Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2021

Mathematics: analysis and approaches

Higher level

Paper 2

22 pages

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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) $a = 0.433156...$, $b = 4.50265...$

$a = 0.433$, $b = 4.50$ ***A1A1***

[2 marks]

(b) attempt to substitute $x = 18$ into their equation ***(M1)***

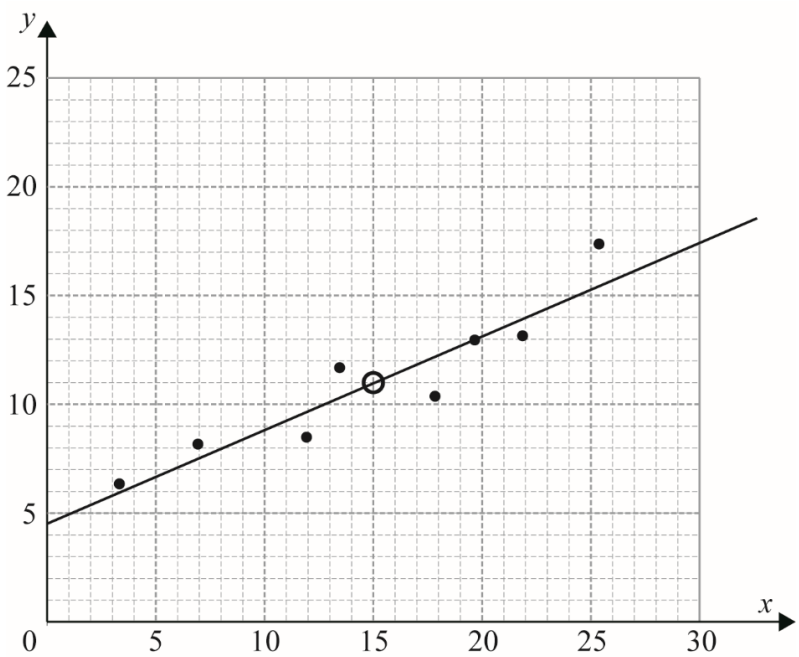
$y = 0.433 \times 18 + 4.50$
$= 12.2994...$
$= 12.3$ ***A1***

[2 marks]

(c) $\bar{x} = 15$, $\bar{y} = 11$ ***A1***

[1 mark]

(d)



A1A1

Note: Award marks as follows:

A1 for a straight line going through $(15, 11)$

A1 for intercepting the y-axis between their $b \pm 1.5$ (when their line is extended), which includes all the data for $3.3 \le x \le 25.3$.

If the candidate does not use a ruler, award ***A0A1*** where appropriate.

[2 marks]
Total [7 marks]

2.

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let X = mass of a bag of sugar

(a) evidence of identifying the correct area ***(M1)***

$\mathrm{P}(X < 995) = 0.0765637...$

$= 0.0766$ ***A1***

[2 marks]

(b) 0.0766×100

≈ 8 ***A1***

[1 mark]

Note: Accept 7.66 .

(c) recognition that $\mathrm{P}\left(X > 1005 | X \geq 995\right)$ is required ***(M1)***

$$\frac{\mathrm{P}\left(X \geq 995 \cap X > 1005\right)}{\mathrm{P}\left(X \geq 995\right)}$$

$$\frac{\mathrm{P}(X > 1005)}{\mathrm{P}(X \geq 995)}$$ ***(A1)***

$$\frac{0.07656\ldots}{1 - 0.07656\ldots}\left(= \frac{0.07656\ldots}{0.9234\ldots}\right)$$

$= 0.0829$ ***A1***

[3 marks]

Total [6 marks]

3. Amplitude is $\frac{110}{2} = 55$ ***(A1)***

$a = -55$ ***A1***

$c = 65$ ***A1***

$\frac{2\pi}{b} = 20$ OR $-55\cos(20b) + 65 = 10$ ***(M1)***

$b = \frac{\pi}{10}(= 0.314)$ ***A1***

Total [5 marks]

4. (a) recognising $v = 0$ ***(M1)***

$t = 6.74416...$

$= 6.74$ (sec) ***A1***

Note: Do not award ***A1*** if additional values are given.

[2 marks]

(b) $\int_0^{10} |v(t)| \mathrm{d}t$ **OR** $-\int_0^{6.74416...} v(t)\,\mathrm{d}t + \int_{6.74416...}^{9.08837...} v(t)\,\mathrm{d}t - \int_{9.08837...}^{10} v(t)\,\mathrm{d}t$ ***(A1)***

$= 37.0968...$

$= 37.1$ (m) ***A1***

[2 marks]

(c) recognising acceleration at $t = 7$ is given by $v'(7)$ ***(M1)***

acceleration $= 5.93430...$

$= 5.93$ (ms^{-2}) ***A1***

[2 marks]

Total [6 marks]

5. METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen ***(M1)***
evidence of correct term chosen ***(A1)***

$^{n+1}C_2 \times 3^{n+1-2} \times \left(x^2\right)^2 \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right)$ OR $n - r = 1$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

EITHER

$^{n+1}C_2 \times 3^{n-1} = 20412$ ***(A1)***

OR

$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$ ***(A1)***

THEN

$n = 7$ ***A1***

METHOD 2

$3^{n+1}\left(1 + \frac{x^2}{3}\right)^{n+1}$

product of a binomial coefficient and a power of $\frac{x^2}{3}$ OR $\frac{1}{3}$ seen ***(M1)***
evidence of correct term chosen ***(A1)***

$3^{n+1} \times {}^{n+1}C_2 \times \left(\frac{x^2}{3}\right)^2 \left(= 3^{n-1}\frac{n(n+1)}{2}x^4\right)$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

$3^{n-1} \times \frac{n(n+1)}{2} = 20412$ ***(A1)***

$n = 7$ ***A1***

Total [5 marks]

6 (a) attempt to find a vector perpendicular to Π_1 and Π_2 using a cross product ***(M1)***

$$\begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}=\left(2-(-2)\right)\boldsymbol{i}+(1-3)\boldsymbol{j}+(-6-2)\boldsymbol{k}$$

$$=\begin{pmatrix}4\\-2\\-8\end{pmatrix}\left(=2\begin{pmatrix}2\\-1\\-4\end{pmatrix}\right)$$ ***(A1)***

equation is $4x-2y-8z=0\left(\Rightarrow 2x-y-4z=0\right)$ ***A1***

[3 marks]

(b) attempt to solve 3 simultaneous equations in 3 variables ***(M1)***

$$\left(\frac{41}{21},-\frac{10}{21},\frac{23}{21}\right)\left(=(1.95,-0.476,1.10)\right)$$ ***A1***

[2 marks]
Total [5 marks]

7. (a) recognition of the need to integrate $\dfrac{x}{\sqrt{\left(x^2+k\right)^3}}$ ***(M1)***

$$\int\frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x\,(=1)$$

EITHER

$u=x^2+k\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=2x$ (or equivalent) ***(A1)***

$$\int\frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x=\frac{1}{2}\int u^{-\frac{3}{2}}\mathrm{d}u$$

$$=-u^{-\frac{1}{2}}(+c)\left(=-\left(x^2+k\right)^{-\frac{1}{2}}(+c)\right)$$ ***A1***

continued…

Question 7 continued

OR

$$\int \frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x$$ ***(A1)***

$$= -\left(x^2+k\right)^{-\frac{1}{2}}(+c)$$ ***A1***

THEN

attempt to use correct limits for their integrand and set equal to 1 ***M1***

$$\left[-u^{-\frac{1}{2}}\right]_k^{16+k} = 1 \text{ OR } \left[-\left(x^2+k\right)^{-\frac{1}{2}}\right]_0^4 = 1$$

$$-(16+k)^{-\frac{1}{2}} + k^{-\frac{1}{2}} = 1\left(\Rightarrow \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1\right)$$ ***A1***

$$\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$$ ***AG***

[5 marks]

(b) attempt to solve $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***(M1)***

$k = 0.645038...$

$= 0.645$ ***A1***

[2 marks]

Total [7 marks]

8. (a) $\left(|zw| = \right)16$ ***A1***

[1 mark]

(b) attempt to find $\arg(z) + \arg(w)$ ***(M1)***

$\arg(zw) = \arg(z) + \arg(w)$

$$= \frac{\pi}{5} - \frac{2k\pi}{5}\left(= \frac{(1-2k)\pi}{5}\right)$$ ***A1***

[2 marks]

continued…

Question 8 continued

(c) (i) $zw \in \mathrm{Z} \Rightarrow \arg(zw)$ is a multiple of π ***(M1)***

$\Rightarrow 1-2k$ is a multiple of 5 ***(M1)***

$k=3$ ***A1***

(ii) $zw = 16(\cos(-\pi)+\mathrm{i}\sin(-\pi))$

-16 ***A1***

[4 marks]
Total [7 marks]

9. (a) $\tan\theta = \dfrac{50}{y-x}$ OR $\cot\theta = \dfrac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to *t* ***(M1)***

$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

attempt to set speed of B equal to double the speed of A (***M1)***

$2\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}t} - 50(\operatorname{cosec}\theta)^2\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -50(\operatorname{cosec}\theta)^2\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

$\theta = \arctan 5 (= 1.373\ldots = 78.69\ldots°)$ OR $\operatorname{cosec}^2\theta = 1+\cot^2\theta = 1+\left(\dfrac{1}{5}\right)^2 = \dfrac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\dfrac{26}{25}\right)\times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]
Total [7 marks]

Section B.

10. (a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $(2\ln 45, 2)$ (accept $(-2\ln\frac{1}{45}, 2)$) ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -(x - 2\ln 45)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

continued…

Question 10 continued

(d) (i) $x = \ln 45 + 1 \left(= 4.81\right)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$$\int_{4.806...}^{5.566...} \left(x - \left(-x + 2\ln 45 + 2\right)\right) \mathrm{d}x + \int_{5.566...}^{7.613...} \left(90\mathrm{e}^{-0.5x} - \left(-x + 2\ln 45 + 2\right)\right) \mathrm{d}x$$ ***(A1)(A1)***

Note: Award ***A1*** for one correct integral expression including correct limits and integrand.
Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]
Total [15 marks]

11. (a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ ***(M1)***

$p = \frac{1}{2}, q = -\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3(4x^2 - 1) - 8x(3x + 2)}{(4x^2 - 1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2 - 1)^2} \right)$$ ***A1A1***

Note: Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x + 2)(4x^2 - 1)^{-2} + 3(4x^2 - 1)^{-1}$$ ***A1A1***

Note: Award ***A1*** for each term.

[3 marks]

(c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ ***(M1)***

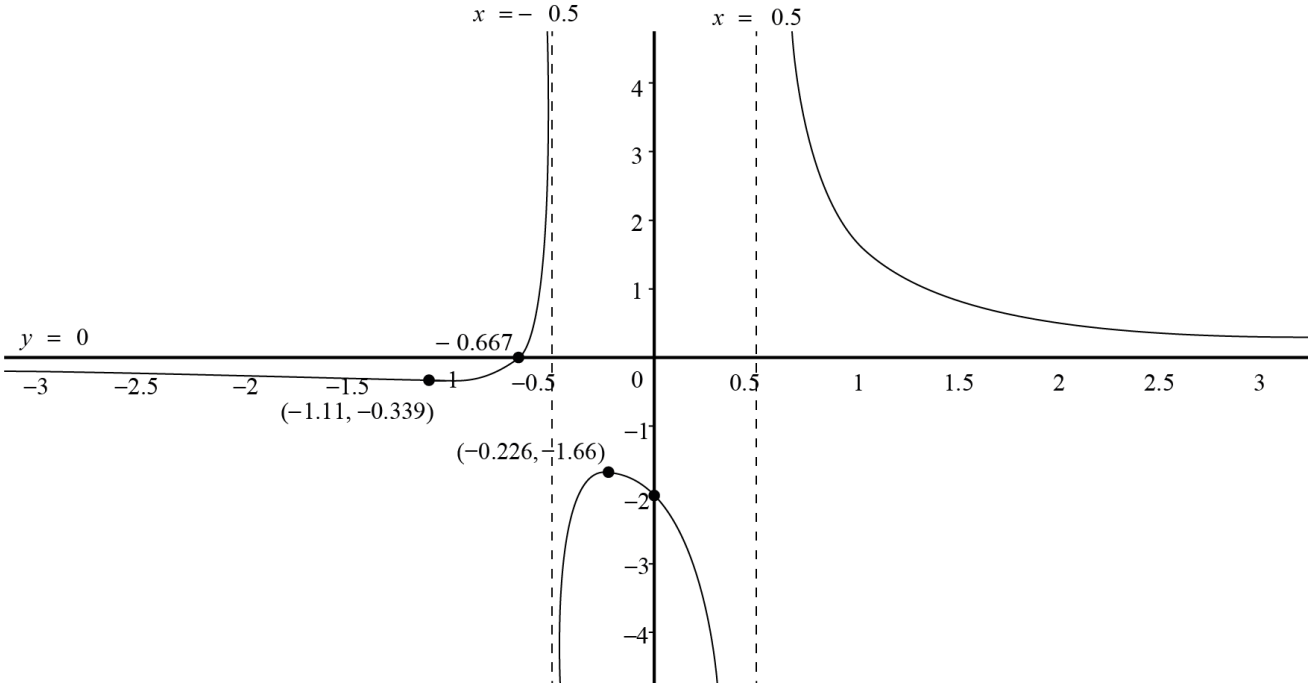
$x = -1.60$ ***A1***

[2 marks]

continued…

Question 11 continued

(d)



A1A1A1A1A1

Note:	Award ***A1*** for both vertical asymptotes with their equations, award ***A1*** for horizontal asymptote with equation, award ***A1*** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts. If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum ***A0A1A0A1A1.***

[5 marks]

continued…

Question 11 continued

e) $x = -\frac{2}{3}\left(= -0.667\right)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3}\left(= 1.33\right)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x+2\overline{\big)\,4x^2 + 0x - 1} \\ \underline{4x^2 + \frac{8}{3}x} \\ -\frac{8}{3}x - 1 \\ \underline{-\frac{8}{3}x - \frac{16}{9}} \\ \frac{7}{9} \end{array}$$

$y = \frac{4}{3}x - \frac{8}{9}\left(= 1.33x - 0.889\right)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $\left(x = -0.568729..., x = 1.31872...\right)$ ***(M1)***

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ ***A1A1A1***

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]
Total [20 marks]

12. (a) $\dfrac{1}{x(k-x)} \equiv \dfrac{a}{x} + \dfrac{b}{k-x}$

$a(k-x) + bx = 1$ ***(A1)***

attempt to compare coefficients OR substitute $x = k$ and $x = 0$ and solve ***(M1)***

$a = \dfrac{1}{k}$ and $b = \dfrac{1}{k}$ ***A1***

$f'(x) = \dfrac{1}{kx} + \dfrac{1}{k(k-x)}$

[3 marks]

(b) attempt to integrate their $\dfrac{a}{x} + \dfrac{b}{k-x}$ ***(M1)***

$f(x) = \dfrac{1}{k}\int\left(\dfrac{1}{x} + \dfrac{1}{k-x}\right)\mathrm{d}x$

$= \dfrac{1}{k}\left(\ln|x| - \ln|k-x|\right)(+c)\left(= \dfrac{1}{k}\ln\left|\dfrac{x}{k-x}\right|(+c)\right)$ ***A1A1***

[3 marks]

Note: Award ***A1*** for each correct term. Award ***A1A0*** for a correct answer without modulus signs. Condone the absence of $+c$.

continued…

Question 12 continued

(c) attempt to separate variables and integrate both sides ***M1***

$$5k\int \frac{1}{P(k-P)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$$

$$5\left(\ln P - \ln(k-P)\right) = t + c$$ ***A1***

Note: There are variations on this which should be accepted, such as $\frac{1}{k}\left(\ln P - \ln(k-P)\right) = \frac{1}{5k}t + c$. Subsequent marks for these variations should be awarded as appropriate.

EITHER

attempt to substitute $t = 0$, $P = 1200$ into an equation involving c ***M1***

$$c = 5\left(\ln 1200 - \ln(k-1200)\right)\left(= 5\ln\left(\frac{1200}{k-1200}\right)\right)$$ ***A1***

$$5\left(\ln P - \ln(k-P)\right) = t + 5\left(\ln 1200 - \ln(k-1200)\right)$$ ***A1***

$$\ln\left(\frac{P(k-1200)}{1200(k-P)}\right) = \frac{t}{5}$$

$$\frac{P(k-1200)}{1200(k-P)} = \mathrm{e}^{\frac{t}{5}}$$ ***A1***

OR

$$\ln\left(\frac{P}{k-P}\right) = \frac{t+c}{5}$$

$$\frac{P}{k-P} = A\mathrm{e}^{\frac{t}{5}}$$ ***A1***

attempt to substitute $t = 0$, $P = 1200$ ***M1***

$$\frac{1200}{k-1200} = A$$ ***A1***

$$\frac{P}{k-P} = \frac{1200\mathrm{e}^{\frac{t}{5}}}{k-1200}$$ ***A1***

continued…

Question 12 continued

THEN

attempt to rearrange and isolate P ***M1***

$Pk - 1200P = 1200k\mathrm{e}^{\frac{t}{5}} - 1200P\mathrm{e}^{\frac{t}{5}}$ OR $Pk\mathrm{e}^{-\frac{t}{5}} - 1200P\mathrm{e}^{-\frac{t}{5}} = 1200k - 1200P$

OR $\dfrac{k}{P} - 1 = \dfrac{k - 1200}{1200\mathrm{e}^{\frac{t}{5}}}$

$P\left(k - 1200 + 1200\mathrm{e}^{\frac{t}{5}}\right) = 1200k\mathrm{e}^{\frac{t}{5}}$ OR $P\left(k\mathrm{e}^{-\frac{t}{5}} - 1200\mathrm{e}^{-\frac{t}{5}} + 1200\right) = 1200k$ ***A1***

$P = \dfrac{1200k}{(k - 1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$ ***AG***

[8 marks]

(d) attempt to substitute $t = 10$, $P = 2400$ ***(M1)***

$2400 = \dfrac{1200k}{(k - 1200)\mathrm{e}^{-2} + 1200}$ ***(A1)***

$k = 2845.34...$

$k = 2845$ ***A1***

Note: Award ***(M1)(A1)A0*** for any other value of k which rounds to 2850

[3 marks]

(e) attempt to find the maximum of the first derivative graph OR zero of the second derivative graph OR that $P = \dfrac{k}{2}\left(= 1422.67...\right)$ ***(M1)***

$t = 1.57814...$

$= 1.58$ (days) ***A2***

Note: Accept any value which rounds to 1.6.

[3 marks]

Total [20 marks]