

APOLLO

Number and algebra

HL

Mathematics: applications and interpretation

First assessment 2021

54 questions | questions

3. [Maximum mark: 8]

Charlie and Daniella each began a fitness programme. On day one, they both ran 500 m. On each subsequent day, Charlie ran 100 m more than the previous day whereas Daniella increased her distance by 2 % of the distance ran on the previous day.

- (a) Calculate how far
 - (i) Charlie ran on day 20 of his fitness programme.
 - (ii) Daniella ran on day 20 of her fitness programme. [5]

On day n of the fitness programmes Daniella runs more than Charlie for the first time.

- (b) Find the value of n . [3]



9. [Maximum mark: 8]

Consider $w = iz + 1$, where $w, z \in \mathbb{C}$.

(a) Find w when

(i) $z = 2i$.

(ii) $z = 1 + i$. [3]

Point z on the Argand diagram can be transformed to point w by two transformations.

(b) Describe these two transformations and give the order in which they are applied. [3]

(c) Hence, or otherwise, find the value of z when $w = 2 - i$. [2]



5. [Maximum mark: 5]

Roger buys a new laptop for himself at a cost of £495. At the same time, he buys his daughter Chloe a higher specification laptop at a cost of £2200.

It is anticipated that Roger’s laptop will depreciate at a rate of 10 % per year, whereas Chloe’s laptop will depreciate at a rate of 15 % per year.

- (a) Estimate the value of Roger’s laptop after 5 years. [2]

Roger and Chloe’s laptops will have the same value k years after they were purchased.

- (b) Find the value of k . [2]
- (c) Comment on the validity of your answer to part (b). [1]

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Turn over

7. [Maximum mark: 6]

A meteorologist models the height of a hot air balloon launched from the ground. The model assumes the balloon travels vertically upwards and travels 450 metres in the first minute.

Due to the decrease in temperature as the balloon rises, the balloon will continually slow down. The model suggests that each minute the balloon will travel only 82 % of the distance travelled in the previous minute.

(a) Find how high the balloon will travel in the first 10 minutes after it is launched. [3]

(b) The balloon is required to reach a height of at least 2520 metres.

Determine whether it will reach this height. [2]

(c) Suggest a limitation of the given model. [1]

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12. [Maximum mark: 8]

It is given that $z_1 = 3 \operatorname{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{n\pi}{16}\right)$, $n \in \mathbb{Z}^+$.

(a) In parts (a)(i) and (a)(ii), give your answers in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$.

(i) Find the value of z_1^3 .

(ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$. [5]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]



Turn over

1. [Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested 37 000 Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

- (a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than 50 000 AUD.

- (b) Find the minimum value of m , where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is 200 000 AUD.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

- (c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of 1700 AUD made by Daisy at the end of each month.

- (d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

- (e) Find the amount of Daisy's final payment. [3]

- (f) Find how much money Daisy saved by making one final payment after 5 years. [3]

Turn over

5. [Maximum mark: 9]

In this question, give all answers correct to 2 decimal places.

Raul and Rosy want to buy a new house and they need a loan of 170 000 Australian dollars (AUD) from a bank. The loan is for 30 years and the annual interest rate for the loan is 3.8 %, compounded monthly. They will pay the loan in fixed monthly instalments at the end of each month.

- (a) Find the amount they will pay the bank each month. [3]
- (b) (i) Find the amount Raul and Rosy will still owe the bank at the end of the first 10 years.
- (ii) Using your answers to parts (a) and (b)(i), calculate how much interest they will have paid in total during the first 10 years. [6]



6. [Maximum mark: 5]

An infinite geometric sequence, with terms u_n , is such that $u_1 = 2$ and $\sum_{k=1}^{\infty} u_k = 10$.

- (a) Find the common ratio, r , for the sequence. [2]
- (b) Find the least value of n such that $u_n < \frac{1}{2}$. [3]

[illegible]

Turn over

5. [Maximum mark: 13](a) Let $z = 1 - i$.(i) Plot the position of z on an Argand Diagram.(ii) Express z in the form $z = ae^{ib}$, where $a, b \in \mathbb{R}$, giving the exact value of a and the exact value of b . [3](b) Let $w_1 = e^{ix}$ and $w_2 = e^{i(x-\frac{\pi}{2})}$, where $x \in \mathbb{R}$.(i) Find $w_1 + w_2$ in the form $e^{ix}(c + id)$.(ii) Hence find $\operatorname{Re}(w_1 + w_2)$ in the form $A \cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \leq \frac{\pi}{2}$. [6]

The current, I , in an AC circuit can be modelled by the equation $I = a \cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_A and I_B . The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_A	12 amps	0
I_B	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_T can be expressed as $I_A + I_B$.

(c) (i) Find the maximum value of I_T .(ii) Find the phase shift of I_T . [4]

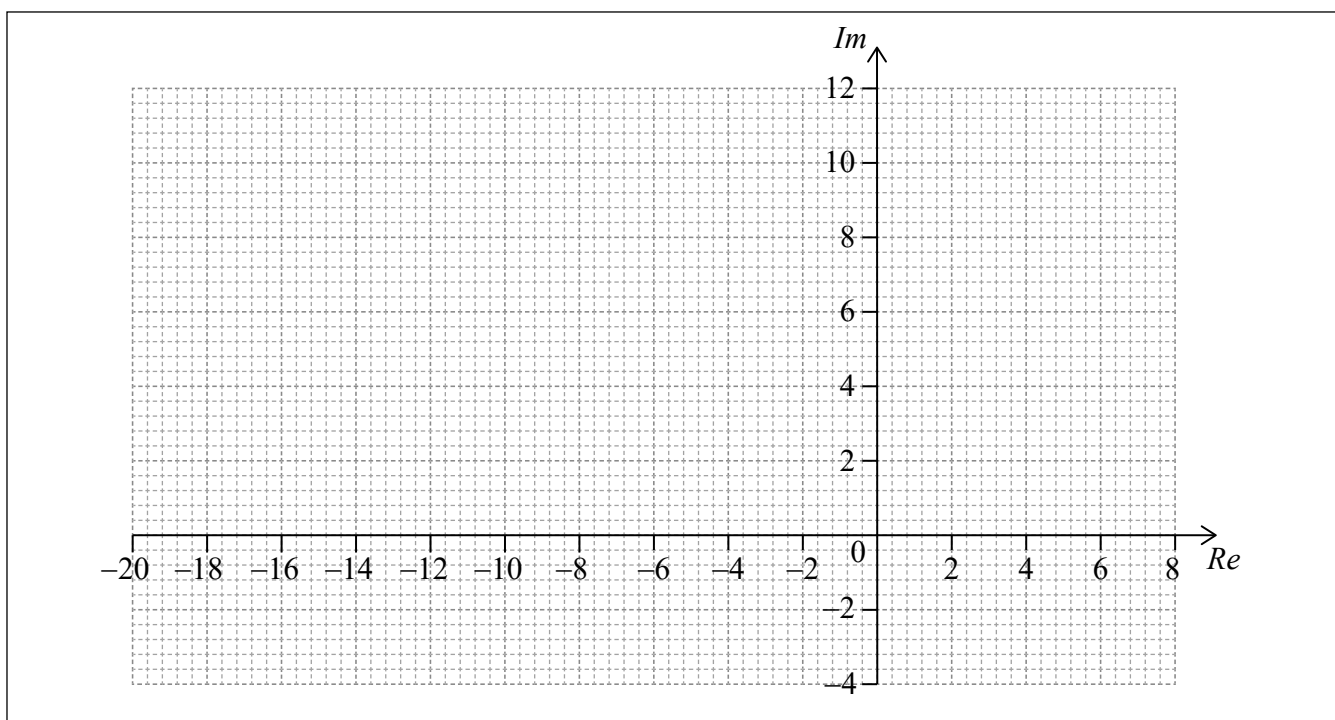
15. [Maximum mark: 7]

Let $w = ae^{\frac{\pi}{4}i}$, where $a \in \mathbb{R}^+$.

(a) For $a = 2$,

- (i) find the values of w^2 , w^3 , and w^4 ;
- (ii) draw w , w^2 , w^3 and w^4 on the following Argand diagram.

[5]



Let $z = \frac{w}{2-i}$.

- (b) Find the value of a for which successive powers of z lie on a circle.

[2]

(This question continues on the following page)



[illegible]

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18. [Maximum mark: 6]

The rate, A , of a chemical reaction at a fixed temperature is related to the concentration of two compounds, B and C , by the equation

$$A = k B^x C^y, \text{ where } x, y, k \in \mathbb{R}.$$

A scientist measures the three variables three times during the reaction and obtains the following values.

Experiment	$A \text{ (mol l}^{-1} \text{ s}^{-1}\text{)}$	$B \text{ (mol l}^{-1}\text{)}$	$C \text{ (mol l}^{-1}\text{)}$
1	5.74	2.1	3.4
2	2.88	1.5	2.4
3	0.980	0.8	1.9

Find x , y and k .

[illegible]

3. [Maximum mark: 18]

In this question, give all answers to two decimal places.

Bryan decides to purchase a new car with a price of €14 000, but cannot afford the full amount. The car dealership offers two options to finance a loan.

Finance option A:

A 6 year loan at a nominal annual interest rate of 14% **compounded quarterly**.

No deposit required and repayments are made each quarter.

(a) (i) Find the repayment made each quarter.

(ii) Find the total amount paid for the car.

(iii) Find the interest paid on the loan.

[7]

Finance option B:

A 6 year loan at a nominal annual interest rate of $r\%$ **compounded monthly**. Terms of the loan require a 10% deposit and monthly repayments of €250.

(b) (i) Find the amount to be borrowed for this option.

(ii) Find the annual interest rate, r .

[5]

(c) State which option Bryan should choose. Justify your answer.

[2]

Bryan chooses option B. The car dealership invests the money Bryan pays as soon as they receive it.

(d) If they invest it in an account paying 0.4% interest per month and inflation is 0.1% per month, calculate the real amount of money the car dealership has received by the end of the 6 year period.

[4]

2. [Maximum mark: 5]

The ticket prices for a concert are shown in the following table.

Ticket Type	Price (in Australian dollars, \$)
Adult	15
Child	10
Student	12

- A total of 600 tickets were sold.
- The total amount of money from ticket sales was \$7816.
- There were twice as many adult tickets sold as child tickets.

Let the number of adult tickets sold be x , the number of child tickets sold be y , and the number of student tickets sold be z .

- (a) Write down three equations that express the information given above. [3]
- (b) Find the number of each type of ticket sold. [2]

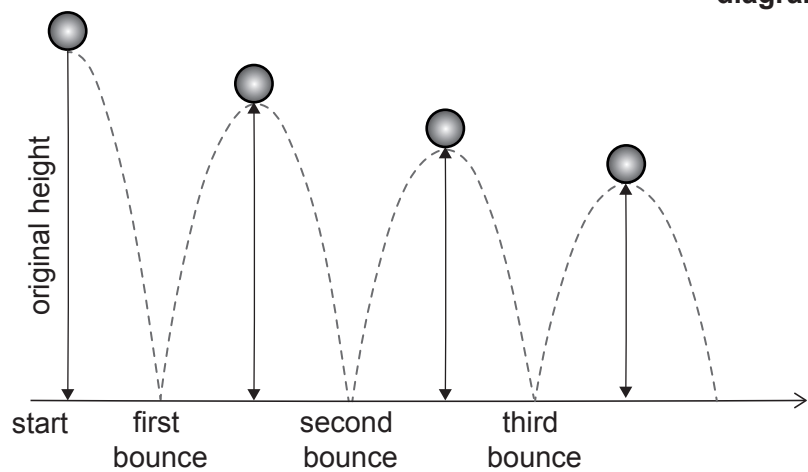


Turn over

5. [Maximum mark: 7]

A ball is dropped from a height of 1.8 metres and bounces on the ground. The maximum height reached by the ball, after each bounce, is 85 % of the previous maximum height.

diagram not to scale



- (a) Show that the maximum height reached by the ball after it has bounced for the sixth time is 68 cm, to the nearest cm.

[2]
- (b) Find the number of times, after the first bounce, that the maximum height reached is greater than 10 cm.

[2]
- (c) Find the total **vertical** distance travelled by the ball from the point at which it is dropped until the fourth bounce.

[3]

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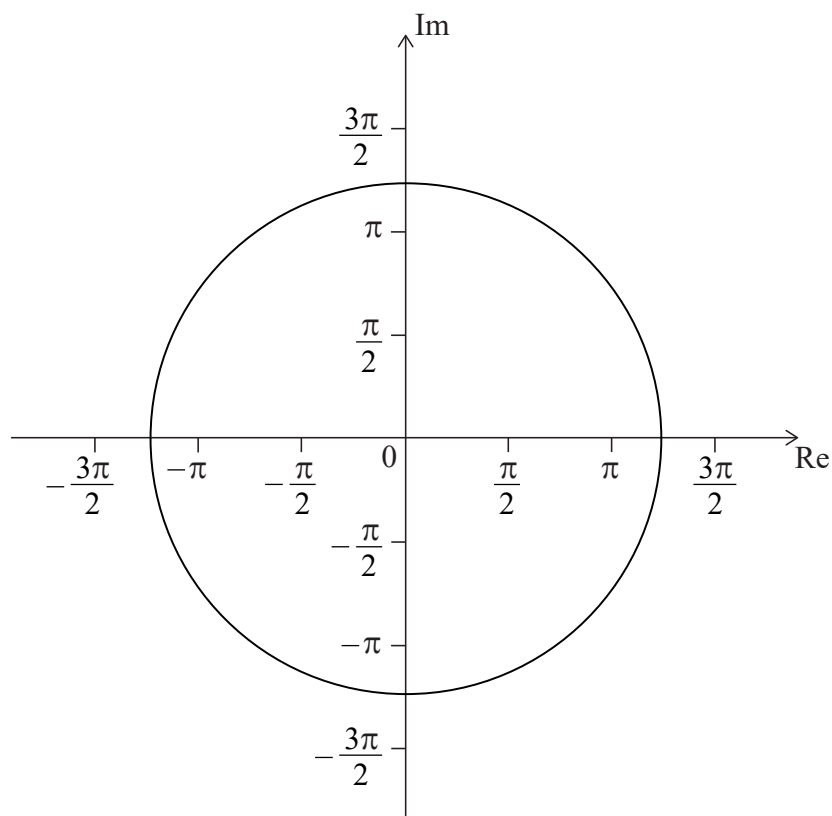
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10. [Maximum mark: 7]

The following Argand diagram shows a circle centre 0 with a radius of 4 units.



A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$$z_\theta = \frac{1}{2} \theta e^{i\theta}, \text{ where } \theta \geq 0.$$

(a) Plot on the Argand diagram the points corresponding to

(i) $\theta = \frac{\pi}{2}$.

(ii) $\theta = \pi$.

(iii) $\theta = \frac{3\pi}{2}$.

[3]

Consider the case where $|z_\theta| = 4$.

(b) (i) Find this value of θ .

(ii) For this value of θ , plot the approximate position of z_θ on the Argand diagram.

[4]

(This question continues on the following page)



(Question 10 continued)

[illegible]

Turn over

The matrix $M = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -0.5 and 1 .

- [3]

[2]

[illegible]

12. [Maximum mark: 8]

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between 0 and 8 where 8 is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, N , which have a magnitude of at least M . For a particular region the equation is

$$\log_{10} N = a - M, \text{ for some } a \in \mathbb{R}.$$

This region has an average of 100 earthquakes per year with a magnitude of at least 3.

- (a) Find the value of a . [2]

The equation for this region can also be written as $N = \frac{b}{10^M}$.

- (b) Find the value of b . [2]

Within this region the most severe earthquake recorded had a magnitude of 7.2.

- (c) Find the average number of earthquakes in a year with a magnitude of at least 7.2. [1]

The number of earthquakes in a given year with a magnitude of at least 7.2 can be modelled by a Poisson distribution, with mean N . The number of earthquakes in one year is independent of the number of earthquakes in any other year.

Let Y be the number of years between the earthquake of magnitude 7.2 and the next earthquake of at least this magnitude.

- (d) Find $P(Y > 100)$. [3]



Turn over

7. [Maximum mark: 5]

The sum of an infinite geometric sequence is 9.

The first term is 4 more than the second term.

Find the third term. Justify your answer.

[illegible]

Turn over

11. [Maximum mark: 6]

Juliana plans to invest money for 10 years in an account paying 3.5 % interest, compounded annually. She expects the annual inflation rate to be 2 % per year throughout the 10-year period.

Juliana would like her investment to be worth a real value of \$4000, compared to current values, at the end of the 10-year period. She is considering two options.

Option 1: Make a one-time investment at the start of the 10-year period.

Option 2: Invest \$1000 at the start of the 10-year period and then invest \$ x into the account at the end of each year (including the first and last years).

- (a) For option 1, determine the minimum amount Juliana would need to invest. Give your answer to the nearest dollar.

[3]
- (b) For option 2, find the minimum value of x that Juliana would need to invest each year. Give your answer to the nearest dollar.

[3]

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Turn over

13. [Maximum mark: 5]

An electric circuit has two power sources. The voltage, V_1 , provided by the first power source, at time t , is modelled by

$$V_1 = \operatorname{Re}(2e^{3ti}).$$

The voltage, V_2 , provided by the second power source is modelled by

$$V_2 = \operatorname{Re}(5e^{(3t+4)i}).$$

The total voltage in the circuit, V_T , is given by

$$V_T = V_1 + V_2.$$

- (a) Find an expression for V_T in the form $A \cos(Bt + C)$, where A , B and C are real constants.

[4]
- (b) Hence write down the maximum voltage in the circuit.

[1]

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Turn over

1. [Maximum mark: 13]

Scott purchases food for his dog in large bags and feeds the dog the same amount of dog food each day. The amount of dog food left in the bag at the end of each day can be modelled by an arithmetic sequence.

On a particular day, Scott opened a new bag of dog food and fed his dog. By the end of the third day there were 115.5 cups of dog food remaining in the bag and at the end of the eighth day there were 108 cups of dog food remaining in the bag.

- (a) Find the number of cups of dog food
- (i) fed to the dog per day;
 - (ii) remaining in the bag at the end of the first day. [4]
- (b) Calculate the number of days that Scott can feed his dog with one bag of food. [2]
- In 2021, Scott spent \$625 on dog food. Scott expects that the amount he spends on dog food will increase at an annual rate of 6.4%.
- (c) Determine the amount that Scott expects to spend on dog food in 2025. Round your answer to the nearest dollar. [3]
- (d) (i) Calculate the value of $\sum_{n=1}^{10} (625 \times 1.064^{(n-1)})$.
- (ii) Describe what the value in part (d)(i) represents in this context. [3]
- (e) Comment on the appropriateness of modelling this scenario with a geometric sequence. [1]

7. [Maximum mark: 7]

On 1 December 2022, Laviola invests 800 euros (EUR) into a savings account which pays a nominal annual interest rate of 7.5% compounded monthly. At the end of each month, Laviola deposits an additional EUR 500 into the savings account.

At the end of k months, Laviola will have saved enough money to withdraw EUR 10 000.

- (a) Find the smallest possible value of k , for $k \in \mathbb{Z}^+$.

[4]
- (b) For this value of k , find the interest earned in the savings account.
Express your answer correct to the nearest EUR.

[3]

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1. [Maximum mark: 5]

On 1 January 2022, Mina deposited \$ 1000 into a bank account with an annual interest rate of 4% , compounded monthly. At the end of January, and the end of every month after that, she deposits \$ 100 into the same account.

- (a) Calculate the amount of money in her account at the start of 2024. Give your answer to two decimal places. [3]
- (b) Find how many complete months, counted from 1 January 2022, it will take for Mina to have more than \$ 5000 in her account. [2]



Turn over

12. [Maximum mark: 5]

Two AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as $V_1 = 6 \sin(at + 30^\circ)$ and $V_2 = 6 \sin(at + 90^\circ)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V \sin(at + \theta^\circ)$.

Determine the value of V and the value of θ .

[illegible]

Turn over

4. [Maximum mark: 6]

Angel has \$ 520 in his savings account. Angel considers investing the money for 5 years with a bank. The bank offers an annual interest rate of 1.2 % compounded quarterly.

- (a) Calculate the amount of money Angel would have at the end of 5 years with the bank.
Give your answer correct to two decimal places. [3]

Instead of investing the money, Angel decides to buy a phone that costs \$ 520. At the end of 5 years, the phone will have a value of \$ 30. It may be assumed that the depreciation rate per year is constant.

- (b) Calculate the annual depreciation rate of the phone. [3]

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Turn over

11. [Maximum mark: 6]

Two AC (alternating current) electrical sources of equal frequencies are combined.

The voltage of the first source is modelled by the equation $V = 30 \sin(t + 60^\circ)$.

The voltage of the second source is modelled by the equation $V = 60 \sin(t + 10^\circ)$.

- (a) Determine the maximum voltage of the combined sources.

[2]
- (b) Using your graphic display calculator, find a suitable equation for the combined voltages, giving your answer in the form $V = V_0 \sin(at + b)$, where a , b and V_0 are constants, $a > 0$ and $0^\circ \leq b < 180^\circ$.

[4]

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Turn over

1. [Maximum mark: 7]

The growth of a particular type of seashell is being studied by Manon. At the end of each month Manon records the increase in the width of a seashell since the end of the previous month.

She models the monthly increase in the width of the seashell by a geometric sequence with common ratio 0.8. In the first month, the width of the seashell increases by 4 mm.

- (a) Find by how much the width of the seashell will increase during the third month, according to her model.

[2]
- (b) Find the total increase in the width of the seashell, predicted by Manon’s model, during the first year.

[2]

Manon’s seashell had a width of 25 mm at the beginning of the first month.

- (c) Find the maximum possible width of the seashell, predicted by Manon’s model.

[3]

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8. [Maximum mark: 6]

Given $z = \sqrt{3} - i$.

(a) Write z in the form $z = re^{\theta i}$, where $r \in \mathbb{R}^+$, $-\pi < \theta \leq \pi$. [2]

Let $z_1 = e^{2ti}$ and $z_2 = 2e^{\left(2t - \frac{\pi}{6}\right)i}$.

(b) Find $\text{Im}(z_1 + z_2)$ in the form $p \sin(2t + q)$, where $p > 0$, $t \in \mathbb{R}$ and $-\pi \leq q \leq \pi$. [4]



Turn over

3. [Maximum mark: 16]

Tiffany wants to buy a house for a price of 285 000 US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to 15 % of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a 4 % nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

- (a) (i) Find the original amount of the loan after the down payment is paid.
Give the exact answer.
- (ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]
- (b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays 1300 USD each month.

- (c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than 1300 USD.

- (d) Determine the amount of Tiffany's final payment, to two decimal places. [4]
- (e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

1. [Maximum mark: 6]

Maan deposited \$100 000 into a savings account with a nominal annual interest rate of $I\%$ **compounded monthly**. At the end of the eighth year, the amount in the account had increased to \$150 000.

- (a) Find the value of I .
- [3]

Maan withdraws the \$150 000 and places it in an annuity, earning a nominal annual interest rate of 6.1% **compounded monthly**. At the end of each month, Maan will receive a payment of \$1000.

- (b) Find the amount of money remaining in the annuity at the end of 10 years. Express your answer to the nearest dollar.
- [3]

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Turn over

6. [Maximum mark: 7]

On 1 January 2025, the Faber Car Company will release a new car to global markets. The company expects to sell 40 cars in January 2025. The number of cars sold each month can be modelled by a geometric sequence where $r = 1.1$.

- (a) Use this model to find the number of cars that will be sold in December 2025. [2]
- (b) Use this model to find the total number of cars that will be sold in the year
 - (i) 2025.
 - (ii) 2026. [5]



13. [Maximum mark: 6]

Let $z_1 = 4 + 5i$.

- (a) (i) Find $|z_1|$.
- (ii) Find $\arg(z_1)$. [2]

Let $z_2 = 3e^{2i}$.

- (b) Find the area of the triangle on an Argand diagram with vertices 0, z_1 and z_2 . [4]



1. [Maximum mark: 9]

Imani invests \$3000 in a bank that pays a nominal annual interest rate of 1.25% compounded monthly.

- (a) Calculate the amount of money Imani will have in the bank at the end of 6 years. Give your answer correct to two decimal places. [3]

- (b) Calculate the number of months it takes until Imani has at least \$3550 in the bank. [2]

Imani uses the \$3550 as a partial payment for a used car costing \$22 000. For the remainder she takes out a loan from a bank.

- (c) Write down the amount of money that Imani takes out as a loan. [1]

The loan is for 8 years and the nominal annual interest rate is 12.6% compounded monthly. Imani will pay the loan in fixed monthly instalments at the end of each month.

- (d) Calculate the amount, correct to the nearest dollar, that Imani will have to pay the bank each month. [3]

(This question continues on the following page)



(Question 1 continued)

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Turn over

5. [Maximum mark: 8]

The annual growth of a tree is 80 % of its growth during the previous year.

This year the tree is 42 m in height and one year ago its height was 37 m.

- (a) Calculate the annual growth of the tree in the coming year. [2]
- (b) Calculate the height of the tree 6 years from now. Give your answer correct to the nearest cm. [4]

If the tree continues to follow this pattern of growth, its height will never exceed k metres.

- (c) Find the smallest possible value of k . [2]



Turn over

15. [Maximum mark: 8]

Consider the complex number $z = -1 + i$.

- (a) Express z in the form $re^{i\theta}$ where $-\pi < \theta \leq \pi$. [2]

A and B are the points on the Argand diagram that represent the complex numbers z and z^2 , respectively.

A is mapped onto B by the composition of a rotation and an enlargement.

- (b) (i) Describe fully this mapping of A onto B, stating the scale factor of the enlargement and the angle of rotation.
- (ii) Find and simplify a matrix that maps A onto B. [5]
- (c) Find the smallest positive integer, n , for which z^n is real and positive. [1]

[illegible]

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Answers written on this page
will not be marked.



4. [Maximum mark: 5]

On 1 January in a particular year, Eva invests \$25 000 in a new bank account. The account earns 5% simple interest, on the original \$25 000, at the start of each subsequent year.

The amounts in the account at the start of each year form an arithmetic sequence.

(a) Find the common difference of this sequence. [2]

After k complete years, the amount in Eva’s account will be greater than \$44 000 for the first time.

(b) Find the value of k . [3]



Turn over

7. [Maximum mark: 6]

When Daniel retires, he invests \$400 000 in an annuity fund that earns interest at a nominal rate of 4.5 % per year, compounded monthly.

Daniel then withdraws \$3600 at the end of every month to pay for his living expenses.

- (a) Find how much is in the annuity fund after 5 years. [3]
- (b) Calculate how many times Daniel is able to make these withdrawals. [3]

[illegible]

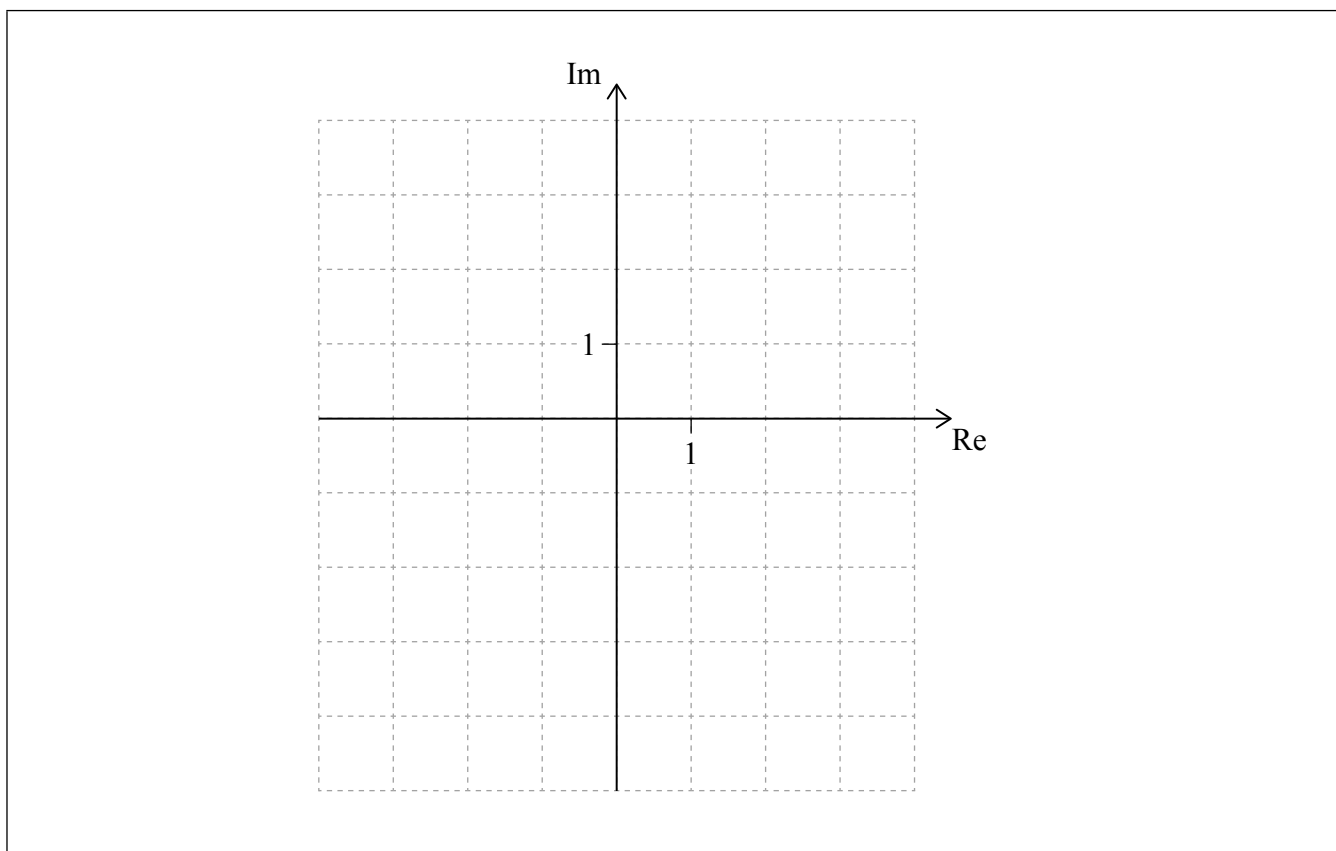
Turn over

9. [Maximum mark: 5]

Let $z = 2 - 3i$.

(a) Plot z on the Argand diagram.

[1]



z can be written in the form $r \operatorname{cis} \theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Find the value of

(i) r

(ii) θ .

[2]

(c) Find the value of zi in the form $a + bi$, where $a, b \in \mathbb{Z}$.

[1]

zi can be obtained from z by a geometric transformation.

(d) By plotting zi on the Argand diagram, or otherwise, describe fully this transformation.

[1]

(This question continues on the following page)



(Question 9 continued)

[illegible]

Turn over

11. [Maximum mark: 8]

Let $M = \begin{pmatrix} -4 & 2 \\ -3 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of M . [3]

M can be written in the form $M = PDP^{-1}$, where D is a diagonal matrix.

(b) (i) Write down D .
(ii) Find P . [5]

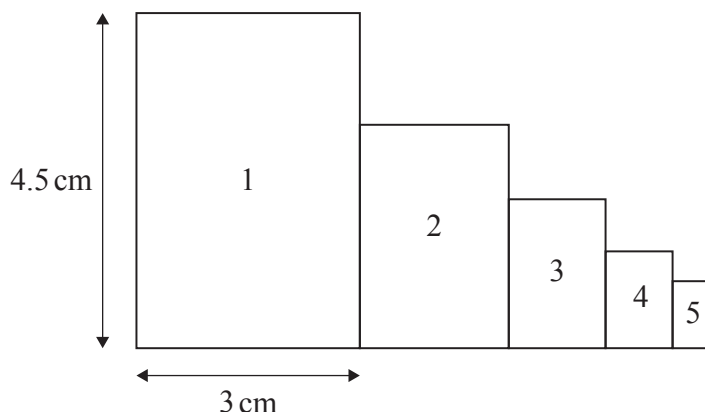


Turn over

3. [Maximum mark: 15]

Ayaka is creating a design made from a sequence of rectangles. The diagram shows part of her design, using 5 rectangles.

diagram not to scale



The first rectangle has the following dimensions: height 4.5 cm and width 3 cm.

The dimensions of each subsequent rectangle are $\frac{2}{3}$ of the dimensions of the previous rectangle.

- (a) Calculate the width of the 5th rectangle. [2]
- (b) Calculate the total width of the design that uses 5 rectangles. [2]

Ayaka continues to add rectangles to her design.

- (c) Find the smallest total width that her design cannot exceed. [3]

The width of Ayaka's final design must be at least 8.5 cm and use the least number of rectangles.

- (d) Find the total number of rectangles in her final design. [3]

The decreasing **areas** of the rectangles form a geometric sequence.

- (e) Find the common ratio for this sequence of areas. [2]
- (f) Find the total area of Ayaka's final design. [3]

Turn over

3. [Maximum mark: 8]

Raul wants to buy a new truck for 42 695 US dollars (USD). Raul is offered two different loan options to pay for the truck. For both options, Raul will pay back the loan over a period of 6 years.

The two options are shown in the following table.

Loan option	Size of loan (% purchase price)	Nominal annual interest rate (compounded monthly)
A	100 % of purchase price (after no initial payment)	7.15 %
B	80 % of purchase price (after a 20 % initial payment from Raul)	6.40 %

For both loan options, Raul will make monthly payments, at the end of each month, until the loan is completely paid off.

- (a) Calculate the amount of Raul’s monthly payment, to two decimal places, if he chooses

(i) option A.

(ii) option B.
- (b) State a reason why Raul might choose

(i) option A.

(ii) option B.

[6]

[2]

(This question continues on the following page)



(Question 3 continued)

[illegible]

Turn over

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Answers written on this page
will not be marked.



Consider the function $f(x) = \ln(2 - x) + \ln(-2 - x) - 2\ln(-x)$, where $x < -2$.

(a) Find an expression for $g(x)$. [2]

The function $f^{-1}(x)$ can be written in the form $\frac{a}{\sqrt{1-e^x}}$.

(c) Find the value of a . [5]

[illegible]

11. [Maximum mark: 8]

Nanthana is completing an exploration on multiplication using complex numbers. She considers a geometric sequence where $u_1 = 9$ and $r = \frac{2}{3} + \frac{2}{3}i$.

(a) Write down the value of

(i) u_2 .

(ii) u_3 . [2]

Nanthana claims that the sequence $|u_1|, |u_2|, |u_3|, \dots$ also forms a geometric sequence.

(b) Show that Nanthana’s claim is correct, stating the exact value of the common ratio for this sequence. [4]

(c) Hence, find the sum of the infinite sequence $|u_1|, |u_2|, |u_3|, \dots$. [2]



Turn over

14. [Maximum mark: 6]

Consider the following functions:

$$L_1(x) = 14.8 \sin(0.35x + 1.27)$$

$$L_2(x) = 18.5 \sin(0.35x - 2.36)$$

- (a) Write down both functions in the form $\operatorname{Im}(ae^{bxi} e^{ci})$. [2]

The sum of $L_1(x)$ and $L_2(x)$ can be expressed in the form $L_3(x) = q \sin(0.35x + r)$, where $0 \leq r < 2\pi$.

- (b) Use your answer to part (a) to find the value of q and of r . [4]



Turn over

1. [Maximum mark: 7]

Give answers to this question correct to two decimal places.

Pierre invests 1500 euros (EUR) at the end of each month for 10 years into a savings plan that pays a nominal annual interest rate of 3.6% compounded monthly.

(a) Calculate the value of Pierre's savings plan at the end of the 10 years. [3]

At the end of the 10 years, Pierre withdraws 100 000 EUR from the savings plan to use as a deposit on a house.

Pierre invests the remainder into another account for 15 years at a nominal annual interest rate of 4.5% compounded quarterly.

(b) Calculate the amount in Pierre's account at the end of this time. [4]

(This question continues on the following page)



(Question 1 continued)

[illegible]

Turn over

5. [Maximum mark: 6]

A speed camera is used to determine whether a car is exceeding a speed limit of 8.3 m s^{-1} .

An exact distance of 10m is marked out.

The car travels this 10m distance in 1.2 seconds, measured to the nearest 0.1 second.

Determine whether it is certain that the car was exceeding the speed limit of 8.3 m s^{-1} .

Justify your answer.

[illegible]

16. [Maximum mark: 7]

An electrical engineer models a circuit using the equation

$$z^2 + 2tz + 8t = 0$$

where t is the time in seconds and $0 \leq t \leq 2$.

- (a) When $t = 1$, find the value of z which satisfies $\frac{\pi}{2} < \arg z < \pi$. Give your answer in the form $a + bi$.

[2]

The power in the circuit is given by $|z|^2$.

- (b) Find the value of t in the interval $0 \leq t \leq 2$ for which the power is maximized.

[5]

[illegible]

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will not be marked.



2. [Maximum mark: 5]

The loudness of a sound, L , measured in decibels (dB) is determined by the intensity of the sound, I , measured in watts per square metre (Wm^{-2}). The relationship between loudness and intensity can be expressed using the logarithmic function

$$L = 10 \log_{10} \left(\frac{I}{I_0} \right), \quad I > 0$$

where I_0 is the reference intensity (the intensity of the least audible sound to the human ear).

The reference intensity I_0 is 10^{-12}Wm^{-2} .

The intensity of sound on a busy street is 10^{-5}Wm^{-2} .

- (a) Calculate the loudness of the sound.
- [2]

The sound of a jet engine reaches a loudness of 185 dB.

- (b) Determine the intensity of its sound. Give your answer in the form $a \times 10^k$ where $1 \leq a < 10, k \in \mathbb{Z}$.
- [3]

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Turn over

5. [Maximum mark: 8]

Sofia takes out a loan for 50 000 euros. She pays a nominal interest rate of 4.2 % per year, compounded monthly. She must pay back the loan in 25 years through regular monthly payments, made at the end of each month.

- (a) Find the amount Sofia must pay back each month. Give your answer correct to two decimal places.

[2]

At the end of eight years, Sofia wins the lottery and pays back the remainder of the loan early in one single payment.

- (b) (i) Find how much Sofia pays back in this single payment. Give your answer correct to two decimal places.

- (ii) Find the amount of money saved by Sofia by paying back the loan early.

[6]

[illegible]

Turn over

1. [Maximum mark: 16]

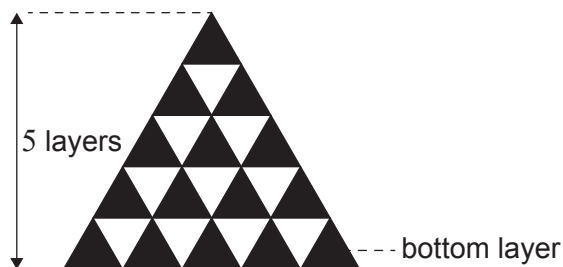
Thai cushions are designed with a triangular cross-section and are made from layers of smaller cushions. These cushions can be modelled as triangular prisms.

This is shown in the diagram.

diagram not to scale



Thai cushion with 4 layers



Cross-section of Thai cushion with 5 layers

(a) Write down the number of triangular prisms in the bottom layer of the cushion with

(i) 4 layers.

(ii) 5 layers.

[2]

Mayumi notices that the number of triangular prisms in the bottom layer of the cushions forms an arithmetic sequence.

(b) (i) Write down the common difference of this sequence.

(ii) Find an expression for the number of triangular prisms in the bottom layer of a cushion with n layers.

[3]

(This question continues on the following page)

(Question 1 continued)

Mayumi wants to extend this design to create a cushion with 9 layers.

- (c) (i) Find the number of triangular prisms in the bottom layer of Mayumi's cushion. [3]
- (ii) Calculate the **total** number of triangular prisms in Mayumi's cushion. [3]
- (d) Find an expression for the **total** number of triangular prisms in a cushion with n layers, giving your answer in its simplest form. [2]

The cross-section of the cushion consists of black triangles and white triangles.



This cushion with 4 layers has a total of 6 white triangles.



This cushion with 5 layers has 4 white triangles in its bottom layer.

- (e) Write down the total number of black triangles in a cushion with 4 layers. [1]

The number of black triangles in each layer forms an arithmetic sequence.

- (f) Find and simplify an expression for the total number of black triangles in a cushion with n layers. [2]

The total number of white triangles in a cushion with n layers is $\frac{n(n-1)}{2}$.

- (g) Using both the given expression and your answer to part (f), find and simplify an expression for the total number of black and white triangles in a cushion with n layers. [3]

Turn over

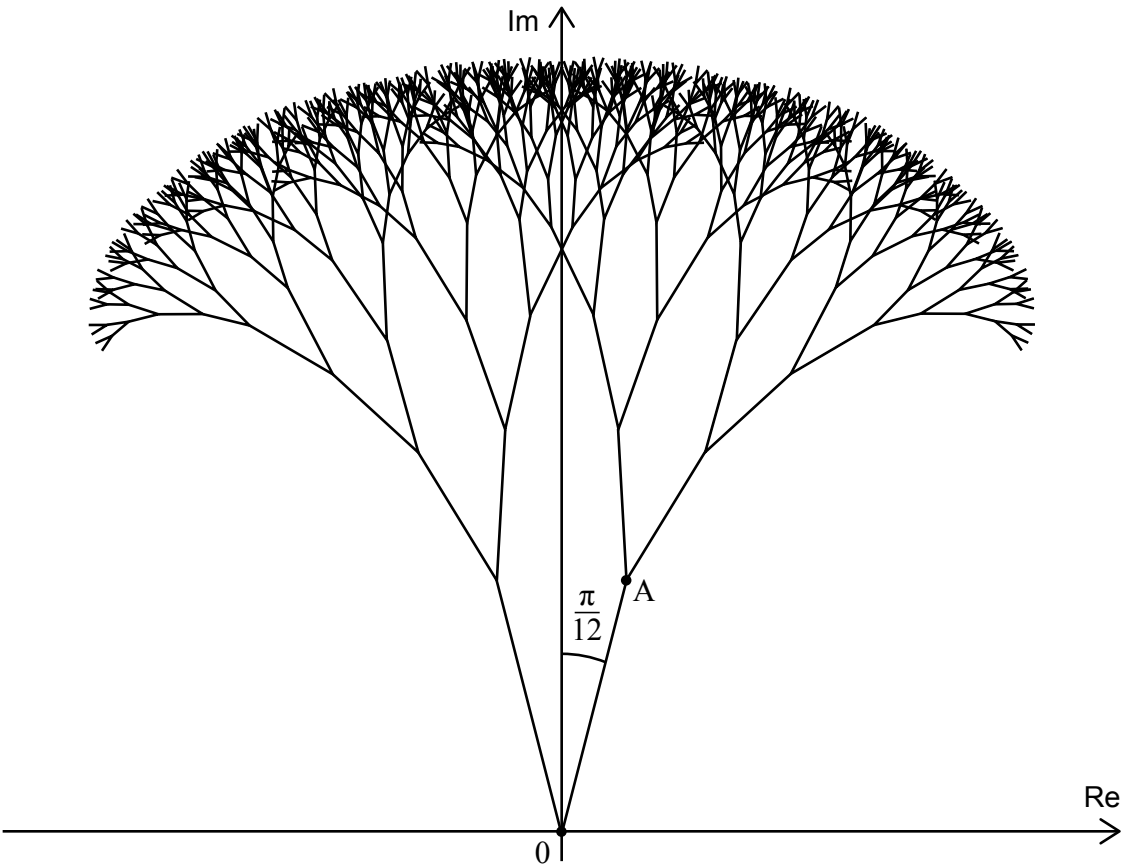
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7. [Maximum mark: 17]

A fractal tree is shown on the following Argand diagram.

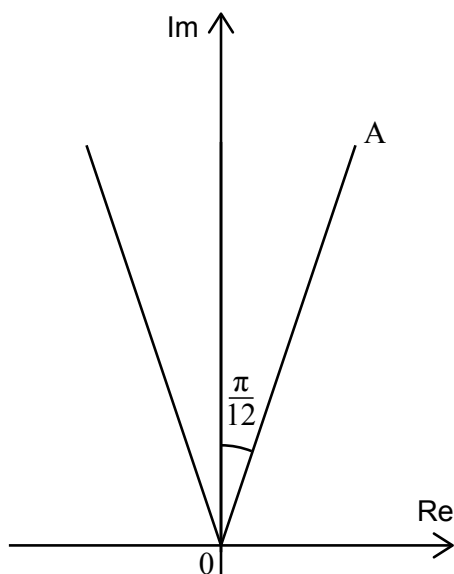


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(Question 7 continued)

The tree is formed in the following manner.

The line segment joining 0 to $5i$ is rotated clockwise by $\frac{\pi}{12}$ about 0 to form the first branch and then anti-clockwise by $\frac{\pi}{12}$ about 0 to form the second branch. The end of one of these branches is shown as A in **Diagram 1**.

Diagram 1

- (a) Find the complex number represented by the point A, giving your answer in exponential form.

[3]

(This question continues on the following page)

Turn over

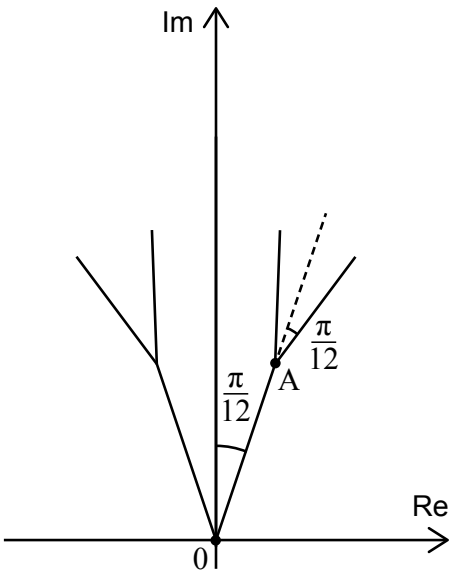
(Question 7 continued)

Subsequent branches are created by adding two branches to the end of each branch created in the previous stage.

Each new branch is 70% the length of the previous one. One new branch is rotated $\frac{\pi}{12}$

clockwise from the direction of the previous branch and the other is rotated anticlockwise. This is shown in **Diagram 2**.

Diagram 2

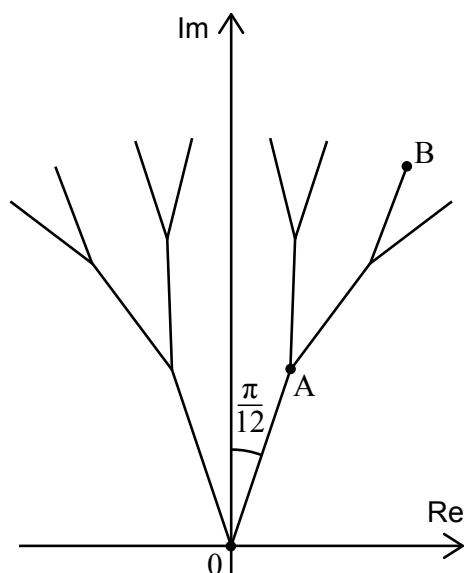
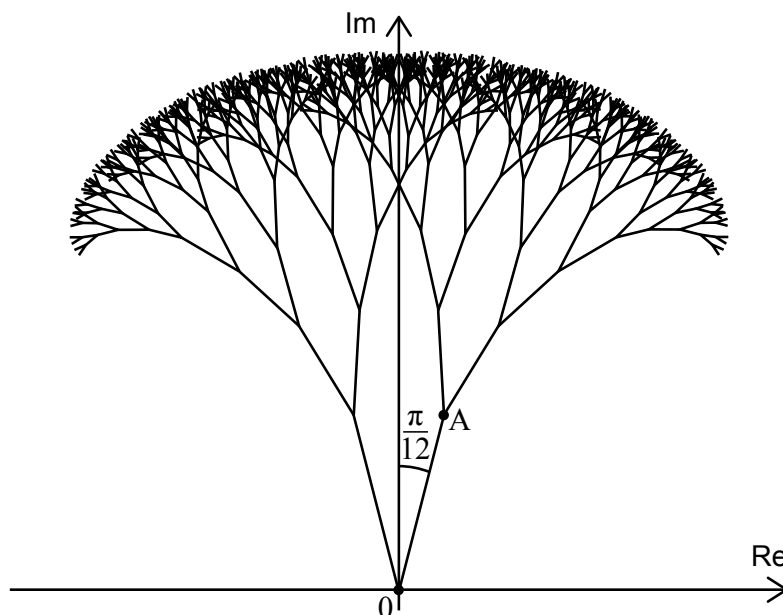


This process is continued using the end of all the branches formed in the previous stage as the centres of rotation for the next stage.

(This question continues on the following page)

(Question 7 continued)

The trees after the 3rd and 10th stage are shown in **Diagrams 3** and **4** respectively.

Diagram 3**Diagram 4**

- (b) (i) Show that the total length of all branches in the tree after the second stage (**Diagram 2**) is equal to 24.
- (ii) Find the total length of all branches after 10 stages (**Diagram 4**).
- (iii) If this process continues indefinitely, state whether there is a limit to the total length of all the branches. Justify your answer.

[9]

Let b be the complex number represented by the point B in **Diagram 3**.

- (c) (i) Find an expression for b as the sum of three complex numbers, each written in exponential form.
- (ii) Hence write down b in the form $a + bi$, $a, b \in \mathbb{R}$.

[5]

2. [Maximum mark: 5]

A bakery sells egg tarts in boxes. Each box contains 6 tarts. On a particular morning, the bakery has 400 freshly baked tarts. The number of tarts **remaining**, after n boxes have been sold, forms an arithmetic sequence, $u_n = u_1 + (n - 1)d$.

- (a) Write down the value of d . [1]
- (b) Find the value of u_1 . [1]
- (c) Find the greatest number of boxes the bakery can sell that morning. [3]



Turn over

Two complex numbers are $z = 2 + ai$ and $w = b + 4i$, where $a, b \in \mathbb{R}$.

- (a) Find an expression for the real part of z^2 , in terms of a . [2]
- (b) Find the value of a and of b , given that $z^2 + 2w = 3 + 6i$. [3]

[illegible]

Turn over

7. [Maximum mark: 18]

The Fibonacci sequence has the initial values 0, 1, 1, 2, 3, 5, 8, ...

To find the next number in the sequence, the previous two numbers are added. This can be written as $f_{n+1} = f_n + f_{n-1}$, together with $f_1 = 0$ and $f_2 = 1$.

Consecutive pairs in the sequence can be written as a matrix, $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$, and used to produce a new sequence of matrices:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \dots$$

(a) Write down the next two terms in this sequence of matrices. [2]

(b) Show that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$. [2]

(c) Show that the characteristic equation of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - \lambda - 1 = 0$. [2]

The eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ are λ_1 and λ_2 , and a corresponding set of eigenvectors is $\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$.

(d) Using part (c), and without finding the value of λ_1 , verify that $\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ satisfies the equation for an eigenvector,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}. \quad [2]$$

(This question continues on the following page)

(Question 7 continued)

It is given that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}$ can be written in the form $\mathbf{PD}^{n-1}\mathbf{P}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

(e) Write down, in terms of λ_1 and λ_2 ,

(i) $\mathbf{D}$;

(ii) $\mathbf{P}$;

(iii) $\mathbf{P}^{-1}$.

[3]

(f) Hence, by multiplying the matrices $\mathbf{PD}^{n-1}\mathbf{P}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, show that

$$f_n = \frac{1}{\lambda_1 - \lambda_2} \left((\lambda_1)^{n-1} - (\lambda_2)^{n-1} \right).$$

[4]

(g) (i) Use part (c) to find the exact value of λ_1 and λ_2 .

(ii) Hence, write down an expression for f_n in terms of n .

[3]