider the box and whisker diagram representing the reaction times after sleeping well.

(a) State the median reaction time after sleeping well. [1]

(b) Verify that the measurement of 0.46 seconds is not an outlier. [3]

(c) State why it appears that the mean reaction time is greater than the median reaction time. [1]

Now consider the two box and whisker diagrams.

(d) Comment on whether these box and whisker diagrams provide any evidence that might suggest that not sleeping well causes an increase in reaction time. [1]

(This question continues on the following page)

(Question 4 continued)

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Rachel and Sophia are competing in a javelin-throwing competition.

The distances, R metres, thrown by Rachel can be modelled by a normal distribution with mean 56.5 and standard deviation 3.

The distances, S metres, thrown by Sophia can be modelled by a normal distribution with mean 57.5 and standard deviation 1.8.

In the first round of competition, each competitor must have five throws. To qualify for the next round of competition, a competitor must record at least one throw of 60 metres or greater in the first round.

Find the probability that only one of Rachel or Sophia qualifies for the next round of competition.

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.8$.

(a) Determine the value of $\mathrm{P}(A \cap B)$ in the case where the events A and B are independent. [1]

(b) Determine the minimum possible value of $\mathrm{P}(A \cap B)$. [3]

(c) Determine the maximum possible value of $\mathrm{P}(A \cap B)$, justifying your answer. [2]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The following table shows the Mathematics test scores (x) and the Science test scores (y) for a group of eight students.

Mathematics scores (x)	64	68	72	75	80	82	85	86
Science scores (y)	67	72	77	76	84	83	89	91

The regression line of y on x for this data can be written in the form $y = ax + b$.

(a) Find the value of a and the value of b. [2]

(b) Write down the value of the Pearson's product-moment correlation coefficient, r. [1]

(c) Use the equation of your regression line to predict the Science test score for a student who has a score of 78 on the Mathematics test. Express your answer to the nearest integer. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} axe^{x}, & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

where $a, b \in \mathbb{R}^{+}$.

(a) Find an expression for a in terms of b. [5]

(b) In the case where $a = b = 1$, find the median of X. [3]

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

The time worked, T, in hours per week by employees of a large company is normally distributed with a mean of 42 and standard deviation 10.7.

(a) Find the probability that an employee selected at random works more than 40 hours per week. [2]

(b) A group of four employees is selected at random. Each employee is asked in turn whether they work more than 40 hours per week. Find the probability that the fourth employee is the only one in the group who works more than 40 hours per week. [3]

(c) A large group of employees work more than 40 hours per week.

(i) An employee is selected at random from this large group.

Find the probability that this employee works less than 55 hours per week.

(ii) Ten employees are selected at random from this large group.

Find the probability that exactly five of them work less than 55 hours per week. [7]

It is known that $\mathrm{P}(a \leq T \leq b) = 0.904$ and that $\mathrm{P}(T > b) = 2\mathrm{P}(T < a)$, where a and b are numbers of hours worked per week. An employee who works fewer than a hours per week is considered to be a part-time employee.

(d) Find the maximum time, in hours per week, that an employee can work and still be considered part-time. [4]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

On a Monday at an amusement park, a sample of 40 visitors was randomly selected as they were leaving the park. They were asked how many times that day they had been on a ride called *The Dragon*. This information is summarized in the following frequency table.

Number of times on *The Dragon*	**Frequency**
0	6
1	16
2	13
3	2
4	3

It can be assumed that this sample is representative of all visitors to the park for the following day.

(a) For the following day, Tuesday, estimate

(i) the probability that a randomly selected visitor will ride *The Dragon*;

(ii) the expected number of times a visitor will ride *The Dragon*. [4]

It is known that 1000 visitors will attend the amusement park on Tuesday. *The Dragon* can carry a maximum of 10 people each time it runs.

(b) Estimate the minimum number of times *The Dragon* must run to satisfy demand. [2]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

Events A and B are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(A \mid B) = 0.25$ and $\mathrm{P}(A \cup B) = 0.55$.

Find $\mathrm{P}(B)$.

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} \frac{1}{2a}, & a \leq x \leq 3a \\ 0, & \text{otherwise} \end{cases}$$

where a is a positive real number.

(a) State $\mathrm{E}(X)$ in terms of a. [1]

(b) Use integration to find $\mathrm{Var}(X)$ in terms of a. [4]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

The total number of children, y, visiting a park depends on the highest temperature, T, in degrees Celsius ($°C$). A park official predicts the total number of children visiting his park on any given day using the model $y = -0.6T^2 + 23T + 110$, where $10 \leq T \leq 35$.

(a) Use this model to estimate the number of children in the park on a day when the highest temperature is $25\,°C$. [2]

An ice cream vendor investigates the relationship between the total number of children visiting the park and the number of ice creams sold, x. The following table shows the data collected on five different days.

Total number of children (y)	81	175	202	346	360
Ice creams sold (x)	15	27	23	35	46

(b) Find an appropriate regression equation that will allow the vendor to predict the number of ice creams sold on a day when there are y children in the park. [3]

(c) Hence, use your regression equation to predict the number of ice creams that the vendor sells on a day when the highest temperature is $25\,°C$. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

A company manufactures metal tubes for bicycle frames. The diameters of the tubes, $D\,\text{mm}$, are normally distributed with mean 32 and standard deviation σ. The interquartile range of the diameters is 0.28.

Find the value of σ.

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Question 4 - working space

Section A

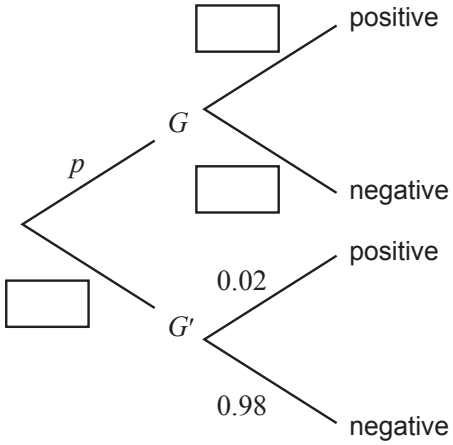
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

A new test has been developed to identify whether a particular gene, G, is present in a population of parrots. The test returns a correct positive result 95% of the time for parrots with the gene, and a false positive result 2% of the time for parrots without the gene.

The proportion of the population with the gene is p.

(a) Complete the tree diagram below. [2]



(b) A random sample of the population was tested. It was found that 150 tests returned a positive result. Out of the 150 parrots with a positive test result, 18 did not actually have the gene. Find an estimate for p. [4]

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

A continuous random variable, X, has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{6}{\pi\sqrt{16 - x^2}}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the exact value of $\mathrm{E}(X)$. [5]

(b) Find $\mathrm{P}(X < 0.5)$. [2]

A laboratory trial may require up to 2 millilitres of reagent. The amount of reagent used has been found to have a probability distribution that can be modelled by $f(x)$, where X is the amount of reagent in millilitres.

Each laboratory trial is independent. A trial is considered a success when $X < 0.5$.

(c) Determine the least number of trials required to be 99% sure of at least one success. [3]

Ten trials were conducted.

(d) Find the probability that exactly three trials were successful. [2]

(e) Write down the number of ways these three successful trials could have occurred consecutively. [1]

Now consider n trials where it is given that exactly three successes have occurred.

(f) (i) Write down an expression for the number of ways these three successful trials could have occurred consecutively.

(ii) Find the greatest value of n such that the probability of three consecutive successful trials is more than 0.05. [6]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

A botanist is conducting an experiment which studies the growth of plants.

The heights of the plants are measured on seven different days.

The following table shows the number of days, d, that the experiment has been running and the average height, $h\,\text{cm}$, of the plants on each of those days.

Number of days (d)	2	5	13	24	33	37	42
Average height (h)	10	16	30	59	76	79	82

The value of Pearson's product-moment correlation coefficient, r, for this data is 0.991, correct to three significant figures.

(a) The regression line of h on d for this data can be written in the form $h = ad + b$.

Find the value of a and the value of b. [2]

(b) Use your regression line to estimate the average height of the plants when the experiment has been running for 20 days. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 8]

The weights, W grams, of bags of rice packaged in a factory can be modelled by a normal distribution with mean 204 grams and standard deviation 5 grams.

(a) A bag of rice is selected at random.

Find the probability that it weighs more than 210 grams. [2]

According to this model, $80\,\%$ of the bags of rice weigh between w grams and 210 grams.

(b) Find the probability that a randomly selected bag of rice weighs less than w grams. [2]

(c) Find the value of w. [2]

(d) Ten bags of rice are selected at random.

Find the probability that exactly one of the bags weighs less than w grams. [2]

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

A game of chance involves drawing **two** balls at random out of a box without replacement. The box initially contains r red balls and y yellow balls.

Let $\mathrm{P}(YY)$ represent the probability of drawing two yellow balls from the box without replacement.

Consider a version of this game where it is known that $\mathrm{P}(YY) = \frac{1}{3}$.

(a) Show that $2y^2 - 2(r+1)y + r - r^2 = 0$. [4]

(b) By solving the equation in part (a), show that $y = \frac{(r+1) + \sqrt{3r^2+1}}{2}$. [4]

(c) Find two pairs of values for r and y that satisfy the condition $\mathrm{P}(YY) = \frac{1}{3}$. [4]

Now consider a similar game of chance that involves drawing **three** balls out of a box without replacement. The box initially contains 10 red balls and y yellow balls.

Let $\mathrm{P}(YYY)$ represent the probability of drawing three yellow balls from the box without replacement.

(d) Find an expression for $\mathrm{P}(YYY)$ in terms of y. [3]

A yellow ball is added so that the box now contains 10 red balls and $(y+1)$ yellow balls. The probability of drawing three yellow balls from the box without replacement is now twice the probability expressed in part (d).

(e) Find the initial number of yellow balls in the box. [5]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 4]

Events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.75$ and $\mathrm{P}(A \cap B) = 0.6$.

(a) Find $\mathrm{P}(A \cup B)$. [2]

(b) Hence, or otherwise, find $\mathrm{P}(A' \cap B')$. [2]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 4]

Events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.75$ and $\mathrm{P}(A \cap B) = 0.6$.

(a) Find $\mathrm{P}(A \cup B)$. [2]

(b) Hence, or otherwise, find $\mathrm{P}(A' \cap B')$. [2]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

The following table shows the probability distribution of a discrete random variable X, where $a, k \in \mathbb{R}^+$.

x	1	2	3	4
$\mathrm{P}(X = x)$	k	k^2	a	k^3

Given that $\mathrm{E}(X) = 2.3$, find the value of a.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

The random variable X is such that $X \sim B(25, p)$ and $\text{Var}(X) = 5.75$.

(a) Find the possible values of p. [3]

The random variable Y is such that $Y = 1 - 2X$.

(b) Find $\text{Var}(Y)$. [2]

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A farmer is growing a field of wheat plants. The height, $H\,\text{cm}$, of each plant can be modelled by a normal distribution with mean μ and standard deviation σ.

It is known that $\text{P}(H < 94.6) = 0.288$ and $\text{P}(H > 98.1) = 0.434$.

(a) Find the probability that the height of a randomly selected plant is between $94.6\,\text{cm}$ and $98.1\,\text{cm}$. [2]

(b) Find the value of μ and the value of σ. [5]

The farmer measures 100 randomly selected plants. Any plant with a height greater than $98.1\,\text{cm}$ is considered ready to harvest. Heights of plants are independent of each other.

(c) (i) Find the probability that exactly 34 plants are ready to harvest.

(ii) Given that fewer than 49 plants are ready to harvest, find the probability that exactly 34 plants are ready to harvest. [6]

In another field, the farmer is growing the same variety of wheat, but is using a different fertilizer. The heights of these plants, $F\,\text{cm}$, are normally distributed with mean 98.6 and standard deviation d. The farmer finds the interquartile range to be $4.82\,\text{cm}$.

(d) Find the value of d. [3]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

The following table shows the probability distribution of a discrete random variable X, where $a, k \in \mathbb{R}^+$.

x	1	2	3	4
$\mathrm{P}(X = x)$	k	k^2	a	k^3

Given that $\mathrm{E}(X) = 2.3$, find the value of a.

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

The random variable X is such that $X \sim B(25, p)$ and $\text{Var}(X) = 5.75$.

(a) Find the possible values of p. [3]

The random variable Y is such that $Y = 1 - 2X$.

(b) Find $\text{Var}(Y)$. [2]

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Question 6 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A farmer is growing a field of wheat plants. The height, $H\,\text{cm}$, of each plant can be modelled by a normal distribution with mean μ and standard deviation σ.

It is known that $\text{P}(H < 94.6) = 0.288$ and $\text{P}(H > 98.1) = 0.434$.

(a) Find the probability that the height of a randomly selected plant is between $94.6\,\text{cm}$ and $98.1\,\text{cm}$. [2]

(b) Find the value of μ and the value of σ. [5]

The farmer measures 100 randomly selected plants. Any plant with a height greater than $98.1\,\text{cm}$ is considered ready to harvest. Heights of plants are independent of each other.

(c) (i) Find the probability that exactly 34 plants are ready to harvest.

(ii) Given that fewer than 49 plants are ready to harvest, find the probability that exactly 34 plants are ready to harvest. [6]

In another field, the farmer is growing the same variety of wheat, but is using a different fertilizer. The heights of these plants, $F\,\text{cm}$, are normally distributed with mean 98.6 and standard deviation d. The farmer finds the interquartile range to be $4.82\,\text{cm}$.

(d) Find the value of d. [3]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

Claire rolls a six-sided die 16 times.

The scores obtained are shown in the following frequency table.

Score	Frequency
1	p
2	q
3	4
4	2
5	0
6	3

It is given that the mean score is 3.

(a) Find the value of p and the value of q. [5]

Each of Claire's scores is multiplied by 10 in order to determine the final score for a game she is playing.

(b) Write down the mean final score. [1]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

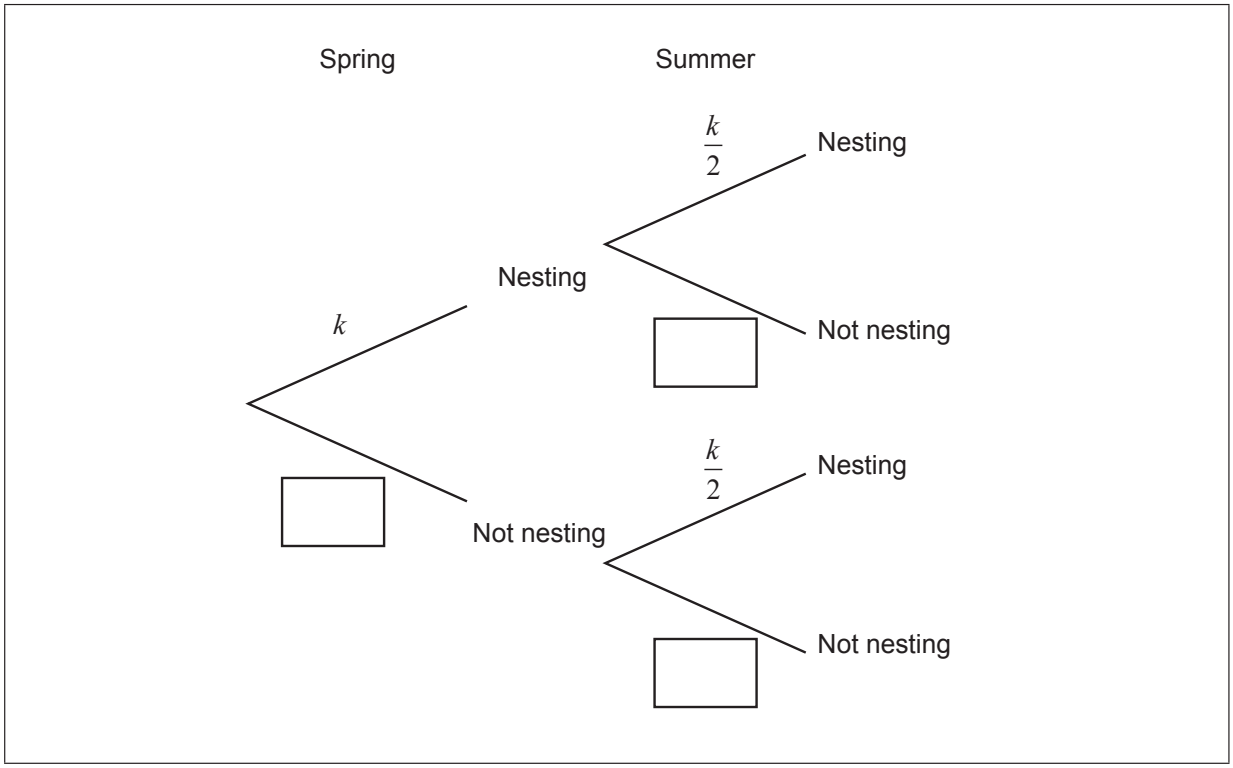
4. [Maximum mark: 6]

A species of bird can nest in two seasons: Spring and Summer.

The probability of nesting in Spring is k.

The probability of nesting in Summer is $\frac{k}{2}$.

This is shown in the following tree diagram.



(a) Complete the tree diagram to show the probabilities of not nesting in each season. Write your answers in terms of k. [2]

It is known that the probability of not nesting in Spring and not nesting in Summer is $\frac{5}{9}$.

(b) (i) Show that $9k^2 - 27k + 8 = 0$.

(ii) Both $k = \frac{1}{3}$ and $k = \frac{8}{3}$ satisfy $9k^2 - 27k + 8 = 0$.

State why $k = \frac{1}{3}$ is the only valid solution. [4]

(This question continues on the following page)

(Question 4 continued)

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

A farmer grows two types of apples—cooking apples and eating apples. The weights of the apples, in grams, can be modelled as normal distributions with the following parameters.

Apple type	Mean μ	Standard deviation σ
Eating	$100\,\text{g}$	$20\,\text{g}$
Cooking	$140\,\text{g}$	$40\,\text{g}$

For each type of apple you can assume that $95\,\%$ of the weights are within two standard deviations of the mean.

(a) Find the percentage of eating apples that have a weight greater than $140\,\text{g}$. [1]

The farmer grows a large number of apples of which $80\,\%$ are eating apples.

Both types of apples are picked and randomly mixed together in a cleaning machine.

After cleaning, the machine separates out those that have a weight greater than $140\,\text{g}$ into a container.

(b) An apple is randomly selected from this container. Find the probability that it is an eating apple. Give your answer in the form $\frac{c}{d}$, where $c, d \in \mathbb{Z}^+$. [4]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 4]

The random variable X is normally distributed with mean 10 and standard deviation 2.

(a) Find the probability that X is more than 1.5 standard deviations above the mean. [2]

The probability that X is more than k standard deviations above the mean is 0.1, where $k \in \mathbb{R}$.

(b) Find the value of k. [2]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

A class is given two tests, Test A and Test B. Each test is scored out of a total of 100 marks. The scores of the students are shown in the following table.

Student	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Test A	52	71	100	93	81	80	88	100	70	61
Test B	58	80	92	98	90	82	100	100	65	74

Let x be the score on Test A and y be the score on Test B.

The teacher finds that the equation of the regression line of y on x for these scores is $y = 0.822x + 18.4$.

(a) Find the value of Pearson's product-moment correlation coefficient, r. [2]

Giovanni was absent for Test A and Paulo was absent for Test B.

The teacher uses the regression line of y on x to estimate the missing scores.

Paulo scored 10 on Test A.

The teacher estimated his score on Test B to be 27 to the nearest integer using the following calculation:

$$y = 0.822(10) + 18.4 \approx 27$$

(b) Give a reason why this method is not appropriate for Paulo. [1]

Giovanni scored 90 on Test B.

The teacher estimated his score on Test A to be 87 to the nearest integer using the following calculation:

$$90 = 0.822x + 18.4, \text{ so } x = \frac{90 - 18.4}{0.822} \approx 87$$

(c) (i) Give a reason why this method is not appropriate for Giovanni.

(ii) Use an appropriate method to show that the estimated Test A score for Giovanni is 86 to the nearest integer. [4]

(This question continues on the following page)

(Question 5 continued)

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

In Happyland, the weather on any given day is independent of the weather on any other day. On any day in May, the probability of rain is 0.2. May has 31 days.

Find the probability that

(a) it rains on exactly 10 days in May; [2]

(b) it rains on at least 10 days in May; [2]

(c) the first day that it rains in May is on the 10th day. [2]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

A continuous random variable X has a probability density function f given by

$$f(x) = \begin{cases} kx & 0 \leq x \leq k \\ 2kx - x^2 & k < x \leq 2k \\ 0, & \text{otherwise} \end{cases}$$

where $k > 0$.

(a) Show that k satisfies the equation $7k^3 = 6$. [2]

(b) Find the median of X. [4]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider the following bivariate data set where $p, q \in \mathbb{Z}^+$.

x	5	6	6	8	10
y	9	13	p	q	21

The regression line of y on x has equation $y = 2.1875x + 0.6875$.

The regression line passes through the mean point $(\overline{x}, \overline{y})$.

(a) Given that $\overline{x} = 7$, verify that $\overline{y} = 16$. [1]

(b) Given that $q - p = 3$, find the value of p and the value of q. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

Consider a random variable X such that $X \sim \mathrm{B}(n, 0.25)$.

Determine the least value of n such that $\mathrm{P}(X \geq 1) > 0.99$.

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

A shop sells chocolates. The weight, in kilograms, of chocolates bought by a random customer can be modelled by a continuous random variable X with probability density function f defined by

$$f(x) = \begin{cases} \frac{6}{85}\left(4 + 3x - x^2\right), & 0.5 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the mode of X. [2]

(b) Find $\mathrm{P}(1 \leq X \leq 2)$. [2]

(c) Find the median of X. [3]

The shop sells chocolates to customers at $\$25$ per kilogram.

However, if the weight of chocolate bought by a customer is at least 0.75 kilograms, the shop sells chocolate at a discounted rate of $\$24$ per kilogram.

(d) Find the probability that a randomly selected customer spends at most $\$48$. [3]

(e) Find the expected amount spent per customer. Give your answer correct to the nearest cent. [5]

Do **not** write solutions on this page.

Question 10 - working space

1. [Maximum mark: 27]

This question considers two possible models for the occurrence of random events in a computer game.

In a new computer game, each time a player performs an action, there is a random chance that the action will be *boosted*, meaning that it provides a benefit to the player.

The designer of this computer game is considering two possible models for when to boost an action.

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1.

(i) Find the probability that the first boost occurs on the third action. [2]

(ii) Find the probability that at least one boost occurs in the first six actions. [3]

(b) Suppose the probability that an action will be boosted is p, where $0 < p < 1$.

(i) Explain why the probability that the first boost occurs on the xth action is $p(1-p)^{x-1}$. [1]

Let X be the number of actions until the first boost occurs.

(ii) Hence, write down an expression, using sigma notation, for $\mathrm{E}(X)$ in terms of x and p. [1]

Consider the sum of an infinite geometric sequence, with first term a and common ratio $r\ (|r| < 1)$,

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}.$$

(c) (i) By differentiating both sides of the above equation with respect to r, find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r. [4]

(ii) Hence, show that $\mathrm{E}(X) = \frac{1}{p}$. [2]

(This question continues on the following page)

(Question 1 continued)

It can be shown that $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

(d) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ when $p = 0.1$. [2]

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

(e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Let Y be the number of actions until the first boost occurs.

(f) Explain why $Y \leq 5$. [1]

The following table shows the probability distribution of Y.

y	1	2	3	4	5
$\mathrm{P}(Y = y)$	0.2	m	0.288	n	0.0384

(g) (i) Find the value of m and the value of n. [2]

(ii) Show that $\mathrm{E}(Y) = 2.5104$. [2]

(iii) Find $\mathrm{Var}(Y)$. [2]

(h) (i) Use the expression given in (c)(ii) to find the value of p for which $\mathrm{E}(X) = \mathrm{E}(Y)$. [1]

(ii) Find $\mathrm{Var}(X)$ for this value of p. [1]

(iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]

Question 1 - working space

2. [Maximum mark: 31]

This question asks you to find the probability of graphs of randomly generated quadratic functions having a specified number of x-intercepts.

In parts (a) – (f), consider quadratic functions, $f(x) = ax^2 + bx + c$, whose coefficients, a, b and c, are randomly generated in turn by rolling an unbiased six-sided die three times and reading off the value shown on the uppermost face of the die.

For example, rolling a 2, 3 and 5 in turn generates the quadratic function $f(x) = 2x^2 + 3x + 5$.

(a) Explain why there are 216 possible quadratic functions that can be generated using this method. [1]

(b) The set of coefficients, $a = 1$, $b = 4$ and $c = 4$, is randomly generated to form the quadratic function $f(x) = x^2 + 4x + 4$.

Verify that this graph of f has only one x-intercept. [2]

(c) By considering the discriminant, or otherwise, show that the probability of the graph of such a randomly generated quadratic function having only one x-intercept is $\frac{5}{216}$. [6]

Now consider randomly generated quadratic functions whose corresponding graphs have two **distinct** x-intercepts.

(d) By considering the discriminant, determine the set of possible values of ac. [3]

(e) (i) For the case where $ac = 1$, show that there are four quadratic functions whose corresponding graphs have two distinct x-intercepts. [1]

(ii) For the case where $ac = 2$, show that there are eight quadratic functions whose corresponding graphs have two distinct x-intercepts. [2]

Let p be the probability of the graph of such a randomly generated quadratic function having two distinct x-intercepts.

(f) Using the approach started in part (e), or otherwise, find the value of p. [6]

(This question continues on the following page)

(Question 2 continued)

In parts (g) and (h), consider a randomly generated quadratic function, $f(x) = x^2 + 2Zx + 1$, where the continuous random variable $Z \sim \mathrm{N}(0, 1)$.

(g) Find the probability that the graph of f has two x-intercepts. [3]

The continuous random variables, X_1 and X_2, represent the x-intercepts of the graph of f where $X_1 = -Z - \sqrt{Z^2 - 1}$ and $X_2 = -Z + \sqrt{Z^2 - 1}$.

(h) Given that the graph of f has two x-intercepts, X_1 and X_2, find the probability that both X_1 and X_2 are greater than 0.5. [7]

Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

Two events A and B are such that $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.45$ and $\mathrm{P}(A \cup B) = 0.85$.

(a) Find $\mathrm{P}(A \cap B)$. [3]

(b) Find $\mathrm{P}(A' \mid B')$. [3]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

A discrete random variable, X, has the following probability distribution:

$$\mathrm{P}(X = x) = \frac{kx}{20} \text{ for } x \in \{3, 5, 8, 11\}.$$

(a) Find the value of k. [2]

(b) Find $\mathrm{E}(X)$. [3]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

A continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{1}{5} & 0 \leq x < 2 \\ -\frac{x}{30} + \frac{4}{15} & 2 \leq x \leq 8 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{E}(X)$. [3]

(b) Given that $\mathrm{E}(c - 2X) = 0$, where c is a constant, determine the value of c. [2]

(c) Find the median of X. [3]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

Consider events A and B such that $\mathrm{P}(A') = \mathrm{P}(A \cup B) = \frac{3}{4}$ and $\mathrm{P}(B|A) = \frac{2}{3}$.

(a) Find $\mathrm{P}(A \cap B)$. [3]

(b) Show that events A and B are independent. [3]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

Events A and B are such that $\mathrm{P}(A \cup B) = \frac{5}{8}$ and $\mathrm{P}(A \cap B') = \frac{7}{24}$.

(a) Find $\mathrm{P}(B)$. [3]

(b) Given that events A and B are independent, find $\mathrm{P}(A' \mid B)$. [4]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

The random variables X and Y are normally distributed with $X \sim \mathrm{N}(7, a^2)$ and $Y \sim \mathrm{N}(19, a^2)$, where $a > 0$.

(a) Find b such that $\mathrm{P}(X > b) = \mathrm{P}(Y > 22)$. [2]

(b) Write down the approximate value of $\mathrm{P}(7 - a < X < 7 + a)$, correct to two significant figures. [1]

(c) Given that $a = 3$, calculate the approximate value of $\mathrm{P}(Y < 22)$, correct to two significant figures. [3]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 8]

In a study, measurements for arm span, $A\,\text{cm}$, and foot length, $F\,\text{cm}$, are taken from a large group of adults.

For this group, the regression line of F on A is found to be $F = 0.335A - 32.6$, and the regression line of A on F is found to be $A = 2.89F + 99.3$. Each regression line passes through the mean point.

(a) By using an appropriate regression line, find an estimate of the arm span for an adult with a foot length of $19.8\,\text{cm}$. [2]

(b) For this group of adults, find the mean arm span and the mean foot length. [3]

The heights, $H\,\text{cm}$, of adults in the group can be modelled by a normal distribution with mean $163\,\text{cm}$ and standard deviation $\sigma\,\text{cm}$.

It is found that 88% of the group have a height between $153\,\text{cm}$ and $173\,\text{cm}$.

(c) Find the value of σ. [3]

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

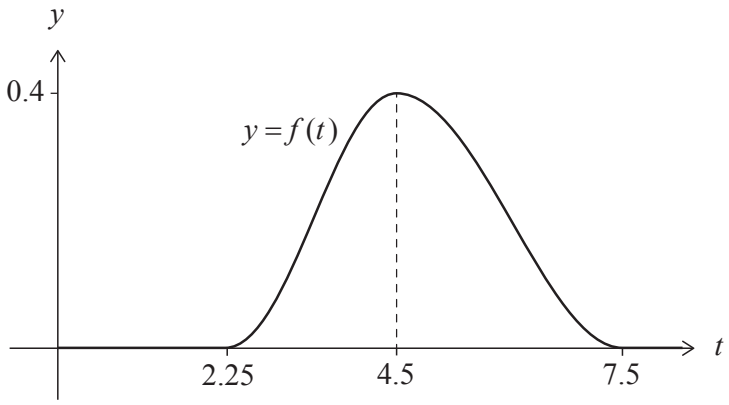
11. [Maximum mark: 19]

In a marathon race, the random variable T represents the time, in hours, taken for a runner to complete the race. No runner completes the race in less than 2.25 hours, and no runner completes it in more than 7.5 hours.

The probability density function for T is modelled by f, defined by

$$f(t) = \begin{cases} \frac{4}{21}\left(1-\cos\left(\frac{4\pi}{9}(t-2.25)\right)\right), & 2.25 \le t < 4.5 \\ \frac{4}{21}\left(1+\cos\left(\frac{\pi}{3}(t-4.5)\right)\right), & 4.5 \le t \le 7.5 \\ 0, & \text{otherwise.} \end{cases}$$

The graph of f has a maximum point at $t = 4.5$ as shown in the following diagram:



(a) (i) Find the value of $\int_{2.25}^{4.5} f(t)\,\mathrm{d}t$.

(ii) Write down the mode of T.

(iii) Determine which is greater, the mode of T or the median of T, justifying your answer. [4]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 11 continued)

The runners who finish the race in 3.5 hours or less are considered to be fast runners.

(b) Find the probability that a runner chosen at random is a fast runner. [2]

(c) Find the probability that a fast runner chosen at random finishes the race in 3 hours or less. [3]

(d) Find the lower quartile of T. [3]

Each runner's time is converted to a score which is calculated as $a - bt$, where t represents their time in hours, and a, $b > 0$.

Consider the random variable P which represents the score of a runner. It is given that $\mathrm{E}(P) = 100$ and the maximum possible score is 150.

(e) Use $\mathrm{E}(T) = 4.723$ to determine the value of a and the value of b, giving your answers to the nearest integer. [5]

(f) Given also that $\mathrm{Var}(T) = 0.906$, find $\mathrm{Var}(P)$. [2]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The following table shows the number of hours of play time, x, and sleep time, y, for a group of six children, over the period of one week.

Play time (x)	11	13	14	17	22	24
Sleep time (y)	62	65	68	75	84	87

The regression line of y on x for this data can be written in the form $y = ax + b$.

(a) Find the value of a and the value of b. [2]

(b) Use the equation of the regression line to estimate the sleep time of a child whose weekly play time is 20 hours. [2]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

The marks obtained by students in a class quiz are shown in the following table where $p, q \in \mathbb{Z}^+$.

Marks	Frequency
20	12
35	q
p	8

The mean and variance of the marks are 31 and 124 respectively.

Find the value of p and the value of q.

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

At Adam's Apple Orchard the weights of apples, W, in grams, are normally distributed with a mean 175 grams and standard deviation 8 grams.

(a) Find the probability that a randomly chosen apple weighs less than 170 grams. [2]

(b) It is found that 20% of the apples weigh more than w grams. Find w, correct to four significant figures. [2]

All orchards classify an apple as premium when its weight is between 170 and 185 grams.

(c) Find the percentage of apples that are classified as premium at Adam's Apple Orchard. [2]

After orders are completed, there are many apples left over. Boxes are filled with randomly chosen left-over apples. Each box contains 40 apples.

(d) Find the probability that a randomly chosen box contains at least 30 premium apples. [3]

(e) If 10 of these boxes are randomly selected, find the probability that exactly 4 boxes have at least 30 premium apples. [2]

At a neighbouring orchard the weights of apples, M, in grams, are normally distributed with mean μ and standard deviation σ. It is known that:

- 82% of their apples are classified as premium
- the percentage of apples that weigh less than 170 grams is twice the percentage of apples that weigh more than 185 grams.

(f) Find the value of μ. [6]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

A supermarket analyses the shopping habits of its customers.

The number of times, X, each customer visits the supermarket in a week is given by the following probability distribution.

x	1	2	3	4	5	≥ 6
$\mathrm{P}(X = x)$	$1.5a$	$2a$	0.281	a	0.026	0

(a) (i) Find the value of a.

(ii) Write down the mode of X. [3]

(b) Find the mean of X. [2]

The manager wants to know why customers come to their supermarket. They survey the first 50 customers to arrive at the supermarket on a particular day.

(c) Identify which one of the following best describes the manager's sampling method. Circle your answer. [1]

Simple random / Systematic / Convenience / Quota / Stratified

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 18]

Amanda enters data from surveys into a database. It can be assumed that the accuracy of any survey entered is independent of all other surveys entered.

From previous records, it is known that Amanda enters 8% of the surveys inaccurately.

(a) On a particular day Amanda enters data from 50 surveys.

(i) Find the probability that Amanda entered at most six surveys inaccurately.

(ii) Given that at most six surveys were entered inaccurately, find the probability that exactly four surveys were entered inaccurately. [5]

On a different day Amanda enters data from n surveys. On this day, the probability that at most six surveys were entered inaccurately is approximately 0.367.

(b) Find the value of n. [3]

Bryce and Carmen also enter data from surveys into the same database. It is known that surveys entered by Bryce and Carmen are inaccurate 6% and 11% of the time respectively. It can again be assumed that the accuracy of any survey entered is independent of all other surveys entered.

From the surveys assigned to the three of them, Amanda enters 55%, Bryce 25% and Carmen 20%.

(c) Find the probability that a randomly selected survey was

(i) entered inaccurately;

(ii) entered by Amanda, given that the survey was entered inaccurately. [6]

The following year, the accuracy of Amanda's and Bryce's work remained the same, as did the percentage of surveys entered by each of the three employees. However, Carmen's accuracy had improved and the probability that she entered a survey inaccurately was now $x\%$.

The probability that a randomly selected survey had been entered inaccurately was now the same as the probability that Carmen made an error when entering a survey.

(d) Find the value of x. [4]

Do **not** write solutions on this page.

Question 11 - working space

1. [Maximum mark: 23]

This question asks you to use polynomial functions to model some situations in probability.

Two unbiased tetrahedral (four-sided) dice with faces labelled 1, 2, 3 and 4 are thrown and the scores recorded.

The random variable M denotes the maximum of these two scores.

The probability distribution of M is given in the following table.

m	1	2	3	4
$\mathrm{P}(M = m)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

(a) Find $\mathrm{E}(M)$. [2]

An alternative way to represent the probability distribution of M is to use a polynomial function, G, where $G(t) = \sum_{m=1}^{4} \mathrm{P}(M = m)t^m$.

Hence, for the distribution of M, $G(t) = \frac{1}{16}t + \frac{3}{16}t^2 + \frac{5}{16}t^3 + \frac{7}{16}t^4$.

(b) Find $G(1)$. [1]

(c) (i) Find $G'(t)$. [2]

(ii) Hence, show that $G'(1) = \mathrm{E}(M)$. [3]

(This question continues on the following page)

(Question 1 continued)

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable X denotes the total number of red balls selected.

The probability distribution of X can be represented by the polynomial function, G_X, where

$$G_X(t) = \sum_{x=0}^{2} \mathrm{P}(X = x)t^x.$$

(d) Show that $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$, making it clear how the coefficients of $G_X(t)$ have been determined. [5]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is p.

The random variable Y denotes the total number of tails obtained from tossing both coins.

The probability distribution of Y can be represented by the polynomial function, G_Y, where

$$G_Y(t) = \sum_{y=0}^{2} \mathrm{P}(Y = y)t^y.$$

(e) Given that the coefficient of t^2 in $G_Y(t)$ is $\frac{1}{3}$, find

(i) the value of p; [2]

(ii) an expression for $G_Y(t)$. [4]

The random variable Z denotes the sum of the total number of red balls selected, X, and the total number of tails obtained from tossing both coins, Y.

The probability distribution of Z can be represented by the function, G_Z, where

$$G_Z(t) = G_X(t)\,G_Y(t)\,.$$

(f) For random variable Z, it can be shown that $G_Z'(1) = \mathrm{E}(Z)$.

Use this result to find $\mathrm{E}(Z)$. [4]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A)=\frac{1}{2}$, $\mathrm{P}(B)=\frac{1}{3}$ and $\mathrm{P}(A\mid B)=\frac{1}{4}$.

(a) Find the probability that both events A and B occur. [2]

(b) Find $\mathrm{P}(A\mid B')$. [4]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 8]

The discrete random variable X has the following probability distribution:

x	1	2	3
$\mathrm{P}(X = x)$	$0.6 - 2a$	$3a$	$0.4 - a$

(a) Find the range of possible values of a. [2]

(b) In the case where $a = 0.2$, determine $\mathrm{Var}(2 - X)$. [6]

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

Data was collected on two variables, x and y, from eight members of a population. The data is shown in the table below.

x	3.8	6.3	2.5	2.4	8.9	10.7	6.7	10.3
y	11.8	9.7	6.1	4.1	p	23.3	13.7	24.5

(a) Find $\overline{x}$. [2]

The regression line of y on x for this data is $y = 0.014375 + 2.1625x$.

(b) Find

(i) $\overline{y}$;

(ii) the value of p. [4]

(c) Hence, find Pearson's product-moment correlation coefficient, r. [1]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

Consider the following grouped data set.

Class	Frequency
$15 < y \leq 20$	31
$20 < y \leq 25$	42
$25 < y \leq 30$	61
$30 < y \leq 35$	46
$35 < y \leq 40$	29

(a) Find an estimate for

(i) the mean;

(ii) the standard deviation. [2]

The data can be approximated by a normal distribution.

(b) Use this information and your mean and standard deviation found in part (a) to find an approximate value for the interquartile range correct to two significant figures. [3]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

Two machines are used in a factory to manufacture semiconductors. Machine A manufactures defective semiconductors 10% of the time, while machine B manufactures defective semiconductors 5% of the time.

A randomly selected machine manufactures ten semiconductors.

Both machines are equally likely to be selected.

(a) Find the probability that exactly two semiconductors are defective. [4]

(b) Given that exactly two semiconductors are defective, find the probability that they were manufactured by machine A. [3]

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 18]

(a) Use integration by parts to find $\int \arccos x \, \mathrm{d}x$. [4]

The probability density function of a random variable X, is given by

$$f(x) = \begin{cases} 3x \arccos\left(x^2\right), & 0 \leq x \leq k; \\ 0, & \text{otherwise} \end{cases}.$$

(b) (i) Show that $k^2 \arccos(k^2) - \sqrt{1-k^4} + \frac{1}{3} = 0$.

(ii) Hence, find the value of k. Give your answer correct to six significant figures. [6]

(c) Find

(i) $\mathrm{E}(X)$;

(ii) $\mathrm{Var}(X)$. [5]

(d) Given that $\mu = \mathrm{E}(X)$ and $\sigma^2 = \mathrm{Var}(X)$, find $\mathrm{P}(\mu - \sigma < X < \mu + \sigma)$. [3]

Do **not** write solutions on this page.

Question 11 - working space

Section A

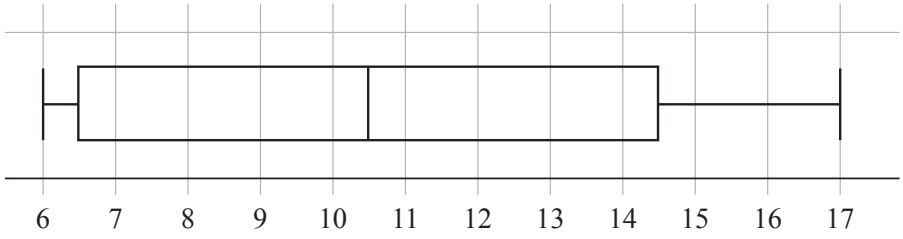
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

A teacher sets her class of 30 pupils a quiz.

Aiden and Brett were absent on the day of the quiz.

The following box and whisker diagram shows the results of the 28 pupils who took the quiz on the day.



Aiden and Brett take the quiz when they return.

Aiden scores less than 6.

Brett scores more than 17.

(a) Explain, briefly, why the median score for all 30 pupils would still be 10.5. [1]

The mean score of the 28 pupils was 10.5.

The mean score for all 30 pupils is now 10.6.

The range of scores for all 30 pupils is 14.

(b) Determine Aiden's score and Brett's score. [5]

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

A garden centre sells seeds for two varieties of sunflowers: large and giant.

The heights of large sunflowers are normally distributed with mean μ and standard deviation σ.

It is known that $25\,\%$ of large sunflowers have a height of over $180\,\text{cm}$.

(a) Given that $\mu + A\sigma = 180$, where $A \in \mathbb{R}$, find the value of A. [3]

The heights of giant sunflowers are also normally distributed.

The giant sunflowers have a mean height that is $35\,\text{cm}$ more than that of the large sunflowers and a standard deviation that is double that of the large sunflowers.

It is known that $98\,\%$ of giant sunflowers have a height greater than $180\,\text{cm}$.

(b) Determine μ and σ. [4]

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

The lengths, in metres, of jumps in a long jump competition can be modelled by a continuous random variable X with probability density function f defined by:

$$f(x) = \begin{cases} 0, & x < 3 \\ a(x-3)^3 + b(x-3)^2, & 3 \le x \le 9 \\ 0, & x > 9 \end{cases}$$

where $a, b \in \mathbb{R}$, $a, b \neq 0$.

(a) Show that $324a + 72b = 1$. [4]

It is given that $f(9) = 0$.

(b) (i) Show that $6a + b = 0$.

(ii) Determine the value of a and the value of b. [4]

(c) Show that the median of X is less than the mode of X. [5]

Matt has the final jump in the competition. He needs to jump at least $8.52\,\text{m}$ to win the competition.

(d) Given that he jumps over $8\,\text{m}$, use the model to find the probability that he wins the competition. [4]

Do **not** write solutions on this page.

Question 11 - working space

APOLLO

Calculus

135 questions

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

Consider the functions $f(x) = -(x-h)^2 + 2k$ and $g(x) = \mathrm{e}^{x-2} + k$ where $h, k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x = 3$.

(b) Show that $h = \frac{\mathrm{e}+6}{2}$. [3]

(c) Hence, show that $k = \mathrm{e} + \frac{\mathrm{e}^2}{4}$. [3]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$.

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Let $f(x) = \sqrt{1+x}$ for $x > -1$.

(a) Show that $f''(x) = -\frac{1}{4\sqrt{(1+x)^3}}$. [3]

(b) Use mathematical induction to prove that $f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$ for $n \in \mathbb{Z}$, $n \geq 2$. [9]

Let $g(x) = e^{mx}$, $m \in \mathbb{Q}$.

Consider the function h defined by $h(x) = f(x) \times g(x)$ for $x > -1$.

It is given that the x^2 term in the Maclaurin series for $h(x)$ has a coefficient of $\frac{7}{4}$.

(c) Find the possible values of m. [8]

References:

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Question 12 - working space

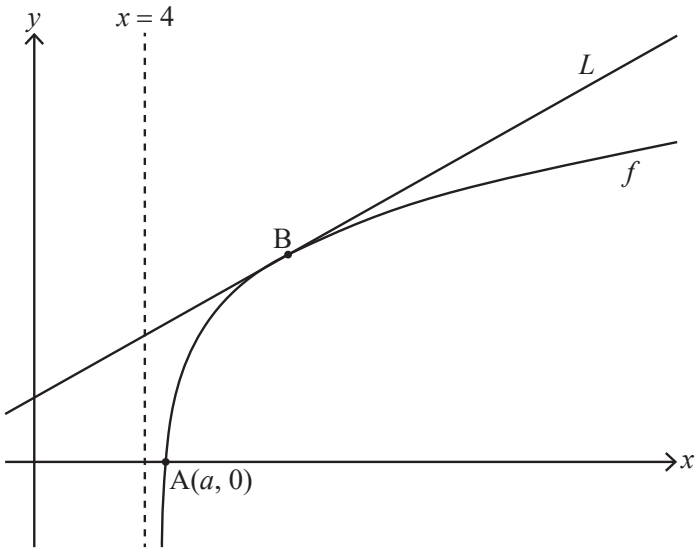
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.



(a) Find the exact value of a. [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2} \mathrm{d}x$.

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Do **not** write solutions on this page.

Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

The acceleration, $a\,\text{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\text{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\text{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\text{e}^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\text{e}^T = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\text{e}^{-(T-k)} - 1.$$

(c) By using the result to part (b) (i), show that $v(T-k) = \text{e}^k - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

(d) Deduce a similar expression for $v(T+k)$ in terms of k. [2]

(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

Do **not** write solutions on this page.

Question 11 - working space

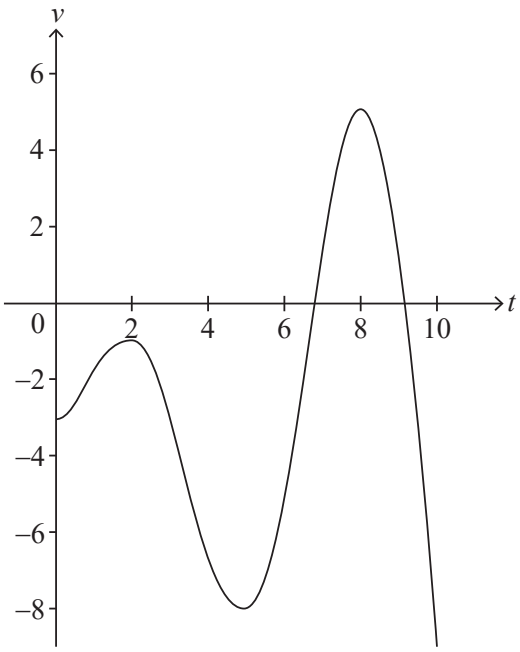
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

A particle moves in a straight line. The velocity, $v\,\text{ms}^{-1}$, of the particle at time t seconds is given by $v(t) = t\sin t - 3$, for $0 \le t \le 10$.

The following diagram shows the graph of v.



(a) Find the smallest value of t for which the particle is at rest. [2]

(b) Find the total distance travelled by the particle. [2]

(c) Find the acceleration of the particle when $t = 7$. [2]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

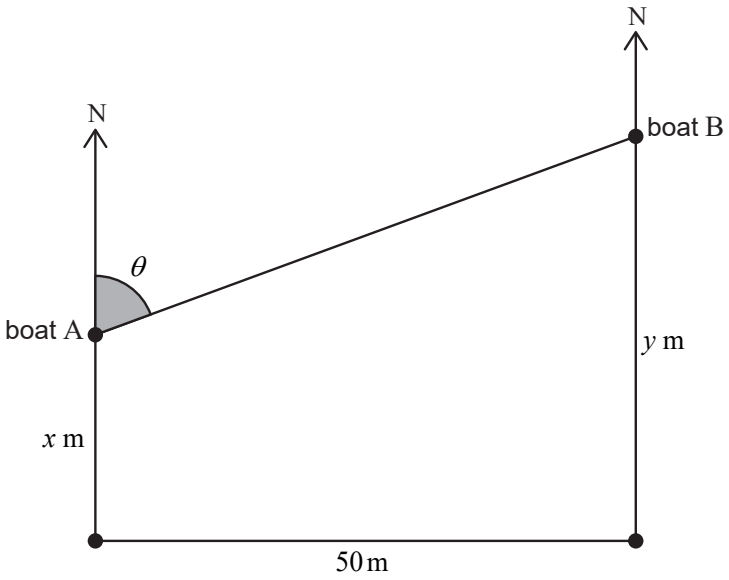
9. [Maximum mark: 7]

Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A.
This information is shown on the following diagram.

diagram not to scale



(a) Show that $y = x + 50\cot\theta$. [1]

At time T, the following conditions are true.

Boat B has travelled 10 metres further than boat A.
Boat B is travelling at double the speed of boat A.
The rate of change of the angle θ is -0.1 radians per second.

(b) Find the speed of boat A at time T. [6]

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Do **not** write solutions on this page.

Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,e^{-0.5x}$ for $x \in \mathbb{R}^+$.

The graph of f and the line $y = x$ intersect at point P.

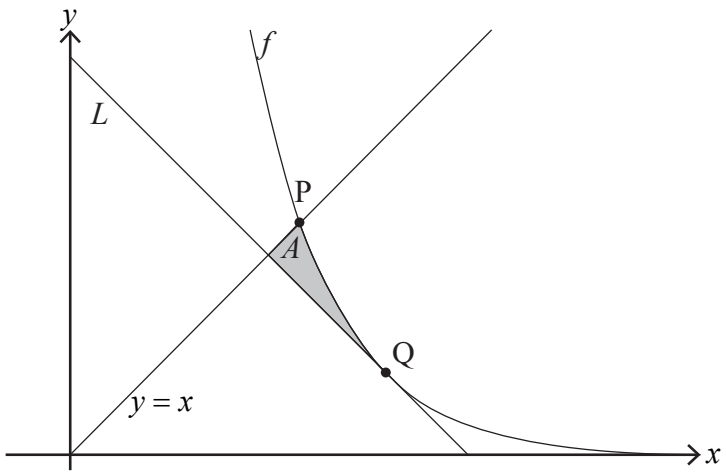
(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.



(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

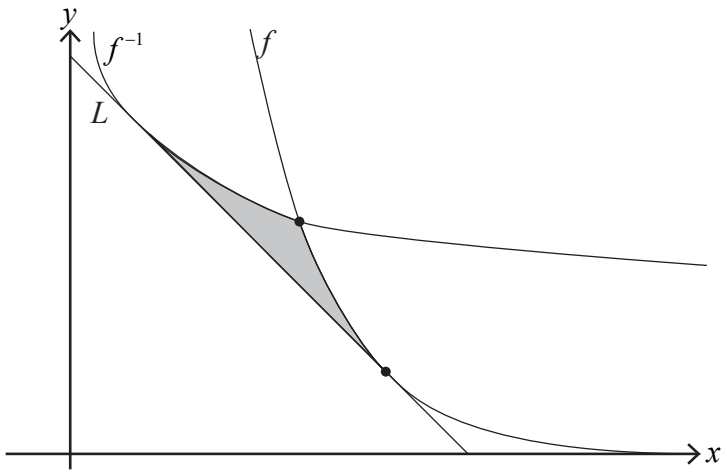
(ii) Hence, find the area of A. [5]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The line L is tangent to the graphs of both f and the inverse function f^{-1}.



(e) Find the shaded area enclosed by the graphs of f and f^{-1} and the line L. [2]

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

The function f has a derivative given by $f'(x) = \frac{1}{x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, $x \neq k$ where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{x} + \frac{b}{k-x}$, where $a, b \in \mathbb{R}$. Find a and b in terms of k. [3]

(b) Hence, find an expression for $f(x)$. [3]

Consider P, the population of a colony of ants, which has an initial value of 1200.

The rate of change of the population can be modelled by the differential equation $\frac{dP}{dt} = \frac{P(k-P)}{5k}$, where t is the time measured in days, $t \geq 0$, and k is the upper bound for the population.

(c) By solving the differential equation, show that $P = \frac{1200k}{(k-1200)e^{-\frac{t}{5}} + 1200}$. [8]

At $t = 10$ the population of the colony has doubled in size from its initial value.

(d) Find the value of k, giving your answer correct to four significant figures. [3]

(e) Find the value of t when the rate of change of the population is at its maximum. [3]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

(a) Write down the first three terms of the binomial expansion of $(1+t)^{-1}$ in ascending powers of t. [1]

(b) By using the Maclaurin series for $\cos x$ and the result from part (a), show that the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\frac{x^2}{2}+\frac{5x^4}{24}$. [4]

(c) By using the Maclaurin series for $\arctan x$ and the result from part (b), find $\lim_{x\to 0}\left(\frac{x\arctan 2x}{\sec x-1}\right)$. [3]

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Question 9 - working space

Section B

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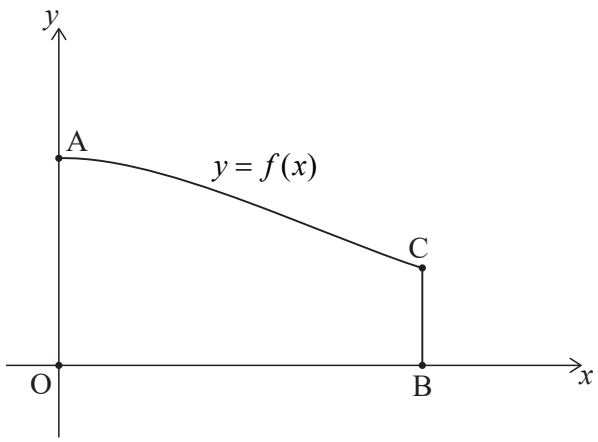
11. [Maximum mark: 18]

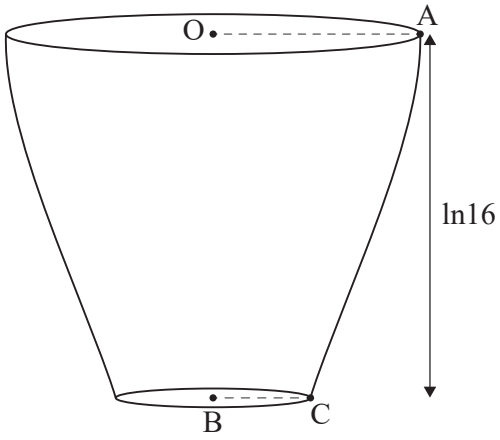
A function f is defined by $f(x)=\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^{x}}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^{+}$.

The region enclosed by the graph of $y=f(x)$, the x-axis, the y-axis and the line $x=\ln 16$ is rotated $360°$ about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of $300\,\mathrm{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.





The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of k that satisfies the requirements of Pedro's design. [2]

(c) Find

(i) OA;

(ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly.

(ii) State the cross-sectional radius of the bowl at this point. [6]

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Question 11 - working space

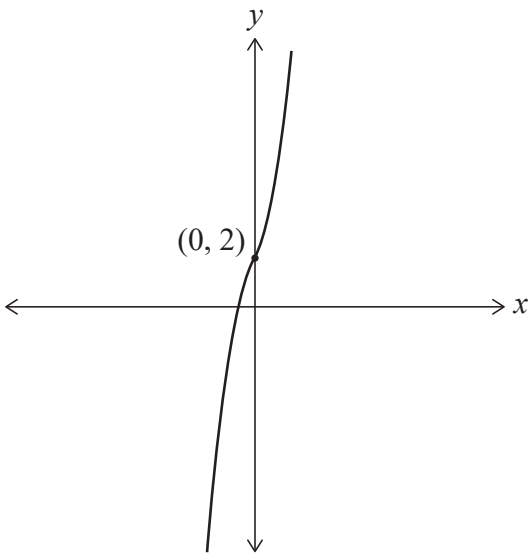
1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

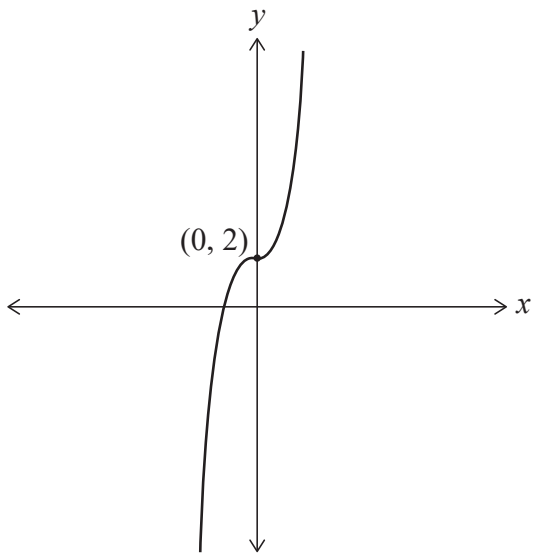
Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.

$c = -1$



$c = 0$



(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y-intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

(c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has

(i) a point of inflexion with zero gradient; [1]

(ii) one local maximum point and one local minimum point; [2]

(iii) no points where the gradient is equal to zero. [1]

(d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that

(i) the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]

(ii) the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]

(e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has

(i) exactly one x-axis intercept; [2]

(ii) exactly two x-axis intercepts; [2]

(iii) exactly three x-axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c, d \in \mathbb{R}$.

(f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x-axis intercept, explaining your reasoning. [6]

Question 1 - working space

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$;
- the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

(c) Show that $f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(This question continues on the following page)

(Question 1 continued)

(f) Hence, or otherwise, show that $f_n'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a - x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a - x)^n = k$ has four solutions for x. [5]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

Given that $\frac{dy}{dx} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find y in terms of x.

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The function f is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of f at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function g is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of h at $x = 4$. [3]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = f(x)$.

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

A particle P moves along the x-axis. The velocity of P is $v\,\text{m s}^{-1}$ at time t seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0$, P is at the origin O.

(a) (i) Find the value of t when P reaches its maximum velocity.

(ii) Show that the distance of P from O at this time is $\frac{88}{27}$ metres. [7]

(b) Sketch a graph of v against t, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by P. [5]

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n\left(n-1\right)\right]\mathrm{e}^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x)=x^2\mathrm{e}^x$ in ascending powers of x, up to and including the term in x^4. [3]

(c) Hence or otherwise, determine the value of $\lim\limits_{x\to 0}\left[\frac{\left(x^2\mathrm{e}^x-x^2\right)^3}{x^9}\right]$. [4]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 8]

Consider the curve C given by $y = x - xy\ln(xy)$ where $x > 0$, $y > 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln\left(xy\right)\right) = 1$. [3]

(b) Hence find the equation of the tangent to C at the point where $x = 1$. [5]

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Question 8 - working space

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = ax + y,$$

where $x, y, t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a) (i) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y$, show that $y = A\mathrm{e}^t$ where A is a constant. [3]

(ii) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$. [1]

(iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

(b) (i) By differentiating $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ with respect to t, show that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2\frac{\mathrm{d}y}{\mathrm{d}t}$. [3]

(ii) By substituting $Y = \frac{\mathrm{d}y}{\mathrm{d}t}$, show that $Y = B\mathrm{e}^{2t}$ where B is a constant. [3]

(iii) Hence find y as a function of t. [2]

(iv) Hence show that $x = -\frac{B}{2}\mathrm{e}^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c) (i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = F\mathrm{e}^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(ii) Find the two values for λ that satisfy $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. [4]

Let the two values found in part (c)(ii) be λ_1 and λ_2.

(iii) Verify that $y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References:

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 5]

Let $f'(x) = \frac{8x}{\sqrt{2x^2 + 1}}$. Given that $f(0) = 5$, find $f(x)$.

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Question 3 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Let $f(x) = \frac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \frac{1 - \ln 5x}{kx^2}$. [3]

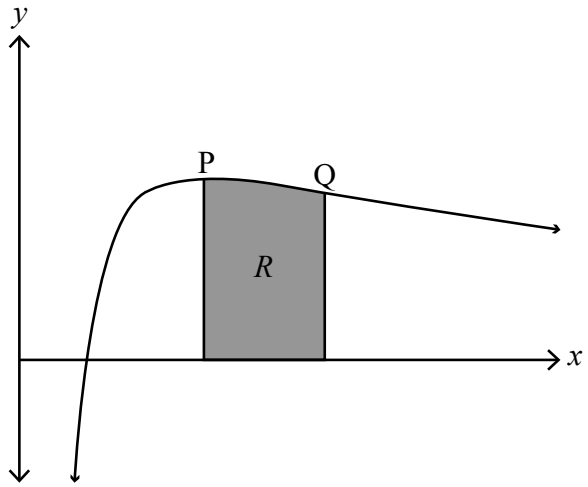
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \frac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\frac{1}{5}\mathrm{e}^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k. [7]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The function f is defined by $f(x) = \mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x} - 1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x} - 1\right)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x \to 0} \dfrac{f(x) - 1}{\arctan\left(\mathrm{e}^{3x} - 1\right)}$. [3]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t + 2t \sin t$, $0 \leq t \leq 5$.

(a) Find the maximum distance of the particle from O. [3]

(b) Find the acceleration of the particle at the instant it first changes direction. [4]

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Question 5 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let x grams represent the amount of salt in the tank and let t minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{\mathrm{d}x}{\mathrm{d}t}$, is described by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 10\mathrm{e}^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t+1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40\mathrm{e}^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of x versus t for $0 \leq t \leq 60$ and hence find the maximum amount of salt in the tank and the value of t at which this occurs. [5]

(d) Find the value of t at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first 60 minutes. [4]

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Find the value of $\int_1^9 \left(\frac{3\sqrt{x} - 5}{\sqrt{x}} \right) \mathrm{d}x$.

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

Consider the curve with equation $y = (2x - 1)\mathrm{e}^{kx}$, where $x \in \mathbb{R}$ and $k \in \mathbb{Q}$.

The tangent to the curve at the point where $x = 1$ is parallel to the line $y = 5\mathrm{e}^{k}x$.

Find the value of k.

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The function f is defined by $f(x) = \mathrm{e}^x \sin x$, where $x \in \mathbb{R}$.

(a) Find the Maclaurin series for $f(x)$ up to and including the x^3 term. [4]

(b) Hence, find an approximate value for $\int_0^1 \mathrm{e}^{x^2} \sin(x^2)\,\mathrm{d}x$. [4]

The function g is defined by $g(x) = \mathrm{e}^x \cos x$, where $x \in \mathbb{R}$.

(c) (i) Show that $g(x)$ satisfies the equation $g''(x) = 2\big(g'(x) - g(x)\big)$.

(ii) Hence, deduce that $g^{(4)}(x) = 2\big(g'''(x) - g''(x)\big)$. [5]

(d) Using the result from part (c), find the Maclaurin series for $g(x)$ up to and including the x^4 term. [5]

(e) Hence, or otherwise, determine the value of $\lim_{x \to 0} \dfrac{\mathrm{e}^x \cos x - 1 - x}{x^3}$. [3]

References:

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Question 12 - working space

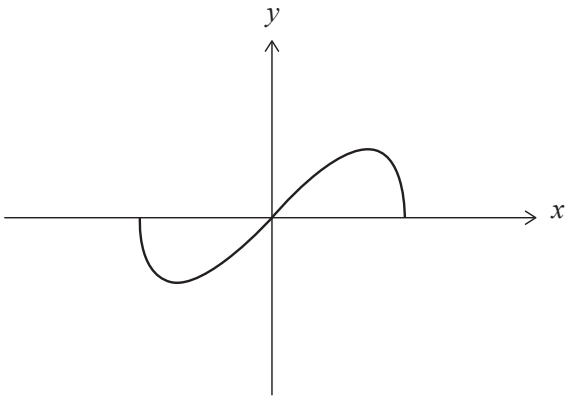
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \leq x \leq 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

The range of f is $a \leq y \leq b$, where $a, b \in \mathbb{R}$.

(b) Find the value of a and the value of b. [6]

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

By using the substitution $u = \sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, \mathrm{d}x$ in terms of n, where n is a non-zero real number.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

A particle moves along a straight line so that its velocity, $v\,\text{m s}^{-1}$, after t seconds is given by $v(t) = e^{\sin t} + 4\sin t$ for $0 \le t \le 6$.

(a) Find the value of t when the particle is at rest. [2]

(b) Find the acceleration of the particle when it changes direction. [3]

(c) Find the total distance travelled by the particle. [2]

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Question 4 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 20]

Consider the function $f(x) = \sqrt{x^2 - 1}$, where $1 \leq x \leq 2$.

(a) Sketch the curve $y = f(x)$, clearly indicating the coordinates of the endpoints. [2]

(b) (i) Show that the inverse function of f is given by $f^{-1}(x) = \sqrt{x^2 + 1}$.

(ii) State the domain and range of f^{-1}. [5]

The curve $y = f(x)$ is rotated 2π about the y-axis to form a solid of revolution that is used to model a water container.

(c) (i) Show that the volume, $V\,\text{m}^3$, of water in the container when it is filled to a height of h metres is given by $V = \pi\left(\frac{1}{3}h^3 + h\right)$.

(ii) Hence, determine the maximum volume of the container. [5]

At $t = 0$, the container is empty. Water is then added to the container at a constant rate of $0.4\,\text{m}^3\,\text{s}^{-1}$.

(d) Find the time it takes to fill the container to its maximum volume. [2]

(e) Find the rate of change of the height of the water when the container is filled to half its maximum volume. [6]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

Consider the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 1.5$. [4]

(b) Use the substitution $y = vx$ to show that $x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - v - 2$. [3]

(c) (i) By solving the differential equation, show that $y = \frac{8x + x^4}{4 - x^3}$.

(ii) Find the actual value of y when $x = 1.5$.

(iii) Using the graph of $y = \frac{8x + x^4}{4 - x^3}$, suggest a reason why the approximation given by Euler's method in part (a) is not a good estimate to the actual value of y at $x = 1.5$. [12]

References:

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

The derivative of a function g is given by $g'(x) = 3x^2 + 5e^x$, where $x \in \mathbb{R}$. The graph of g passes through the point $(0, 4)$. Find $g(x)$.

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

A particle moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v = \frac{\left(t^2+1\right)\cos t}{4}$, $0 \le t \le 3$.

(a) Determine when the particle changes its direction of motion. [2]

(b) Find the times when the particle's acceleration is $-1.9\,\text{m s}^{-2}$. [3]

(c) Find the particle's acceleration when its speed is at its greatest. [2]

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Question 5 - working space

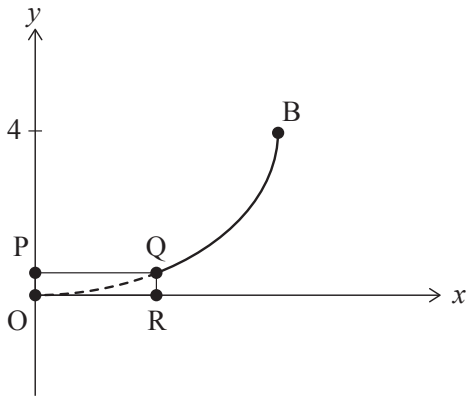
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

The following diagram shows the curve $\frac{x^2}{36}+\frac{(y-4)^2}{16}=1$, where $h \le y \le 4$.

diagram not to scale



The curve from point Q to point B is rotated $360°$ about the y-axis to form the interior surface of a bowl. The rectangle OPQR, of height h cm, is rotated $360°$ about the y-axis to form a solid base.

The bowl is assumed to have negligible thickness.

Given that the interior volume of the bowl is to be $285\,\mathrm{cm}^3$, determine the height of the base.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 8]

Consider $\lim_{x \to 0} \frac{\arctan(\cos x) - k}{x^2}$, where $k \in \mathbb{R}$.

(a) Show that a finite limit only exists for $k = \frac{\pi}{4}$. [2]

(b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

A scientist conducted a nine-week experiment on two plants, A and B, of the same species. He wanted to determine the effect of using a new plant fertilizer. Plant A was given fertilizer regularly, while Plant B was not.

The scientist found that the height of Plant A, $h_A \text{cm}$, at time t weeks can be modelled by the function $h_A(t) = \sin(2t + 6) + 9t + 27$, where $0 \leq t \leq 9$.

The scientist found that the height of Plant B, $h_B \text{cm}$, at time t weeks can be modelled by the function $h_B(t) = 8t + 32$, where $0 \leq t \leq 9$.

(a) Use the scientist's models to find the initial height of

(i) Plant B;

(ii) Plant A correct to three significant figures. [3]

(b) Find the values of t when $h_A(t) = h_B(t)$. [3]

(c) For $t > 6$, prove that Plant A was always taller than Plant B. [3]

(d) For $0 \leq t \leq 9$, find the total amount of time when the rate of growth of Plant B was greater than the rate of growth of Plant A. [6]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

The population, P, of a particular species of marsupial on a small remote island can be modelled by the logistic differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$$

where t is the time measured in years and k, N are positive constants.

The constant N represents the maximum population of this species of marsupial that the island can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{\mathrm{d}P}{\mathrm{d}t}$. [1]

(b) Show that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. [4]

(c) Hence show that the population of marsupials will increase at its maximum rate when $P = \frac{N}{2}$. Justify your answer. [5]

(d) Hence determine the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ in terms of k and N. [2]

Let P_0 be the initial population of marsupials.

(e) By solving the logistic differential equation, show that its solution can be expressed in the form

$$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right).$$ [7]

After 10 years, the population of marsupials is $3P_0$. It is known that $N = 4P_0$.

(f) Find the value of k for this population model. [2]

References:

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Answers written on this page
will not be marked.

Question 12 - working space

2. [Maximum marks: 28]

This question asks you to explore cubic polynomials of the form $(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r=1$, $a=4$ and $b=1$.

Consider the equation $(z-1)(z^2-8z+17)=0$ for $z \in \mathbb{C}$.

(a) (i) Given that 1 and $4+\mathrm{i}$ are roots of the equation, write down the third root. [1]

(ii) Verify that the mean of the two complex roots is 4. [1]

Consider the function $f(x)=(x-1)(x^2-8x+17)$ for $x \in \mathbb{R}$.

(b) Show that the line $y=x-1$ is tangent to the curve $y=f(x)$ at the point $\mathrm{A}(4, 3)$. [4]

(c) Sketch the curve $y=f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x-axis. [2]

Consider the function $g(x)=(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(d) (i) Show that $g'(x)=2(x-r)(x-a)+x^2-2ax+a^2+b^2$. [2]

(ii) Hence, or otherwise, prove that the tangent to the curve $y=g(x)$ at the point $\mathrm{A}(a, g(a))$ intersects the x-axis at the point $\mathrm{R}(r, 0)$. [6]

The equation $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$ has roots r and $a \pm b\mathrm{i}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

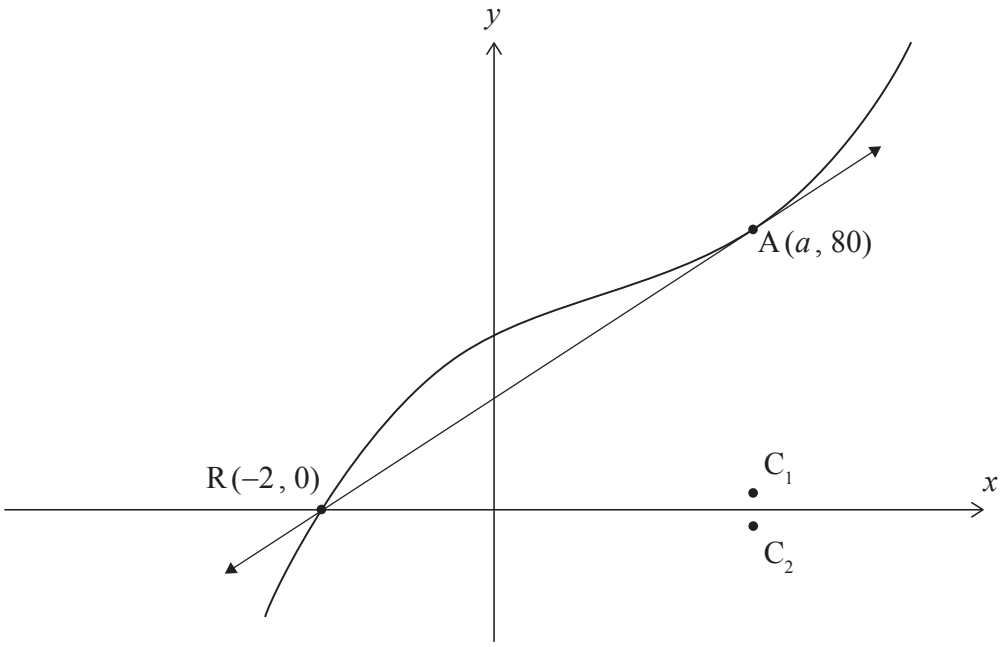
(e) Deduce from part (d)(i) that the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$ can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$. [1]

(This question continues on the following page)

(Question 2 continued)

On the Cartesian plane, the points $C_1\left(a, \sqrt{g'(a)}\right)$ and $C_2\left(a, -\sqrt{g'(a)}\right)$ represent the real and imaginary parts of the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$.

The following diagram shows a particular curve of the form $y=(x-r)(x^2-2ax+a^2+16)$ and the tangent to the curve at the point $A(a, 80)$. The curve and the tangent both intersect the x-axis at the point $R(-2, 0)$. The points C_1 and C_2 are also shown.



(f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z-r)(z^2-2az+a^2+16)=0$ for $z \in \mathbb{C}$. [4]

(ii) State the coordinates of C_2. [1]

Consider the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a \neq r$, $b>0$. The points $A(a, g(a))$ and $R(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

(g) (i) Show that the x-coordinate of P is $\frac{1}{3}(2a+r)$.

You are **not** required to demonstrate a change in concavity. [2]

(ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A. [1]

Consider the special case where $a=r$ and $b>0$.

(h) (i) Sketch the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a=r=1$ and $b=2$. [2]

(ii) For $a=r$ and $b>0$, state in terms of r, the coordinates of points P and A. [1]

References:

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Question 2 - working space

1. [Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b, where $a, b \in \mathbb{N}$.

(a) On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3,\ x \ge 0$ [2]

(ii) $y^2 = x^3 + 1,\ x \ge -1$ [2]

(b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

(ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \ge -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

(c) By varying the value of b, suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x,\ x \ge 0$.

(d) (i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2 + 1}{2\sqrt{x^3 + x}}$, for $x > 0$. [3]

(ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

(e) Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3} + q}{r}}$, where $p, q, r \in \mathbb{Z}$. [7]

(This question continues on the following page)

(Question 1 continued)

$\mathrm{P}(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $\mathrm{P}(-1, -1)$ lies on C.

(f) (i) Find the equation of the tangent to C at P. [2]

(ii) Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. [2]

(g) The point $\mathrm{S}(-1, 1)$ also lies on C. The line $[\mathrm{QS}]$ intersects C at a further point. Determine the coordinates of this point. [5]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 4]

The function g is defined by $g(x) = e^{x^2+1}$, where $x \in \mathbb{R}$.

Find $g'(-1)$.

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

The derivative of the function f is given by $f'(x) = \frac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 10]

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{xy}$, where $x, y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of m. [2]

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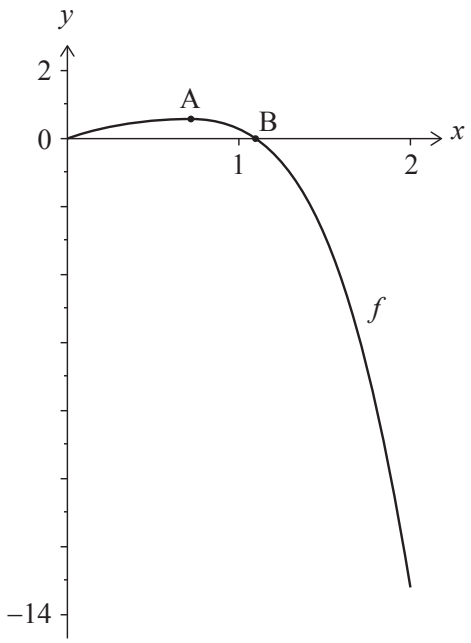
Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 6]

The function f is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 \le x \le 2$. The graph of f is shown in the following diagram.



The graph of f has a local maximum at point A. The graph intersects the x-axis at the origin and at point B.

(a) Find the coordinates of A. [2]

(b) Find the x-coordinate of B. [1]

(c) Find the total area enclosed by the graph of f, the x-axis and the line $x = 2$. [3]

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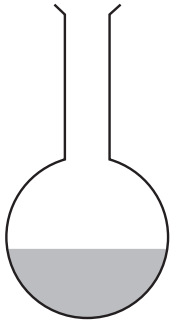
Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

The following diagram shows liquid in a round-bottomed glass flask, which is made of a sphere and a cylindrical neck.



Initially, the flask is empty. Liquid is poured into the flask at a rate of $2\,\text{cm}^3\text{s}^{-1}$. You may assume that the liquid does not reach the cylindrical neck.

The volume $V\,\text{cm}^3$ and the height $h\,\text{cm}$ of the liquid in the flask satisfy the equation $V = 5\pi h^2 - \frac{1}{3}\pi h^3$.

Find the rate of change of the height of the liquid in the flask at the instant when the volume of the liquid is $200\,\text{cm}^3$.

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 15]

The function f is defined by $f(x) = e^{2x}(3x - 4)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [3]

(b) Hence or otherwise, find the coordinates of the point on the graph of $y = f(x)$ where the tangent is parallel to the line $y = x$. [3]

The region enclosed by the curve $y = f(x)$, the x-axis and the y-axis is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of this solid. [4]

Consider a function g such that $g(0) = 1$ and $g'(0) = 2$.

(d) Find the value of

(i) $(f \circ g)(0)$;

(ii) $(f \circ g)'(0)$. [5]

Question 11 - working space

1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^{n} i^q = 1^q + 2^q + 3^q + \ldots + n^q \text{ where } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n, for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^{n} i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^{n} i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p. [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^{n} i^2 = a_1 n + a_2 n^2 + a_3 n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1, a_2 and a_3. [3]

(iii) Hence, find the values of a_1, a_2 and a_3. [2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q.

Consider the function $f(x) = 1 + x + x^2 + \ldots + x^n,\ n \in \mathbb{Z}^+$.

(c) Show that $xf'(x) = x + 2x^2 + 3x^3 + \ldots + nx^n$. [1]

Let $f_1(x) = xf'(x)$ and consider the following family of functions:

$$f_2(x) = xf_1'(x)$$

$$f_3(x) = xf_2'(x)$$

$$f_4(x) = xf_3'(x)$$

$$\ldots$$

$$f_q(x) = xf_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^{n} i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^{n} i^q x^i,\ q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \ldots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \dfrac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

(g) (i) Show that $\lim_{x \to 1} f_1(x)$ is in indeterminate form. [1]

(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \to 1} f_1(x) = \dfrac{1}{2}n(n+1)$. [5]

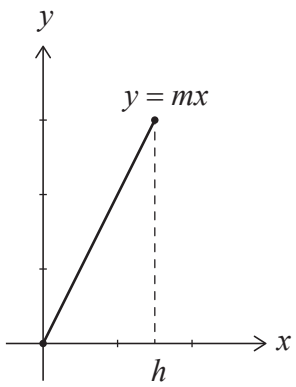
Turn over

Question 1 - working space

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \leq x \leq h$ and m, h are positive constants.



When this line is rotated through $360°$ about the x-axis, a cone is formed with a curved surface area A given by:

$$A = 2\pi \int_0^h y\sqrt{1+m^2}\,\mathrm{d}x\,.$$

(a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\,\pi$. [2]

(b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \leq x \leq h$ through $360°$ about the x-axis.

(i) Deduce an expression for the radius of this cone r in terms of h and m. [1]

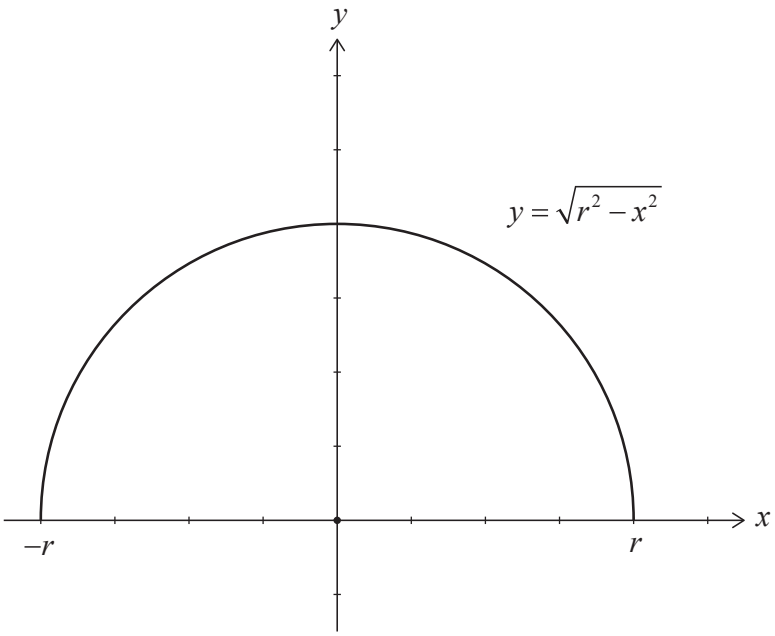
(ii) Deduce an expression for the slant height l in terms of h and m. [2]

(iii) Hence, by using the above integral, show that $A = \pi rl$. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the semi-circle, with radius r, defined by $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.



(c) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

A differentiable curve $y = f(x)$ is defined for $x_1 \leq x \leq x_2$ and $y \geq 0$. When any such curve is rotated through $360°$ about the x-axis, the surface formed has an area A given by:

$$A = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x\ .$$

(d) A sphere is formed by rotating the semi-circle $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$ through $360°$ about the x-axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(This question continues on the following page)

(Question 2 continued)

(e) Let $f(x) = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.

The graph of $y = f(x)$ is transformed to the graph of $y = f(kx)$, $k > 0$. This forms a different curve, called a semi-ellipse.

(i) Describe this geometric transformation. [2]

(ii) Write down the x-intercepts of the graph $y = f(kx)$ in terms of r and k. [1]

(iii) For $y = f(kx)$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, r and k. [2]

(iv) The semi-ellipse $y = f(kx)$ is rotated $360°$ about the x-axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A, of the ellipsoid.

Give your answer in the form $2\pi \int_{x_1}^{x_2} \sqrt{p(x)}\, \mathrm{d}x$, where $p(x)$ is a polynomial. [4]

(v) Planet Earth can be modelled as an ellipsoid. In this model:

- the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
- the distance from the North Pole to the South Pole is $12\,714\,\mathrm{km}$.
- the diameter of the equator is $12\,756\,\mathrm{km}$.

By choosing suitable values for r and k, find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a \times 10^q$ where $1 \leq a < 10$ and $q \in \mathbb{Z}^+$. [4]

References:

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Question 2 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 5]

The side lengths, $x\,\text{cm}$, of an equilateral triangle are increasing at a rate of $4\,\text{cm}\,\text{s}^{-1}$.

Find the rate at which the area of the triangle, $A\,\text{cm}^2$, is increasing when the side lengths are $5\sqrt{3}\,\text{cm}$.

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Question 6 - working space

Section A

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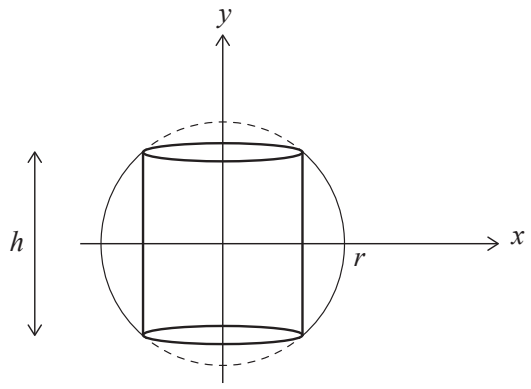
9. [Maximum mark: 7]

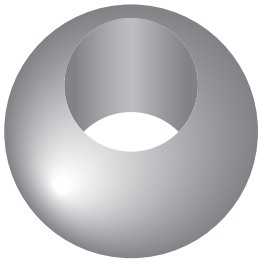
The function f is defined by $f(y)=\sqrt{r^2-y^2}$ for $-r \leq y \leq r$.

The region enclosed by the graph of $x=f(y)$ and the y-axis is rotated by $360°$ about the y-axis to form a solid sphere. The sphere is drilled through along the y-axis, creating a cylindrical hole. The resulting spherical ring has height, h.

This information is shown in the following diagrams.

diagram not to scale





The spherical ring has a volume of π cubic units. Find the value of h.

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Question 9 - working space

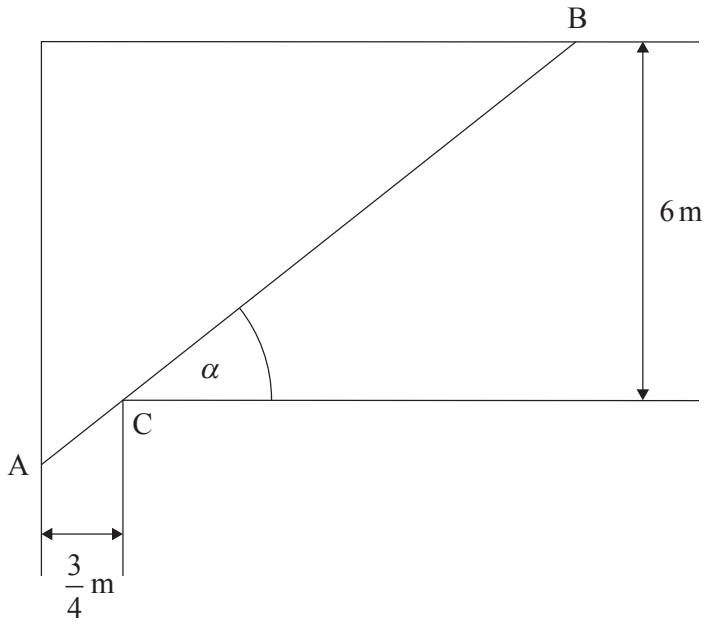
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with width $\frac{3}{4}\,\text{m}$ is perpendicular to a room of width $6\,\text{m}$. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that $[\text{AB}]$ makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$. [2]

(b) (i) Find $\frac{\text{d}L}{\text{d}\alpha}$.

(ii) When $\frac{\text{d}L}{\text{d}\alpha} = 0$, show that $\alpha = \arctan 2$. [5]

(This question continues on the following page)

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(Question 11 continued)

(c) (i) Find $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$.

(ii) When $\alpha = \arctan 2$, show that $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5}$. [7]

(d) (i) Hence, justify that L is a minimum when $\alpha = \arctan 2$.

(ii) Determine this minimum value of L. [3]

Two people need to carry a pole of length $11.25\,\mathrm{m}$ from the passageway into the room. It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]

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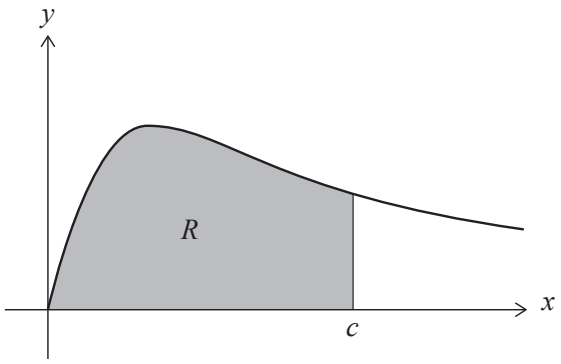
Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

The following diagram shows part of the graph of $y = \frac{x}{x^2 + 2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

Find the value of c.

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Question 4 - working space

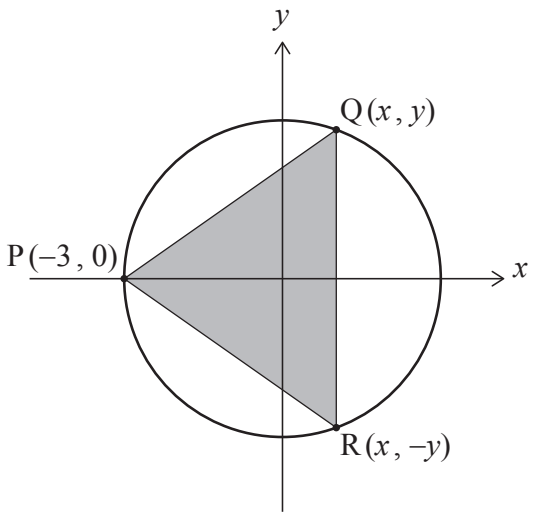
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0, 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $\mathrm{P}(-3, 0)$, $\mathrm{Q}(x, y)$ and $\mathrm{R}(x, -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



(a) For point Q, show that $y = \sqrt{9 - x^2}$. [1]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

(c) Show that $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$. [4]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

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Question 10 - working space

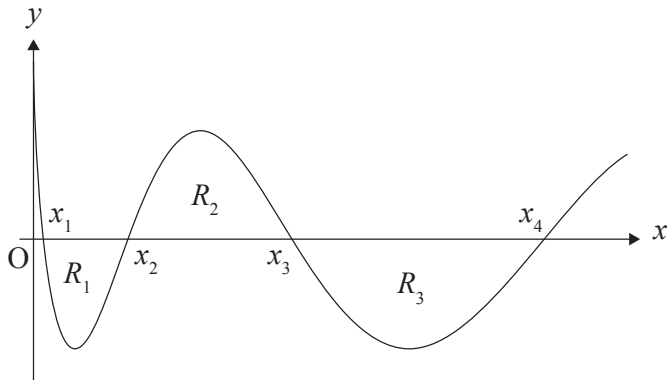
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 17]

(a) By using an appropriate substitution, show that $\int \cos\sqrt{x}\ \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos\sqrt{x}$ for $x \geq 0$.



The curve intersects the x-axis at $x_1, x_2, x_3, x_4, \ldots$.

The nth x-intercept of the curve, x_n, is given by $x_n = \frac{(2n-1)^2\pi^2}{4}$, where $n \in \mathbb{Z}^+$.

(b) Write down a similar expression for x_{n+1}. [1]

The regions bounded by the curve and the x-axis are denoted by $R_1, R_2, R_3, \ldots$, as shown on the above diagram.

(c) Calculate the area of region R_n.

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

(d) Hence, show that the areas of the regions bounded by the curve and the x-axis, $R_1, R_2, R_3, \ldots$, form an arithmetic sequence. [3]

References:

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Question 12 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

(a) Use Euler's method, with a step length of 0.03, to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [4]

(b) (i) Write down the value of $\frac{dy}{dx}$ when $x = 0$.

(ii) Show that $\frac{d^2y}{dx^2} = 3$ when $x = 0$. [5]

(c) (i) Given that $\frac{d^3y}{dx^3} = 9$ when $x = 0$, find the first four terms of the Maclaurin series for y.

(ii) Use the Maclaurin series to find an approximate value for y when $x = 0.15$. Give your answer correct to six significant figures. [3]

(d) (i) It is given that $\frac{x^2 - 1}{x^2 + 1} \equiv 1 - \frac{2}{x^2 + 1}$.

Solve the differential equation $\frac{dy}{dx} = \frac{x^2y - y}{x^2 + 1}$, where $y > 0$ and $y = 3$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(ii) Hence, find the value of y when $x = 0.15$. Give your answer correct to six significant figures. [7]

(e) For $0 \leq x < 1$, explain why the approximate value for y obtained using Euler's method will always be less than the actual value for y. [2]

References:

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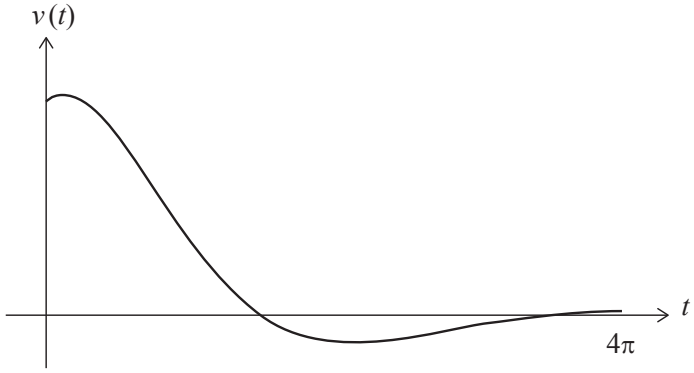
Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v(t) = 4e^{-\frac{t}{3}}\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \le t \le 4\pi$. The graph of v is shown in the following diagram.



Let t_1 be the first time when the particle's **acceleration** is zero.

(a) Find the value of t_1. [2]

Let t_2 be the **second** time when the particle is instantaneously at rest.

(b) Find the value of t_2. [2]

(c) Find the distance travelled by the particle between $t = t_1$ and $t = t_2$. [2]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

A function f is defined as $f(x) = \arctan(x-2)$, where $2 \leq x \leq 2+\sqrt{3}$.

The region bounded by the curve, the y-axis, the x-axis and the line $y = \frac{\pi}{3}$ is rotated 360° about the y-axis to form a solid of revolution.

Find the volume of the solid.

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2 + 3y^2}{xy}$, where $x > 0$, $y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.1$. [2]

(b) By solving the differential equation, show that $y = x\sqrt{\frac{9x^4 - 1}{2}}$. [8]

(c) Find the value of y when $x = 1.1$. [1]

(d) With reference to the concavity of the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $1 \leq x \leq 1.1$, explain why the value of y found in part (c) is greater than the approximate value of y found in part (a). [2]

The graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ for $\frac{\sqrt{3}}{3} < x < 1$ has a point of inflexion at the point P.

(e) By sketching the graph of an appropriate derivative of y, determine the x-coordinate of P. [2]

It can be shown that $\frac{d^2y}{dx^2} = \frac{-x^4 + x^2y^2 + 6y^4}{x^2y^3}$, where $x > 0$, $y > 0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point P lies on the straight line $y = mx$ where the exact value of m is to be determined. [6]

References:

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Question 12 - working space

1. [Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = xe^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x-axis and the line $x = b$, where $b > 0$, is given by $\frac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, A_n, of the region between the graph $y = f_n(x)$ and the x-axis can be written as $A_n = \int_0^\infty f_n(x)\,dx$ and is given by $\lim_{b\to\infty} \int_0^b f_n(x)\,dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b\to\infty} \frac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1. [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4; [2]

(ii) A_5. [1]

(e) Suggest an expression for A_n in terms of n, where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m, $\lim_{x\to\infty} x^m e^{-x} = 0$. [8]

Turn over

Question 1 - working space

2. [Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers x_1, $x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of x_1 and x_2 as a function, f, of x_1 only. [2]

(b) (i) Find the value of x_1 for which the function is maximum. [1]

(ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S, where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, $x_1, x_2, \ldots, x_n$.

The geometric mean is defined as $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1 + x_2 + \ldots + x_n)}{n}$.

(d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \ldots = x_n$. [2]

(ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]

(e) Hence determine the value of

(i) $M_3(12)$; [1]

(ii) $M_4(12)$; [1]

(iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n.

(f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]

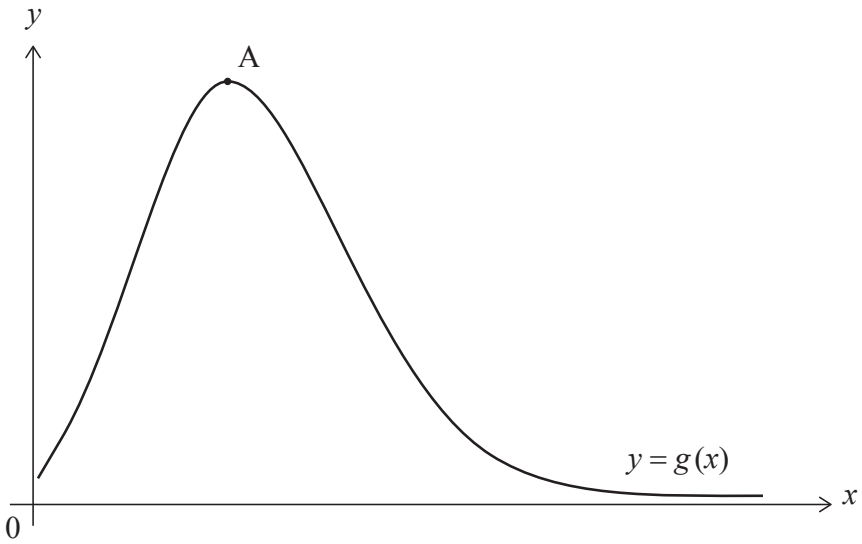
(g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the function g, defined by $\ln\big(g(x)\big) = x\ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



(h) Find, in terms of S, the x-coordinate of point A. [6]

(i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]

(j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}^+$. [3]

References:

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Question 2 - working space

1. [Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a.

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator "logarithm to any base feature".

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+,\ a \neq 1.$$

(a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x-axis intercepts. [4]

(This question continues on the following page)

(Question 1 continued)

In parts (b) and (c), consider the case where $a = \mathrm{e}$. Note that $\ln x \equiv \log_{\mathrm{e}} x$.

(b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a minimum. [5]

(c) Hence deduce that $x > \ln x$. [1]

(d) There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a.

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a, write down the values of p, q and r. [4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

(e) Find the exact coordinates of P and the exact value of a. [8]

(f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have

(i) two intersection points; [1]

(ii) no intersection points. [1]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 9]

(a) Find $\int x(\ln x)^2 \mathrm{d}x$. [6]

(b) Hence, show that $\int_1^4 x(\ln x)^2 \mathrm{d}x = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$. [3]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x \to 0} f(x) = 16$, find the value of k. [6]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4.

(ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos 2x - 1}$, up to and including the term in x^4.

(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [6]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$. [3]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 9]

(a) Find $\int x(\ln x)^2 \mathrm{d}x$. [6]

(b) Hence, show that $\int_1^4 x(\ln x)^2 \mathrm{d}x = 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$. [3]

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Question 8 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

(b) Given that $\lim_{x \to 0} f(x) = 16$, find the value of k. [6]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Consider the function $f(x) = \mathrm{e}^{\cos 2x}$, where $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find the coordinates of the points on the curve $y = f(x)$ where the gradient is zero. [5]

(b) Using the second derivative at each point found in part (a), show that the curve $y = f(x)$ has two local maximum points and one local minimum point. [4]

(c) Sketch the curve of $y = f(x)$ for $0 \leq x \leq \pi$, taking into consideration the relative values of the second derivative found in part (b). [3]

(d) (i) Find the Maclaurin series for $\cos 2x$, up to and including the term in x^4.

(ii) Hence, find the Maclaurin series for $\mathrm{e}^{\cos 2x - 1}$, up to and including the term in x^4.

(iii) Hence, write down the Maclaurin series for $f(x)$, up to and including the term in x^4. [6]

(e) Use the first two non-zero terms in the Maclaurin series for $f(x)$ to show that $\int_0^{1/10} \mathrm{e}^{\cos 2x} \mathrm{d}x \approx \frac{149\mathrm{e}}{1500}$. [3]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

A particle moves along a straight line. Its displacement, s metres, from a fixed point O after time t seconds is given by $s(t) = 4.3\sin\left(\sqrt{3t+5}\right)$, where $0 \le t \le 10$.

The particle first comes to rest after q seconds.

(a) Find the value of q. [2]

(b) Find the total distance that the particle travels in the first q seconds. [3]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 9]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a). [1]

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

A particle moves along a straight line. Its displacement, s metres, from a fixed point O after time t seconds is given by $s(t) = 4.3\sin\left(\sqrt{3t+5}\right)$, where $0 \le t \le 10$.

The particle first comes to rest after q seconds.

(a) Find the value of q. [2]

(b) Find the total distance that the particle travels in the first q seconds. [3]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 9]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a). [1]

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Question 9 - working space

2. [Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α.

(a) Consider a family of straight lines, L, with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C, at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

(i) Write down an expression for the gradient of L in terms of x and y. [1]

(ii) Hence show that the gradient of C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. [1]

(iii) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, show that the family of curves, C, has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

(b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x-intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

(c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $\mathrm{M}\left(a-b, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b, -2\sqrt{ab}\right)$. [6]

(d) At point M, show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

(This question continues on the following page)

(Question 2 continued)

Consider two families of curves, F and G.

The gradient of F is denoted by $f(x, y)$.

The gradient of G is denoted by $g(x, y)$.

Each member of F intersects every member of G at an acute angle, α.

It can be shown that

$$g(x, y) = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}.$$

In part (e), consider the specific case where $f(x, y) = -\frac{x}{y}$, for $x \neq 0$, $y \neq 0$ and $\alpha = \frac{\pi}{4}$.

(e) (i) Show that $g(x, y) = \frac{y - x}{y + x}$. [2]

(ii) Hence, by solving the homogeneous differential equation $\frac{dy}{dx} = \frac{y - x}{y + x}$, find a general equation that represents this family of curves, G. Give your answer in the form $h(x, y) = d$ where d is a parameter. [9]

(f) By considering $\lim_{\alpha \to \frac{\pi}{2}} \tan\alpha$, show that, for all finite $f(x, y)$,

$$\lim_{\alpha \to \frac{\pi}{2}} g(x, y) = -\frac{1}{f(x, y)}.$$ [2]

Question 2 - working space

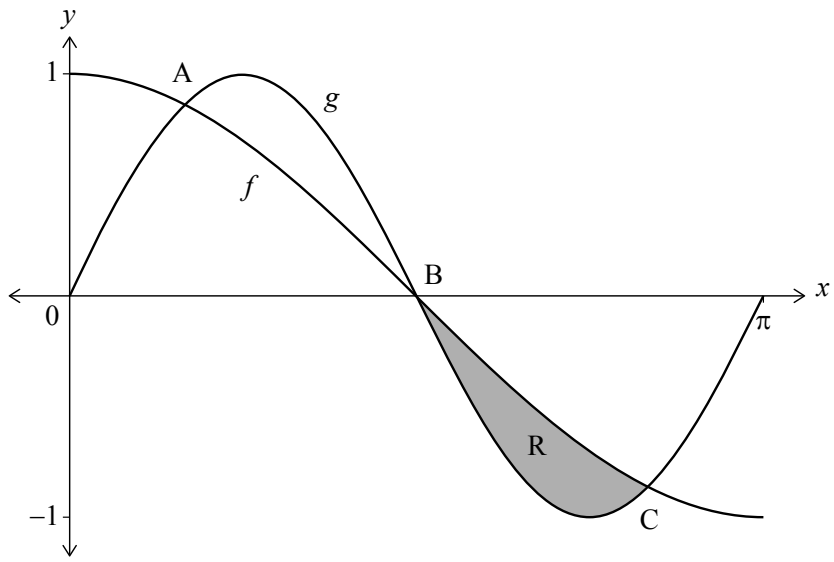
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 7]

Consider the functions $f(x) = \cos x$ and $g(x) = \sin 2x$, where $0 \le x \le \pi$.

The graph of f intersects the graph of g at the point A, the point $\mathrm{B}\left(\frac{\pi}{2}, 0\right)$ and the point C as shown on the following diagram.



(a) Find the x-coordinate of point A and the x-coordinate of point C. [3]

The shaded region R is enclosed by the graph of f and the graph of g between the points B and C.

(b) Find the area of R. [4]

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Question 4 - working space

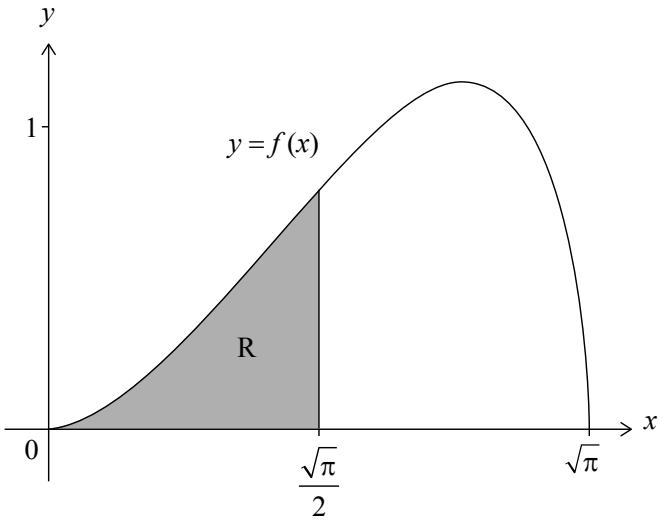
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

The function f is defined as $f(x) = \sqrt{x\sin(x^2)}$, where $0 \le x \le \sqrt{\pi}$.

Consider the shaded region R enclosed by the graph of f, the x-axis and the line $x = \frac{\sqrt{\pi}}{2}$, as shown in the following diagram.



The shaded region R is rotated by 2π radians about the x-axis to form a solid.

Show that the volume of the solid is $\frac{\pi(2-\sqrt{2})}{4}$.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

(a) Find the first two non-zero terms in the Maclaurin series of

(i) $\sin\left(x^2\right)$;

(ii) $\sin^2\left(x^2\right)$. [5]

(b) Hence, or otherwise, find the first two non-zero terms in the Maclaurin series of $4x\sin\left(x^2\right)\cos\left(x^2\right)$. [2]

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

Consider the polynomial $P(x) = 3x^3 + 5x^2 + x - 1$.

(a) Show that $(x + 1)$ is a factor of $P(x)$. [2]

(b) Hence, express $P(x)$ as a product of three linear factors. [3]

Now consider the polynomial $Q(x) = (x + 1)(2x + 1)$.

(c) Express $\frac{1}{Q(x)}$ in the form $\frac{A}{x+1} + \frac{B}{2x+1}$, where $A, B \in \mathbb{Z}$. [3]

(d) Hence, or otherwise, show that $\frac{1}{(x+1)Q(x)} = \frac{4}{2x+1} - \frac{2}{x+1} - \frac{1}{(x+1)^2}$. [2]

(e) Hence, find $\int \frac{1}{(x+1)^2(2x+1)} \mathrm{d}x$. [4]

Consider the function defined by $f(x) = \frac{P(x)}{(x+1)Q(x)}$, where $x \neq -1$, $x \neq -\frac{1}{2}$.

(f) Find

(i) $\lim_{x \to -1} f(x)$;

(ii) $\lim_{x \to \infty} f(x)$. [3]

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Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

Use l'Hôpital's rule to find $\lim_{x \to 0} \frac{\sec^4 x - \cos^2 x}{x^4 - x^2}$.

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

(a) Let $f(x) = (1-ax)^{-\frac{1}{2}}$, where $ax < 1$, $a \neq 0$.

The nth derivative of $f(x)$ is denoted by $f^{(n)}(x)$, $n \in \mathbb{Z}^+$.

Prove by induction that $f^{(n)}(x) = \dfrac{a^n (2n-1)!(1-ax)^{-\frac{2n+1}{2}}}{2^{2n-1}(n-1)!}$, $n \in \mathbb{Z}^+$. [8]

(b) By using part (a) or otherwise, show that the Maclaurin series for $f(x) = (1-ax)^{-\frac{1}{2}}$ up to and including the x^2 term is $1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2$. [2]

(c) Hence, show that $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \dfrac{2+6x+19x^2}{2}$. [4]

(d) Given that the series expansion for $(1-ax)^{-\frac{1}{2}}$ is convergent for $|ax| < 1$, state the restriction which must be placed on x for the approximation $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \dfrac{2+6x+19x^2}{2}$ to be valid. [1]

(e) Use $x = \dfrac{1}{10}$ to determine an approximate value for $\sqrt{3}$.

Give your answer in the form $\dfrac{c}{d}$, where $c, d \in \mathbb{Z}^+$. [5]

Question 12 - working space

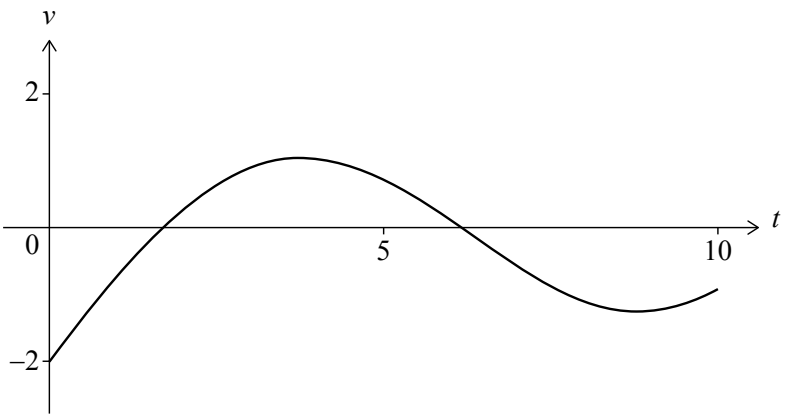
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

A particle moves in a straight line such that it passes through a fixed point O at time $t = 0$, where t represents time measured in seconds after passing O. For $0 \le t \le 10$ its velocity, v metres per second, is given by $v = 2\sin(0.5t) + 0.3t - 2$.

The graph of v is shown in the following diagram.



(a) Find the smallest value of t when the particle changes direction. [2]

The displacement of the particle is measured in metres from O.

(b) Find the range of values of t for which the displacement of the particle is increasing. [2]

(c) Find the displacement of the particle relative to O when $t = 10$. [2]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$, given that $y = 2$ when $x = 0$.

Give your answer in the form $y = f(x)$.

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Question 7 - working space

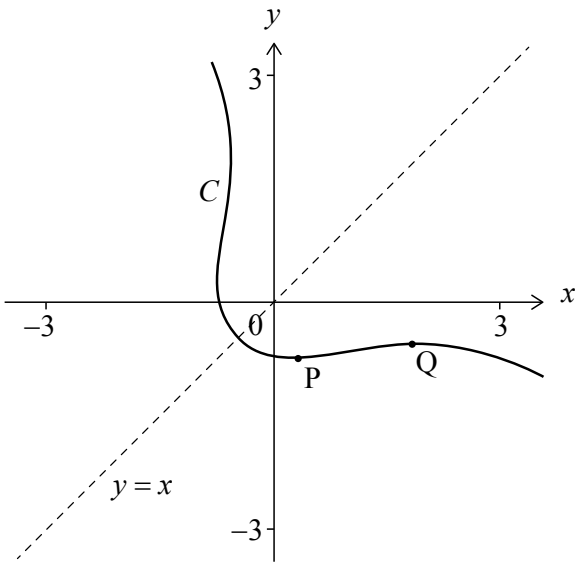
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the curve C defined by the equation $e^{x+y} = x^2 + y^2$, shown on the following diagram. The curve has a line of symmetry $y = x$.

There are two points on the curve C where the tangent is horizontal. These points are labelled P and Q.



(a) Show that $\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - 2y}$. [5]

(b) (i) Show that the x-coordinates of points P and Q satisfy the equation $2x^2 + (\ln(2x))^2 - 2x\ln(2x) - 2x = 0$.

(ii) Hence, find the coordinates of P and the coordinates of Q. [9]

(c) Using the line of symmetry, write down the coordinates of the points on the curve C where the tangent is vertical. [1]

(d) Find the coordinates of the point on the curve C where the tangent has a gradient of -1. [4]

Do **not** write solutions on this page.

Question 11 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The functions f and g are both defined for $-1 \leq x \leq 0$ by

$$f(x) = 1 - x^2$$

$$g(x) = \mathrm{e}^{2x}.$$

The graphs of f and g intersect at $x = a$ and $x = b$, where $a < b$.

(a) Find the value of a and the value of b. [3]

(b) Find the area of the region enclosed by the graphs of f and g. [3]

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 6]

A particle moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v(t) = 1 + \text{e}^{-t} - \text{e}^{-\sin 2t}$ for $0 \leq t \leq 2$.

(a) Find the velocity of the particle at $t = 2$. [1]

(b) Find the maximum velocity of the particle. [2]

(c) Find the acceleration of the particle at the instant it changes direction. [3]

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Question 4 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 6]

The volume of a spherical bubble increases at a constant rate of $5\,\text{cm}^3\,\text{s}^{-1}$.

The initial volume of the bubble can be assumed to be zero.

Find the rate, in $\text{cm}\,\text{s}^{-1}$, at which the radius of the bubble is increasing when the volume of the bubble is $20\,\text{cm}^3$.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 5]

The curve $y = 4\ln(x-2)$ for $0 \leq y \leq 4$ is rotated $360°$ about the y-axis to form a solid of revolution.

Find the volume of the solid formed.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider the curve $y = \frac{x-4}{ax^2+bx+c}$, where a, b and c are non-zero constants.

The curve has a local minimum point at $(2, 1)$ and a vertical asymptote with equation $x = 1$.

Find the values of a, b and c.

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Do **not** write solutions on this page.

Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec} 2x = \sqrt{\tan x}$, where $0 < x < \frac{\pi}{2}$ and $y = \frac{\pi}{4}$ at $x = \frac{\pi}{4}$.

(a) Use Euler's method with step length $\frac{\pi}{12}$ to find an approximate value of y when $x = \frac{5\pi}{12}$.

Give your answer correct to three significant figures. [3]

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln\left(\cot x\right)\right) = -\operatorname{cosec} 2x$. [4]

(c) Show that $\sqrt{\cot x}$ is an integrating factor for this differential equation. [4]

(d) Hence, by solving the differential equation, show that $y = x\sqrt{\tan x}$. [5]

(e) Consider the curve $y = x\sqrt{\tan x}$ for $0 < x < \frac{\pi}{2}$ and the Euler's method approximation calculated in part (a).

(i) Find the y-coordinate at $x = \frac{5\pi}{12}$. Give your answer correct to three significant figures.

(ii) By considering the gradient of the curve, suggest a reason why Euler's method does not give a good approximation for the y-coordinate at $x = \frac{5\pi}{12}$.

(iii) State why this approximation is less than the y-coordinate at $x = \frac{5\pi}{12}$. [3]

(f) By considering $\frac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec} 2x = \sqrt{\tan x}$, deduce that the curve $y = x\sqrt{\tan x}$ has a positive gradient for $0 < x < \frac{\pi}{2}$. [2]

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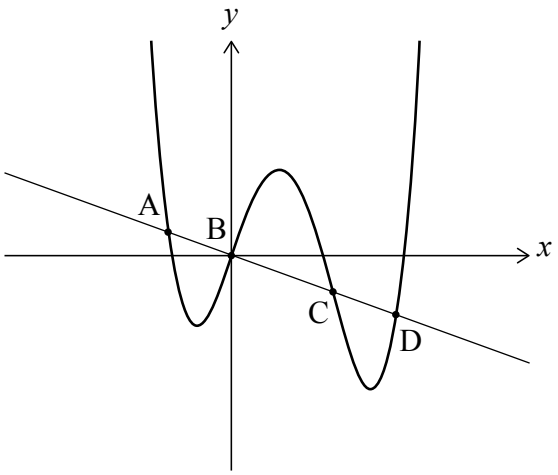
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Question 12 - working space

2. [Maximum mark: 28]

This question investigates a ratio of lengths found from the line passing through the points of inflexion of a quartic curve of the form $y = x^4 - mx^3 + nx$.

The curve $y = x^4 - 3x^3 + 3x$ has points of inflexion at B and C. The line passing through B and C intersects the curve again at points A and D. This is shown in the following graph.



(a) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [3]

(b) Find the coordinates of B and C. [4]

(c) Show that the equation of the line through B and C is $y = -0.375x$. [2]

(d) Find, correct to three decimal places, the x-coordinate of D. [2]

Now consider the general curve $y = x^4 - mx^3 + nx$, where m, $n \in \mathbb{R}$ and $m > 0$.

(e) Find the x-coordinates of the two points of inflexion in terms of m. [3]

Let these points of inflexion be B and C. The line passing through B and C intersects the curve again at points A and D. Let x_A be the x-coordinate of point A, and similarly for x_B, x_C and x_D. It is given that $x_A < x_B < x_C < x_D$.

(f) (i) Write down the coordinates of B. [1]

(ii) Find, in terms of m and n, the coordinates of C. [2]

(This question continues on the following page)

(Question 2 continued)

(g) Show that the equation of the line through B and C is $y = \left(-\frac{m^3}{8} + n\right)x$. [2]

(h) Show that $x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$. [7]

(i) Hence, find the exact value of $\frac{x_B - x_A}{x_C - x_B}$. [2]

Question 2 - working space

1. [Maximum mark: 24]

If two functions $f(x)$ and $g(x)$ are differentiable, then their product is differentiable and the two functions satisfy the product rule: $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$.

In this question, you will meet examples of pairs of differentiable functions, $f(x)$ and $g(x)$, that also satisfy $(f(x)g(x))' = f'(x)g'(x)$.

In part (a), consider $f(x) = \frac{1}{(2-x)^2}$, where $x \in \mathbb{R}$, $x \neq 2$, and $g(x) = x^2$, where $x \in \mathbb{R}$.

(a) (i) Find an expression for $f'(x)$. [2]

(ii) Show that $f'(x)g'(x) = \frac{4x}{(2-x)^3}$. [2]

(iii) Show that $f(x)g'(x) + g(x)f'(x) = \frac{4x}{(2-x)^3}$. [4]

In parts (b) and (c), consider two non-constant functions, $f(x)$ and $g(x)$, where $f(x) > 0$ and $g(x) \neq g'(x)$.

(b) By rearranging the equation $f(x)g'(x) + g(x)f'(x) = f'(x)g'(x)$, show that $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$. [2]

(This question continues on the following page)

(Question 1 continued)

(c) Hence, by integrating both sides of $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$, show that $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\,\mathrm{d}x\right)}$, where A is an arbitrary positive constant. [2]

The result from part (c) can be used to find pairs of functions, $f(x)$ and $g(x)$, which satisfy **both** of the following:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \text{ and } \left(f(x)g(x)\right)' = f'(x)g'(x).$$

In parts (d) and (e), use the result in part (c) with $A = 1$.

(d) Consider $g(x) = x\mathrm{e}^x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations. [5]

(e) Consider $g(x) = \sin x + \cos x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations over the domain $0 < x < \pi$.

Give your answer in the form $f(x) = \sqrt{\mathrm{e}^x h(x)}$, where $h(x)$ is a function to be determined. [7]

Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

Consider the function $f(x) = \sec\left(x - \frac{\pi}{4}\right)$, for $0 \le x \le \frac{\pi}{2}$.

(a) Determine the range of f. [3]

The region bounded by the graph of $y = f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$ is rotated 2π radians about the x-axis.

(b) Find the volume of revolution generated. [4]

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Question 7 - working space

Section B

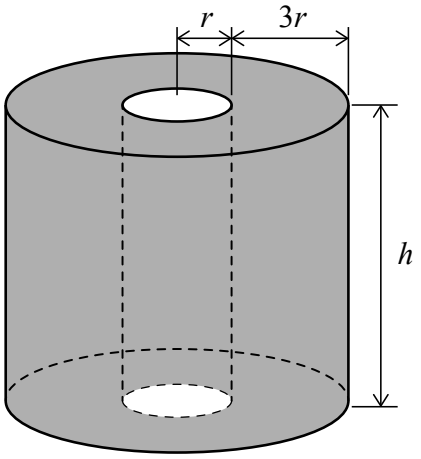
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

Consider a cylinder of radius $4r$ and height h. A smaller cylinder of radius r is removed from the centre to form a hollow cylinder. This is shown in the following diagram.

All lengths are measured in centimetres.

diagram not to scale



The total surface area of the hollow cylinder, in cm^2, is given by S.

The volume of the hollow cylinder, in cm^3, is given by V.

(a) Show that $S = 30\pi r^2 + 10\pi rh$. [3]

(b) The total surface area of the hollow cylinder is $240\pi\,\text{cm}^2$.

Show that $V = 360\pi r - 45\pi r^3$. [6]

(c) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}r}$. [2]

The hollow cylinder has its maximum volume when $r = p\sqrt{\frac{2}{3}}$, where $p \in \mathbb{Z}^+$.

(d) Find the value of p. [3]

(e) Hence, find this maximum volume, giving your answer in the form $q\pi\sqrt{\frac{2}{3}}$, where $q \in \mathbb{Z}^+$. [3]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

A curve is given by the equation $y = \frac{e^{2x} - 1}{e^{2x} + 1}$, $x \in \mathbb{R}$.

(a) By applying l'Hôpital's rule or otherwise, show that $\lim_{x \to \infty} \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right) = 1$. [2]

(b) (i) Show that $\frac{dy}{dx} = \frac{4e^{2x}}{\left(e^{2x} + 1\right)^2}$.

(ii) Hence, show that $1 - y^2 = \frac{dy}{dx}$. [6]

(c) (i) By using implicit differentiation and the result in part (b)(ii), show that $\frac{d^2y}{dx^2} = 2y^3 - 2y$.

(ii) Hence, find an expression for $\frac{d^3y}{dx^3}$ in terms of y. [5]

(d) By using your results from parts (b) and (c), find the Maclaurin series for $\frac{e^{2x} - 1}{e^{2x} + 1}$ up to and including the term in x^3. [4]

Do **not** write solutions on this page.

Question 11 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The following table shows the population of Canada t years after the year 2000.

t (years after 2000)	0	5	10	15	20
p (population in millions)	30.7	32.2	34.0	35.7	37.9

A student uses linear regression to model the population of Canada using these data. The student model is $p = at + b$.

(a) (i) Write down the value of a and the value of b.

(ii) Interpret, in context, the value of a. [3]

The student uses this model to predict the population of Canada in the year 2030, where $t = 30$, and calculates a population of approximately 41.3 million people.

(b) Comment on the reliability of the student's prediction. [1]

A data scientist, Benoit, uses additional information to develop an exponential model for Canada's future population.

In this model, $B(t) = 33.5(1.005)^t$ represents the millions of people in Canada t years after the year 2000, where $25 \leq t \leq 100$.

(c) (i) Use Benoit's model to predict the population of Canada in the year 2100.

(ii) Interpret, in context, the value 1.005 in Benoit's model. [3]

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(Question 10 continued)

Another data scientist, Cecilia, develops a third model for the Canadian population.

In this model, $C(t) = \frac{62}{1 + e^{-0.02t}}$ represents the millions of people in Canada t years after the year 2000, where $25 \leq t \leq 100$.

(d) Use Cecilia's model to predict the population of Canada in the year 2100. [1]

(e) Determine the year in which the difference between the predictions from Benoit's model and Cecilia's model is greatest. [3]

(f) Find the value of

(i) $B'(75)$;

(ii) $C'(75)$. [2]

(g) Compare and interpret, in context, the values of $B'(75)$ and $C'(75)$. [2]

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Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Consider the function $f(x) = \frac{4x^3}{3} - 16x$, where $x \in \mathbb{R}$.

The graph of $y = f(x)$ has a local minimum point at (p, q) where $p > 0$.

Find the value of p and the value of q.

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Question 1 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 4]

Find the area completely enclosed by the curves $y = \mathrm{e}^x$, $y = -\mathrm{e}^x$, and the lines $x = -1$ and $x = 1$.

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y}$, where $x > 0$ and $y \neq -x$.

It is given that $y = 0$ when $x = 2$.

By using the substitution $y = vx$, show that the solution of the differential equation is $x^2 - 2xy - y^2 = 4$.

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

Consider the family of functions $f_n(x) = \cos^n x$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

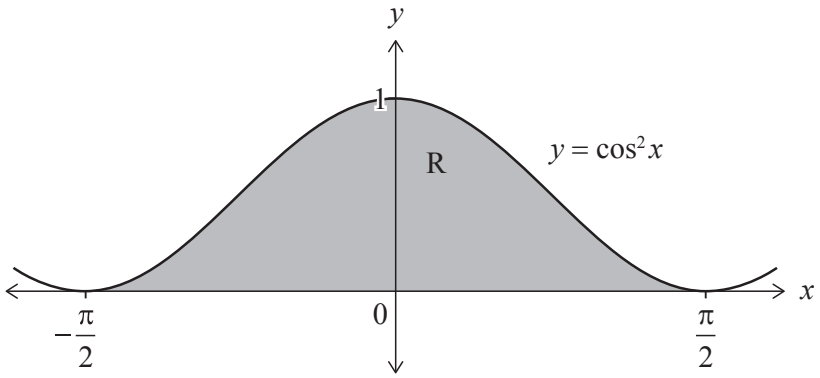
(a) By writing $\cos^n x$ as $\cos^{n-1} x \cos x$, show that

$$\int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n x \, dx \text{ for } n > 1. \quad [4]$$

(b) Hence, show that $\int f_n(x) dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int f_{n-2}(x) dx$ for $n > 1$. [2]

(c) Hence, find an expression for $\int \cos^4 x \, dx$, giving your answer in the form $p\cos^3 x \sin x + q \cos x \sin x + rx + c$ where $p, q, r \in \mathbb{Q}^+$. [4]

The region R is enclosed by the graph of $y = \cos^2 x$ and the x-axis where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as shown in the following diagram.



The region R is rotated by 2π radians around the x-axis to form a solid of revolution.

(d) Find the volume of the solid. [4]

(e) (i) Find the Maclaurin series of $f_n(x)$ up to the term in x^2.

(ii) Hence or otherwise, find $\lim_{x \to 0} \frac{f_n(x) - 1}{x^2}$ in terms of n. [5]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Consider the function $f(x) = \sqrt{x^2 \ln x + 4 - x^2}$, where $x \in \mathbb{R}$, $x > 0$.

(a) Show that the distance, l, between the origin and any point on the graph of f is given by $l = \sqrt{x^2 \ln x + 4}$. [1]

(b) Hence, find the x-coordinate of the point on the graph of f which is closest to the origin. [6]

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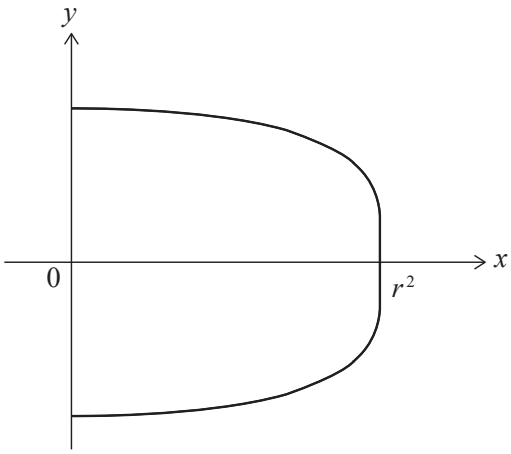
Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

The curve $x^2 + y^4 = r^4$, where $0 \leq x \leq r^2$, is shown in the following diagram.



The region enclosed by the curve and the y-axis is rotated through 2π radians about the y-axis to form a solid of revolution.

Find an expression for the volume of the solid in the form $V = a\pi r^b$, where $a, b \in \mathbb{Q}^+$.

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Question 7 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 6]

Determine the value of $\lim_{x \to 0}\left(\frac{x \sin x}{1 - \cos x}\right)$.

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

A population, P, has a rate of change modelled by $\frac{\mathrm{d}P}{\mathrm{d}t} = -104000\mathrm{e}^{-0.0145t}$, where t is the time measured in years since the **start** of 2022.

At the start of 2022, the population was 6.78×10^6.

Based on this model, find the predicted population at the start of 2026.

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Question 3 - working space

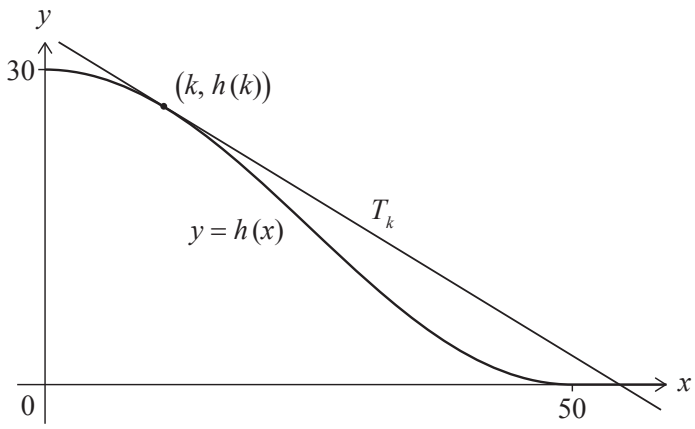
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 6]

Consider the function $h(x) = 15\cos\left(\frac{\pi x}{50}\right) + 15$, where $0 \le x \le 50$.

The tangent, T_k, to the curve $y = h(x)$ at the point $(k, h(k))$ is shown on the following diagram.



(a) Find the gradient of T_k in terms of k. [3]

Consider the case where the angle between T_k and the x-axis is $\frac{\pi}{8}$ radians.

(b) Find the possible values of k. [3]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

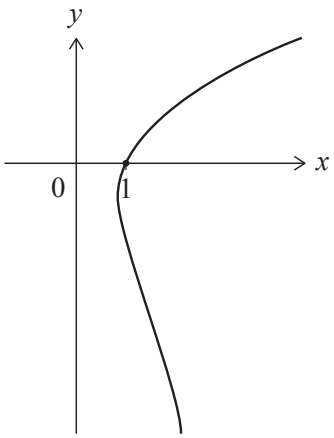
9. [Maximum mark: 6]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2 + y}$.

The solution curve passes through the point $(1, 0)$.

(a) Use Euler's method with a step value of 0.25 to estimate the value of y when $x = 2$. [3]

Part of the solution curve is shown in the following diagram.



(b) (i) Determine whether your answer to part (a) is an overestimate or an underestimate, justifying your answer.

(ii) Justify why the use of Euler's method starting at $(1, 0)$ does not lead to an estimate of the negative value of y when $x = 2$. [3]

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Question 9 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 6]

The derivative of a function f is given by $f'(x) = 4 + 2x - 3e^x$, where $x \in \mathbb{R}$.

(a) Find the values of x for which f is decreasing. [3]

(b) Find the values of x for which the graph of f is concave-up. [3]

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

A particle P moves in a straight line. The velocity $v\,\mathrm{m\,s^{-1}}$ of P, at time t seconds is given by $v(t) = \mathrm{e}^{-\sin t}\cos(2t)$, for $0 \le t \le 5$.

(a) Find the maximum speed of P. [2]

(b) Find the total distance travelled by P. [2]

(c) Find the acceleration when P changes direction for the **second** time. [3]

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Question 5 - working space

Section B

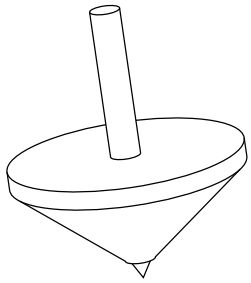
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 14]

A mathematics class of 15 students plays a game which requires three equal size teams.

(a) Find the total number of ways that the three teams can be chosen. [3]

The game involves the spinning of a top.



The time, T, in minutes that the spinning top is in motion can be modelled by the probability density function f where

$$f(t) = \begin{cases} kte^{-3t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $k \in \mathbb{Z}^+$.

(b) Show that $\int_0^a f(t)\,dt = \frac{k}{9}\left[1-(3a+1)e^{-3a}\right]$, where $a \in \mathbb{R}^+$. [4]

(c) (i) Use l'Hôpital's rule to find $\lim_{x\to\infty}(3x+1)e^{-3x}$.

(ii) Hence, by considering $\lim_{a\to\infty}\int_0^a f(t)\,dt$, find the value of k. [5]

(d) Find the median length of time that a spinning top is in motion. [2]

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Question 11 - working space

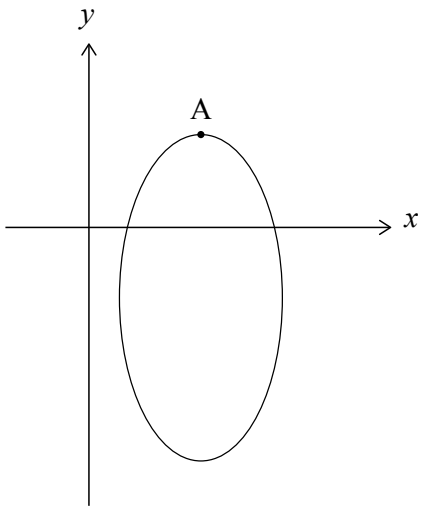
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

The curve C has equation $4x^2 + y^2 - 24x + 4y + 20 = 0$.

The following diagram shows C with a maximum point at A.



(a) Use implicit differentiation to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(3-x)}{y+2}$. [4]

(b) Hence, determine the domain of C. Give your answer in the form $3 - \sqrt{a} \leq x \leq 3 + \sqrt{a}$, where $a \in \mathbb{Z}^+$. [4]

(c) Find $(x_\mathrm{A}, y_\mathrm{A})$, the coordinates of A. [3]

A line $y = mx$ is a tangent to C, where $m \in \mathbb{Z}$.

(d) Find the possible values of m. [4]

The line $y = -4x$ touches C at point B.

(e) Find y_B, the y-coordinate of B. [3]

The region bounded by the curve C, the y-axis and the lines $y = y_\mathrm{A}$ and $y = y_\mathrm{B}$, is rotated $360°$ about the y-axis to form a solid of revolution.

(f) Find the volume of the solid formed. [4]

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Question 12 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

Consider the function $f(x) = \frac{(2x+a)^3}{(x+5)^2}$, where $x \neq -5$ and $a \in \mathbb{R}^+$.

(a) Find an expression for $f'(x)$, in terms of a. [3]

When $x = 1$, the tangent to the graph of f makes an angle of $70°$ to the horizontal.

(b) Find the two possible values of a. [4]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

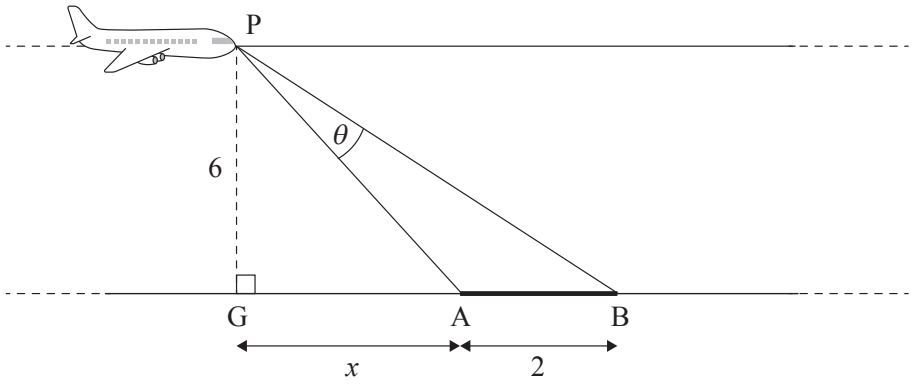
9. [Maximum mark: 9]

An airplane, P, is flying over horizontal ground at a constant height of $6\,\mathrm{km}$ and travelling at a constant speed. It is approaching a runway, $[\mathrm{AB}]$, which is $2\,\mathrm{km}$ in length.

Let G be the point on the ground directly below the airplane. When $\mathrm{GA} = x\,\mathrm{km}$, the pilot's viewing angle of the runway, $\mathrm{A\hat{P}B}$, is θ.

This is shown in the following diagram.

diagram not to scale



(a) Show that $\theta = \arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right)$. [2]

When the viewing angle is 0.178 radians, the rate at which the viewing angle is changing is 12.5 radians per hour.

(b) Find the speed of the airplane. [7]

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Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

Two athletes, Fiona and Lucy, compete in a 200 metres race along a straight track.

Fiona's velocity, in m s^{-1}, during the race can be modelled by $v(t) = \dfrac{8.14\,t}{\sqrt{t^2 + 0.2}}$, where $t \geq 0$. Time, t, is measured in seconds from when the race starts.

(a) (i) Write down the value of $v(1)$.

(ii) Find the time when Fiona's velocity is $5\,\text{m s}^{-1}$. [3]

(b) Find the time when Fiona's acceleration is $4\,\text{m s}^{-2}$. [2]

(c) (i) Write down the limit of $v(t)$ as t approaches infinity.

(ii) State a reason why the value in part (c)(i) is not valid in the context of this question. [3]

Lucy's velocity, in m s^{-1}, during the race can be modelled by $w(t) = \dfrac{8\,t}{\sqrt{t^2 + 0.3}}$, where $t \geq 0$.

Fiona completes the race and crosses the finishing line in front of Lucy.

(d) Find the distance Lucy is from the finishing line when Fiona completes the 200 metres. [6]

Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

(a) Find $\int (x^2 - 5)\mathrm{e}^x \,\mathrm{d}x$. [6]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y - 5$.

(b) By solving the differential equation, show that its solution can be expressed in the form $y = x^2 - 2x - 3 + C\mathrm{e}^{-x}$, where C is a constant. [4]

(c) Sketch the curve of the particular solution which passes through the point $(-3, 2)$, for $-4 \leq x \leq 4$, clearly labelling the coordinates of any local maximum and minimum points. [5]

Consider the family of curves that are solutions of the differential equation.

The tangent at $x = -3$ is drawn for each of these curves.

(d) By considering the curve which passes through the point $(-3, p)$ and the curve which passes through the point $(-3, q)$, where $p, q \in \mathbb{R}$, $p \neq q$, show that all these tangents intersect at a common point, and state its coordinates. [6]

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Question 12 - working space

2. [Maximum mark: 32]

Informally, the curvature of a curve can be thought of as the amount by which the curve deviates from being a straight line. In this question, you will investigate the curvature of a variety of functions.

Consider any function f that can be differentiated twice.

The curvature, k, of any function f is defined by $k(x) = \dfrac{|f''(x)|}{\left(1+\left(f'(x)\right)^2\right)^{\frac{3}{2}}}$.

Consider the family of linear functions $g(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

(a) Show that $k(x) = 0$ for this family of linear functions. [2]

Consider the family of quadratic functions $h(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b, c \in \mathbb{R}$.

For this family of quadratic functions, it is given that

$$k(x) = \frac{2|a|}{\left(1+(2ax+b)^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = -\frac{12a|a|(2ax+b)}{\left(1+(2ax+b)^2\right)^{\frac{5}{2}}}.$$

The maximum value of $k(x)$ is denoted by $k_{\max}$.

(b) (i) By solving $k'(x) = 0$, find the value of x where $k_{\max}$ occurs.
You are **not** required to justify that this value of x leads to a maximum. [1]

(ii) Determine an expression for $k_{\max}$, in terms of a only. [2]

(iii) State the value of $\lim\limits_{x\to\infty} k(x)$ and explain briefly the significance of this result. [2]

(iv) Consider the quadratic functions

$p(x) = -2x^2 + 2x - 10$ where $x \in \mathbb{R}$ and

$q(x) = 2x^2 + 5x + 25$ where $x \in \mathbb{R}$.

State which one of the following statements is true and justify your answer.

A. $k_{\max}$ of $p > k_{\max}$ of q

B. $k_{\max}$ of $p < k_{\max}$ of q

C. $k_{\max}$ of $p = k_{\max}$ of q [2]

(This question continues on the following page)

(Question 2 continued)

Consider the function $v(x) = \ln x$, where $x \in \mathbb{R}, x > 0$.

For v, it is given that

$$k(x) = \frac{x}{\left(1+x^2\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{1-2x^2}{\left(1+x^2\right)^{\frac{5}{2}}}.$$

(c) (i) Determine the exact value of x where $k_{\max}$ occurs.

You are **not** required to justify that this value of x leads to a maximum. [2]

(ii) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [4]

Consider the function $w(x) = \mathrm{e}^x$, where $x \in \mathbb{R}$.

For w, it is given that

$$k(x) = \frac{\mathrm{e}^x}{\left(1+\mathrm{e}^{2x}\right)^{\frac{3}{2}}} \text{ and}$$

$$k'(x) = \frac{\mathrm{e}^x\left(1-2\mathrm{e}^{2x}\right)}{\left(1+\mathrm{e}^{2x}\right)^{\frac{5}{2}}}.$$

(d) (i) Show that $k_{\max} = \frac{2\sqrt{3}}{9}$. [5]

(ii) Suggest a reason why v and w have the same $k_{\max}$. [1]

Consider a family of curves $y = \sqrt{r^2 - x^2}$, where $-r < x < r$, $y > 0$ and r is a positive constant.

(e) (i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$. [6]

(ii) Hence, show that the curvature, k, is constant for this family of curves. [5]

Question 2 - working space

1. [Maximum mark: 28]

The following question explores features of a family of curves. The family is then linked to a homogeneous differential equation.

Consider the curve given by $y = \frac{x\left(x^2 - 16\right)}{x^2 + 16}$.

(a) (i) Sketch the curve of y for $-10 \leq x \leq 10$. [1]

(ii) State the coordinates of the points where the curve crosses the x-axis. [1]

(iii) State the coordinates of the local maximum point and the coordinates of the local minimum point. [2]

(b) State whether the function $f(x) = \frac{x\left(x^2 - 16\right)}{x^2 + 16}$ is odd, even or neither. Justify your answer. [2]

Now consider the general curve given by $y = \frac{x\left(x^2 - A\right)}{x^2 + A}$, where A is a positive constant and $x \in \mathbb{R}$.

(c) Given $f\left(x\right) = \frac{x\left(x^2 - A\right)}{x^2 + A}$, prove that $f'\left(\sqrt{A}\right)$ is independent of A. [4]

(d) (i) Show that $x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$. [2]

(ii) Hence, determine the equation of the oblique asymptote to the curve. [1]

(iii) Write down the coordinates of a point on the curve where the oblique asymptote is parallel to the tangent to the curve at that point. [1]

(This question continues on the following page)

(Question 1 continued)

Now consider the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = x(x+y) - y^2$, where $x \neq 0$, $y \neq \pm x$.

Using the substitution $y = vx$, the differential equation can be written as $x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - v^2$.

(e) Using partial fractions, show that $\int \frac{1}{1-v^2}\,\mathrm{d}v = \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$, where A is a positive constant. [5]

(f) Hence, show that a solution to the original differential equation may be expressed in the form $x^2 = \left|\frac{A(x+y)}{x-y}\right|$, where A is a positive constant. [5]

Now consider only the case where $\frac{A(x+y)}{x-y} > 0$.

(g) Show that a solution to the original differential equation is $y = \frac{x(x^2 - A)}{x^2 + A}$. [4]

Question 1 - working space

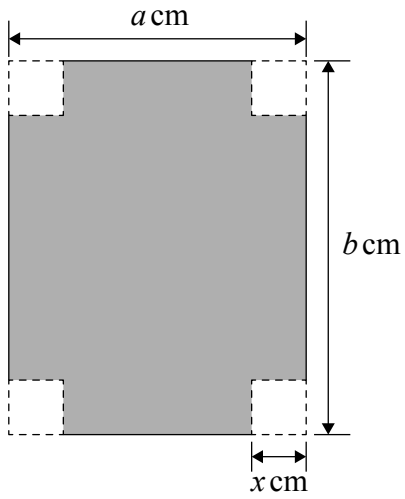
2. [Maximum mark: 28]

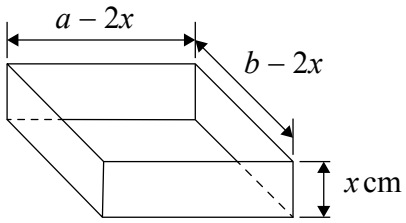
This question asks you to consider open-topped boxes formed by cutting square corners from an $a \times b$ sheet of cardboard and folding the sides. In particular, you are asked to investigate conditions for which the maximum volume of a box is an integer.

A rectangular sheet of cardboard measures $a\,\text{cm}$ by $b\,\text{cm}$ where $a < b$.

A square of side length $x\,\text{cm}$ is cut from each corner and the resulting sheet of cardboard is then folded to form an open-topped box with a rectangular base.

This is shown in the following diagrams.





The volume, $V\,\text{cm}^3$, of a box is given by $V = x(a-2x)(b-2x)$ where $0 \le x \le \frac{a}{2}$.

(a) Write down the value of V when

(i) $x = 0$; [1]

(ii) $x = \frac{a}{2}$. [1]

(b) Show that $V = 4x^3 - 2(a+b)x^2 + abx$. [2]

(c) Show that the only solutions of $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ are $x = \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6}$. [6]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), the following three results are given.

$x = \dfrac{(a+b)+\sqrt{a^2-ab+b^2}}{6}$ lies outside the interval $0 \le x \le \dfrac{a}{2}$.

$x = \dfrac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$ lies inside the interval $0 \le x \le \dfrac{a}{2}$.

$\sqrt{a^2-ab+b^2} > 0$.

Let $x_m = \dfrac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$.

(d) Use $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2}$ to show that there is a local maximum of V at $x = x_m$. [5]

(e) Use your answers to parts (a) and (d) to explain why x_m will give the maximum volume, V_m, of the box. [1]

In parts (f) to (h), consider a method for generating sets of positive integer values for x_m and hence V_m.

Consider the case where $a = 2st - s^2$, $b = t^2 - s^2$, and $s, t \in \mathbb{Z}^+$.

(f) Given that $a < b$, show that $t > 2s$. [2]

It is given that $\sqrt{a^2-ab+b^2} = s^2 - st + t^2$. You are **not** required to show this result.

(g) By substituting for each of a, b and $\sqrt{a^2-ab+b^2}$ in terms of s and t, show that $x_m = \dfrac{s(t-s)}{2}$. [3]

(h) (i) Given that $a, b \in \mathbb{Z}^+$, use $V_m = x_m(a-2x_m)(b-2x_m)$ to explain why s being even implies that V_m is an integer. [2]

(ii) State another condition on s and a corresponding condition on t, that implies that both x_m and V_m are integers. [1]

(iii) Hence, or otherwise, find an integer value of x_m and the corresponding integer value of V_m. [4]

Question 2 - working space

Section A

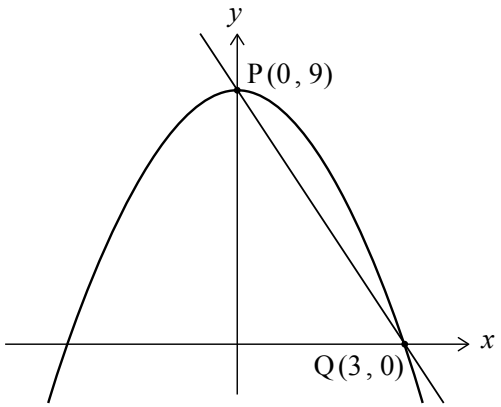
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 7]

A line with equation $y = -3x + 9$ intersects the axes at the points $\mathrm{P}(0, 9)$ and $\mathrm{Q}(3, 0)$.

A parabola of the form $y = ax^2 + c$, where $a, c \in \mathbb{Z}$, also passes through the points P and Q.

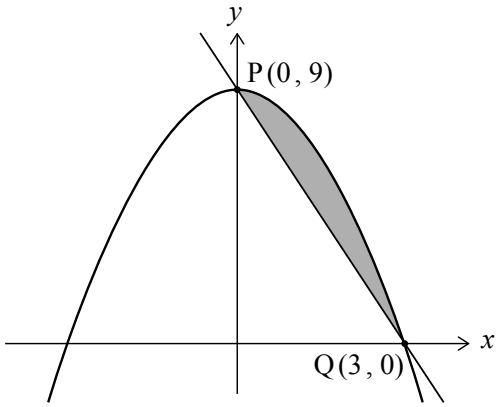
This is shown in the following diagram.



(a) (i) Write down the value of c.

(ii) Find the value of a. [3]

The region enclosed by the line and the parabola is shaded in the following diagram.



(b) Find the area of the shaded region. [4]

(This question continues on the following page)

(Question 3 continued)

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Question 3 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 7]

The first four terms of the Maclaurin series expansion of $(1-x)^{-4}$ are

$$1 + ax + bx^2 + 20x^3,\ a, b \in \mathbb{Z}^+.$$

(a) (i) Show that $a = 4$.

(ii) Find the value of b. [5]

A car was purchased four years ago. The car depreciated in value by 10% each year. The value of the car today is $\$1000$.

(b) Using the results of part (a), estimate the value of the car four years ago. [2]

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Question 5 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 8]

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \cos^2 x\,,\ y \neq 0$$

The solutions to this differential equation can be expressed in the form

$$y = \frac{a}{2x + \sin 2x + b}\,,$$

where a and b are constants such that $a \in \mathbb{Z}$ and $b \in \mathbb{R}$.

(a) By solving the differential equation, find the value of a. [6]

(b) Given that $y = 1$ when $x = \frac{\pi}{2}$, find the value of b. [2]

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Question 7 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

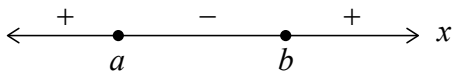
9. [Maximum mark: 14]

The function f has a derivative given by $f'(x) = 3x^2 + 12x - 15$.

The graph of $y = f(x)$ has horizontal tangents at the points where $x = a$ and $x = b$, $a < b$.

(a) Find the value of a and the value of b. [3]

The following diagram shows three intervals along the x-axis defined by a and b. The sign of the **first derivative** of f is shown in each interval.

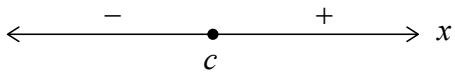


(b) State, with a reason, whether there is a local maximum point or a local minimum point on the graph of $y = f(x)$ at $x = a$. [2]

The second derivative $f''(x)$ is zero at $x = c$.

(c) Find the value of c. [3]

The following diagram shows two intervals along the x-axis defined by c. The sign of the **second derivative** is shown in each interval.



(d) State, with a reason, whether there is a point of inflexion on the graph of $y = f(x)$ at $x = c$. [2]

(e) Given that $f(-2) = 36$, find $f(x)$. [4]

Do **not** write solutions on this page.

Question 9 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the expression $\frac{1}{1+x^2}$, where $|x| < 1$.

(a) (i) Show that the first four terms in the binomial expansion of $\frac{1}{1+x^2}$ are $1 - x^2 + x^4 - x^6$. [3]

(ii) Hence, find an approximation for $\int \frac{1}{1+x^2}\,\mathrm{d}x$ up to and including the term in x^7. [2]

Now consider the expression $\frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$.

(b) (i) Expand $\frac{1}{\sqrt{1-x^2}}$ up to and including the term in x^4.

(ii) Hence, find a polynomial expression for $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ up to and including the term in x^5. [5]

The expression $\frac{25}{48} + \frac{k}{1280}$, where $k \in \mathbb{Z}^+$, can be used to approximate $\arcsin\frac{1}{2}$.

(c) Use the result from part (b)(ii) to find the value of k. [4]

(d) Hence, find an approximation for π in the form $\frac{p}{640}$ where $p \in \mathbb{Z}^+$. [4]

Do **not** write solutions on this page.

Question 10 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

In this question you may assume the result $\int x\operatorname{cosec}^2 x\,\mathrm{d}x = \ln|\sin x| - x\cot x + c$.

Consider the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3y = \dfrac{\operatorname{cosec}^2 x}{x}$ for $0 < x < \pi$.

Given that $y = \dfrac{8}{\pi^2}$ when $x = \dfrac{\pi}{2}$, find the particular solution for this differential equation.

Express your answer in the form $y = f(x)$.

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Question 6 - working space

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

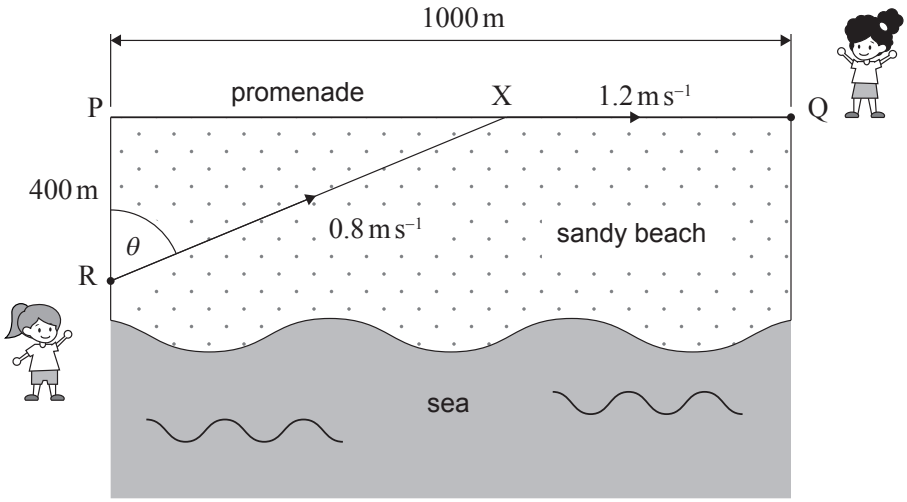
8. [Maximum mark: 9]

Astrid and Bronwyn are on vacation at Blackpool beach.

Astrid is standing on the beach at point R and sees Bronwyn standing on the promenade at point Q. $\mathrm{PR} = 400\,\mathrm{m}$, $\mathrm{PQ} = 1000\,\mathrm{m}$, $\mathrm{P\hat{R}X} = \theta$ where $0 < \tan\theta \leq \frac{5}{2}$.

Astrid walks in a straight line from R with speed $0.8\,\mathrm{m\,s^{-1}}$ until reaching point X on the promenade. She then jogs along the promenade with speed $1.2\,\mathrm{m\,s^{-1}}$.

This is shown in the following diagram.



It is given that $\mathrm{RX} = 400\sec\theta$ and $\mathrm{XQ} = 1000 - 400\tan\theta$.

The time, in seconds, for Astrid to reach Bronwyn is given by T.

(a) Show that $T = 500\sec\theta + \frac{2500 - 1000\tan\theta}{3}$. [2]

(b) Find $\frac{\mathrm{d}T}{\mathrm{d}\theta}$. [3]

Astrid chooses the angle θ of her walk across the beach to point X, in order to reach Bronwyn in the shortest time possible. You may assume that T has exactly one minimum value.

(c) Show that in this case $\mathrm{PX} = 160\sqrt{5}\,\mathrm{m}$. [4]

(This question continues on the following page)

(Question 8 continued)

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Question 8 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

9. [Maximum mark: 15]

Consider the function defined by $f(x) = \frac{1}{2}x^2 + kx + 13$, where $x \in \mathbb{R}$ and $k \in \mathbb{Z}^+$.

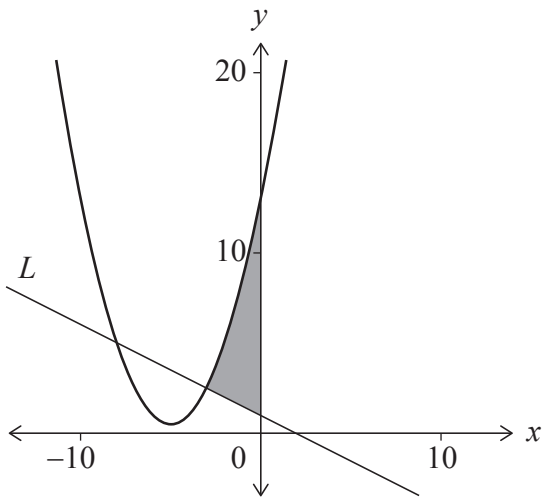
(a) Given that the equation $f(x) = 0$ has no real roots, show that the greatest possible value of k is 5. [2]

For the remainder of this question, consider the case $k = 5$.

(b) (i) Write down the equation of the axis of symmetry of the graph of f.

(ii) Hence, or otherwise, determine the coordinates of the minimum point on the graph of f. [3]

The following diagram shows the graph of f and a line L which is normal to the curve at $x = -3$. The shaded area shown is bounded by the curve, the line L and the y-axis.



(c) Show that the equation of L is given by $y = -\frac{1}{2}x + 1$. [5]

(d) Hence, find the shaded area. [5]

Do **not** write solutions on this page.

Question 9 - working space

Section A

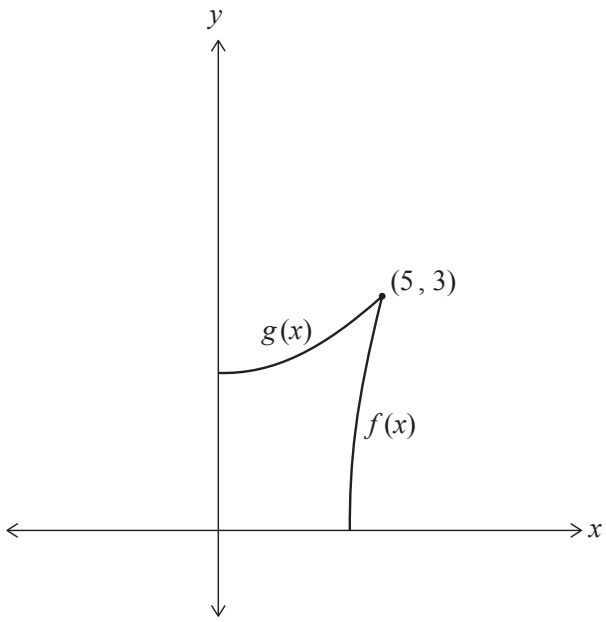
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 6]

The following diagram shows the graphs of two functions, f and g, where

$$f(x) = \sqrt{x^2 - 16} \text{ for } 4 \leq x \leq 5, \text{ and}$$

$$g(x) = 2 + \frac{x^2}{25} \text{ for } 0 \leq x \leq 5.$$



The graphs of f and g intersect at the point $(5, 3)$.

The region bounded by the graphs of f, g and both axes is rotated $360°$ about the y-axis to form a solid of revolution.

Find the volume of the solid formed.

(This question continues on the following page)

(Question 8 continued)

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Question 8 - working space

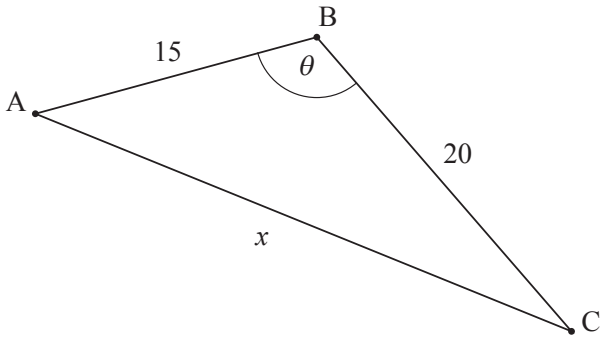
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 7]

Consider triangle ABC with $\mathrm{AB} = 15\,\mathrm{cm}$, $\mathrm{BC} = 20\,\mathrm{cm}$, $\mathrm{AC} = x\,\mathrm{cm}$ and $\mathrm{A\hat{B}C} = \theta$ where $0 < \theta < \frac{\pi}{2}$.

diagram not to scale



Angle θ is decreasing at the constant rate of $\frac{\pi}{60}$ radians per minute.

Determine how fast the length of AC is changing when the area of the triangle is $140\,\mathrm{cm}^2$.

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Question 9 - working space

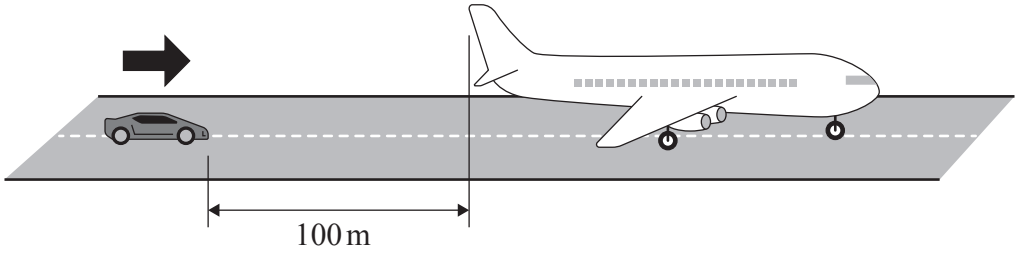
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

An airplane lands on a runway 100 metres in front of a stationary car. At the instant the airplane lands, the car begins to travel in the same direction towards the airplane.

diagram not to scale



Let t represent the number of seconds after the airplane lands. For $t \geq 0$, the velocities of the airplane and the car in m s^{-1}, can be modelled by the following equations:

$$v_{air} = 60\text{e}^{-0.1t}$$

$$v_{car} = 5t$$

(a) When the airplane lands, write down the speed of

(i) the airplane;

(ii) the car. [2]

(b) Find

(i) the value of t when the airplane and the car have the same speed;

(ii) the speed at this time. [3]

Let $d(t)$ represent the distance, in metres, between the car and the back of the airplane after t seconds.

(c) If $d(0) = 100$, find $d(t)$. [7]

(d) Hence, find how long it takes for the car to reach the back of the airplane. [2]

(e) Find the distance travelled by the car when it reaches the back of the airplane. [3]

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Question 10 - working space

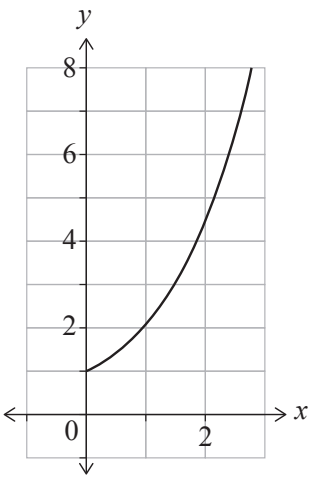
Section A

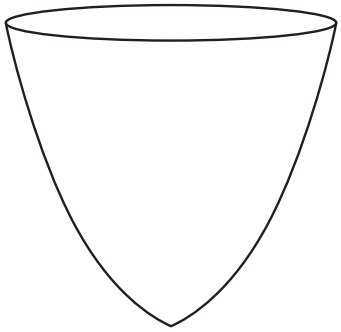
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

Consider the function $f(x) = e^{\frac{3x}{4}}$, where $x \geq 0$.

The region enclosed by the curve $y = f(x)$, the line $y = 8$ and the y-axis is rotated by 2π radians about the y-axis to form a solid. This is shown in the following diagrams.





Find the volume of the solid formed.

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Question 5 - working space

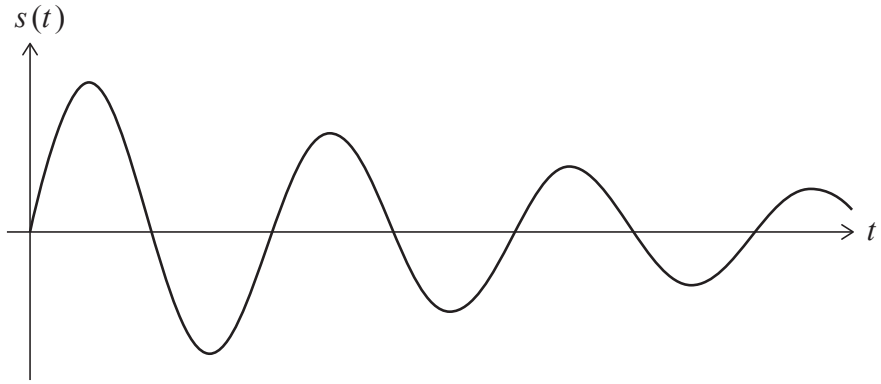
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

A particle P moves in a straight line so that its displacement, $s\,\text{cm}$, from a fixed point O at time t seconds is given by $s(t) = 2^{\left(1-\frac{t}{5}\right)}\sin\left(\frac{2\pi t}{3}\right)$, where $t \geq 0$.

The following diagram shows part of the graph of $y = s(t)$.



(a) Find

(i) the maximum displacement of P from O;

(ii) the maximum velocity of P. [5]

(b) Find

(i) the minimum value of the displacement function $s(t)$;

(ii) the displacement of P from O when $t = 3.5$. [3]

(c) Hence, determine the **total distance** travelled by P in the first 3.5 seconds. [3]

The first time that P returns and passes through O is when $t = T$.

(d) Write down the value of T. [1]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The particle passes through O every T seconds.

A sequence $u_1, u_2, u_3 \ldots$ is formed where $u_1, u_2, u_3 \ldots$ are the largest **distances** from O in each of the intervals $0 < t < T$, $T < t < 2T$, $2T < t < 3T \ldots$ respectively.

It is known that $u_1, u_2, u_3 \ldots$ form a geometric sequence.

(e) (i) Determine the value of the common ratio r of this geometric sequence.

(ii) Calculate the **total distance** travelled by the particle if it were to continue to move in this way indefinitely. [5]

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Do **not** write solutions on this page.

Question 10 - working space

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

Consider the following homogenous differential equation

$$\left(x^2+xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+xy-3y^2 \text{ where } x>0 \text{ and } y>\frac{x}{2}.$$

It is given that $y=\frac{3}{2}$ when $x=1$.

(a) (i) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=1$.

(ii) Use Euler's method with two equal steps to estimate the value of y when $x=1.4$.

(iii) Hence, state the concavity of the solution curve for $1\le x\le 1.4$. You may assume that the concavity does not change in this interval. Give a reason for your answer. [6]

(b) (i) Show that

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2x+y-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-(x+7y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(ii) Find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x=1$. [6]

(c) Determine constants $A, B\in\mathbb{R}$ such that $\frac{1+v}{1-4v^2}\equiv\frac{A}{1-2v}+\frac{B}{1+2v}$. [2]

(d) By solving the differential equation $\left(x^2+xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+xy-3y^2$ where $x>0$, $y>\frac{x}{2}$ and $y=\frac{3}{2}$ when $x=1$, show that $x^6(2y-x)^3=2(x+2y)$. [8]

Please **do not** write on this page.

Answers written on this page
will not be marked.

Question 12 - working space

1. [Maximum mark: 26]

This question asks you to investigate lines normal to curves of the form $y = \frac{k^2}{x}$.

The curve H has equation $y = \frac{1}{x}$ where $x \in \mathbb{R}, x \neq 0$.

(a) A line N is normal to H at $x = t$.

(i) Show that the gradient of N is t^2. [2]

(ii) Hence, show that the equation of N is $y = t^2x + \frac{1}{t} - t^3$. [1]

(b) The equation for N given in part (a)(ii) is of the form $y = mx + c$.

(i) Show that either $c = \frac{1}{\sqrt{m}}\left(1 - m^2\right)$ or $c = \frac{1}{\sqrt{m}}\left(m^2 - 1\right)$. [4]

(ii) Determine the set of values of m for which there exists at least one line normal to H. [1]

(c) Hence, or otherwise, determine the set of values of m for which there exists exactly

(i) one line normal to H; [1]

(ii) two lines normal to H. [2]

(d) On the same set of axes, sketch

(i) the curve H; [1]

(ii) for an appropriate value of m, two lines that satisfy the result found in part (c)(ii). Clearly indicate the point at which each line is normal to H. [2]

You are **not** required to state the equations of these lines nor determine where they intersect H or the coordinate axes.

(This question continues on the following page)

(Question 1 continued)

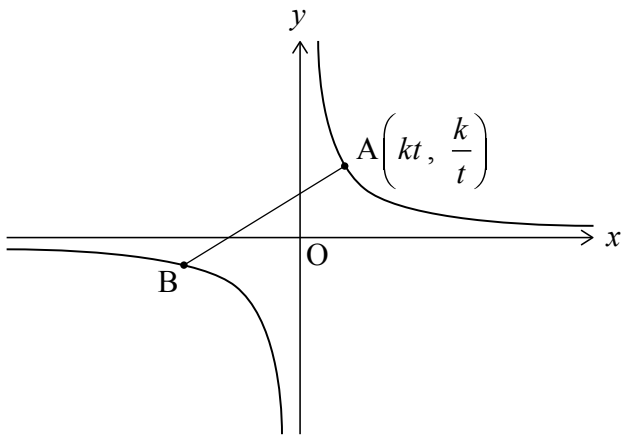
The curve F has equation $y = \frac{k^2}{x}$ where $x \in \mathbb{R}, x \neq 0$ and $k \in \mathbb{R}, k \neq 0$.

The point $\mathrm{A}\left(kt, \frac{k}{t}\right)$, where $t \in \mathbb{R}, t \neq \pm 1$, lies on F.

The line normal to F at A intersects F again at point B.

The line segment $[\mathrm{AB}]$ is shown in the following diagram.

diagram not to scale



(e) The equation of the line normal to F at A is given by $y = t^2x - kt^3 + \frac{k}{t}$.

(i) Show that the x-coordinates of A and B satisfy the quadratic equation

$$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0.$$ [3]

(ii) Hence, by considering either the sum or product of the roots of this quadratic equation, or otherwise, determine the coordinates of B. [3]

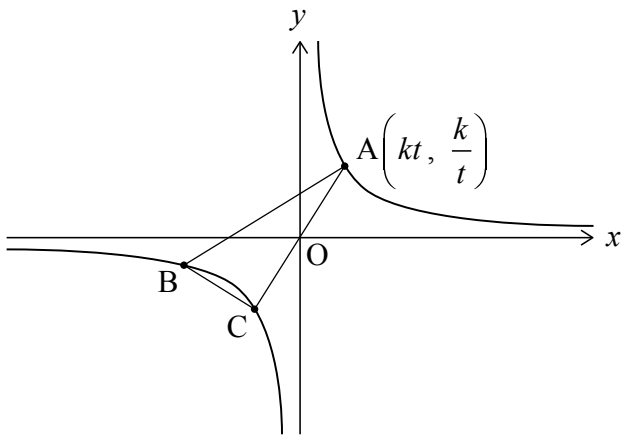
(This question continues on the following page)

(Question 1 continued)

From A, the line passing through the origin O intersects F again at point C.

Points A, B and C form triangle ABC as shown in the following diagram.

diagram not to scale



(f) Prove that $\mathrm{B\hat{C}A}$ is a right angle. [6]

Question 1 - working space

2. [Maximum mark: 29]

In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population, x, as a function of t, where t is the time, in years, since the area became protected.

(a) Initially, the researchers consider using the logistic model

$$x = \frac{L}{1 + C\mathrm{e}^{-kt}}, \text{ where } L, C, k \in \mathbb{R}^+.$$

The researchers decide to let $L = 200$.

(i) State the assumption being made in assuming $L = 200$. [1]

At $t = 0$, the population of wolves is 40.

(ii) Find the value of C. [2]

At $t = 5$, the population of wolves is found to have increased to 70.

(iii) Find the value of k. [2]

(iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

(b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax\ln\left(\frac{200}{x}\right),\ a \in \mathbb{R}^+.$$

(i) Write down the value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 200$. [1]

(ii) Interpret your answer to part (b)(i) in context. [1]

(This question continues on the following page)

(Question 2 continued)

Consider the function $f(x) = \ln(\ln 200 - \ln x)$, where $0 < x < 200$.

(iii) Show that $f'(x) = \dfrac{-1}{x \ln\left(\dfrac{200}{x}\right)}$. [2]

(iv) Hence, use separation of variables to show that the general solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax \ln\left(\frac{200}{x}\right), \text{ where } 0 < x < 200,$$

can be written as

$$\ln x = \ln 200 - A\mathrm{e}^{-at},$$

where A is an arbitrary positive constant. [5]

(v) Use the size of the wolf population at $t = 0$ to find the value of A.
Give your answer in the form $A = \ln p$, where $p \in \mathbb{Z}^+$. [2]

(vi) Use the size of the wolf population at $t = 5$, given in part (a), to show that $a = 0.0855$, correct to three significant figures. [2]

(vii) Use the Gompertz model to predict the size of the wolf population at $t = 10$.
Give your answer correct to the nearest whole number. [3]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.0855x \ln\left(\frac{200}{x}\right) - 0.03x.$$

(d) (i) Use Euler's method, with a step size of 0.5 years and an initial value of $x_0 = 70$ when $t = 5$, to find an estimate for the size of the wolf population when $t = 10$.
Give your answer correct to the nearest whole number. [4]

(ii) Comment on your answer. [1]

Question 2 - working space

1. [Maximum mark: 26]

The following question explores features of composed trigonometric functions, such as $\sin(\sin x)$**,** $\sin\big(\sin(\sin x)\big)$.

Suppose $S_n(x)$ denotes the function $\sin x$ composed within itself $n-1$ times, defined for $n \geq 1$, $n \in \mathbb{Z}^+$, where $0 \leq x \leq 2\pi$.

For example, $S_1(x) = \sin x$ and $S_2(x) = \sin(\sin x)$ where $0 \leq x \leq 2\pi$.

(a) On the same axes, sketch and label the graphs of $y = S_1(x)$ and $y = S_2(x)$. On your sketch, show the values of the intercepts with the axes. [4]

(b) Determine the maximum value of

(i) $S_1(x)$; [1]

(ii) $S_2(x)$; [1]

(iii) $S_3(x)$. [1]

(c) Find the least value of n for which the maximum value of $S_n(x)$ is less than 0.6. [3]

Consider the graph of $y = S_2(x)$.

(d) By considering the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, show that there are exactly two points of zero gradient, one at $x = \frac{\pi}{2}$ and one at $x = \frac{3\pi}{2}$. [6]

(This question continues on the following page)

(Question 1 continued)

The derivative $S_n'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(S_n(x)\right)$ can be expressed as a product of cosine functions, as follows:

$$S'_n(x) = \cos\left(S_{n-1}(x)\right)\cos\left(S_{n-2}(x)\right) \ldots \cos\left(S_1(x)\right)\cos x\,.$$

(e) Hence, show that $S'_3(x) = \cos\left(\sin\left(\sin x\right)\right)\cos\left(\sin x\right)\cos x\,.$ [1]

(f) Use mathematical induction to prove that for all $n \in \mathbb{Z}^+$

$$S'_n(x) = \cos\left(S_{n-1}(x)\right)\cos\left(S_{n-2}(x)\right) \ldots \cos\left(S_1(x)\right)\cos x.$$ [6]

(g) Use l'Hôpital's rule to show that $\lim_{x\to 0}\frac{S_n(x)}{x} = 1$, for $n \in \mathbb{Z}^+$. [3]

Question 1 - working space

2. [Maximum mark: 29]

The following question uses Maclaurin series to investigate approximations of mathematical constants and the accuracies of such approximations.

(a) Given $|x| < 1$, find the sum to infinity of the geometric series $1 - x^2 + x^4 - x^6 + \ldots$. [2]

(b) Hence, use integration to show that the Maclaurin series of $\arctan x$ may be expressed as $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$. [3]

(c) Using $x = \frac{1}{\sqrt{3}}$ and the first **three** (non-zero) terms of the Maclaurin series of $\arctan x$, find an approximation for π to three decimal places. [3]

The Maclaurin series of $\arctan x$ is an example of an alternating series, *ie* a series where consecutive terms are positive and negative. Consider the following theorem.

> Theorem: For alternating series with terms of decreasing magnitude, the error obtained in using a finite number of terms is less than or equal to the absolute value of the next term in the sequence.

Using the theorem, the maximum error in using the first three (non-zero) terms as an approximation to $\arctan x$ is given by $\left|-\frac{x^7}{7}\right|$. In other words, $\left|\arctan x - \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)\right| \leq \left|-\frac{x^7}{7}\right|$.

(d) Determine how many (non-zero) terms of the series would need to be used, such that the error in approximating $\arctan\left(\frac{1}{\sqrt{3}}\right)$ is less than 0.0001. [3]

(e) By using integration by parts, show that $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}x = \frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3}$. [4]

(f) Determine the value of $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) \mathrm{d}x$, giving your answer to six decimal places. [2]

(g) Use the results in parts (e) and (f) to find an approximation for π. Give your answer to four decimal places. [2]

(This question continues on the following page)

(Question 2 continued)

$\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \right) \mathrm{d}x$ may be considered as the sum of alternating terms.

Hence, $\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \right) \mathrm{d}x = \int_0^{\frac{1}{\sqrt{3}}} x \ \mathrm{d}x + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^3}{3} \right) \mathrm{d}x + \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^5}{5} \right) \mathrm{d}x + \int_0^{\frac{1}{\sqrt{3}}} \left(-\frac{x^7}{7} \right) \mathrm{d}x + \ldots .$

$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \ \mathrm{d}x$ is approximated using the sum of the first four definite integrals.

(h) Verify that the theorem used in part (d) holds in this case. [5]

Suppose that the maximum error in approximating $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \ \mathrm{d}x$ is required to be at most 1×10^{-6}.

(i) Determine the smallest number of (non-zero) terms of the Maclaurin series for $\arctan x$ that should be used. [5]

Question 2 - working space