

APOLLO

Number and algebra

HL

Mathematics: analysis and approaches

First assessment 2021

102 questions | questions

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8. [Maximum mark: 7]

(a) Find the modulus of zw . [1]

(b) Find the argument of zw in terms of k . [2]

(c) (i) Find the minimum value of k .

(ii) For the value of k found in part (i), find the value of zw . [4]

[illegible]

7. [Maximum mark: 5]

Find the total number of possible ways in which the eight runners can finish if Jack finishes

- in the position immediately after Andrea; [2]
- in any position after Andrea. [3]

[illegible]

2. [Maximum mark: 24]

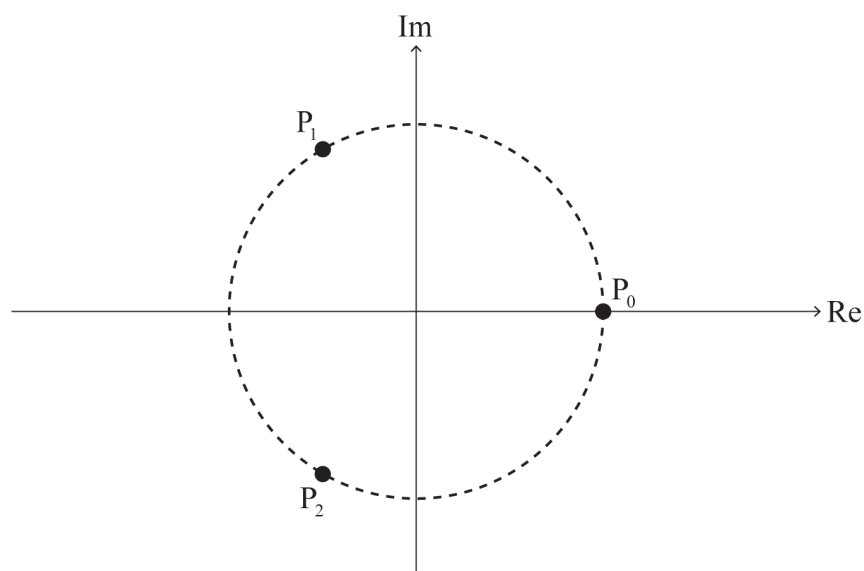
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \dots, \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$. Each root can be represented by a point $P_0, P_1, P_2, \dots, P_{n-1}$, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω . On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1 .

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are $1, \omega$ and ω^2 . On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $O(0, 0)$.



(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$.

[2]

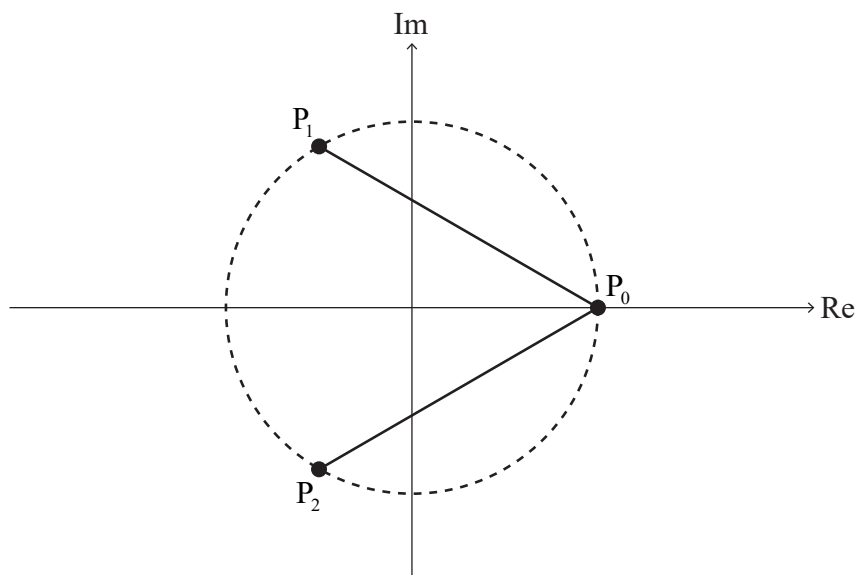
(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$.

[2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1 , ω , ω^2 and ω^3 .

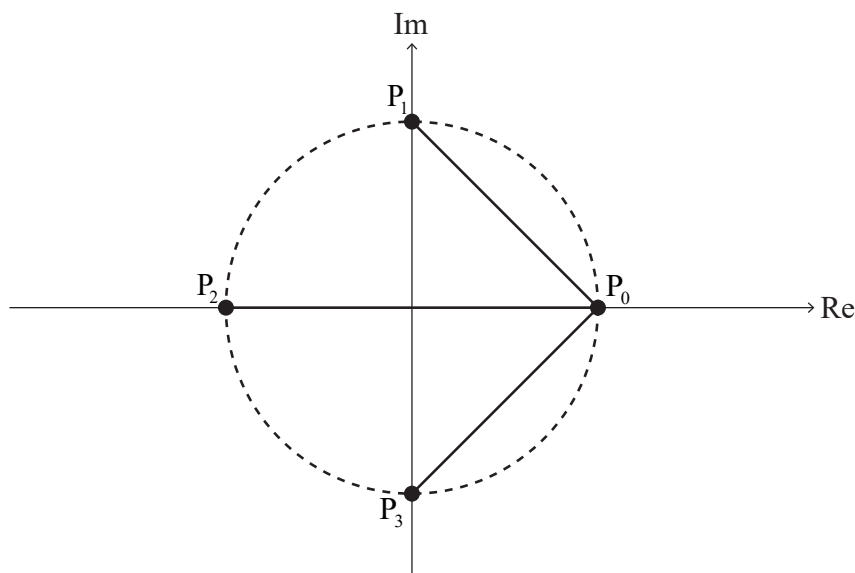
(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

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(Question 2 continued)

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $O(0, 0)$. $[P_0P_1]$, $[P_0P_2]$ and $[P_0P_3]$ are line segments.



- (d) Show that $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots $1, \omega, \omega^2, \omega^3$ and ω^4 .

It can be shown that $P_0P_1 \times P_0P_2 \times P_0P_3 \times P_0P_4 = 5$.

Now consider the general case for integer values of n , where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are $1, \omega, \omega^2, \dots, \omega^{n-1}$. On an Argand diagram, these roots can be represented by the points $P_0, P_1, P_2, \dots, P_{n-1}$ respectively where $[P_0P_1], [P_0P_2], \dots, [P_0P_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $O(0, 0)$.

- (e) Suggest a value for $P_0P_1 \times P_0P_2 \times \dots \times P_0P_{n-1}$. [1]

P_0P_1 can be expressed as $|1 - \omega|$.

- (f) (i) Write down expressions for P_0P_2 and P_0P_3 in terms of ω . [2]
 (ii) Hence, write down an expression for P_0P_{n-1} in terms of n and ω . [1]

Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \dots + z + 1)$ where $z \in \mathbb{C}$.

- (g) (i) Express $z^{n-1} + z^{n-2} + \dots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]
 (ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

References:

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 22]

Consider the equation $(z - 1)^3 = i$, $z \in \mathbb{C}$. The roots of this equation are ω_1 , ω_2 and ω_3 , where $\text{Im}(\omega_2) > 0$ and $\text{Im}(\omega_3) < 0$.

- (a) (i) Verify that $\omega_1 = 1 + e^{i\frac{\pi}{6}}$ is a root of this equation.
- (ii) Find ω_2 and ω_3 , expressing these in the form $a + e^{i\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots ω_1 , ω_2 and ω_3 are represented by the points A, B and C respectively on an Argand diagram.

- (b) Plot the points A, B and C on an Argand diagram. [4]
- (c) Find AC. [3]

Consider the equation $(z - 1)^3 = iz^3$, $z \in \mathbb{C}$.

- (d) By using de Moivre's theorem, show that $\alpha = \frac{1}{1 - e^{i\frac{\pi}{6}}}$ is a root of this equation. [3]
- (e) Determine the value of $\text{Re}(\alpha)$. [6]

References:

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^n \frac{2}{3} \left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1 . [2]

(b) Find S_∞ . [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]



6. [Maximum mark: 8]

The equation $2x^2 - 5x + 1 = 0$ has two real roots, α and β .

(b) Without solving $2x^2 - 5x + 1 = 0$, determine the values of m and n . [6]

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1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z) = \frac{e^z + e^{-z}}{2}$ and $g(z) = \frac{e^z - e^{-z}}{2}$, where $z \in \mathbb{C}$.

Consider t and u , such that $t, u \in \mathbb{R}$.

(a) Verify that $u = f(t)$ satisfies the differential equation $\frac{d^2u}{dt^2} = u$. [2]

(b) Show that $(f(t))^2 + (g(t))^2 = f(2t)$. [3]

(c) Using $e^{iu} = \cos u + i \sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

(i) $f(iu)$; [3]

(ii) $g(iu)$. [2]

(d) Hence find, and simplify, an expression for $(f(iu))^2 + (g(iu))^2$. [2]

(e) Show that $(f(t))^2 - (g(t))^2 = (f(iu))^2 - (g(iu))^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos \theta, \sin \theta)$ defines points on the unit circle with equation $x^2 + y^2 = 1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $(f(\theta), g(\theta))$ defines points on a curve known as a hyperbola with equation $x^2 - y^2 = 1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2 - y^2 = 1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2 - y^2 = 1$ can be rotated to coincide with the curve defined by $xy = k$, $k \in \mathbb{R}$.

(g) Find the possible values of k . [5]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 18]

- (a) Express $-3 + \sqrt{3}i$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [5]

Let the roots of the equation $z^3 = -3 + \sqrt{3}i$ be u , v and w .

- (b) Find u , v and w expressing your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [5]

On an Argand diagram, u , v and w are represented by the points U, V and W respectively.

- (c) Find the area of triangle UVW. [4]

- (d) By considering the sum of the roots u , v and w , show that

$$\cos \frac{5\pi}{18} + \cos \frac{7\pi}{18} + \cos \frac{17\pi}{18} = 0. \quad [4]$$

Consider the expansion of $\left(8x^3 - \frac{1}{2x}\right)^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the series $\ln x + p \ln x + \frac{1}{3} \ln x + \dots$, where $x \in \mathbb{R}, x > 1$ and $p \in \mathbb{R}, p \neq 0$.

(a) Consider the case where the series is geometric.

(i) Show that $p = \pm \frac{1}{\sqrt{3}}$.

(ii) Hence or otherwise, show that the series is convergent.

(iii) Given that $p > 0$ and $S_{\infty} = 3 + \sqrt{3}$, find the value of x . [6]

(b) Now consider the case where the series is arithmetic with common difference d .

(i) Show that $p = \frac{2}{3}$.

(ii) Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$.

(iii) The sum of the first n terms of the series is $\ln\left(\frac{1}{x^3}\right)$.

Find the value of n . [12]

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The n^{th} term of an arithmetic sequence is given by $u_n = 15 - 3n$.

- (a) State the value of the first term, u_1 . [1]
- (b) Given that the n^{th} term of this sequence is -33 , find the value of n . [2]
- (c) Find the common difference, d . [2]



(b) Prove that the sum of the squares of these three integers is never divisible by 3. [4]

[illegible]

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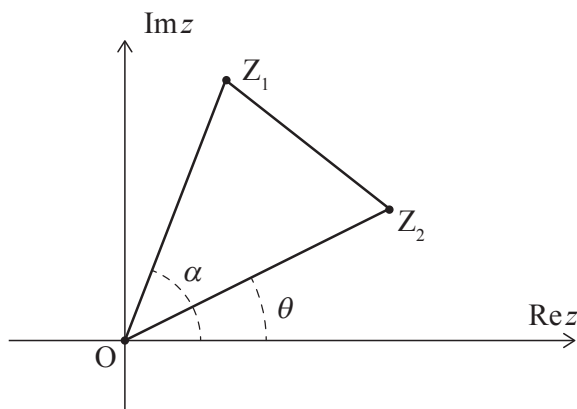
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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 18]

In the following Argand diagram, the points Z_1 , O and Z_2 are the vertices of triangle Z_1OZ_2 described anticlockwise.



The point Z_1 represents the complex number $z_1 = r_1 e^{i\alpha}$, where $r_1 > 0$. The point Z_2 represents the complex number $z_2 = r_2 e^{i\theta}$, where $r_2 > 0$.

Angles α , θ are measured anticlockwise from the positive direction of the real axis such that $0 \leq \alpha$, $\theta < 2\pi$ and $0 < \alpha - \theta < \pi$.

(a) Show that $z_1 z_2^* = r_1 r_2 e^{i(\alpha - \theta)}$ where z_2^* is the complex conjugate of z_2 . [2]

(b) Given that $\operatorname{Re}(z_1 z_2^*) = 0$, show that Z_1OZ_2 is a right-angled triangle. [2]

In parts (c), (d) and (e), consider the case where Z_1OZ_2 is an equilateral triangle.

(c) (i) Express z_1 in terms of z_2 .

(ii) Hence show that $z_1^2 + z_2^2 = z_1 z_2$. [6]

Let z_1 and z_2 be the distinct roots of the equation $z^2 + az + b = 0$ where $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$.

(d) Use the result from part (c)(ii) to show that $a^2 - 3b = 0$. [5]

Consider the equation $z^2 + az + 12 = 0$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

(e) Given that $0 < \alpha - \theta < \pi$, deduce that only one equilateral triangle Z_1OZ_2 can be formed from the point O and the roots of this equation. [3]

References:

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9. [Maximum mark: 7]

(a) Find the number of ways these five people can be seated in this row. [3]

(b) Find the number of ways these five people can now be seated in this row. [4]

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(b) the digits are distinct and are in increasing order. [2]

[illegible]

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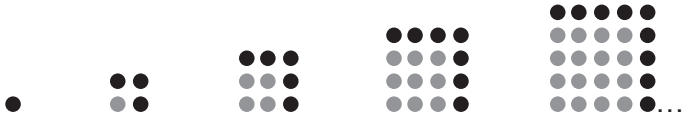
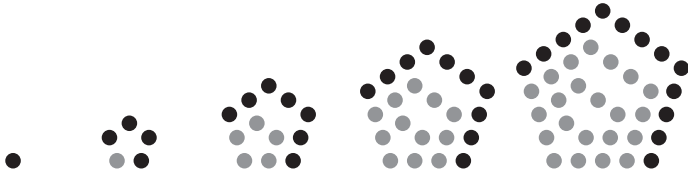
1. [Maximum marks: 27]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are 1, 3, 6, 10 and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	Geometric representation	Values
Triangular numbers		1, 3, 6, 10, 15, ...
Square numbers		1, 4, 9, 16, 25, ...
Pentagonal numbers		1, 5, 12, 22, 35, ...

For an r -sided regular polygon, where $r \in \mathbb{Z}^+$, $r \geq 3$, the n th polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

(This question continues on the following page)

(Question 1 continued)

Hence, for square numbers, $P_4(n) = \frac{(4-2)n^2 - (4-4)n}{2} = n^2$.

- (a) (i) For triangular numbers, verify that $P_3(n) = \frac{n(n+1)}{2}$. [2]
- (ii) The number 351 is a triangular number. Determine which one it is. [2]
- (b) (i) Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. [2]
- (ii) State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. [1]
- (iii) For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). [1]
- (c) Show that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3]

The n th pentagonal number can be represented by the arithmetic series

$$P_5(n) = 1 + 4 + 7 + \dots + (3n - 2).$$

- (d) Hence show that $P_5(n) = \frac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. [3]
- (e) By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. [5]

A polygonal number, $P_r(n)$, can be represented by the series

$$\sum_{m=1}^n (1 + (m-1)(r-2)) \text{ where } r \in \mathbb{Z}^+, r \geq 3.$$

- (f) Use mathematical induction to prove that $P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}$ where $n \in \mathbb{Z}^+$. [8]

Turn over

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 18]

Let z_n be the complex number defined as $z_n = (n^2 + n + 1) + i$ for $n \in \mathbb{N}$.

- (a) (i) Find $\arg(z_0)$.
- (ii) Write down an expression for $\arg(z_n)$ in terms of n . [3]

Let $w_n = z_0 z_1 z_2 z_3 \dots z_{n-1} z_n$ for $n \in \mathbb{N}$.

- (b) (i) Show that $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$ for $a, b \in \mathbb{R}^+$, $ab < 1$.
- (ii) Hence or otherwise, show that $\arg(w_1) = \arctan(2)$. [5]
- (c) Prove by mathematical induction that $\arg(w_n) = \arctan(n+1)$ for $n \in \mathbb{N}$. [10]
-

References:

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

Consider the arithmetic sequence $u_1, u_2, u_3, \dots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a) (i) Find the sum of the first five terms.

(ii) Given that $S_6 = 60$, find u_6 . [4]

(b) Find u_1 . [2]

(c) Hence or otherwise, write an expression for u_n in terms of n . [3]

Consider a geometric sequence, v_n , where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r . [3]

(e) Given that $v_{99} < 0$, find v_5 . [2]



Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}i$.

(a) By finding the modulus and argument of u , show that $u = 2e^{i\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w , or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + bi$, find the roots of the equation. [5]



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2. [Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where $m, c \in \mathbb{R}$ and $m, c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c , in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

- (a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

- (b) (i) Show that $r = -\frac{c}{m}$. [1]

- (ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \frac{m^2}{m+2}$. [4]

- (iii) State any further restrictions on the value of m . [1]

There are only three **integer** sets of values of m, r and c , that form an AS-linear function. One of these is $L(x) = -x - 1$.

- (c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c . [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}$, $a \neq 0$ and $b, c \in \mathbb{R}$.

Let $r_1, r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

- (d) Write down an expression for
- (i) the sum of roots, $r_1 + r_2$, in terms of a and b . [1]
- (ii) the product of roots, $r_1 r_2$, in terms of a and c . [1]

(This question continues on the following page)

(Question 2 continued)

If a , r_1 , b , r_2 and c , in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(i) write down an expression for $r_2 - r_1$ in terms of a and b ; [1]

(ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \frac{a^2 - ab - b}{2a}$; [2]

(iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\frac{1}{2}$. [3]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where $a = -\frac{1}{2}$.

(g) (i) Find an expression for r_1 in terms of b . [2]

(ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\frac{-p \pm q\sqrt{s}}{2}$ where $p, q, s \in \mathbb{Z}^+$. [5]

References:

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Find the value of p and the value of q .

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Find the value of p and the value of q .

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

A junior baseball team consists of six boys and three girls.

The team members are to be placed in a line to have their photograph taken.

- (a) In how many ways can the team members be placed if
 - (i) there are no restrictions;
 - (ii) the girls must be placed next to each other. [3]
- (b) Five members of the team are selected to attend a baseball summer camp. Find the number of possible selections that contain at least two girls. [3]



Turn over

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider $\phi = (a + bi)^3$, where $a, b \in \mathbb{R}$.

(a) In terms of a and b , find

(i) the real part of ϕ ;

(ii) the imaginary part of ϕ . [3]

(b) Hence, or otherwise, show that $(1 + \sqrt{3}i)^3 = -8$. [2]

The roots of the equation $z^3 = -8$ are u , v and w , where $u = 1 + \sqrt{3}i$ and $v \in \mathbb{R}$.

(c) Write down v and w , giving your answers in Cartesian form. [2]

On an Argand diagram, u , v and w are represented by the points U , V and W respectively.

(d) Find the area of the triangle UVW . [3]

Each of the points U , V and W is rotated counter-clockwise (anticlockwise) about 0 through an angle of $\frac{\pi}{4}$ to form three new points U' , V' and W' . These points represent the complex numbers u' , v' and w' respectively.

(e) Find u' , v' and w' , giving your answers in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$. [4]

(f) Given that u' , v' and w' are the solutions of $z^3 = c + di$, where $c, d \in \mathbb{R}$, find the value of c and the value of d . [3]

It is given that u , v , w , u' , v' and w' are all solutions of $z^n = \alpha$ for some $\alpha \in \mathbb{C}$, where $n \in \mathbb{N}$.

(g) Find the smallest positive value of n . [3]



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7. [Maximum mark: 7]

(a) Write down the value of $\alpha + \beta + \gamma$. [1]

(b) Show that $p = 1$. [3]

(ii) Write down the value of α and the value of β . [3]

[illegible]

9. [Maximum mark: 7]

(a) Write down an expression for the number of ways that the students could be assigned. [1]

The teacher agrees and now finds that the number of ways to assign the students is halved.

(b) Determine the value of n . [6]

[illegible]

— 17 —

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Consider the arithmetic sequence $a, p, q \dots$, where $a, p, q \neq 0$.

(a) Show that $2p - q = a$. [2]

Consider the geometric sequence $a, s, t \dots$, where $a, s, t \neq 0$.

(b) Show that $s^2 = at$. [2]

The first term of both sequences is a .

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Consider the case where $a = 9$, $s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(i) arithmetic sequence;

(ii) geometric sequence. [4]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n .

The first three terms of u_n are $u_1 = 9 + \ln 9$, $u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e) (i) Find the common difference of the new sequence in terms of $\ln 3$.

(ii) Show that $\sum_{i=1}^{10} u_i = -90 - 25\ln 3$. [6]



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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

Darren buys a car for \$35 000. The value of the car decreases by 15 % in the first year.

(a) Find the value of the car at the end of the first year. [2]

After the first year, the value of the car decreases by 11 % in each subsequent year.

(b) Find the value of Darren’s car 10 years after he buys it, giving your answer to the nearest dollar. [2]

When Darren has owned the car for n complete years, the value of the car is less than 10 % of its original value.

(c) Find the least value of n . [3]



Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider the equation $z^4 = 16i$, where $z \in \mathbb{C}$.

The equation has four roots z_1, z_2, z_3, z_4 , where $z_i = r(\cos \theta_i + i \sin \theta_i)$, $r > 0$ and $0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$.

- (a) Find z_1, z_2, z_3 and z_4 . [6]

The roots z_1, z_2, z_3 and z_4 form a geometric sequence.

- (b) Find the common ratio of the sequence, expressing your answer in Cartesian form. [3]

The roots z_1, z_2, z_3 and z_4 are represented by the points A, B, C and D respectively on an Argand diagram.

- (c) Plot the points A, B, C and D on an Argand diagram. [3]

The equation $v^4 = a + bi$, where $v \in \mathbb{C}$ and $a, b \in \mathbb{R}$ has roots z_1^*, z_2^*, z_3^* and z_4^* .

- (d) Determine the value of a and the value of b . [3]

The midpoint of $[AB]$ is A' , the midpoint of $[BC]$ is B' , the midpoint of $[CD]$ is C' and the midpoint of $[DA]$ is D' .

Consider the equation $w^p = 2^q$, where $w \in \mathbb{C}$ and $p, q \in \mathbb{Z}^+$.

Four of the roots of $w^p = 2^q$ are represented by the points A', B', C' and D' .

- (e) Find the least possible value of p and the corresponding value of q . [5]



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1. [Maximum mark: 27]

This question asks you to investigate models for the population of trout in a lake.

Trout is a type of fish. At the start of a year, a lake is estimated to contain 6000 trout.

The owner of the lake estimates that the number of trout will increase by 10% per year.

At the end of each year, the owner proposes to remove 500 trout from the lake to prevent overpopulation.

Therefore, the relationship between T_n , the predicted number of trout at the start of year n , and T_{n+1} , the predicted number of trout at the start of year $n + 1$, is given by

$$T_{n+1} = 1.1T_n - 500 \text{ and } T_1 = 6000.$$

For example, the predicted number of trout at the start of the second year is given by

$$T_2 = 1.1T_1 - 500.$$

(a) Use this formula to verify that $T_2 = 6100$. [2]

(b) (i) Verify that $T_3 = 6210$. [1]

(ii) Find T_4 . [2]

It is also known that $T_n = 6000(1.1)^{n-1} - \frac{500((1.1)^{n-1} - 1)}{1.1 - 1}$.

(c) (i) Show that $T_n = 1000(1.1)^{n-1} + 5000$. [2]

(ii) Hence, or otherwise, find T_6 . Give your answer to the nearest whole number. [2]

(This question continues on the following page)

(Question 1 continued)

After deciding that the trout population would increase too quickly, the lake owner proposes instead to remove 750 trout at the end of each year.

The relationship between D_n , the predicted number of trout at the start of year n , and D_{n+1} , the predicted number of trout at the start of year $n+1$, is now given by

$$D_{n+1} = 1.1D_n - 750 \text{ and } D_1 = 6000.$$

It is also known that $D_n = -1500(1.1)^{n-1} + 7500$.

- (d) (i) Show that $D_{n+1} - D_n = -150(1.1)^{n-1}$. [3]

- (ii) Use the result in part (d)(i) to deduce that the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year. [1]

- (e) Determine the first year during which there will be no trout remaining in the lake. [4]

The lake owner now considers a more general approach where d trout are removed at the end of each year.

Let C_n denote the predicted number of trout in the lake at the start of the n th year where

$$C_n = 6000(1.1)^{n-1} - 10d((1.1)^{n-1} - 1).$$

- (f) Find the value of d such that the predicted number of trout at the start of each year is constant. [3]

To model predicted numbers of trout, the lake owner has been using sequences generated by

$$u_{n+1} = ru_n - d, \text{ where } d, r \in \mathbb{R}^+ \text{ and } r \neq 1.$$

- (g) Use mathematical induction to prove that $u_n = u_1 r^{n-1} - \frac{d(r^{n-1} - 1)}{r - 1}$, for $n \in \mathbb{Z}^+$. [7]

Turn over

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 8]

Consider a sequence of ten rectangular picture frames $F_1, F_2, \dots, F_9, F_{10}$.

Picture frame F_1 has width 4 cm and height 5 cm.

The width and height of picture frame F_n , are each increased by 50% to generate the width and height of the next picture frame F_{n+1} , for $n \in \mathbb{Z}^+$, $1 \leq n \leq 9$.

- (a) (i) Show that the area of picture frame F_n is $20\left(\frac{9}{4}\right)^{n-1} \text{ cm}^2$.
- (ii) Hence, find the mean area of the ten picture frames, giving your answer in the form $p\left(\left(\frac{9}{4}\right)^a - 1\right) \text{ cm}^2$, where $p \in \mathbb{Q}^+$, $a \in \mathbb{Z}^+$. [5]
- (b) Find the median area of the ten picture frames, giving your answer in the form $q\left(\frac{9}{4}\right)^4 \text{ cm}^2$, where $q \in \mathbb{Q}^+$. [3]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 17]

- (a) Find the first four terms in the binomial expansion of $\sqrt{1+5x}$ in ascending powers of x . [4]

Consider the expression $(1+px)(1+qx)^{-1}$, where $p, q \in \mathbb{Q}$.

- (b) Find the expansion of $(1+px)(1+qx)^{-1}$ in ascending powers of x , up to and including the term in x^2 . [3]

The expansions found in parts (a) and (b) are identical up to the first three terms, for a value of p and a value of q .

- (c) Show that $q = \frac{5}{4}$. [4]

- (d) The expression $\frac{1+px}{1+qx}$, with $p = \frac{15}{4}$ and $q = \frac{5}{4}$, can be used as an approximation

for $\sqrt{1+5x}$ where $|x| < \frac{1}{5}$.

- (i) Hence, by finding a suitable value for x , find the approximation for $\sqrt{1.2}$ in the form $\frac{m}{n}$, where $m, n \in \mathbb{Z}$.

- (ii) Now consider the approximation for $\frac{\sqrt{5}}{2}$. Explain why the approximation for $\frac{\sqrt{5}}{2}$ is not as accurate as the approximation for $\sqrt{1.2}$. [6]



Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 19]

- (a) Solve $z^2 = -1 - \sqrt{3}i$, giving your answers in the form $z = r(\cos \theta + i \sin \theta)$. [4]

Let z_1 and z_2 be the square roots of $-1 - \sqrt{3}i$, where $\operatorname{Re}(z_1) > 0$.

Let z_3 and z_4 be the square roots of $-1 + \sqrt{3}i$, where $\operatorname{Re}(z_3) > 0$.

- (b) Expressing your answers in the form $z = a + bi$, where $a, b \in \mathbb{R}$,

(i) find z_1 and z_2 ;

(ii) deduce z_3 and z_4 . [4]

The four roots z_1, z_2, z_3 and z_4 are represented by the points A, B, C and D respectively on an Argand diagram.

- (c) (i) Plot the points A, B, C and D on an Argand diagram.

(ii) Find the area of the polygon formed by these four points. [4]

The four roots z_1, z_2, z_3 and z_4 satisfy the equation $z^4 + 2z^2 + 4 = 0$.

The four roots $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ and $\frac{1}{z_4}$ satisfy the equation $pw^4 + qw^2 + r = 0$ where $p, q, r \in \mathbb{Z}$.

- (d) Find the value of p, q and r . [3]

The four roots $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ and $\frac{1}{z_4}$ are represented by the points E, F, G and H respectively on an Argand diagram.

- (e) (i) Find $\frac{1}{z_1}$ in the form $z = a + bi$, where $a, b \in \mathbb{R}$.

(ii) Hence, deduce the area of the polygon formed by these four points. [4]



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1. [Maximum mark: 5]

(a) Find an expression for $\log_{10} 24$ in terms of p and q . [3]

(b) Find an expression for $\log_3 8$ in terms of p and q . [2]

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8. [Maximum mark: 8]

(a) Express z_1 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

The cube roots of $\frac{z_2}{z_1}$ are denoted by u , v and w .

(b) Find u , v and w . [5]

[illegible]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Consider the sequence $\{u_n\}$, with n th term given by u_n . The first three terms are

$$u_1 = k - 5, \quad u_2 = 3 - 2k \quad \text{and} \quad u_3 = 5k + 3, \quad \text{where } k \in \mathbb{R}.$$

(a) Consider the case when $\{u_n\}$ is arithmetic.

(i) Find the value of k .

(ii) Hence, or otherwise, find u_3 .

[5]

(b) Consider the case where $k = 12$.

(i) Show that the first three terms of $\{u_n\}$ form a geometric sequence.

(ii) Given that $\{u_n\}$ is geometric, state a reason why the sum of an infinite number of terms of this sequence does not exist.

[4]

(c) The sequence, $\{u_n\}$, is geometric for a second value of k .

(i) Show that $k^2 - 10k - 24 = 0$.

(ii) Find the first three terms of $\{u_n\}$ for this second value of k .

(iii) Hence, write down the value of S_{2m} , the sum of the first $2m$ terms, for this second value of k .

[7]



Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

Consider the complex number $z = x + yi$, where $x, y \in \mathbb{R}$, such that $|z - (2 + i)| = 3$.

(a) Show that $x^2 + y^2 - 4x - 2y - 4 = 0$. [3]

The argument of $\frac{z+p}{z-1}$ is $\frac{\pi}{4}$, where $p \in \mathbb{R}$.

(b) Show that $x^2 + y^2 + (p-1)x + (p+1)y - p = 0$. [7]

Two roots of the equation $z^4 + az^3 + bz^2 + cz + d = 0$ are z_1 and z_2 , where $z \in \mathbb{C}$ and $a, b, c, d \in \mathbb{R}$.

Both z_1 and z_2 satisfy the conditions $|z - (2 + i)| = 3$ and $\operatorname{Re}\left(\frac{z+4}{z-1}\right) = \operatorname{Im}\left(\frac{z+4}{z-1}\right)$.

(c) Use the results from parts (a) and (b) to find z_1 and z_2 . [7]

(d) Find the value of a . [4]



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8. [Maximum mark: 7]

$$f(t) = \sin(2t + 1)$$

$$g(t) = \sin(2t + 3)$$

$$h(t) = f(t) + g(t)$$

- (a) Show that $h(t) = \text{Im}(e^{2ti}(e^i + e^{3i}))$. [2]

- (b) Write $e^i + e^{3i}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

- (c) Hence or otherwise, write $h(t)$ in the form $p \sin(2t + q)$, where $p > 0$ and $0 < q < 2\pi$. [3]

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Section B

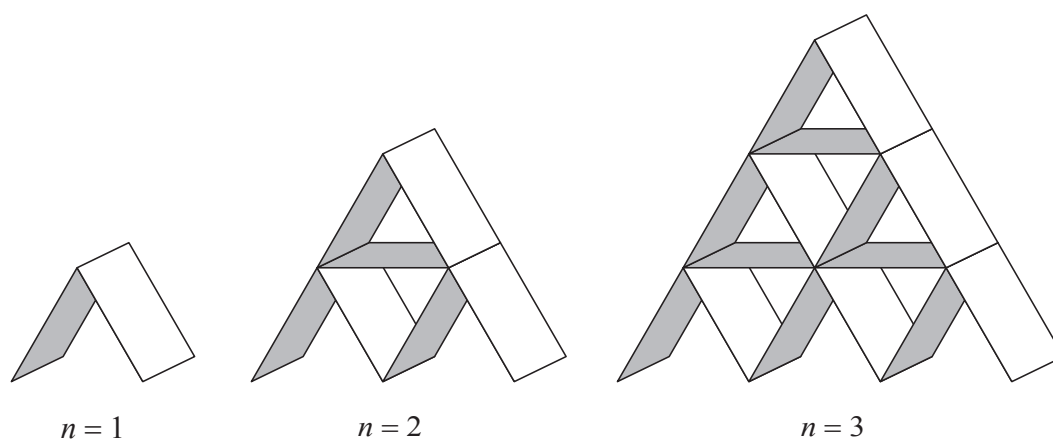
Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Rectangular playing cards are stacked in the shape of a pyramid with n rows, where $n \geq 1$.

Some cards are placed horizontally and some cards are stacked at an angle of 60° to the horizontal.

The following diagrams represent pyramid stacks for $n = 1$, $n = 2$ and $n = 3$.



Let t_n represent the number of cards used to create a pyramid stack with n rows.

- (a) Write down t_3 . [1]
- (b) Find t_4 . [2]
- (c) Show that $t_n = \frac{n(3n+1)}{2}$. [3]

There are 52 cards in a full pack of playing cards.

- (d) A complete pyramid stack is created using playing cards taken from 14 full packs. Find the maximum number of rows in this stack. [3]
- (e) A complete pyramid stack is created using playing cards taken from full packs with no cards left over. Find the minimum number of rows in this stack. [2]

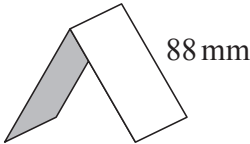
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(Question 10 continued)

The long edge of each playing card measures 88 mm as illustrated in the following diagram.



- (f) Find the minimum number of cards needed to create a complete pyramid stack with a vertical height of more than 2 metres. The thickness of the cards may be ignored. [5]



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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 20]

Consider the family of functions f_n defined by $f_n(x) = \sum_{r=0}^n (-2x^2)^r$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(a) Show that f_n is an even function for all values of n . [3]

(b) (i) Show that $f_3(x) = 1 - 2x^2 + 4x^4 - 8x^6$.

(ii) Write down a similar expression for $f_4(x)$ in ascending powers of x . [2]

Consider the function $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ defined over the domain $-k < x < k$ where $k > 0$.

The largest possible value of k is K .

(c) (i) Find the value of K , giving your answer in exact form.

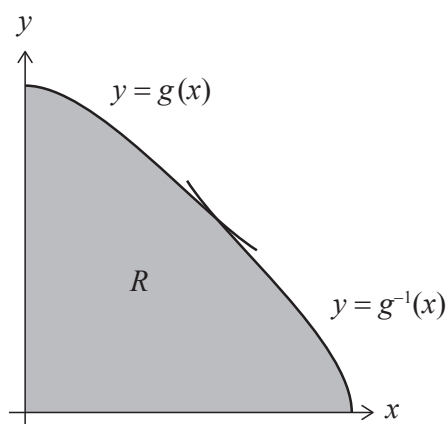
(ii) Express $f(x)$ as a rational function in the form $\frac{1}{a + bx^2}$, where a and b are constants to be determined. [5]

The function g is defined as $g(x) = f(x)$ for $0 \leq x < K$.

(d) (i) Justify that g^{-1} exists.

(ii) Find $g^{-1}(x)$, giving its domain. [6]

The region R is completely enclosed by the curves $y = g(x)$, $y = g^{-1}(x)$ and the x - and y -axes, as shown on the following diagram.



(e) Find the area of R . [4]



7. [Maximum mark: 7]

(a) Find the second term. [3]

Let S_n denote the sum of the first n terms of the arithmetic sequence.

(b) Find the greatest value of S_n , giving your answer to two decimal places. [4]

[illegible]

1. [Maximum mark: 6]

(a) Find the 13th term of this arithmetic sequence.

[4]

(b) Find the value of n .

[2]

[illegible]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 22]

Consider the complex number $w = \frac{4i}{1+i\sqrt{3}}$.

(a) Show that $w = \sqrt{3} + i$. [2]

(b) Find

(i) $|w|$;

(ii) $\arg w$. [3]

(c) Use De Moivre's theorem to find the two square roots of w . Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

The complex number w can be expressed in the form $(a + bi)^2$, where $a, b \in \mathbb{R}^+$.

(d) (i) Show that $a^2 = \frac{\sqrt{3}}{2} + 1$.

(ii) Find the value of b^2 . [8]

(e) Hence, express $\tan \frac{\pi}{12}$ in the form $p + q\sqrt{3}$, where $p, q \in \mathbb{Z}$. [5]



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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

(a) It is given that $w = 1 + i\sqrt{3}$.

(i) Express w in the form $w = re^{i\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$.

(ii) Show that $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -256$. [6]

It is given that $(1 + i \tan \theta)^n \equiv \frac{\cos n\theta + i \sin n\theta}{\cos^n \theta}$ for $\cos \theta \neq 0$, $n \in \mathbb{Z}^+$.

(b) Show that $(1 + i \tan \theta)^4 + (1 - i \tan \theta)^4 \equiv \frac{2 \cos 4\theta}{\cos^4 \theta}$. [3]

(c) Hence, determine the four roots of the equation $(1 + z)^4 + (1 - z)^4 = 0$, giving your answers in the form $z = i \tan\left(\frac{k\pi}{8}\right)$, where $k \in \mathbb{Z}^+$. [4]

(d) By considering the binomial expansion of $(1 + z)^4 + (1 - z)^4$, show that $\tan \frac{\pi}{8} = \sqrt{3 - 2\sqrt{2}}$. [7]



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[illegible]

6. [Maximum mark: 5]

$$\sum_{k=0}^{\infty} \left(\frac{53}{1000} \right) \left(\frac{1}{100} \right)^k.$$

- (a) Find the exact value of S_{∞} . [3]

(b) Use your answer from part (a) to express $0.\overline{253}$ as a fraction such that the numerator and denominator have no common factors. [2]

[illegible]

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 6]

Consider the following two equations.

$$2x + 6y - 8z = 13$$

$$3x - y + 3z = 12$$

(a) Determine the general solution giving the answer in parametric form. [2]

Consider a third equation $ax + 12y - 16z = k$, where $a, k \in \mathbb{R}$.

(b) In the case where $a = 3$ and $k = -3$, determine the unique solution to the system of three equations. [2]

(c) (i) Write down the value of a for which there is no unique solution to the system of three equations.

(ii) Hence, write down the corresponding value of k for which there is an infinite number of solutions to the system of three equations. [2]



Turn over

8. [Maximum mark: 7]

(a) Find the number of different ways of arranging the books in a line on a shelf. [1]

Six books are from Asia, five from Europe and four from Africa.

(b) Determine the number of different ways that Malik can do this. [3]

(c) Determine the number of different choices Malik could make. [3]

[illegible]

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