

APOLLO

Calculus

HL

Mathematics: applications and interpretation

First assessment 2021

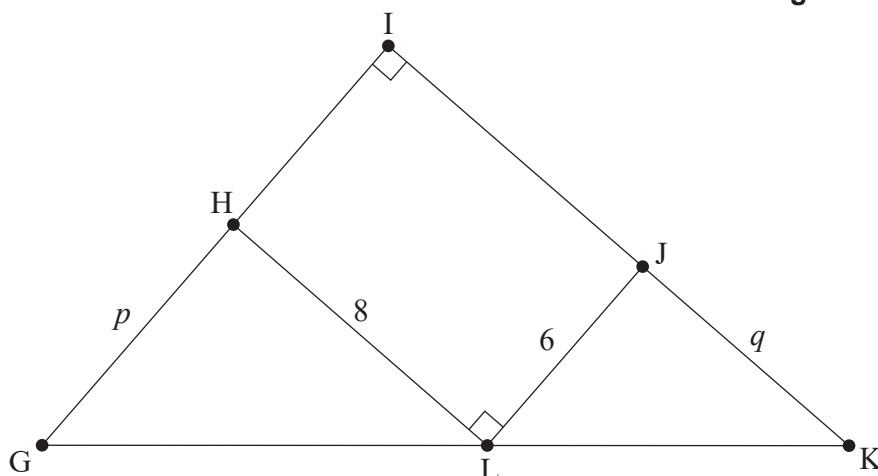
104 questions | questions

7. [Maximum mark: 8]

Ellis designs a gift box. The top of the gift box is in the shape of a right-angled triangle GIK.

A rectangular section HIJL is inscribed inside this triangle. The lengths of GH, JK, HL, and LJ are p cm, q cm, 8 cm and 6 cm respectively.

diagram not to scale



The area of the top of the gift box is $A \text{ cm}^2$.

(a) (i) Find A in terms of p and q .

(ii) Show that $A = \frac{192}{q} + 3q + 48$.

[4]

(b) Find $\frac{dA}{dq}$.

[2]

Ellis wishes to find the value of q that will minimize the area of the top of the gift box.

(c) (i) Write down an equation Ellis could solve to find this value of q .

(ii) Hence, or otherwise, find this value of q .

[2]

(This question continues on the following page)



(Question 7 continued)

[illegible]

Turn over

12. [Maximum mark: 8]

A tank of water initially contains 400 litres. Water is leaking from the tank such that after 10 minutes there are 324 litres remaining in the tank.

The volume of water, V litres, remaining in the tank after t minutes, can be modelled by the differential equation

$$\frac{dV}{dt} = -k\sqrt{V}, \text{ where } k \text{ is a constant.}$$

- (a) Show that $V = \left(20 - \frac{t}{5}\right)^2$. [6]
- (b) Find the time taken for the tank to empty. [2]

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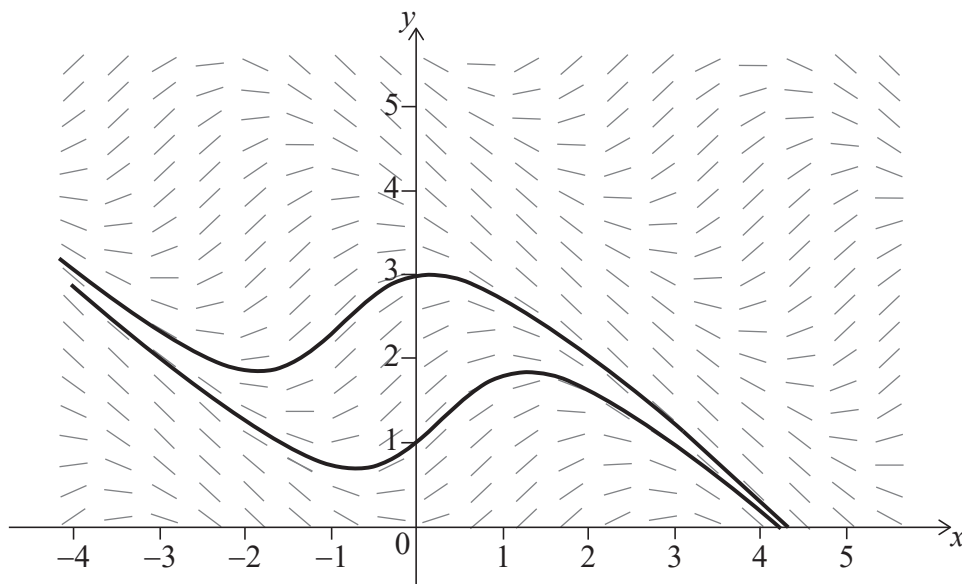
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15. [Maximum mark: 5]

The diagram shows the slope field for the differential equation

$$\frac{dy}{dx} = \sin(x + y), \quad -4 \leq x \leq 5, \quad 0 \leq y \leq 5.$$

The graphs of the two solutions to the differential equation that pass through points $(0, 1)$ and $(0, 3)$ are shown.



For the two solutions given, the local minimum points lie on the straight line L_1 .

- (a) Find the equation of L_1 , giving your answer in the form $y = mx + c$.

[3]

For the two solutions given, the local maximum points lie on the straight line L_2 .

- (b) Find the equation of L_2 .

[2]

(This question continues on the following page)



(Question 15 continued)

[illegible]

Turn over

16. [Maximum mark: 8]

A particle P moves in a straight line, such that its displacement x at time t ($t \geq 0$) is defined by the differential equation $\dot{x} = x \cos t (e^{-\sin t})$. At time $t = 0$, $x = \frac{1}{e}$.

- By using Euler's method with a step length of 0.1, find an approximate value for x when $t = 0.3$.
- By solving the differential equation, find the percentage error in your approximation for x when $t = 0.3$.

[illegible]

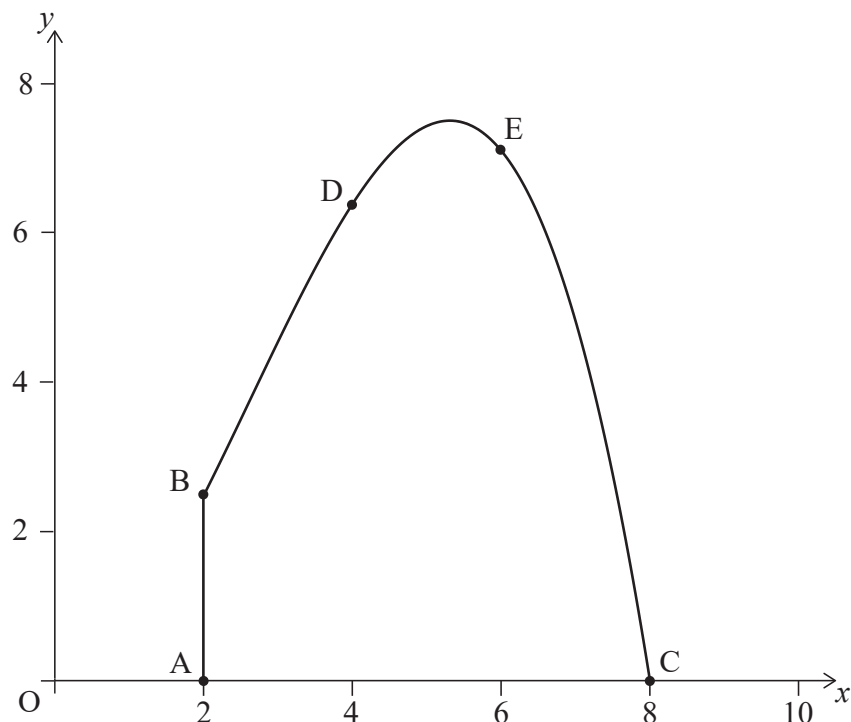
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2. [Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line $[AB]$ represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2$, $2 \leq x \leq 8$, relative to an origin O .



Point A has coordinates $(2, 0)$, point B has coordinates $(2, 2.4)$, and point C has coordinates $(8, 0)$.

- (a) (i) Find $\frac{dy}{dx}$.
- (ii) Hence find the maximum height of the tunnel. [6]
- (b) Find the height of the tunnel when
- (i) $x = 4$.
- (ii) $x = 6$. [3]
- (c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]
- (d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.
- (ii) Hence find the cross-sectional area of the tunnel. [4]

6. [Maximum mark: 18]

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a e^{bt} \cos t \\ a e^{bt} \sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t . [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a e^{bt} \sqrt{1 + b^2}. \quad [4]$$

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b . [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

At a point P, the ice-skater is skating parallel to the y -axis for the first time.

(e) Find OP. [6]

Turn over

7. [Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{dx}{dt} = 2x$$

- (a) Find the population of rabbits 1 year after they were introduced.

[5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t , can be approximately modelled by the coupled equations:

$$\frac{dx}{dt} = x(2 - 0.01y)$$

$$\frac{dy}{dt} = y(0.0002x - 0.8)$$

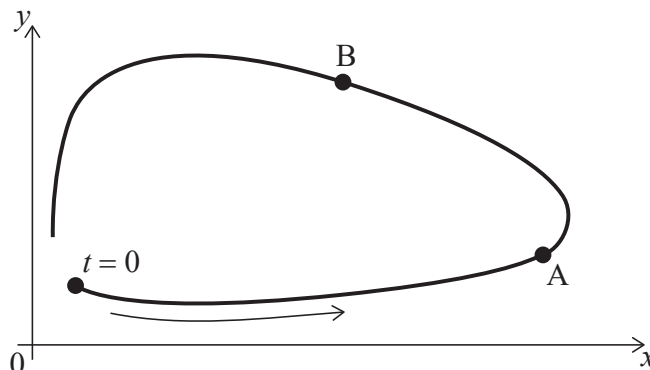
- (b) Use Euler's method with a step size of 0.25, to find

(i) the population of rabbits 1 year after the foxes were introduced.

(ii) the population of foxes 1 year after the foxes were introduced.

[6]

- (c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.



Describe the changes in the populations of rabbits and foxes for these 4 years at

(i) point A.

(ii) point B.

[3]

- (d) Find the non-zero equilibrium point for the populations of rabbits and foxes.

[3]

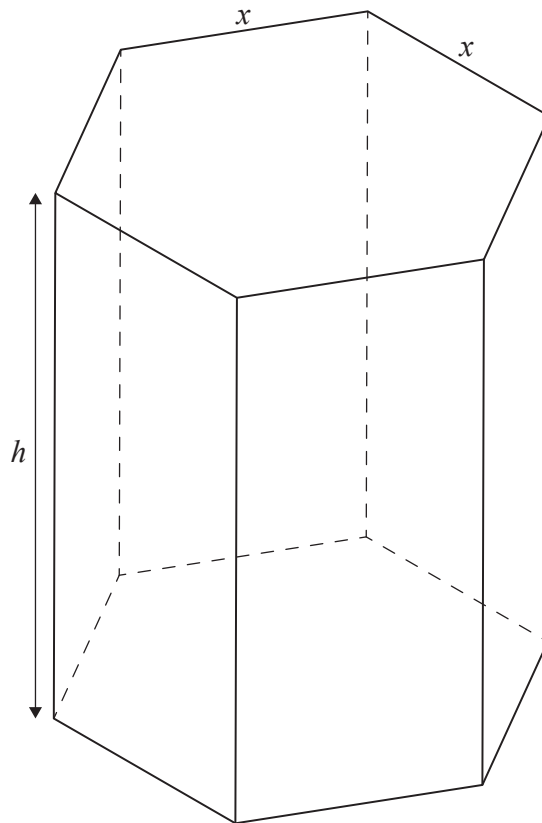
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3. [Maximum mark: 15]

A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is h cm, and the top and base of the prism have sides of length x cm.

diagram not to scale



- Given that $\sin 60^\circ = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]
- Given that the total external surface area of the box is 1200 cm^2 , show that the volume of the box may be expressed as $V = 300\sqrt{3}x - \frac{9}{4}x^3$. [5]
- Sketch the graph of $V = 300\sqrt{3}x - \frac{9}{4}x^3$, for $0 \leq x \leq 16$. [2]
- Find an expression for $\frac{dV}{dx}$. [2]
- Find the value of x which maximizes the volume of the box. [2]
- Hence, or otherwise, find the maximum possible volume of the box. [2]

6. [Maximum mark: 15]

A particle P moves along the x -axis. The velocity of P is $v \text{ m s}^{-1}$ at time t seconds, where $v = -2t^2 + 16t - 24$ for $t \geq 0$.

- (a) Find the times when P is at instantaneous rest. [2]
- (b) Find the magnitude of the particle's acceleration at 6 seconds. [4]
- (c) Find the greatest speed of P in the interval $0 \leq t \leq 6$. [2]
- (d) The particle starts from the origin O. Find an expression for the displacement of P from O at time t seconds. [4]
- (e) Find the total distance travelled by P in the interval $0 \leq t \leq 4$. [3]

Turn over

7. [Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{dx}{dt} = -4x$$

$$\frac{dy}{dt} = 3x - 2y$$

- (a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]
- (b) Hence, write down the general solution of the system. [2]
- (c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]
- (d) Find the value of $\frac{dy}{dx}$
- (i) at $(4, 0)$.
- (ii) at $(-4, 0)$. [3]
- (e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \leq x \leq 6$, $-6 \leq y \leq 6$. [4]
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2. [Maximum mark: 31]

Alessia is an ecologist working for Mediterranean fishing authorities. She is interested in whether the mackerel population density is likely to fall below 5000 mackerel per km³, as this is the minimum value required for sustainable fishing. She believes that the primary factor affecting the mackerel population is the interaction of mackerel with sharks, their main predator.

The population densities of mackerel (M thousands per km³) and sharks (S per km³) in the Mediterranean Sea are modelled by the coupled differential equations:

$$\frac{dM}{dt} = \alpha M - \beta MS$$

$$\frac{dS}{dt} = \gamma MS - \delta S$$

where t is measured in years, and α , β , γ and δ are parameters.

This model assumes that no other factors affect the mackerel or shark population densities.

The term αM models the population growth rate of the mackerel in the absence of sharks. The term βMS models the death rate of the mackerel due to being eaten by sharks.

(a) Suggest similar interpretations for the following terms.

(i) γMS [1]

(ii) δS [1]

(b) An equilibrium point is a set of values of M and S , such that $\frac{dM}{dt} = 0$ and $\frac{dS}{dt} = 0$.

Given that both species are present at the equilibrium point,

(i) show that, at the equilibrium point, the value of the mackerel population density is $\frac{\delta}{\gamma}$; [3]

(ii) find the value of the shark population density at the equilibrium point. [2]

(c) The equilibrium point found in part (b) gives the average values of M and S over time.

Use the model to predict how the following events would affect the average value of M . Justify your answers.

(i) Toxic sewage is added to the Mediterranean Sea. Alessia claims this reduces the shark population growth rate and hence the value of γ is halved. No other parameter changes. [2]

(ii) Global warming increases the temperature of the Mediterranean Sea. Alessia claims that this promotes the mackerel population growth rate and hence the value of α is doubled. No other parameter changes. [2]

(This question continues on the following page)

(Question 2 continued)

- (d) To estimate the value of α , Alessia considers a situation where there are no sharks and the initial mackerel population density is M_0 .
- (i) Write down the differential equation for M that models this situation. [1]
 - (ii) Show that the expression for the mackerel population density after t years is $M = M_0 e^{\alpha t}$. [4]
 - (iii) Alessia estimates that the mackerel population density increases by a factor of three every two years. Show that $\alpha = 0.549$ to three significant figures. [3]

Based on additional observations, it is believed that

$$\alpha = 0.549,$$

$$\beta = 0.236,$$

$$\gamma = 0.244,$$

$$\delta = 1.39.$$

Alessia decides to use Euler's method to estimate future mackerel and shark population densities. The initial population densities are estimated to be $M_0 = 5.7$ and $S_0 = 2$. She uses a step length of 0.1 years.

- (e) (i) Write down expressions for M_{n+1} and S_{n+1} in terms of M_n and S_n . [3]
- (ii) Use Euler's method to find an estimate for the mackerel population density after one year. [2]
- (f) Alessia will use her model to estimate whether the mackerel population density is likely to fall below the minimum value required for sustainable fishing, 5000 per km³, during the first nine years.
- (i) Use Euler's method to sketch the trajectory of the phase portrait, for $4 \leq M \leq 7$ and $1.5 \leq S \leq 3$, over the first nine years. [3]
 - (ii) Using your phase portrait, or otherwise, determine whether the mackerel population density would be sufficient to support sustainable fishing during the first nine years. [2]
 - (iii) State **two** reasons why Alessia's conclusion, found in part (f)(ii), might not be valid. [2]

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2. [Maximum mark: 5]

Inspectors are investigating the carbon dioxide emissions of a power plant. Let R be the rate, in tonnes per hour, at which carbon dioxide is being emitted and t be the time in hours since the inspection began.

When R is plotted against t , the total amount of carbon dioxide produced is represented by the area between the graph and the horizontal t -axis.

The rate, R , is measured over the course of two hours. The results are shown in the following table.

t	0	0.4	0.8	1.2	1.6	2
R	30	50	60	40	20	50

- (a) Use the trapezoidal rule with an interval width of 0.4 to estimate the total amount of carbon dioxide emitted during these two hours.

[3]

The real amount of carbon dioxide emitted during these two hours was 72 tonnes.

- (b) Find the percentage error of the estimate found in part (a).

[2]

[illegible]

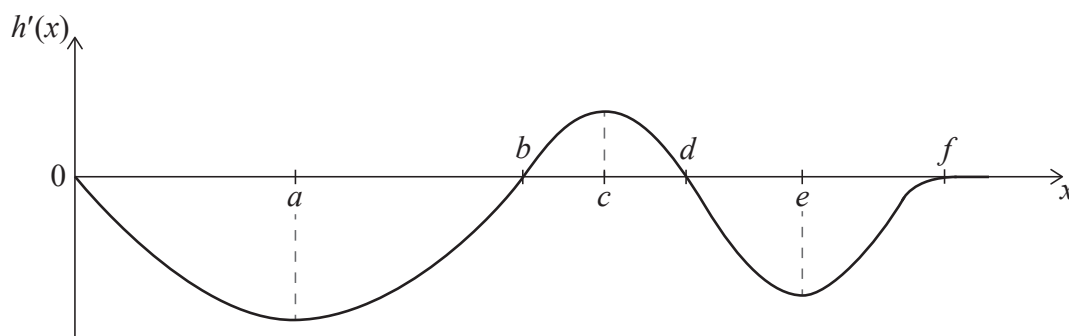
8. [Maximum mark: 5]

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).



Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h'(x)$ has local minima and maxima when x is equal to a , c and e . The graph of $h'(x)$ intersects the x -axis when x is equal to b , d , and f .



- (a) (i) Identify the x value of the point where $|h'(x)|$ has its maximum value.
- (ii) Interpret this point in the given context.

[2]

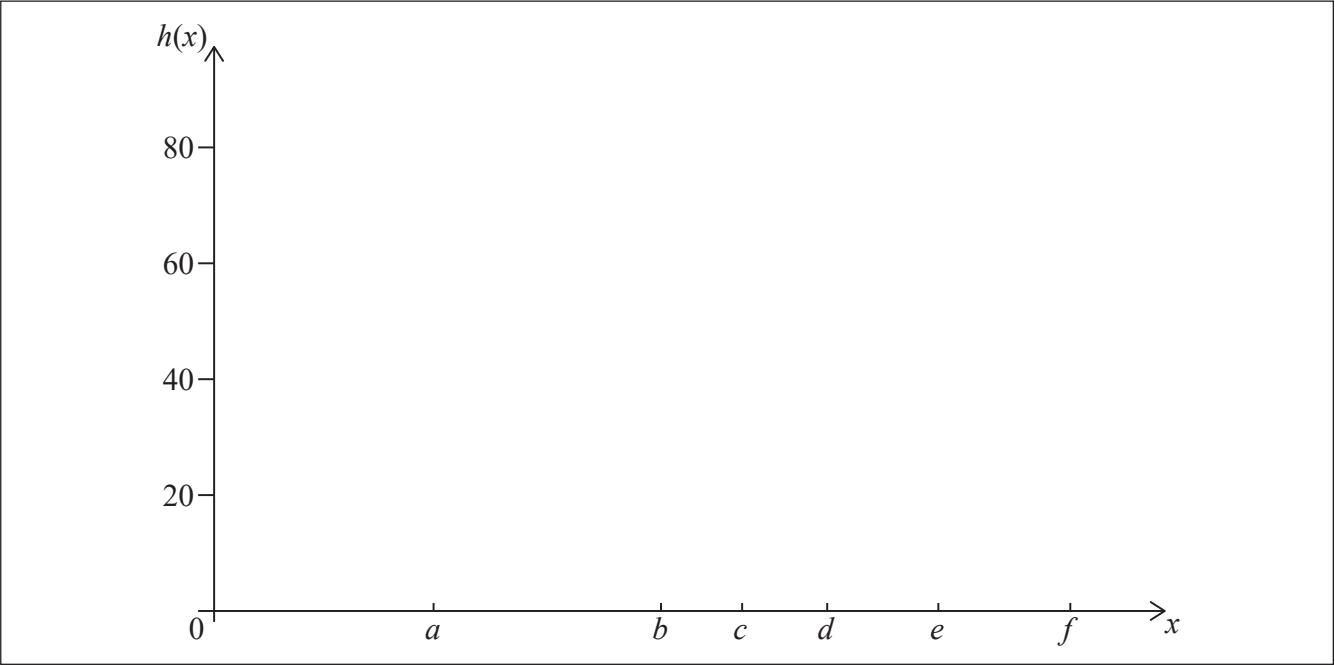
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(Question 8 continued)

Juri starts at a height of 60 metres and finishes at F, where $x = f$.

- (b) Sketch a possible diagram of the hill on the following pair of coordinate axes. [3]



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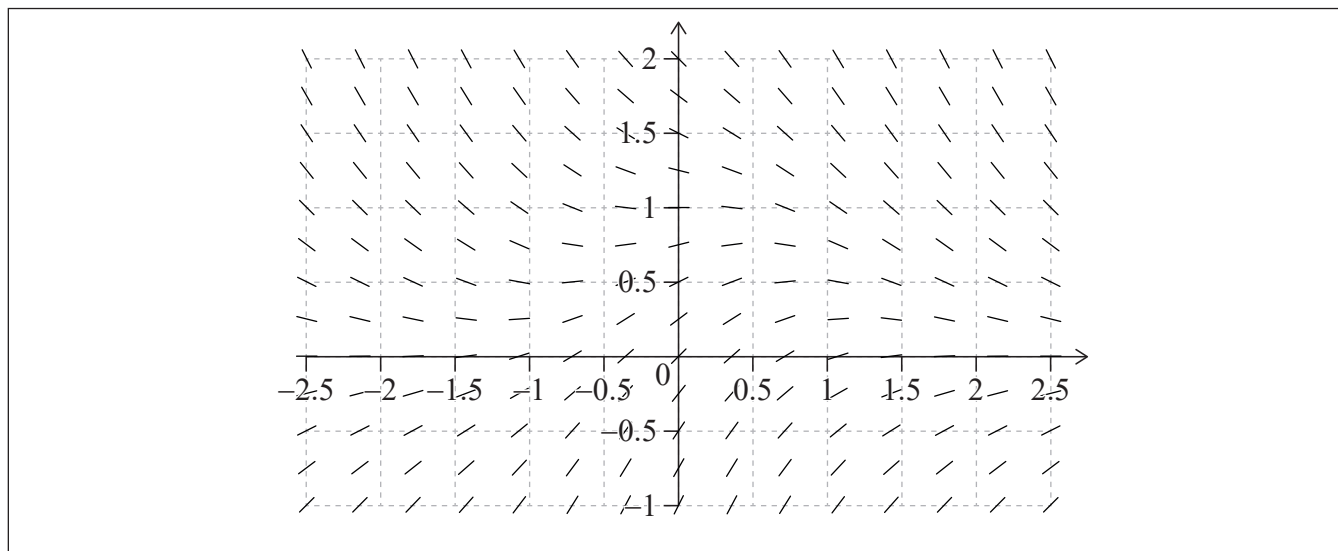


13. [Maximum mark: 7]

The slope field for the differential equation $\frac{dy}{dx} = e^{-x^2} - y$ is shown in the following two graphs.

(a) Calculate the value of $\frac{dy}{dx}$ at the point $(0, 1)$. [1]

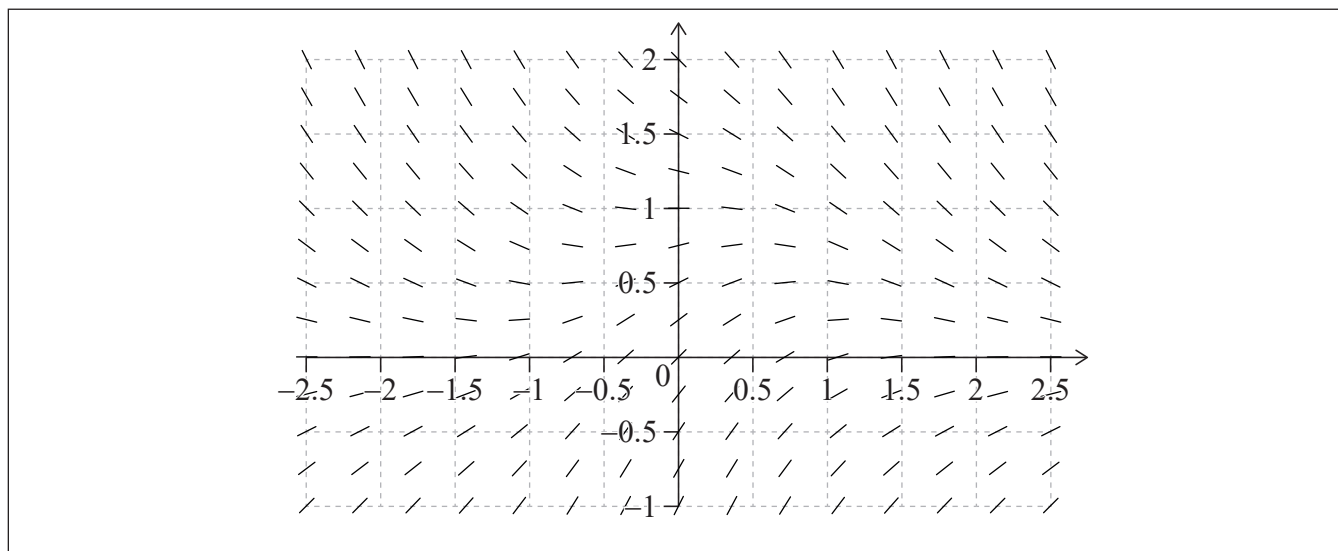
(b) Sketch, on the first graph, a curve that represents the points where $\frac{dy}{dx} = 0$. [2]



(c) On the second graph,

(i) sketch the solution curve that passes through the point $(0, 0)$.

(ii) sketch the solution curve that passes through the point $(0, 0.75)$. [4]



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(Question 13 continued)

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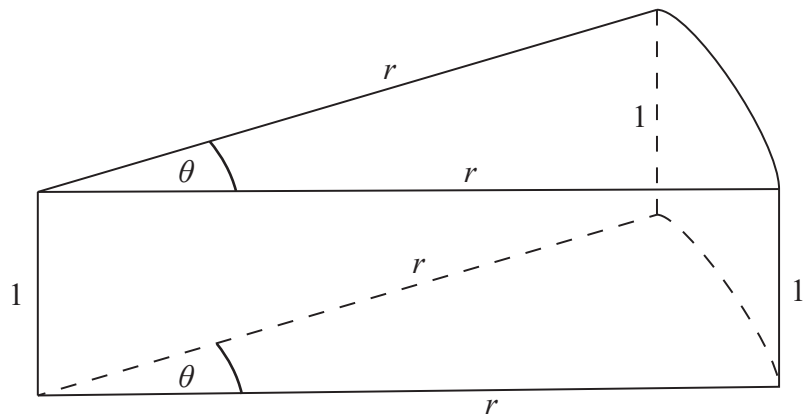
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15. [Maximum mark: 9]

The following diagram shows a frame that is made from wire. The total length of wire is equal to 15 cm. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius r cm. They are connected by 1 cm lengths of wire perpendicular to the sectors. This is shown in the diagram below.



- (a) Show that $r = \frac{6}{2 + \theta}$. [2]

The faces of the frame are covered by paper to enclose a volume, V .

- (b) (i) Find an expression for V in terms of θ .
(ii) Find the expression $\frac{dV}{d\theta}$.
(iii) Solve algebraically $\frac{dV}{d\theta} = 0$ to find the value of θ that will maximize the volume, V . [7]



Turn over

17. [Maximum mark: 7]

The sides of a bowl are formed by rotating the curve $y = 6 \ln x$, $0 \leq y \leq 9$, about the y -axis, where x and y are measured in centimetres. The bowl contains water to a height of h cm.

- (a) Show that the volume of water, V , in terms of h is $V = 3\pi(e^{\frac{h}{3}} - 1)$. [5]
- (b) Hence find the maximum capacity of the bowl in cm^3 . [2]

[illegible]

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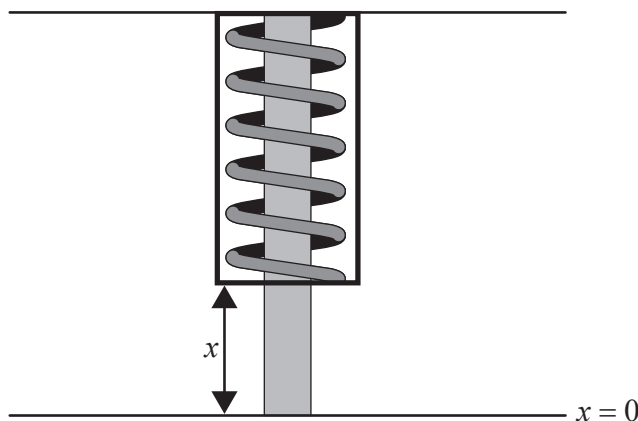
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6. [Maximum mark: 15]

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x , of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

- (a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, where A is a 2×2 matrix.

- (b) Write down the matrix A . [1]

- (c) (i) Find the eigenvalues of matrix A .

- (ii) Find the eigenvectors of matrix A . [6]

- (d) Given that when $t = 0$ the shock absorber is displaced 8 cm and its velocity is zero, find an expression for x in terms of t . [6]

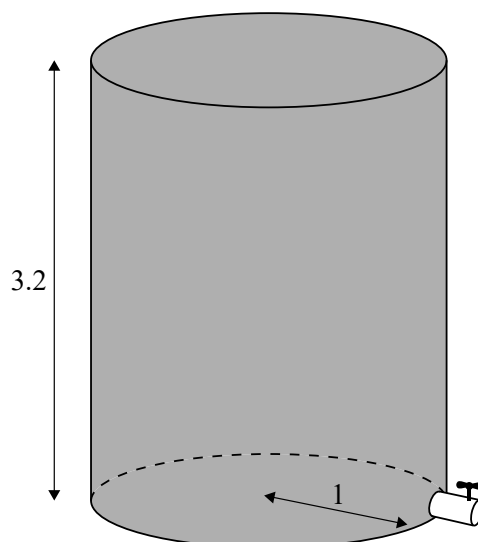
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2. [Maximum mark: 30]

This question explores models for the height of water in a cylindrical container as water drains out.

The diagram shows a cylindrical water container of height 3.2 metres and base radius 1 metre. At the base of the container is a small circular valve, which enables water to drain out.

diagram not to scale



Eva closes the valve and fills the container with water.

At time $t = 0$, Eva opens the valve. She records the height, h metres, of water remaining in the container every 5 minutes.

Time, t (minutes)	Height, h (metres)
0	3.2
5	2.4
10	1.6
15	1.1
20	0.5

Eva first tries to model the height using a linear function, $h(t) = at + b$, where $a, b \in \mathbb{R}$.

(a) (i) Find the equation of the regression line of h on t . [2]

(ii) Interpret the meaning of parameter a in the context of the model. [1]

(This question continues on the following page)

(Question 2 continued)

Eva uses the equation of the regression line of h on t , to predict the time it will take for all the water to drain out of the container.

- (iii) Suggest why Eva's use of the linear regression equation in this way could be unreliable. [1]

Eva thinks she can improve her model by using a quadratic function, $h(t) = pt^2 + qt + r$, where $p, q, r \in \mathbb{R}$.

- (b) (i) Find the equation of the least squares quadratic regression curve. [1]

Eva uses this equation to predict the time it will take for all the water to drain out of the container and obtains an answer of k minutes.

- (ii) Find the value of k . [2]

- (iii) Hence, write down a suitable domain for Eva's function $h(t) = pt^2 + qt + r$. [1]

Let V be the volume, in cubic metres, of water in the container at time t minutes.

Let R be the radius, in metres, of the circular valve.

Eva does some research and discovers a formula for the rate of change of V .

$$\frac{dV}{dt} = -\pi R^2 \sqrt{70560h}$$

- (c) Show that $\frac{dh}{dt} = -R^2 \sqrt{70560h}$. [3]

- (d) By solving the differential equation $\frac{dh}{dt} = -R^2 \sqrt{70560h}$, show that the general solution is given by $h = 17640(c - R^2t)^2$, where $c \in \mathbb{R}$. [5]

Eva measures the radius of the valve to be 0.023 metres. Let T be the time, in minutes, it takes for all the water to drain out of the container.

- (e) Use the general solution from part (d) and the initial condition $h(0) = 3.2$ to predict the value of T . [4]

Eva wants to use the container as a timer. She adjusts the initial height of water in the container so that all the water will drain out of the container in 15 minutes.

- (f) Find this new height. [3]

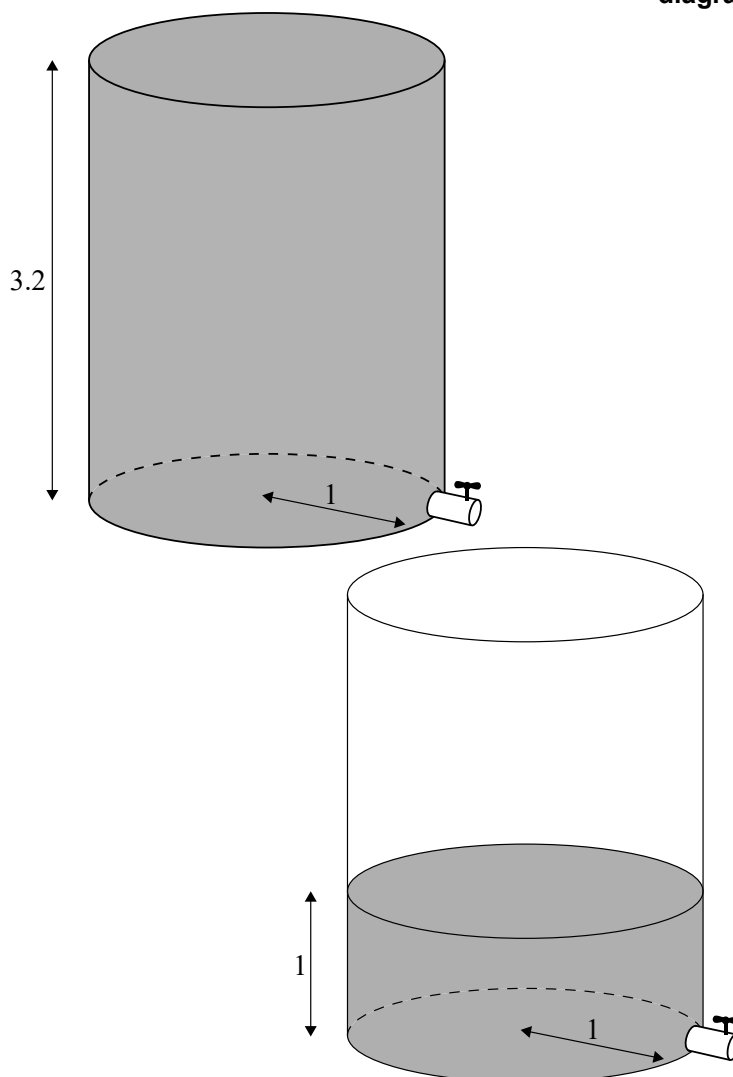
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(Question 2 continued)

Eva has another water container that is identical to the first one. She places one water container above the other one, so that all the water from the highest container will drain into the lowest container. Eva completely fills the highest container, but only fills the lowest container to a height of 1 metre, as shown in the diagram.

diagram not to scale



At time $t = 0$ Eva opens both valves. Let H be the height of water, in metres, in the lowest container at time t .

- (g) (i) Show that $\frac{dH}{dt} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H}$, where $0 \leq t \leq T$. [4]
- (ii) Use Euler's method with a step length of 0.5 minutes to estimate the maximum value of H . [3]

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7. [Maximum mark: 5]

A particle, A, moves so that its velocity ($v\text{ ms}^{-1}$) at time t is given by $v = 2 \sin t$, $t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5v^2$.

- (a) Write down an expression for E as a function of time. [1]
- (b) Hence find $\frac{dE}{dt}$. [2]
- (c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of 5 J s^{-1} . [2]



13. [Maximum mark: 6]

The rates of change of the area covered by two types of fungi, X and Y, on a particular tree are given by the following equations, where x is the area covered by X and y is the area covered by Y.

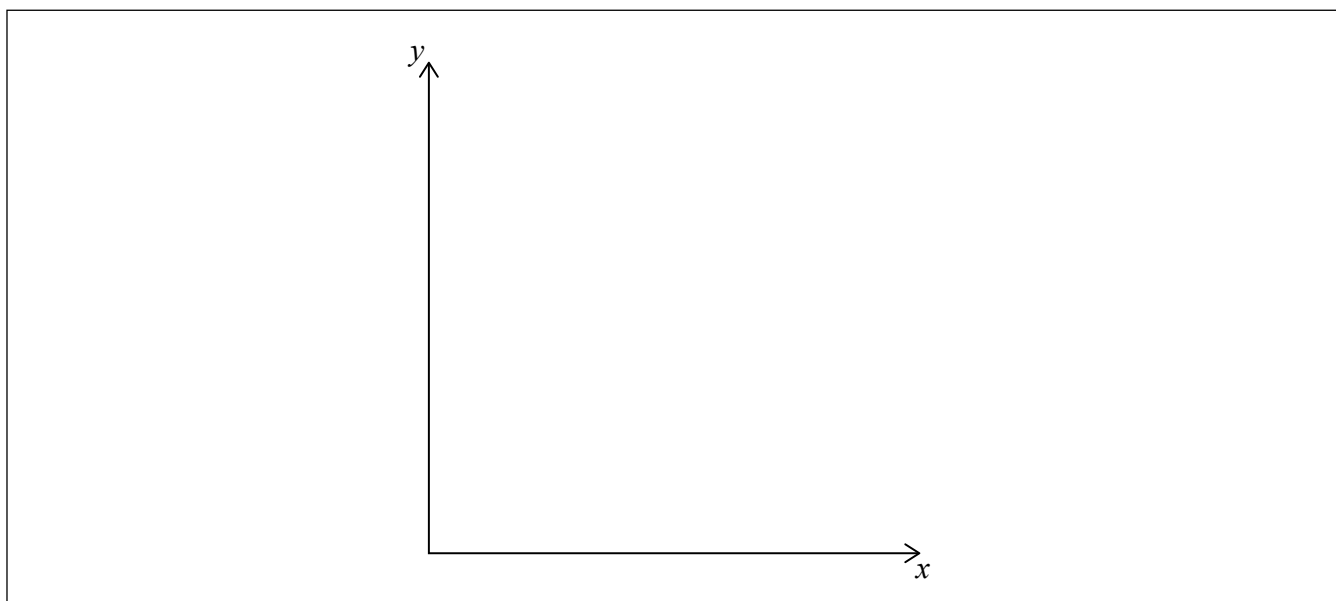
$$\frac{dx}{dt} = 3x - 2y$$

$$\frac{dy}{dt} = 2x - 2y$$

The matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues of 2 and -1 with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Initially $x = 8 \text{ cm}^2$ and $y = 10 \text{ cm}^2$.

- (a) Find the value of $\frac{dy}{dx}$ when $t = 0$. [2]
- (b) On the following axes, sketch a possible trajectory for the growth of the two fungi, making clear any asymptotic behaviour. [4]



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(Question 13 continued)

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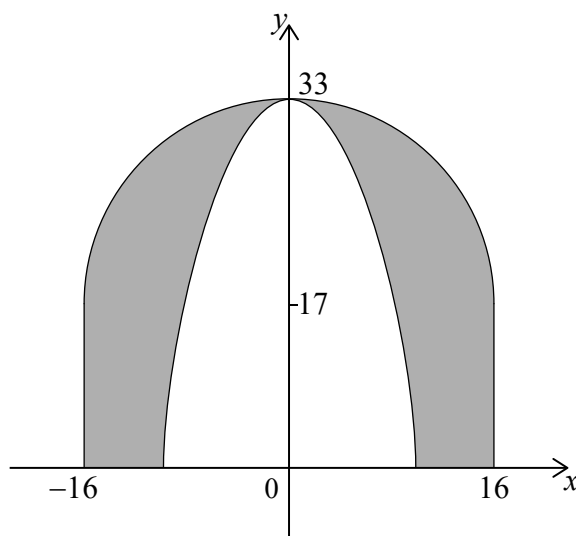
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14. [Maximum mark: 8]

- (a) The graph of $y = -x^3$ is transformed onto the graph of $y = 33 - 0.08x^3$ by a translation of a units vertically and a stretch parallel to the x -axis of scale factor b .
- (i) Write down the value of a .
- (ii) Find the value of b .
- (b) The outer dome of a large cathedral has the shape of a hemisphere of diameter 32 m, supported by vertical walls of height 17 m. It is also supported by an inner dome which can be modelled by rotating the curve $y = 33 - 0.08x^3$ through 360° about the y -axis between $y = 0$ and $y = 33$, as indicated in the diagram.

[3]



Find the volume of the space between the two domes.

[5]

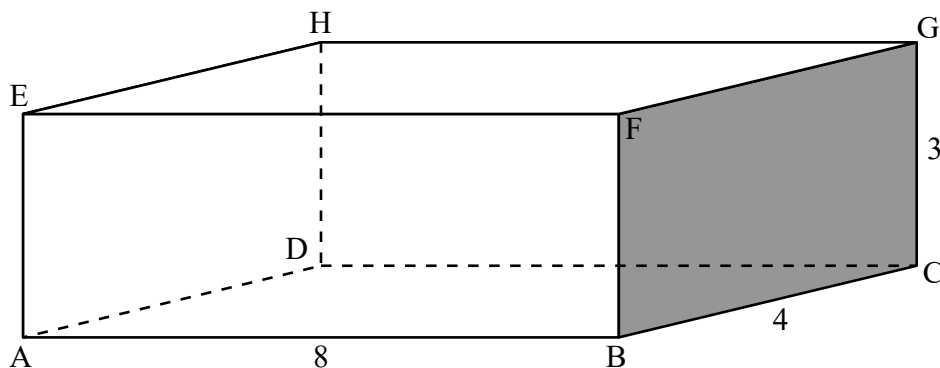
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1. [Maximum mark: 15]

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length 8 cm, width 4 cm and height 3 cm. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2 . [2]

(b) Calculate the length AG. [2]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{dP}{dx} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

The profit from the sale of 20 000 boxes is \$1700.

(d) Find $P(x)$. [5]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

7. [Maximum mark: 20]

An object is placed into the top of a long vertical tube, filled with a thick viscous fluid, at time $t = 0$.

Initially it is thought that the resistance of the fluid would be proportional to the velocity of the object. The following model was proposed, where the object's displacement, x , from the top of the tube, measured in metres, is given by the differential equation

$$\frac{d^2x}{dt^2} = 9.81 - 0.9 \left(\frac{dx}{dt} \right).$$

- (a) By substituting $v = \frac{dx}{dt}$ into the equation, find an expression for the velocity of the particle at time t . Give your answer in the form $v = f(t)$. [7]

The maximum velocity approached by the object as it falls is known as the terminal velocity.

- (b) From your solution to part (a), or otherwise, find the terminal velocity of the object predicted by this model. [2]

An experiment is performed in which the object is placed in the fluid on a number of occasions and its terminal velocity recorded. It is found that the terminal velocity was consistently smaller than that predicted by the model used. It was suggested that the resistance to motion is actually proportional to the velocity squared and so the following model was set up.

$$\frac{d^2x}{dt^2} = 9.81 - 0.9 \left(\frac{dx}{dt} \right)^2$$

- (c) Write down the differential equation as a system of first order differential equations. [2]
- (d) Use Euler's method, with a step length of 0.2, to find the displacement and velocity of the object when $t = 0.6$. [4]
- (e) By repeated application of Euler's method, find an approximation for the terminal velocity, to five significant figures. [1]

At terminal velocity the acceleration of an object is equal to zero.

- (f) Use the differential equation to find the terminal velocity for the object. [2]
- (g) Use your answers to parts (d), (e) and (f) to comment on the accuracy of the Euler approximation to this model. [2]

2. [Maximum mark: 28]

The number of brown squirrels, x , in an area of woodland can be modelled by the following differential equation.

$$\frac{dx}{dt} = \frac{x}{1000}(2000 - x), \text{ where } x > 0$$

- (a) (i) Find the equilibrium population of brown squirrels suggested by this model.
- (ii) Explain why the population of squirrels is increasing for values of x less than this value.

[3]

One year conservationists notice that some black squirrels are moving into the woodland. The two species of squirrel are in competition for the same food supplies. Let y be the number of black squirrels in the woodland.

Conservationists wish to predict the likely future populations of the two species of squirrels. Research from other areas indicates that when the two populations come into contact the growth can be modelled by the following differential equations, in which t is measured in tens of years.

$$\begin{aligned}\frac{dx}{dt} &= \frac{x}{1000}(2000 - x - 2y), \quad x, y \geq 0 \\ \frac{dy}{dt} &= \frac{y}{1000}(3000 - 3x - y), \quad x, y \geq 0\end{aligned}$$

An equilibrium point for the populations occurs when both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

- (b) (i) Verify that $x = 800$, $y = 600$ is an equilibrium point.
- (ii) Find the other three equilibrium points.

[6]

(This question continues on the following page)

(Question 2 continued)

When the two populations are small the model can be reduced to the linear system

$$\begin{aligned}\frac{dx}{dt} &= 2x \\ \frac{dy}{dt} &= 3y.\end{aligned}$$

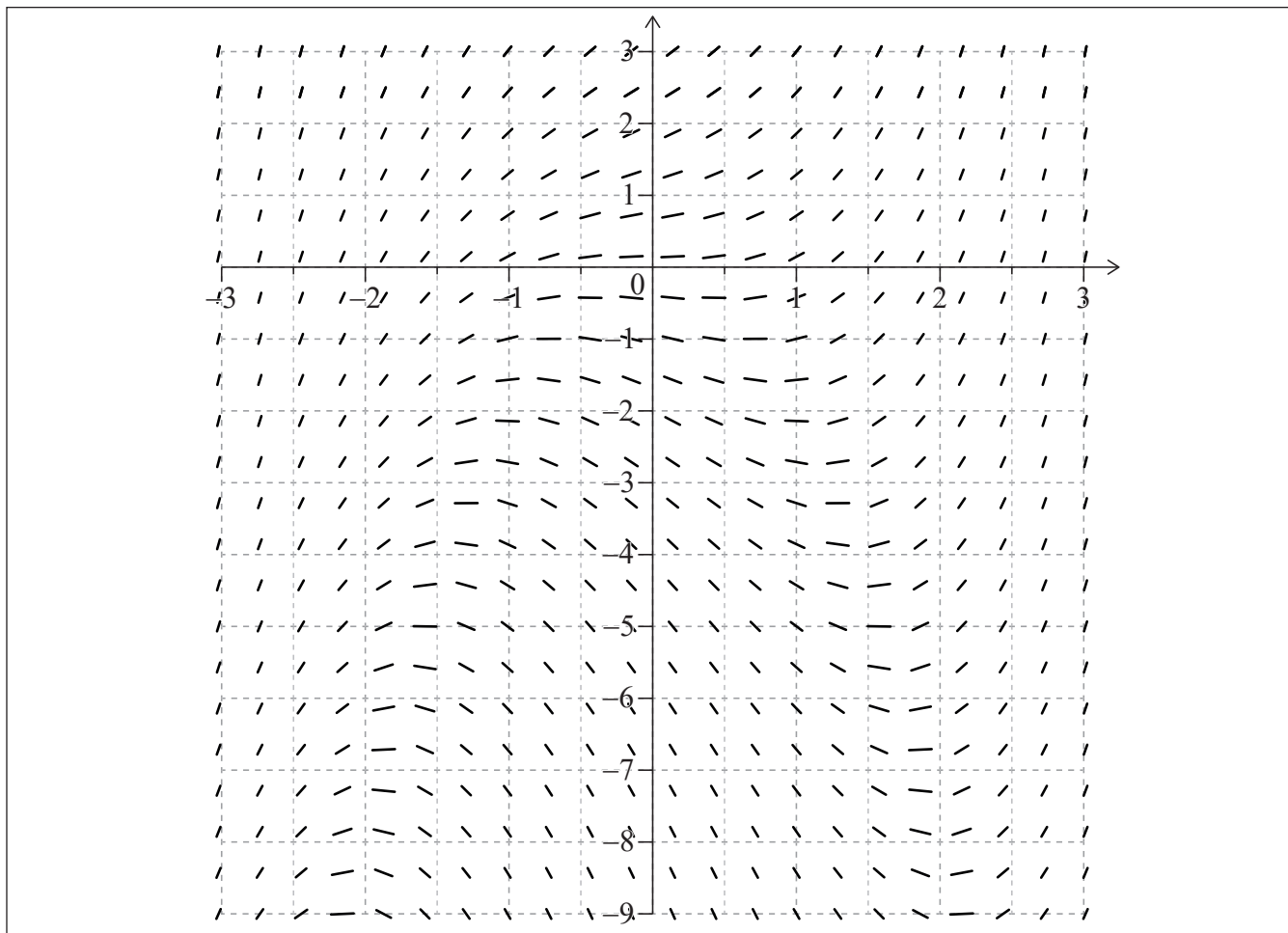
- (c) (i) By using separation of variables, show that the general solution of $\frac{dx}{dt} = 2x$ is $x = Ae^{2t}$.
- (ii) Write down the general solution of $\frac{dy}{dt} = 3y$.
- (iii) If both populations contain 10 squirrels at $t = 0$ use the solutions to parts (c) (i) and (ii) to estimate the number of black and brown squirrels when $t = 0.2$. Give your answers to the nearest whole numbers. [7]

For larger populations, the conservationists decide to use Euler's method to find the long-term outcomes for the populations. They will use Euler's method with a step length of 2 years ($t = 0.2$).

- (d) (i) Write down the expressions for x_{n+1} and y_{n+1} that the conservationists will use.
- (ii) Given that the initial populations are $x = 100$, $y = 100$, find the populations of each species of squirrel when $t = 1$.
- (iii) Use further iterations of Euler's method to find the long-term population for each species of squirrel from these initial values.
- (iv) Use the same method to find the long-term populations of squirrels when the initial populations are $x = 400$, $y = 100$. [7]
- (e) Use Euler's method with step length 0.2 to sketch, on the same axes, the approximate trajectories for the populations with the following initial populations.
- (i) $x = 1000$, $y = 1500$
- (ii) $x = 1500$, $y = 1000$ [3]
- (f) Given that the equilibrium point at $(800, 600)$ is a saddle point, sketch the phase portrait for $x \geq 0$, $y \geq 0$ on the same axes used in part (e). [2]

7. [Maximum mark: 4]

A slope field for the differential equation $\frac{dy}{dx} = x^2 + \frac{y}{2}$ is shown.



Some of the solutions to the differential equation have a local maximum point and a local minimum point.

- (a) (i) Write down the equation of the curve on which all these maximum and minimum points lie.

- (ii) Sketch this curve on the slope field.

[2]

The solution to the differential equation that passes through the point $(0, -2)$ has both a local maximum point and a local minimum point.

- (b) On the slope field, sketch the solution to the differential equation that passes through $(0, -2)$.

[2]

(This question continues on the following page)



(Question 7 continued)

[illegible]

Turn over

8. [Maximum mark: 6]

Consider the curve $y = 2x(4 - e^x)$.

(a) Find

(i) $\frac{dy}{dx}$.

(ii) $\frac{d^2y}{dx^2}$.

[4]

The curve has a point of inflexion at (a, b) .

(b) Find the value of a .

[2]



14. [Maximum mark: 8]

(a) (i) Expand $\left(\frac{1}{u} + 1\right)^2$.

(ii) Find $\int \left(\frac{1}{(x+2)} + 1\right)^2 dx$.

[4]

The region bounded by $y = \frac{1}{(x+2)} + 1$, $x = 0$, $x = 2$ and the x -axis is rotated through 2π about the x -axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a + b \ln(c))$, where $a, b, c \in \mathbb{Z}$.

[4]



Turn over

16. [Maximum mark: 7]

The wind chill index W is a measure of the temperature, in $^{\circ}\text{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour (km h^{-1}) is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

- (a) Find an expression for $\frac{dW}{dv}$. [2]

When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5 \text{ km h}^{-1} \text{ minute}^{-1}$.

- (b) Find the rate of change of W at this time. [5]

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Turn over

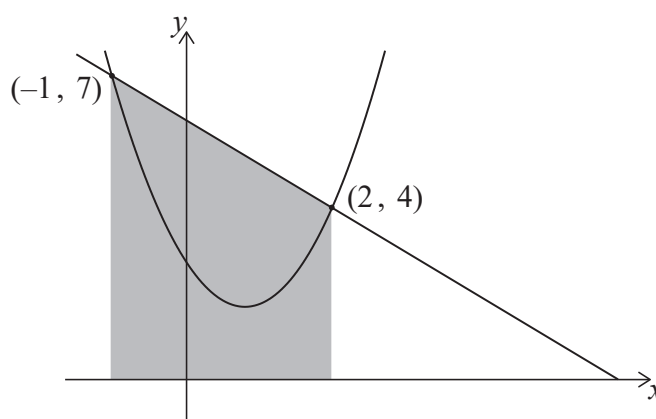
4. [Maximum mark: 7]

The graphs of $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$ intersect at $(2, 4)$ and $(-1, 7)$, as shown in the following diagrams.

In **diagram 1**, the region enclosed by the lines $y = 6 - x$, $x = -1$, $x = 2$ and the x -axis has been shaded.

diagram not to scale

Diagram 1



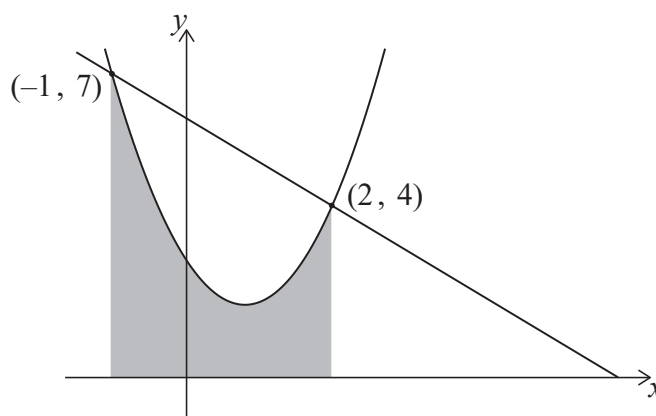
- (a) Calculate the area of the shaded region in **diagram 1**.

[2]

In **diagram 2**, the region enclosed by the curve $y = 1.5x^2 - 2.5x + 3$, and the lines $x = -1$, $x = 2$ and the x -axis has been shaded.

diagram not to scale

Diagram 2



- (b) (i) Write down an integral for the area of the shaded region in **diagram 2**.

- (ii) Calculate the area of this region.

[3]

- (c) Hence, determine the area enclosed between $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$.

[2]

(This question continues on the following page)



[illegible]

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14. [Maximum mark: 4]

The shape of a vase is formed by rotating a curve about the y -axis.

The vase is 10 cm high. The internal radius of the vase is measured at 2 cm intervals along the height:

Height (cm)	Radius (cm)
0	4
2	6
4	8
6	7
8	3
10	5

Use the trapezoidal rule to estimate the volume of water that the vase can hold.

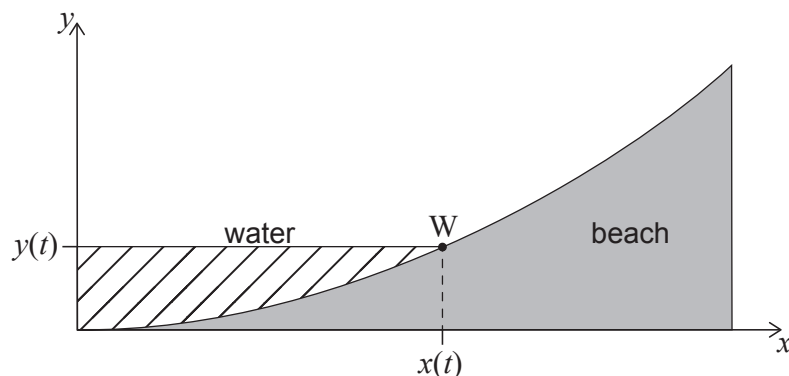
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17. [Maximum mark: 8]

The cross-section of a beach is modelled by the equation $y = 0.02x^2$ for $0 \leq x \leq 10$ where y is the height of the beach (in metres) at a horizontal distance x metres from an origin. t is the time in hours after low tide.

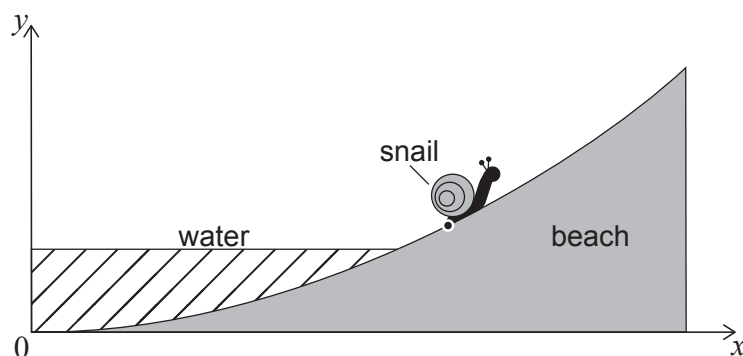
At $t = 0$ the water is at the point $(0, 0)$. The height of the water rises at a rate of 0.2 metres per hour. The point $W(x(t), y(t))$ indicates where the water level meets the beach at time t .



- (a) When W has an x -coordinate equal to 1, find the horizontal component of the velocity of W .

[3]

A snail is modelled as a single point. At $t = 0$ it is positioned at $(1, 0.02)$. The snail travels away from the incoming water at a speed of 1 metre per hour in the direction along the curve of the cross-section of the beach. The following diagram shows this for a value of t , such that $t > 0$.



- (b) (i) Find the time taken for the snail to reach the point $(10, 2)$.
 (ii) Hence show that the snail reaches the point $(10, 2)$ before the water does.

[5]

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4. [Maximum mark: 13]

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

- (a) (i) Use the substitution $y = \frac{dx}{dt}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- (ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

- (iii) Hence state the long-term velocity of the particle.

[5]

The equation for the motion of the particle is amended to

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 3t + 4.$$

- (b) (i) Use the substitution $y = \frac{dx}{dt}$ to write the differential equation as a system of coupled, first order differential equations.

When $t = 0$ the particle is stationary at O.

- (ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$.

- (iii) Find the long-term velocity of the particle.

[8]

6. [Maximum mark: 18]

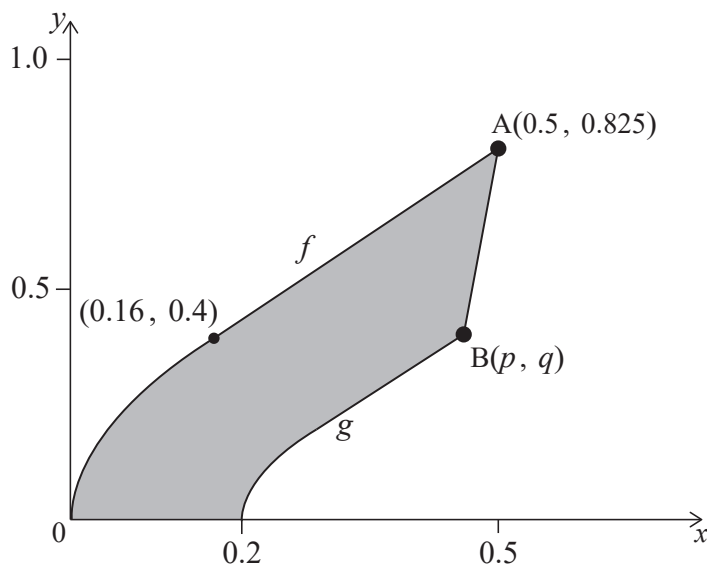
Consider the curve $y = \sqrt{x}$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence show that the equation of the tangent to the curve at the point $(0.16, 0.4)$ is $y = 1.25x + 0.2$.

[4]

The shape of a piece of metal can be modelled by the region bounded by the functions f , g , the x -axis and the line segment $[AB]$, as shown in the following diagram. The units on the x and y axes are measured in metres.



The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \leq x \leq 0.16 \\ 1.25x + 0.2 & 0.16 < x \leq 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5, 0.825)$. Point B is the image of A under the given transformations and has coordinates (p, q) .

(b) Find the value of p and the value of q .

[2]

(This question continues on the following page)

(Question 6 continued)

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \leq x \leq a \\ 1.25x + b & a < x \leq p \end{cases}$$

(c) Find

(i) an expression for $h(x)$.

(ii) the value of a .

(iii) the value of b .

[5]

(d) (i) Find the area enclosed by $y = f(x)$, the x -axis and the line $x = 0.5$.

The area enclosed by $y = g(x)$, the x -axis and the line $x = p$ is 0.0627292 m^2 correct to six significant figures.

(ii) Find the area of the shaded region on the diagram.

[7]

Turn over

2. [Maximum mark: 15]

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C , measured in dollars, is modelled by the following equation

$$C = \frac{x}{10} \left(k^2 - \frac{3}{100} x^2 \right)$$

where k is a positive constant.

(a) Find an expression for $\frac{dC}{dx}$ in terms of k and x . [3]

(b) Hence find the maximum value of C in terms of k . Give your answer in the form pk^3 , where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of \$426 when 20 litres of coffee are made in a morning.

(c) (i) Find the value of k .
(ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [3]

(d) Sketch the graph of C against x , labelling the maximum point and the x -intercepts with their coordinates. [3]

The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

Turn over

7. [Maximum mark: 16]

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{d^2x}{dt^2} + 3 \frac{dx}{dt} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(a) Show that the system of coupled first order equations:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2x - 3y \end{aligned}$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t , labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

References:

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1. [Maximum mark: 28]

This question is about modelling the spread of a computer virus to predict the number of computers in a city which will be infected by the virus.

A systems analyst defines the following variables in a model:

- t is the number of days since the first computer was infected by the virus.
- $Q(t)$ is the total number of computers that have been infected up to and including day t .

The following data were collected:

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146

- (a) (i) Find the equation of the regression line of $Q(t)$ on t . [2]
- (ii) Write down the value of r , Pearson's product-moment correlation coefficient. [1]
- (iii) Explain why it would not be appropriate to conduct a hypothesis test on the value of r found in (a)(ii). [1]

A model for the early stage of the spread of the computer virus suggests that

$$Q'(t) = \beta N Q(t)$$

where N is the total number of computers in a city and β is a measure of how easily the virus is spreading between computers. Both N and β are assumed to be constant.

- (b) (i) Find the general solution of the differential equation $Q'(t) = \beta N Q(t)$. [4]
- (ii) Using the data in the table write down the equation for an appropriate non-linear regression model. [2]
- (iii) Write down the value of R^2 for this model. [1]
- (iv) Hence comment on the suitability of the model from (b)(ii) in comparison with the linear model found in part (a). [2]
- (v) By considering large values of t write down one criticism of the model found in (b)(ii). [1]

(This question continues on the following page)

(Question 1 continued)

- (c) Use your answer from part (b)(ii) to estimate the time taken for the number of infected computers to double. [2]

The data above are taken from city X which is estimated to have 2.6 million computers. The analyst looks at data for another city, Y. These data indicate a value of $\beta = 9.64 \times 10^{-8}$.

- (d) Find in which city, X or Y, the computer virus is spreading more easily. Justify your answer using your results from part (b). [3]

An estimate for $Q'(t)$, $t \geq 5$, can be found by using the formula:

$$Q'(t) \approx \frac{Q(t+5) - Q(t-5)}{10}.$$

The following table shows estimates of $Q'(t)$ for city X at different values of t .

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146
$Q'(t)$		a	171.6	766.7	b	11 207.6	

- (e) Determine the value of a and of b . Give your answers correct to one decimal place. [2]

An improved model for $Q(t)$, which is valid for large values of t , is the logistic differential equation

$$Q'(t) = kQ(t) \left(1 - \frac{Q(t)}{L} \right)$$

where k and L are constants.

Based on this differential equation, the graph of $\frac{Q'(t)}{Q(t)}$ against $Q(t)$ is predicted to be a straight line.

- (f) (i) Use linear regression to estimate the value of k and of L . [5]

- (ii) The solution to the differential equation is given by

$$Q(t) = \frac{L}{1 + Ce^{-kt}}$$

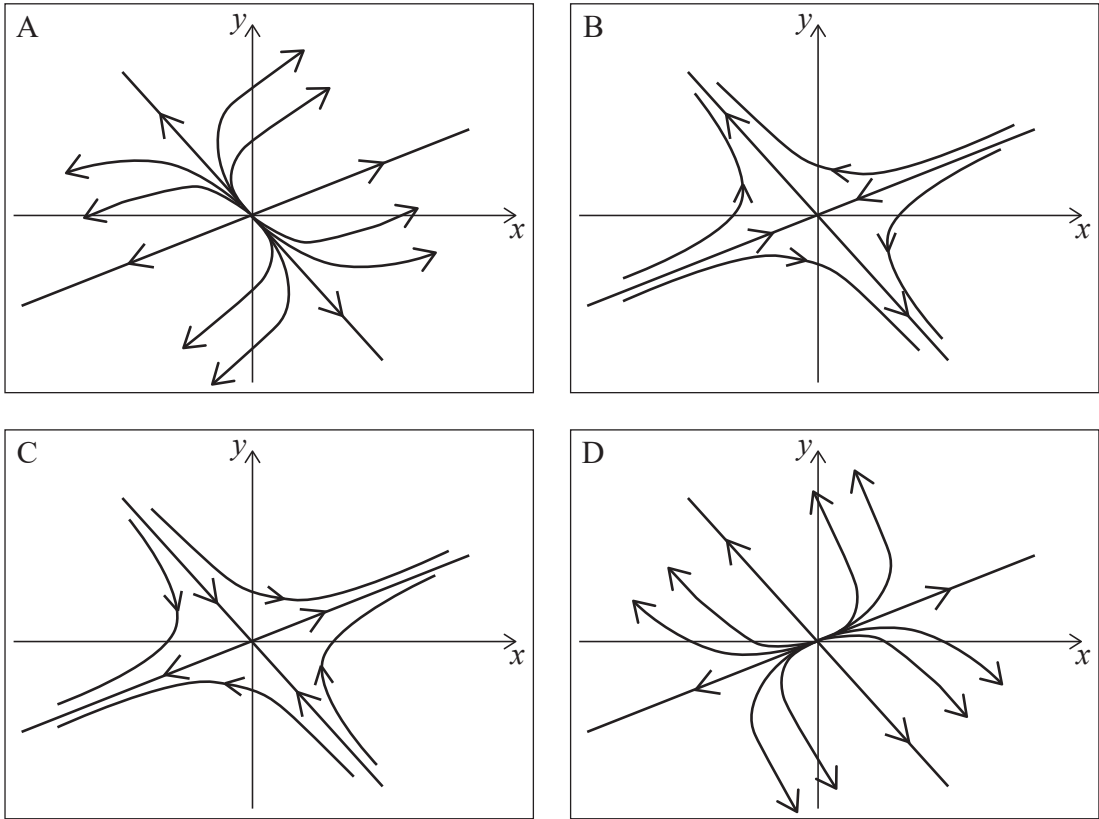
where C is a constant.

Using your answer to part (f)(i), estimate the percentage of computers in city X that are expected to have been infected by the virus over a long period of time. [2]

Turn over

12. [Maximum mark: 5]

Four possible phase portraits for the coupled differential equations $\frac{dx}{dt} = ax + by$ and $\frac{dy}{dt} = cx + dy$ are shown, labelled A, B, C and D.



The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2 .

- (a) Complete the following table by writing down the letter of the phase portrait that best matches the description.

[3]

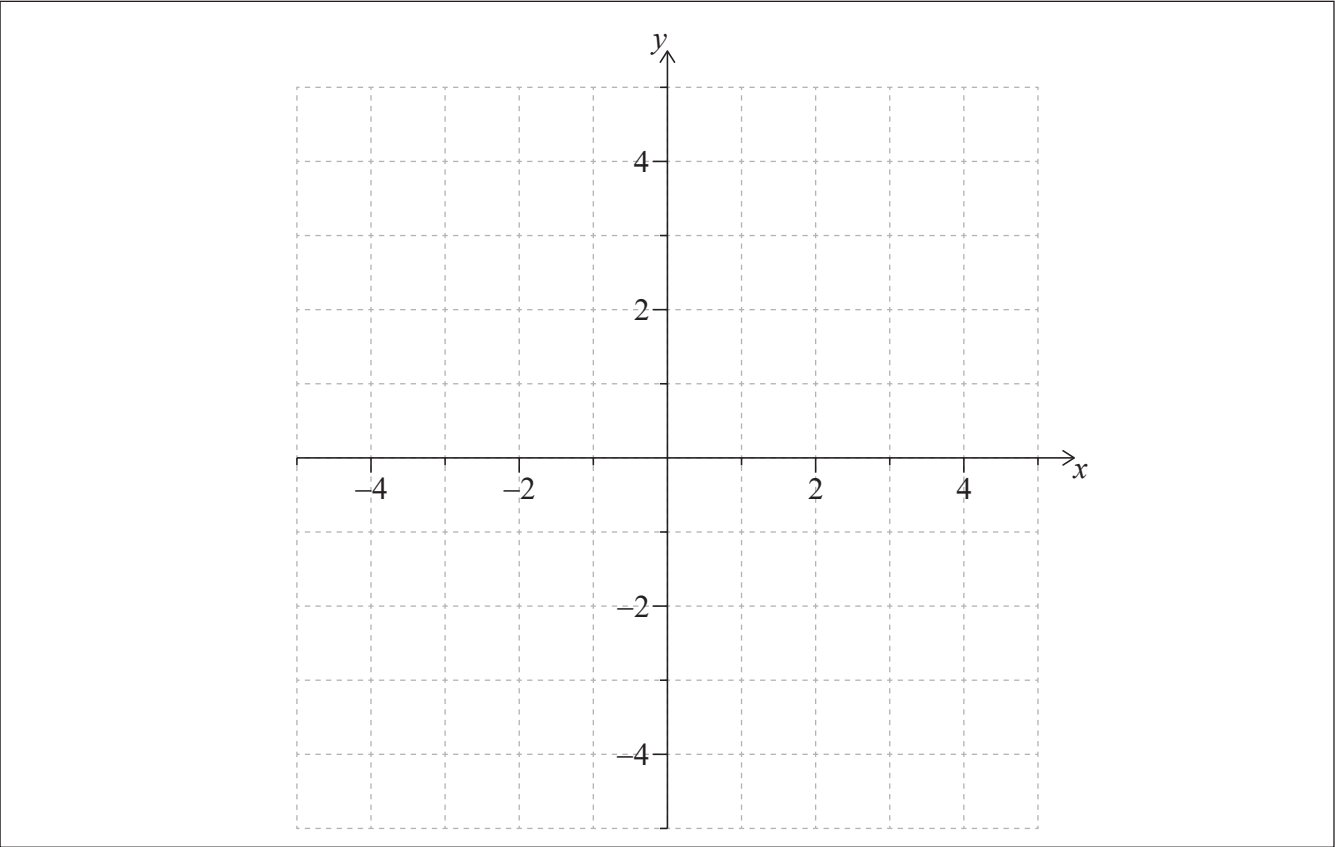
Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	

(This question continues on the following page)



(Question 12 continued)

- (b) On the following axes, sketch the phase portrait that corresponds to $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$, given that $\frac{dy}{dt} = -12$ at $(3, 0)$. [2]



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Turn over

A particle moves such that its velocity, v metres per second, at time t seconds, is given by $v = t \sin(t^2)$.

- The particle starts at the origin.

- [illegible]



15. [Maximum mark: 5]

An electrical circuit contains a capacitor. The charge on the capacitor, q Coulombs, at time t seconds, satisfies the differential equation

$$\frac{d^2q}{dt^2} + 5\frac{dq}{dt} + 20q = 200.$$

Initially $q = 1$ and $\frac{dq}{dt} = 8$.

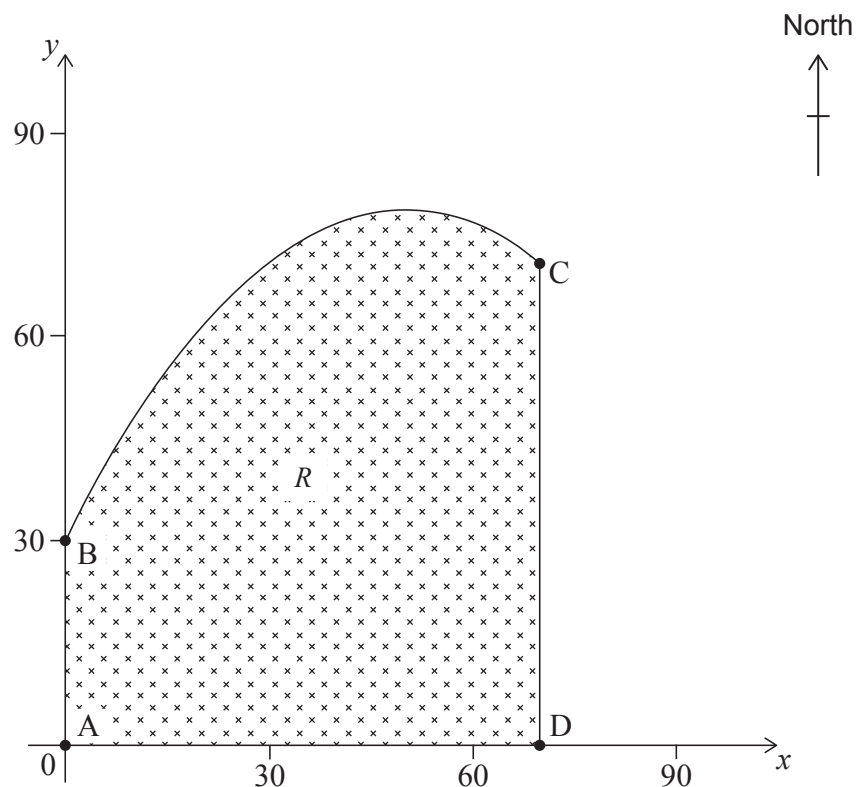
Use Euler's method with $h = 0.1$ to estimate the maximum charge on the capacitor during the first second.

[illegible]

Turn over

3. [Maximum mark: 17]

Linda owns a field, represented by the shaded region R . The plan view of the field is shown in the following diagram, where both axes represent distance and are measured in metres.



The segments $[AB]$, $[CD]$ and $[AD]$ respectively represent the western, eastern and southern boundaries of the field. The function, $f(x)$, models the northern boundary of the field between points B and C and is given by

$$f(x) = \frac{-x^2}{50} + 2x + 30, \text{ for } 0 \leq x \leq 70.$$

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the point on the field that is furthest north.

[5]

Point A has coordinates $(0, 0)$, point B has coordinates $(0, 30)$, point C has coordinates $(70, 72)$ and point D has coordinates $(70, 0)$.

(b) (i) Write down the integral which can be used to find the area of the shaded region R .

(ii) Find the area of Linda's field.

[4]

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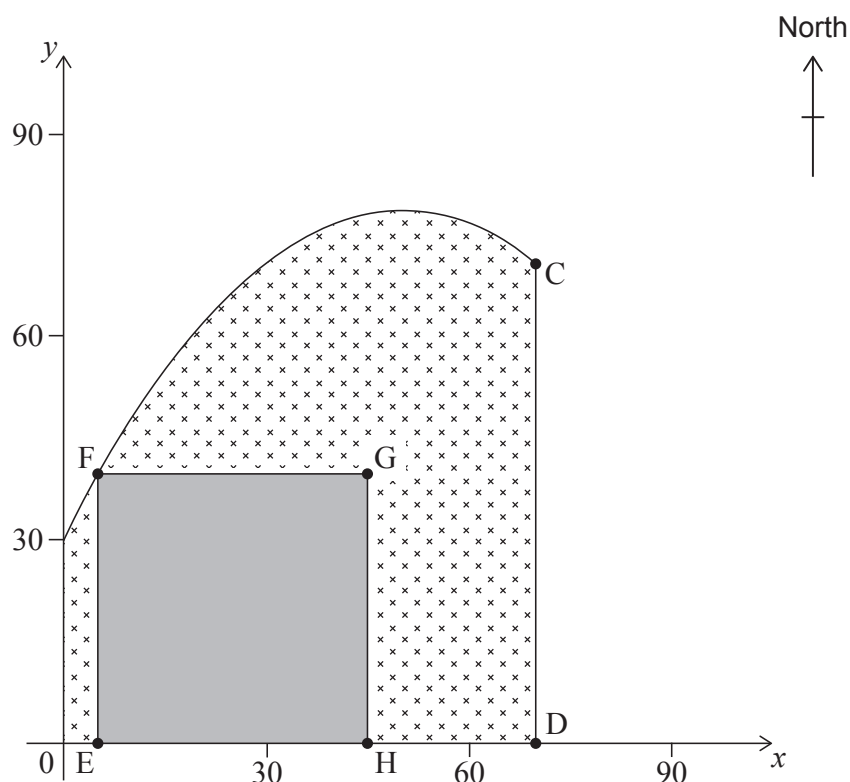
(Question 3 continued)

Linda used the trapezoidal rule with ten intervals to estimate the area. This calculation underestimated the area by 11.4 m^2 .

- (c) (i) Calculate the percentage error in Linda's estimate.
- (ii) Suggest how Linda might be able to reduce the error whilst still using the trapezoidal rule.

[3]

Linda would like to construct a building on her field. The **square** foundation of the building, EFGH, will be located such that [EH] is on the southern boundary and point F is on the northern boundary of the property. A possible location of the foundation of the building is shown in the following diagram.



The area of the square foundation will be largest when [GH] lies on [CD].

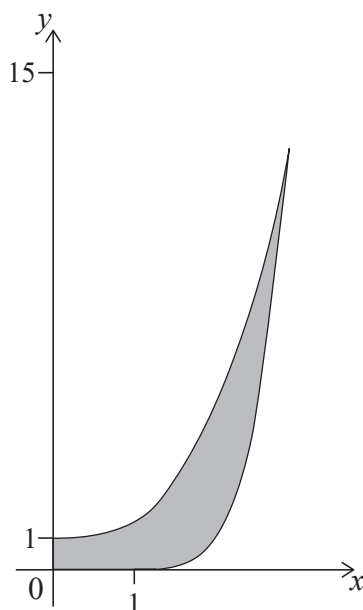
- (d) (i) Find the x -coordinate of point E for the largest area of the square foundation of building EFGH.
- (ii) Find the largest area of the foundation.

[5]

Turn over

5. [Maximum mark: 13]

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents 1 cm.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \leq x \leq p$.

The outer surface is modelled by $g(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ (x-1)^4 & \text{for } 1 \leq x \leq p \end{cases}$.

- (a) Find the value of p . [2]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y -axis.

- (b) Find the volume of liquid that can be contained inside the finished glass. [5]
- (c) Find the volume of the region between the two surfaces of the finished glass. [6]

2. [Maximum mark: 26]

Some medical conditions require patients to take medication regularly for long periods of time. In this question, you will explore the concentration of a medicinal drug in the body, when the drug is given repeatedly.

Once a drug enters the body, it is absorbed into the blood. As the body breaks down the drug over time, the concentration of the drug decreases. Let $C(t)$, measured in milligrams per millilitre (mg ml^{-1}), be the concentration of the drug, t hours after the drug is given to the patient. The rate at which the drug is broken down is modelled as directly proportional to its concentration, leading to the differential equation

$$\frac{dC}{dt} = -kC, \text{ where } k \in \mathbb{R}^+.$$

The initial concentration is $d \text{ mg ml}^{-1}$, $d > 0$.

- (a) By solving the differential equation, show that $C = de^{-kt}$. [3]

For the remainder of this question, you will consider a particular drug where it is known that $k = 0.2$. The first dose is given at time $t = 0$ and it is assumed that before this there is no drug present in the blood.

- (b) Find the time, in hours, for this drug to reach 5% of its initial concentration. [2]

The drug is to be given every T hours and in constant doses, such that the concentration of the drug is increased by an amount $d \text{ mg ml}^{-1}$. To simplify the model, it is assumed that each time the drug is given the concentration of the drug in the blood increases instantaneously.

- (c) Show that the concentration of the drug is $d(1 + e^{-0.2T} + e^{-0.4T})$ immediately after the third dose is given. [4]

Immediately after the n th dose is given, the concentration of the drug is

$$d(1 + e^{-0.2T} + e^{-0.4T} + \dots + e^{-0.2(n-1)T}).$$

- (d) Show that this concentration can be expressed as $d\left(\frac{1 - e^{-0.2nT}}{1 - e^{-0.2T}}\right)$. [2]

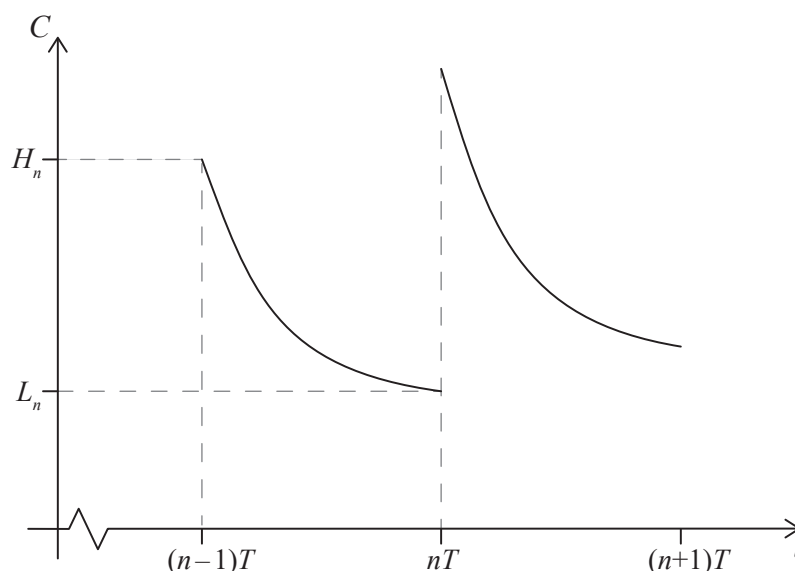
(This question continues on the following page)

(Question 2 continued)

After a patient has been taking this drug for a long time, it is required to keep the concentration within a particular range so that it is both safe and effective.

Let H_n be the highest concentration of the drug in the body for the interval $(n-1)T \leq t < nT$.

Let L_n be the lowest concentration of the drug in the body for the interval $(n-1)T \leq t < nT$. This is shown in the following graph.



H_∞ is defined as $\lim_{n \rightarrow \infty} H_n$ and L_∞ is defined as $\lim_{n \rightarrow \infty} L_n$.

(e) Find, in terms of d and T , an expression for

(i) H_∞ . [2]

(ii) L_∞ . [3]

(f) Show that

(i) $H_\infty - L_\infty = d$. [2]

(ii) $5 \ln \left(\frac{H_\infty}{L_\infty} \right) = T$. [3]

(This question continues on the following page)

Turn over

(Question 2 continued)

It is known that this drug is ineffective if the long-term concentration is less than 0.06 mg ml^{-1} and safe if it never exceeds 0.28 mg ml^{-1} .

(g) Hence, for this drug, find a suitable value for

(i) d . [1]

(ii) T . [1]

(h) For the values of d and T found in part (g), find the proportion of time for which the concentration of the drug is at least 0.06 mg ml^{-1} between the first and second doses. [2]

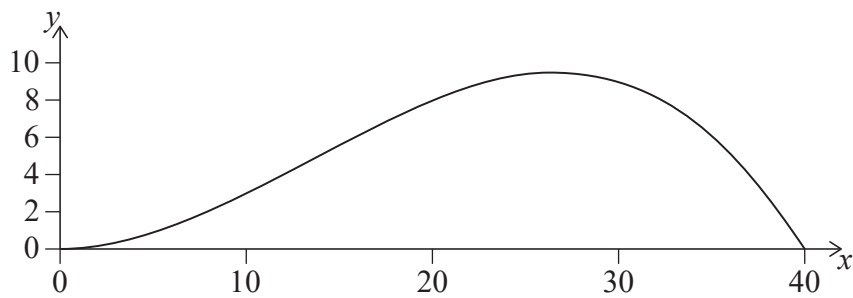
(i) Suggest a reason why the instructions on the label of the drug might use a different value for T to that found in part (g)(ii). [1]

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4. [Maximum mark: 8]

The cross section of a scale model of a hill is modelled by the following graph.



The heights of the model are measured at horizontal intervals and are given in the table.

Horizontal distance, x cm	0	10	20	30	40
Vertical distance, y cm	0	3	8	9	0

- (a) Use the trapezoidal rule with $h = 10$ to find an approximation for the cross-sectional area of the model. [2]

It is given that the equation of the curve is $y = 0.04x^2 - 0.001x^3$, $0 \leq x \leq 40$.

- (b) (i) Write down an integral to find the exact cross-sectional area.
- (ii) Calculate the value of the cross-sectional area to two decimal places. [4]
- (c) Find the percentage error in the area found using the trapezoidal rule. [2]

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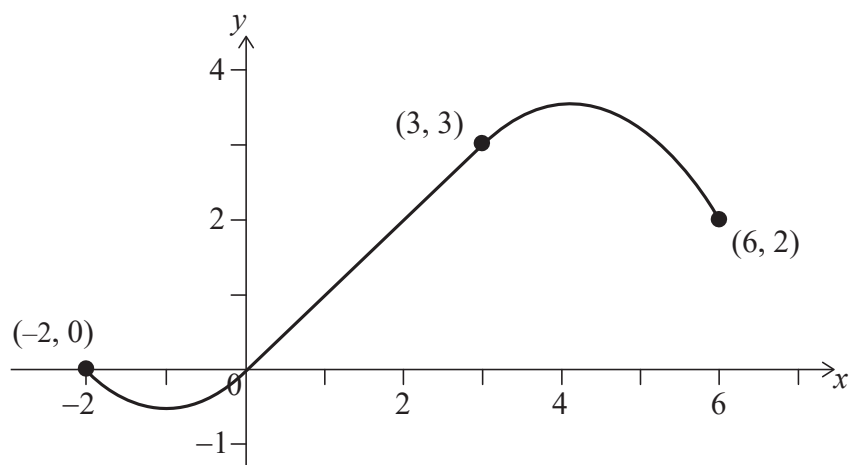
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10. [Maximum mark: 9]

A decorative hook can be modelled by the curve with equation $y = f(x)$. The graph of $y = f(x)$ is shown and consists of a line segment from $(0, 0)$ to $(3, 3)$ and two sections formed by quadratic curves.



- (a) Write down the equation of the line segment for $0 \leq x \leq 3$. [1]

The quadratic curve, with endpoints $(-2, 0)$ and $(0, 0)$, has the same gradient at $(0, 0)$ as the line segment.

- (b) Find the equation of the curve between $(-2, 0)$ and $(0, 0)$. [3]

The second quadratic curve, with endpoints $(3, 3)$ and $(6, 2)$, has the same gradient at $(3, 3)$ as the line segment.

- (c) Find the equation of this curve. [4]

- (d) Write down f as a piecewise function. [1]

(This question continues on the following page)



(Question 10 continued)

[illegible]

Turn over

13. [Maximum mark: 6]

The displacement, $x(\text{cm})$, of the end of a spring, at time t (seconds), is given by

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0.$$

At $t = 0$, $x = 0.75$ and $\frac{dx}{dt} = 0$.

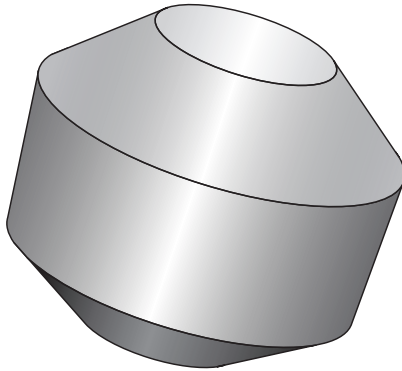
Use Euler's method, with a step length 0.1 seconds, to estimate the value of x when $t = 0.5$.

[illegible]

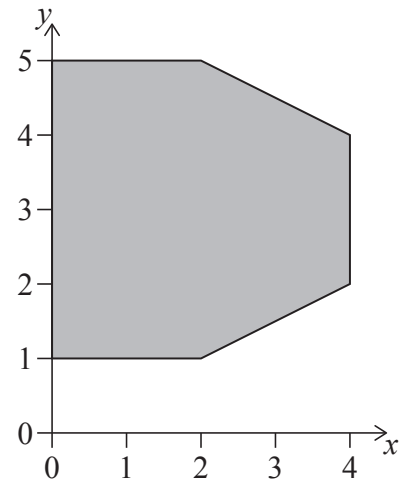
15. [Maximum mark: 6]

The solid shown is formed by rotating the hexagon with vertices $(2, 1)$, $(0, 1)$, $(0, 5)$, $(2, 5)$, $(4, 4)$ and $(4, 2)$ about the y -axis.

Solid



Hexagon



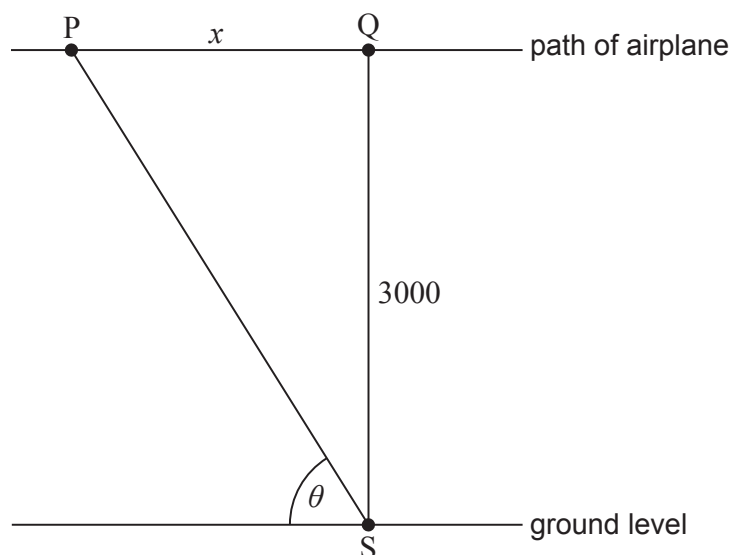
Find the volume of this solid.

[illegible]

17. [Maximum mark: 9]

An airplane, P, is flying at a constant altitude of 3000 m at a speed of 250 m s^{-1} . Its path passes over a tracking station, S, at ground level. Let Q be the point 3000 m directly above the tracking station.

At a particular time, T , as the airplane is flying towards Q, the angle of elevation, θ , of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x .



- Use related rates to show that, at time T , $\frac{dx}{d\theta} = -\frac{10\,000}{3}$. [2]
- Find $x(\theta)$, x as a function of θ . [1]
- Find an expression for $\frac{dx}{d\theta}$ in terms of $\sin \theta$. [3]
- Hence find the horizontal distance from the station to the plane at time T . [3]

(This question continues on the following page)



[illegible]

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12. [Maximum mark: 5]

A spherical balloon is being inflated such that its volume is increasing at a rate of $15 \text{ cm}^3 \text{ s}^{-1}$.

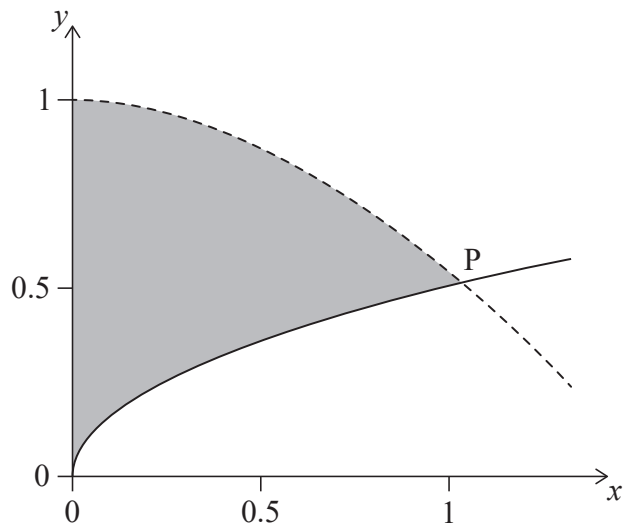
- (a) Find the radius of the balloon when its volume is $288\pi \text{ cm}^3$. [2]
- (b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

[illegible]

16. [Maximum mark: 9]

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.



- (a) Use your graphic display calculator to find the coordinates of P. [2]

The shaded region is rotated 360° **about the y-axis** to form a volume of revolution V .

- (b) Express V as the sum of two definite integrals. [5]
(c) Hence find the value of V . [2]



17. [Maximum mark: 6]

Consider the differential equation

$$(x^2 + 1) \frac{dy}{dx} = \frac{x}{2y - 2}, \text{ for } x \geq 0, y \geq 1,$$

where $y = 1$ when $x = 0$.

- (a) Explain why Euler’s method cannot be used to find an approximate value for y when $x = 0.1$.

[1]
- (b) By solving the differential equation, show that $y = 1 + \sqrt{\frac{\ln(x^2 + 1)}{2}}$.

[4]
- (c) Hence deduce the value of y when $x = 0.1$.

[1]

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References:

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6. [Maximum mark: 15]

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{dx}{dt} = 5y - 0.05x, \quad \frac{dy}{dt} = -5x - 0.05y,$$

where x metres is the distance east and y metres is the distance north of a fixed point O .

- (a) Find the eigenvalues of $A = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + bi$, where $a \neq 0$, $b \neq 0$. [4]

- (b) (i) State what $a \neq 0$ indicates about the path of the speedboat.

- (ii) State what the sign of a indicates about the path of the speedboat. [2]

At time $t = 0$, the speedboat has position $(20, 0)$.

- (c) At time $t = 0$, find the value of

(i) $\frac{dy}{dt}$.

(ii) $\frac{dy}{dx}$. [5]

- (d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

Turn over

3. [Maximum mark: 16]

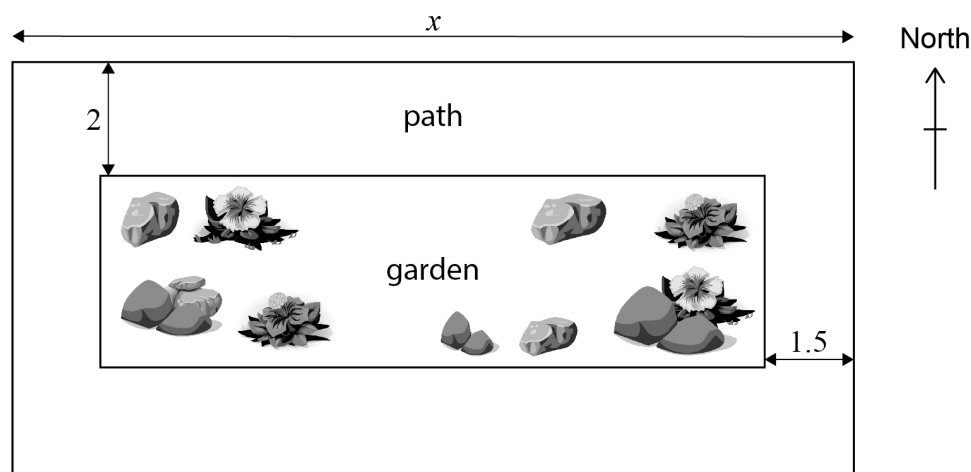
A particular park consists of a rectangular garden, of area $A \text{ m}^2$, and a concrete path surrounding it. The park has a total area of 1200 m^2 .

The width of the path at the north and south side of the park is 2 m.

The width of the path at the west and east side of the park is 1.5 m.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



- Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]
- Find the possible dimensions of the park if the area of the garden is 800 m^2 . [4]
- Find an expression for $\frac{dA}{dx}$. [3]
- Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]
- Find the maximum possible area of the garden. [2]

Turn over

7. [Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{dx}{dt} = -4x + 6y$$

$$\frac{dy}{dt} = 9x - y$$

- (a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]
- (b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathbf{p}_1e^{\lambda_1 t} + B\mathbf{p}_2e^{\lambda_2 t}$, where $A, B, \lambda_1, \lambda_2$ ($\lambda_2 > \lambda_1$) are scalar constants and $\mathbf{p}_1, \mathbf{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

- (c) Determine the value of A and the value of B . [2]
- (d) State the long-term ratio of male fruit flies to female fruit flies. [1]
- (e) Find the value of $\frac{dy}{dx}$ at time $t = 0$. [3]
- (f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]

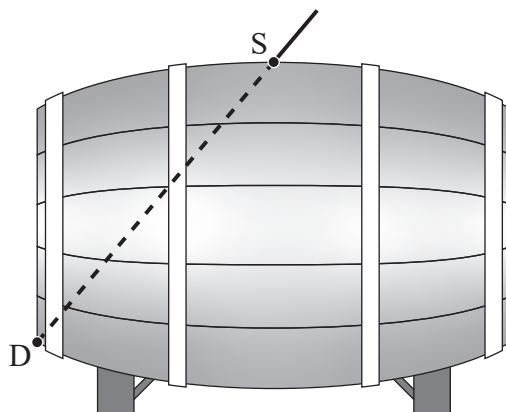
References:

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1. [Maximum mark: 26]

In this question you will use a historic method of calculating the cost of a barrel of wine to determine which shape of barrel gives the best value for money.

In Austria in the 17th century, one method for measuring the volume of a barrel of wine, and hence determining its cost, was by inserting a straight stick into a hole in the side, as shown in the following diagram, and measuring the length SD . The longer the length, the greater the cost to the customer.



Let SD be d metres and the cost be C gulden (the local currency at the time). When the length of SD was 0.5 metres, the cost was 0.80 gulden.

- (a) Given that C was directly proportional to d , find an equation for C in terms of d . [3]

A particular barrel of wine cost 0.96 gulden.

- (b) Show that $d = 0.6$. [1]

(This question continues on the following page)

Turn over

(Question 1 continued)

This method of determining the cost was noticed by a mathematician, Kepler, who decided to try to calculate the dimensions of a barrel which would give the maximum volume of wine for a given length SD .

Initially he modelled the barrel as a cylinder, with S at the midpoint of one side. He took the length of the cylinder as $2h$ metres and its radius as r metres, as shown in the following diagram of the cross-section.

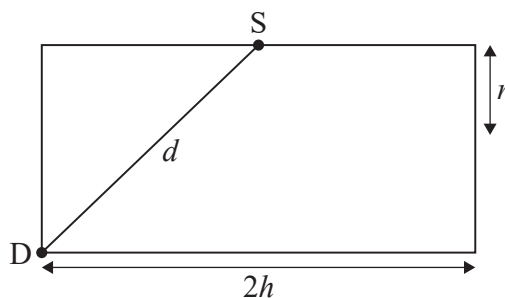


diagram not to scale

- (c) Find an expression for r^2 in terms of d and h . [3]

Let the volume of this barrel be $V \text{ m}^3$.

- (d) Show that $V = \frac{\pi}{2}(d^2h - h^3)$. [2]

The remainder of this question considers the shape of barrel that gives the best value when $d = 0.6$.

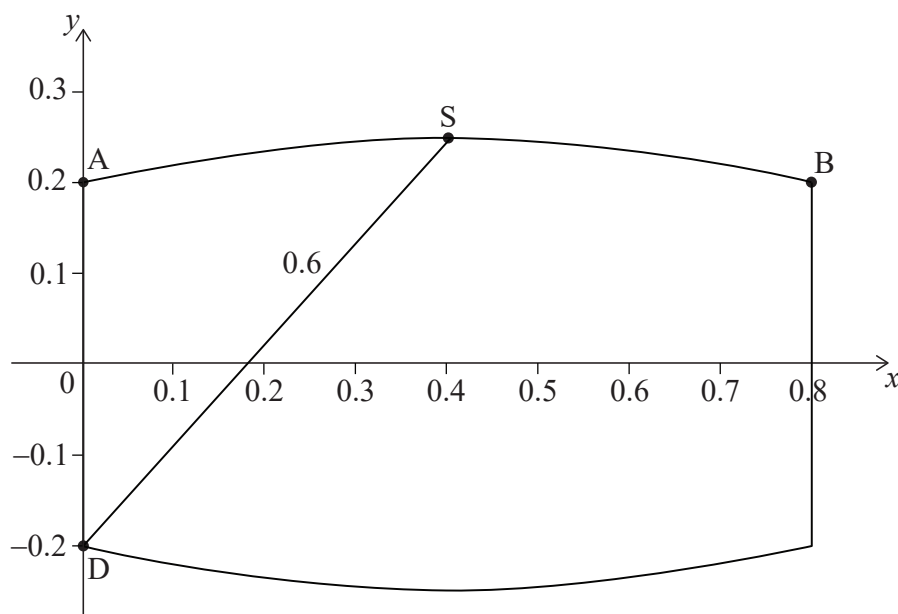
- (e) (i) Use the formula from part (d) to find the volume of this barrel when $h = 0.4$. [2]
- (ii) Use differentiation to show that $h = \sqrt{0.12}$ when $\frac{dV}{dh} = 0$. [3]
- (iii) Given that this value of h maximizes the volume, find the largest possible volume of this barrel. [2]

(This question continues on the following page)

(Question 1 continued)

Kepler then considered a non-cylindrical barrel whose base and lid are circles with radius 0.2 m and whose length is 0.8 m.

He modelled the curved surface of this barrel by rotating a quadratic curve, ASB, with equation $y = ax^2 + bx + c$, $0 \leq x \leq 0.8$, about the x -axis. The origin of the coordinate system is at the centre of one of the circular faces as shown in the following diagram. S is at the vertex of the quadratic curve and $SD = 0.6$.



Kepler wished to find out if his barrel would give him more wine than any cylindrical barrel with $d = 0.6$.

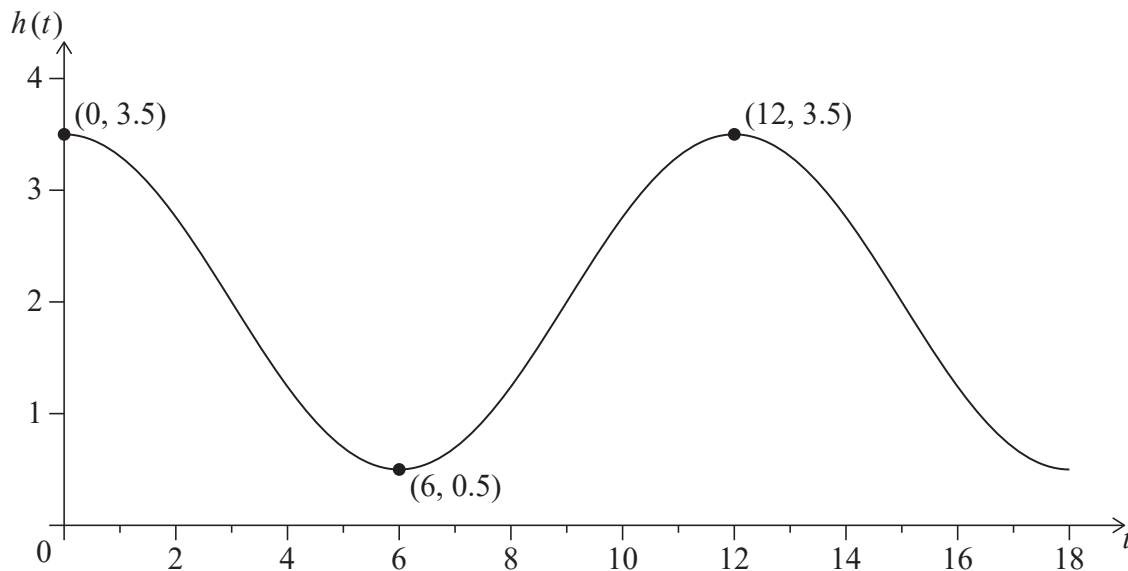
The coordinates of A and B are $(0, 0.2)$ and $(0.8, 0.2)$ respectively.

- (f) Find the equation of the quadratic curve, ASB. [6]
- (g) Show that the volume of this barrel is greater than the maximum volume of any cylindrical barrel with $d = 0.6$. [3]
- (h) State one assumption, not already given, that has been made in using these models to find the shape of the barrel that gives the best value. [1]

Turn over

9. [Maximum mark: 8]

Joon is a keen surfer and wants to model waves passing a particular point P, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and t , the time in seconds from when he begins recording the height of the water at point P.



The function has the form $h(t) = p \cos\left(\frac{\pi}{6}t\right) + q$, $t \geq 0$.

(a) Find the values of p and q . [2]

(b) Find

(i) $h'(t)$.

(ii) $h''(t)$. [3]

Joon will begin to surf the wave when the rate of change of h with respect to t , at P, is at its maximum. This will first occur when $t = k$.

(c) (i) Find the value of k .

(ii) Find the height of the water at this time. [3]

(This question continues on the following page)



(Question 9 continued)

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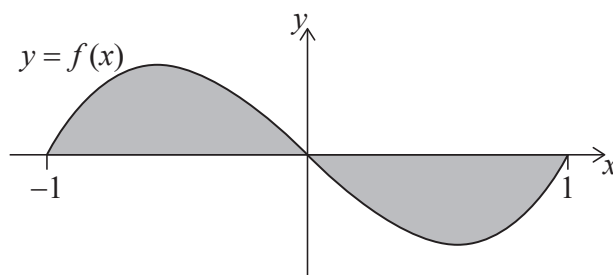
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Turn over

11. [Maximum mark: 7]

Consider the function $f(x) = x^3 - x$, for $-1 \leq x \leq 1$. The shaded region, R , is bounded by the graph of $y = f(x)$ and the x -axis.



(a) (i) Write down an integral that represents the area of R .

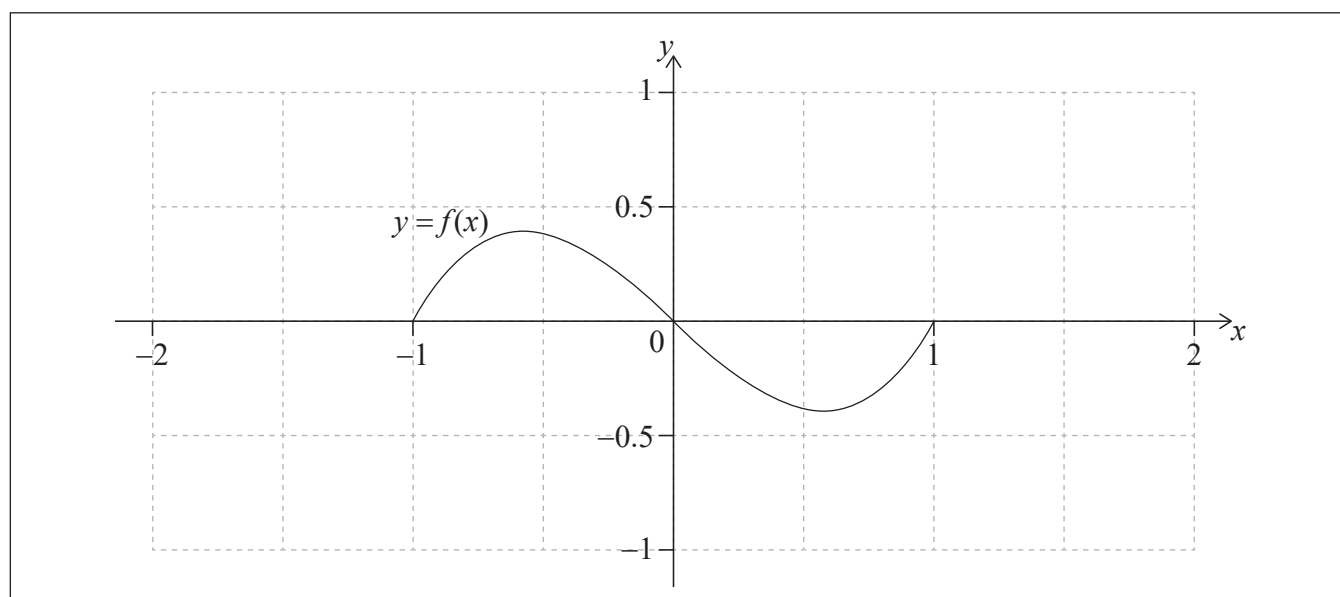
(ii) Find the area of R .

[2]

Another function, g , is defined such that $g(x) = 2f(x - 1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$.

[2]



The region R from the original graph $y = f(x)$ is rotated through 2π radians about the x -axis to form a solid.

(c) Find the volume of the solid.

[3]

(This question continues on the following page)



(Question 11 continued)

[illegible]

Turn over

13. [Maximum mark: 8]

The velocity v of a particle at time t , as it moves along a straight line, can be modelled by the piecewise function

$$v(t) = \begin{cases} u_1(t), & 0 \leq t \leq T \\ u_2(t), & t \geq T \end{cases}$$

where $u_1(t) = 2t^2 - t^3$ and $u_2(t) = 8 - 4t$. It is required that $u_1(T) = u_2(T)$.

- (a) Find the value of T . [2]
- (b) Show that $u_1'(T) = u_2'(T)$. [2]

The displacement of the particle at time $t = 0$ is zero.

- (c) Find the time when the particle returns to its initial position. [4]

[illegible]

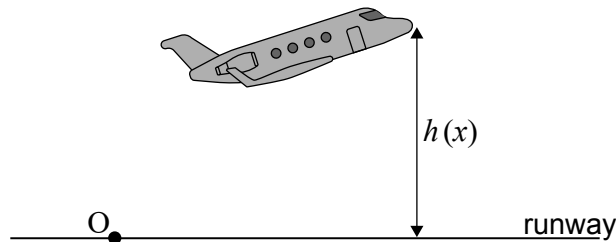
Turn over

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1 + 150e^{-0.07x}} - 0.06, \quad x \geq 0.$$

diagram not to scale



- (a) (i) Find $h(0)$.
 (ii) Interpret this value in terms of the context. [2]
- (b) (i) Find the horizontal asymptote of the graph of $y = h(x)$.
 (ii) Interpret this value in terms of the context. [2]
- (c) Find $h'(x)$ in terms of x . [4]
- A safety regulation recommends that $h'(x)$ never exceed 0.2.
- (d) Given that this plane flies a distance of at least 200 km horizontally from point O, determine whether the plane is following this safety regulation. [4]

Turn over

1. [Maximum mark: 28]

This question uses differential equations to model the maximum velocity of a skydiver in free fall.

In 2012, Felix Baumgartner jumped from a height of 40 000 m. He was attempting to travel at the speed of sound, 330 m s^{-1} , whilst free-falling to the Earth.

Before making his attempt, Felix used mathematical models to check how realistic his attempt would be. The simplest model he used suggests that

$$\frac{dv}{dt} = g$$

where $v \text{ m s}^{-1}$ is Felix's velocity and $g \text{ m s}^{-2}$ is the acceleration due to gravity. The time since he began to free-fall is t seconds and the displacement from his initial position is s metres.

Throughout this question, the direction towards the centre of the Earth is taken to be positive and v is a positive quantity.

When $s = 0$, it is given that Felix jumps with an initial velocity $v = 10$.

- (a) (i) Use the chain rule to show that $\frac{dv}{dt} = v \frac{dv}{ds}$. [1]

- (ii) Assuming that g is a constant, solve the differential equation $v \frac{dv}{ds} = g$ to find v as a function of s . [4]

- (iii) Using $g = 9.8$, determine whether the model predicts that Felix will succeed in travelling at the speed of sound at some point before $s = 40\,000$. Justify your answer. [3]

- (b) To test the model

$$\frac{dv}{dt} = g,$$

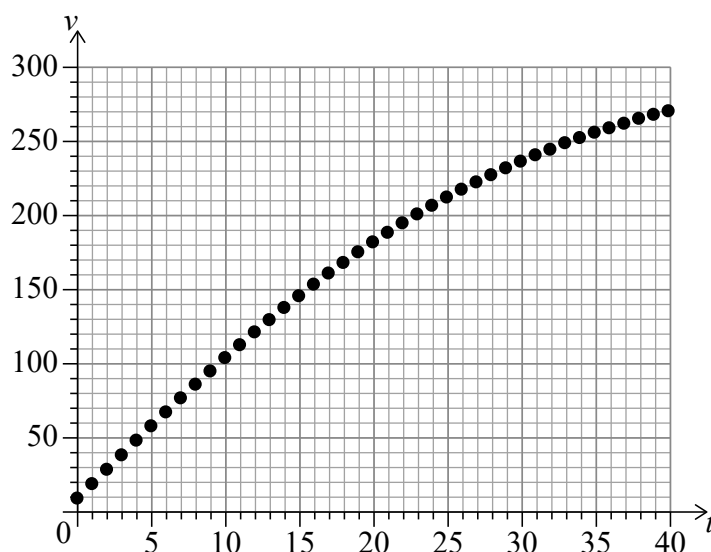
Felix conducted a trial jump from a lower height, and data for v against t was found.

- (i) If the model is correct, describe the shape of the graph of v against t . [2]

(This question continues on the following page)

(Question 1 continued)

Felix's data are plotted on the following graph.



(ii) Use the plot to comment on the validity of the model in part (a).

[1]

(c) An improved model considers air resistance, using

$$\frac{dv}{dt} = g - kv^2$$

where k is a positive constant. You are reminded that initially $s = 0$ and $v = 10$.

(i) By using $\frac{dv}{dt} = v \frac{dv}{ds}$, solve the differential equation to find v in terms of s , g and k .

You may assume that $g - kv^2 > 0$.

[5]

Felix uses the graph of v against t shown in part (b) to estimate the value of k .

(ii) The gradient is estimated to be 9.672 when $v = 40$. Taking g to be 9.8, use this information to show that Felix found that $k = 8 \times 10^{-5}$.

[2]

(iii) Hence, find the value of v predicted by this model, as s tends to infinity.

[2]

(iv) Find the upper bound for the velocity according to this model, given that $0 < s \leq 40\,000$. Give your answer to four significant figures.

[2]

(This question continues on the following page)

Turn over

(Question 1 continued)

The assumption that the value of g is constant is not correct. It can be shown that

$$g = \frac{3.98 \times 10^{14}}{(6.41 \times 10^6 - s)^2}.$$

Hence, the new model is given by

$$v \frac{dv}{ds} = \frac{3.98 \times 10^{14}}{(6.41 \times 10^6 - s)^2} - (8 \times 10^{-5})v^2.$$

When $s = 0$, it is known that $v = 10$.

- (d) Use Euler's method with a step length of 4000 to estimate the value of v when $s = 40\,000$. [4]
- (e) After Felix completed his record-breaking jump, he found that the answer from part (d) was not supported by data collected during the jump.
- (i) Suggest **one** improvement to the use of Euler's method which might increase the accuracy of the prediction of the model. [1]
- (ii) Suggest **one** factor **not** explicitly considered by the model in part (d) which might lead to a difference between the model's prediction and the data collected. [1]

4. [Maximum mark: 4]

Consider the differential equation $\frac{dy}{dx} = \log_{10}(x+y)$, where $x \geq 0$ and $y > 0$.

Given that $y = 1$ when $x = 0$, use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 2$.

A series of horizontal dotted lines for writing.



Turn over

9. [Maximum mark: 7]

(a) Find $\int \frac{8}{2x+3} \mathrm{d}x$. [3]

(b) Hence find the exact area between the curve $y = \frac{8}{2x+3}$, the x -axis and the lines $x = 0$ and $x = 6$. Give your answer in the form $a \ln b$, where $a, b \in \mathbb{N}$. [4]



15. [Maximum mark: 6]

A system of differential equations of the form $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$ has

eigenvalues $\lambda = -1$ and $\lambda = 2$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

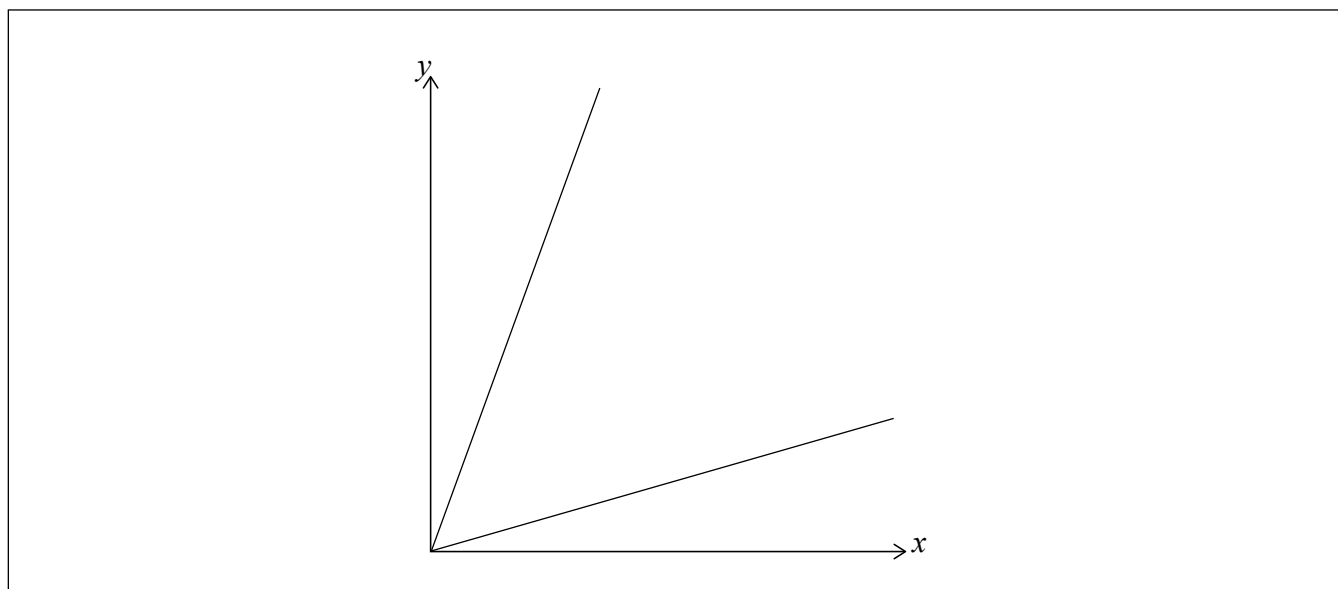
The following incomplete phase portrait for this system, with $x, y \geq 0$, shows lines through $(0, 0)$ parallel to the eigenvectors.

(a) On the phase portrait

(i) show the direction of motion along the eigenvectors.

(ii) sketch one trajectory in each of the three regions.

[3]



In the system described above, x and y are the population sizes of two species, X and Y. The population of Y is vulnerable, so it will be increased by adding more animals from a different area. Currently, $x = 252$ and $y = 60$.

(b) Find the minimum number of new animals from species Y that need to be added for the population not to reduce to 0 over time.

[3]

(This question continues on the following page)



(Question 15 continued)

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Turn over

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7. [Maximum mark: 4]

Consider the differential equation $\frac{dy}{dx} = xy - 1$, given that $y = 2$ when $x = 1$.

Use Euler's method with step size 0.1 to find the approximate value of y when $x = 1.5$.

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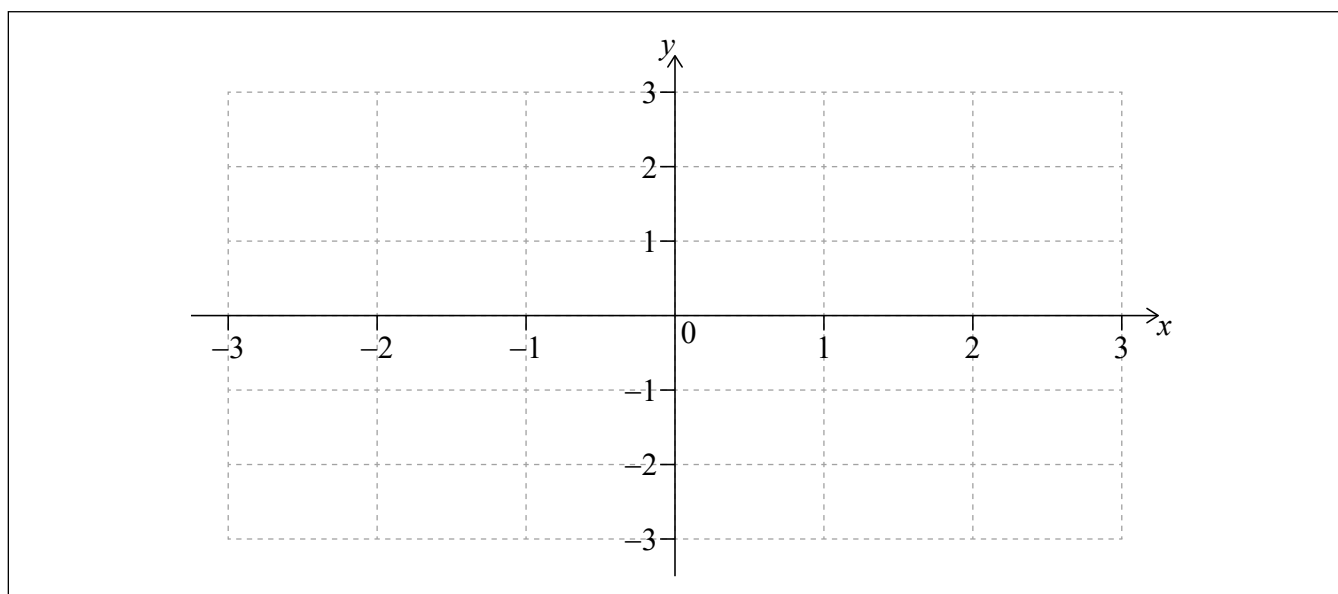
Turn over

10. [Maximum mark: 9]

Consider the function $f(x) = x\sqrt{3-x^2}$, $-\sqrt{3} \leq x \leq \sqrt{3}$.

(a) Sketch the graph of $y = f(x)$ on the following pair of axes.

[2]



The area between the graph of $y = f(x)$ and the x -axis is rotated through 360° about the x -axis.

(b) (i) Write down an integral that represents this volume.

(ii) Calculate the value of this integral.

[4]

The graph of the function f is transformed, to give the graph of the function g , in the following way:

- It is first stretched by scale factor 2, parallel to the x -axis with the y -axis invariant.
- It is then stretched by scale factor 0.5, parallel to the y -axis with the x -axis invariant.

(c) Find the volume obtained when the area between the graph of $y = g(x)$ and the x -axis is rotated through 360° about the x -axis.

[3]

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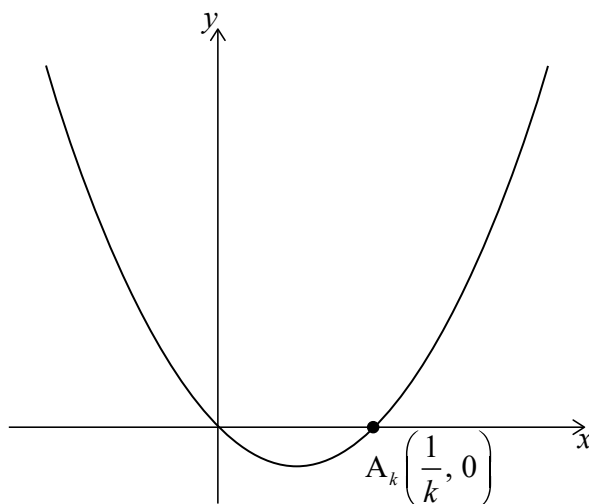
(Question 10 continued)

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11. [Maximum mark: 9]

The diagram shows the curve with equation $y_k = kx^2 - x$, $k > 0$, which intersects the x -axis at the origin and at the point $A_k\left(\frac{1}{k}, 0\right)$.



The normal to the curve at A_k intersects the curve again at point B_k .

- (a) Show that the x -coordinate of B_k is $-\frac{1}{k}$. [6]

Consider the case where $k = 2$.

- (b) Calculate the finite area of the region between the curve with equation $y_2 = 2x^2 - x$ and the normal at A_2 . [3]

(This question continues on the following page)



(Question 11 continued)

[illegible]

Turn over

13. [Maximum mark: 8]

A particle starts from rest at point O and moves in a straight line with velocity, v , given by

$$v = 3 \sin(t)(1 + \cos(t)) \text{ , } t \geq 0$$

where v is measured in metres per second and time, t (radians), is measured in seconds.

The particle next comes to instantaneous rest when $t = a$.

- (a) Determine the value of a .

[2]
- (b) Find the maximum velocity of the particle during the interval $0 \leq t \leq a$.

[2]
- (c) By finding the total distance travelled between $t = 0$ and $t = a$, find the average speed of the particle during the interval $0 \leq t \leq a$.

[4]

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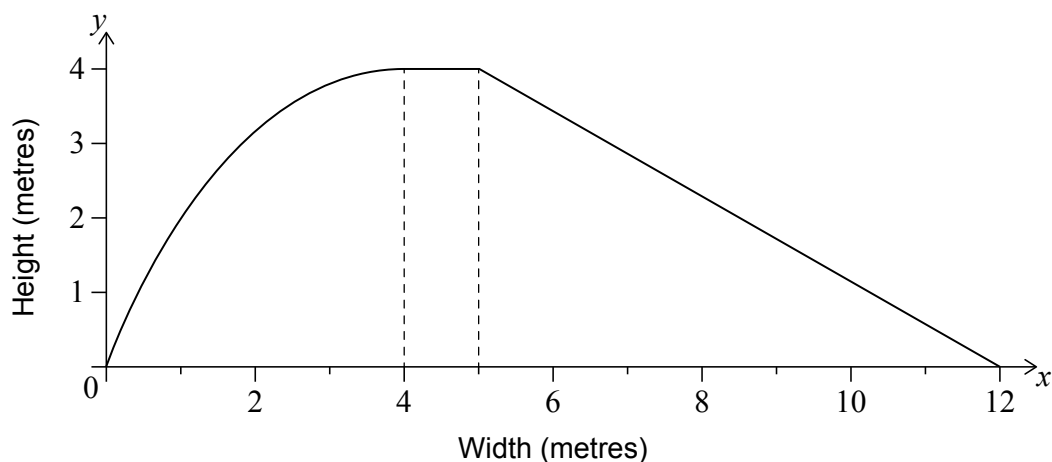
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Turn over

2. [Maximum mark: 15]

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$$f(x) = -\frac{1}{4}x^2 + 2x \text{ for } 0 \leq x \leq 4.$$

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, & \text{for } 4 \leq x \leq 5 \\ \frac{48}{7} - \frac{4x}{7}, & \text{for } 5 \leq x \leq 12 \end{cases}$$

- (a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \leq x \leq 4$. [5]
- (b) Find $\int \left(-\frac{1}{4}x^2 + 2x \right) dx$. [3]
- (c) Calculate the exact area under the entire model of the slide, for $0 \leq x \leq 12$. [4]
- (d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a). [3]

Turn over

2. [Maximum mark: 31]

The following question explores a possible method of drawing phase portraits for non-linear coupled systems, taking a predator-prey model as a particular example.

Gander Green wildlife park contains a population of Czech geese (x , measured in hundreds), and a population of gray foxes (y , measured in hundreds).

Research indicates that the population growth of both geese and foxes can be modelled by the following differential equations, in which t is measured in years.

$$\left. \begin{aligned} \frac{dx}{dt} &= 2x - \frac{xy}{2} \\ \frac{dy}{dt} &= -3y + xy \end{aligned} \right\} \text{ for } x, y \geq 0$$

- (a) At a specific time, there are 500 geese and 500 foxes, represented here by the coordinate pair $(5, 5)$. At this time, determine the rate of change of

(i) geese. [2]

(ii) foxes. [1]

There are two equilibrium points for the populations: $A(0, 0)$ and $B(p, q)$.

- (b) (i) Explain why A is an equilibrium point. [1]

(ii) Find the value of p and the value of q . [3]

At points close to $A(0, 0)$, we can ignore the xy terms, so that the system can be approximated by:

$$\left. \begin{aligned} \frac{dx}{dt} &= 2x \\ \frac{dy}{dt} &= -3y \end{aligned} \right\} \text{ for } x, y \geq 0.$$

- (c) By solving these two differential equations,

(i) find an expression for x in terms of t . [4]

(ii) find an expression for y in terms of t . [1]

- (d) (i) Using your answers from part (c), show that phase portrait trajectories close to A may be given by the equation $x^3 y^2 = k$, where k is a positive constant. [3]

(ii) Hence sketch, on a phase portrait, one possible trajectory for small values of x and y . [3]

(This question continues on the following page)

(Question 2 continued)

Now consider points (x, y) close to B on the phase plane. These coordinates can be rewritten as $x = p + X$ and $y = q + Y$, where p and q are the values from part (b)(ii).

- (e) By substituting into the original model, show that, for small values of X and Y :

$$\dot{X} \approx -\frac{3Y}{2}. \quad [3]$$

Similarly, it can be shown that $\dot{Y} \approx 4X$.

- (f) Given that $\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \mathbf{M} \begin{pmatrix} X \\ Y \end{pmatrix}$, where $\mathbf{M}$ is a square matrix, write down $\mathbf{M}$. [1]
- (g) By finding the eigenvalues of $\mathbf{M}$, describe the path of the trajectories close to point B. [4]
- (h) Hence sketch a complete set of trajectories in the phase plane for the original model, clearly indicating both equilibrium points. [3]
- (i) In this wildlife park, at a specific time, there are 500 Czech geese and 500 gray foxes.

Based on the values found in part (a), the park's wildlife keeper is worried and assumes that the geese will quickly die out. Suggest whether this assumption is supported by the model. Justify your answer. [2]

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References:

1. Panimoni, 2022. *Volume Target icon in flat style on color background. Darts game. Arrow in the center aim. Vector design element for you business projects.* [image online] Available at: <https://www.gettyimages.co.uk/detail/illustration/volume-target-icon-in-flat-style-on-color-royalty-free-illustration/1044319572> [Accessed 21 February 2022]. Source adapted.

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14. [Maximum mark: 8]

While playing on the beach, Sabine builds a mound of sand. The shape of the cross section through the centre of this mound can be modelled by the function

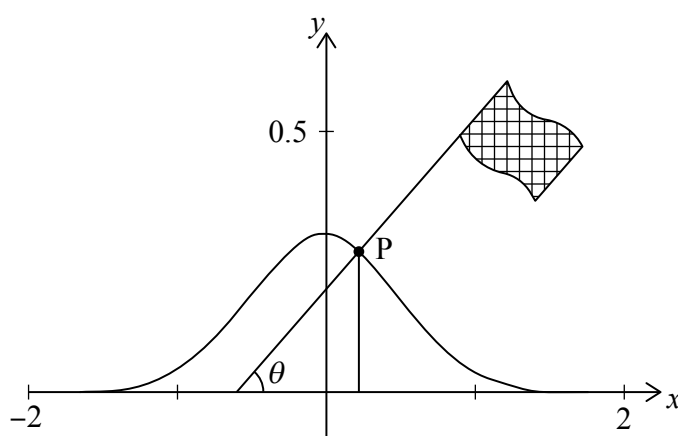
$$f(x) = 0.3e^{-2x^2},$$

where $-2 < x < 2$.

Sabine places a flagpole into this mound at a point, P, which is 0.25 m above the level beach. The pole is perpendicular to the mound of sand at point P.

This information is shown in the diagram, where the x -axis represents the level beach and the y -axis represents the height above the beach. The scale on the axes is 1 unit = 1 metre.

diagram not to scale



The coordinates of P are $(a, 0.25)$, where $a > 0$.

- (a) Calculate the value of a . [2]
- (b) Find an expression for $f'(x)$. [2]
- (c) Hence, or otherwise, find the angle, θ , the flagpole makes with the level beach. [4]

(This question continues on the following page)



(Question 14 continued)

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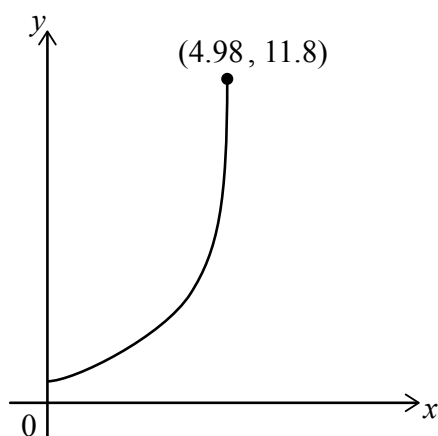
16. [Maximum mark: 7]

As part of his mathematics exploration, Jules models the shape of part of a wine glass to find the capacity (volume) of the glass.

He finds that the edge of half the glass can be modelled by the function

$$f(x) = 4 - 2\ln(5 - x), \text{ where } 0 \leq x \leq 4.98.$$

A graph of $y = f(x)$ is shown, with a scale of 1 unit = 1 cm.



- (a) Find the y -intercept.

[1]

The point $(4.98, 11.8)$ represents a point at the top of the glass.

Let R be the region enclosed by the graph of f , the y -axis and the line $y = 11.8$.

Jules finds the capacity by rotating the region, R , 360° about the y -axis.

- (b) Calculate the capacity of the glass that Jules obtains.

[6]

(This question continues on the following page)



(Question 16 continued)

[illegible]

Turn over

17. [Maximum mark: 7]

The fish in a lake feed on insects. Let F be the population of fish and L be the population of insects at time t (weeks).

The populations can be modelled using the coupled differential equations

$$\begin{aligned}\frac{dF}{dt} &= 0.000004FL - 0.2F \\ \frac{dL}{dt} &= 0.06L - 0.00003FL\end{aligned}$$

where $F > 0$ and $L > 0$.

When $t = 0$, it is estimated that $F = 5000$ and $L = 80\,000$.

- (a) For $0 \leq t \leq 52$, use Euler's method, with a step length $h = 4$, to find an approximate value for

(i) the maximum value of F

(ii) the minimum value of L .

[5]

- (b) (i) State what will happen to the number of fish in the lake, as predicted by the model.

(ii) Suggest a reason why this prediction may not occur.

[2]

(This question continues on the following page)



[illegible]

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Figure 1 animatedfunk, n.d. *Beaver Illustrations – stock illustration* [image online] Available at: <https://www.gettyimages.co.uk/detail/illustration/beaver-illustrations-royalty-free-illustration/1453231718> [Accessed 30 November 2023]. Source adapted.

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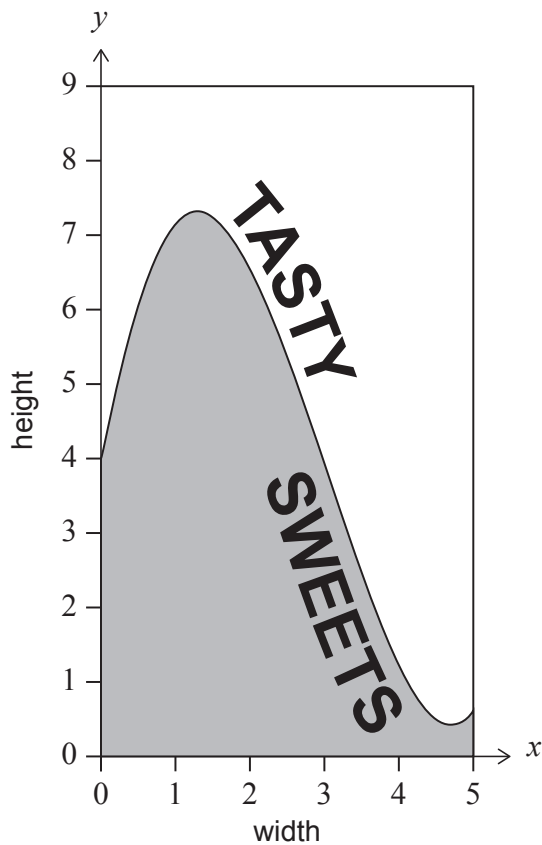
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2. [Maximum mark: 18]

Sweets are sold in cylindrical containers. A new label for the container is being considered. The label will be a rectangle that is 5 cm wide and 9 cm high.

The design on the label is a curve, as shown on the following axes, where one unit represents 1 cm for both axes.



The values in the table approximate points on the curve, correct to one decimal place.

Width, x	0	1	2	3	4	5
Height, y	4	7.3	6.7	4.0	1.3	0.7

- (a) Use the trapezoidal rule with five intervals, and the values given in the table, to estimate the shaded area below the curve.

[3]

The curve used in the label design can be modelled by:

$$y = \frac{x^3}{3} - 3x^2 + 6x + 4, \text{ for } 0 \leq x \leq 5.$$

- (b) Use this equation to find the area of the shaded region.

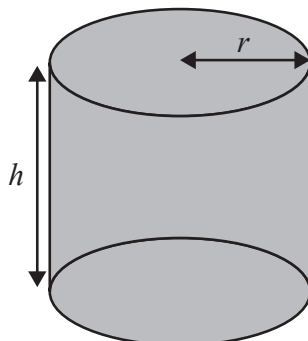
[2]

(This question continues on the following page)

(Question 2 continued)

The sweets are sold in closed cylindrical containers, with radius r and height h .

diagram not to scale



The whole container is made from one type of material, and it is assumed that the thickness of the material is negligible.

Each container has a volume of 600 cm^3 .

- (c) Write down an equation, in terms of r and h , that shows this information. [1]

The amount of material used for each container can be modelled by the external surface area of the container.

The external surface area, A , of the container can be expressed as

$$A = 2\pi r^2 + \frac{k}{r}, \text{ where } r > 0.$$

- (d) Find the value of k . [4]

- (e) (i) Find $\frac{dA}{dr}$.
 (ii) Given that A has a minimum value, find the value of r that will minimize the material used. [5]

The containers are made so that the surface area is minimized. The 5 cm by 9 cm rectangular label is to be glued to the curved surface of the container.

- (f) Show that the label will fit on the container. [3]

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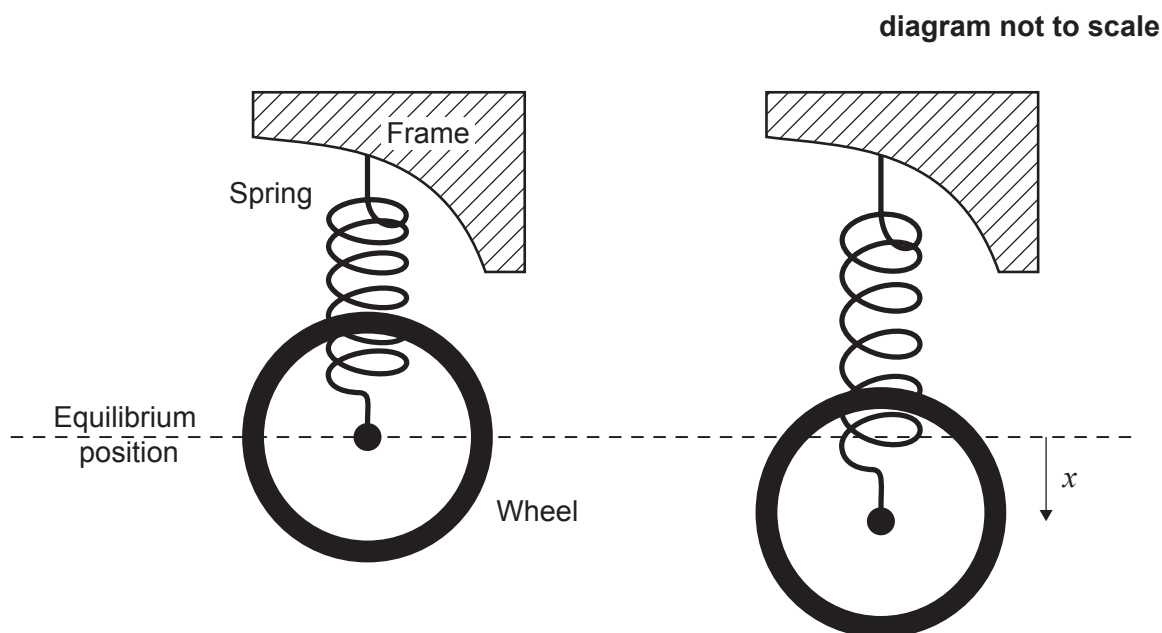
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7. [Maximum mark: 18]

The wheel on a motorbike is attached to the frame by a spring. The movement of the spring acts as a **shock absorber**. When the rider sits on the motorbike, the spring compresses, and this position is called the equilibrium position.

When the wheel goes into holes or over bumps in the road, the spring will extend or compress to ensure a smooth ride.

Let x denote the vertical displacement, in centimetres, of the centre of the wheel below the equilibrium position, as shown in the diagram.



The vertical displacement of the centre of the wheel, at time t seconds, can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0, \text{ where } a, b \in \mathbb{R}.$$

Let $y = \frac{dx}{dt}$.

(a) Show that $\frac{dy}{dt} = -bx - ay$.

[2]

(This question continues on the following page)

(Question 7 continued)

The equations $y = \frac{dx}{dt}$ and $\frac{dy}{dt} = -bx - ay$ can be written in matrix form as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \mathbf{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}.$$

A manufacturer wants to compare two springs, Spring 1 and Spring 2, that could be used as shock absorbers.

The differential equation for Spring 1 has $a = 18$ and $b = 77$.

For these values, the eigenvalues of $\mathbf{M}$ are -7 and -11 .

(b) Find the corresponding eigenvectors. [3]

The manufacturer models **both** springs using the same initial conditions:

$$t = 0, \quad x = 5 \text{ cm and } \frac{dx}{dt} = 2 \text{ cm s}^{-1}.$$

(c) Hence, for Spring 1

- (i) find the exact solution for $x(t)$
- (ii) sketch the graph of $x(t)$, for $0 \leq t \leq 1$. [7]

The differential equation for Spring 2 has $a = 18$ and $b = 85$.

For these values, the eigenvalues of $\mathbf{M}$ are $-9 \pm 2i$.

- (d) (i) Sketch the phase portrait for Spring 2, indicating the direction of the trajectory.
- (ii) Hence, sketch the graph of x against t . [5]
- (e) Using your answers to parts (c)(ii) and (d)(ii), give a reason why Spring 1 might make a better shock absorber than Spring 2. [1]

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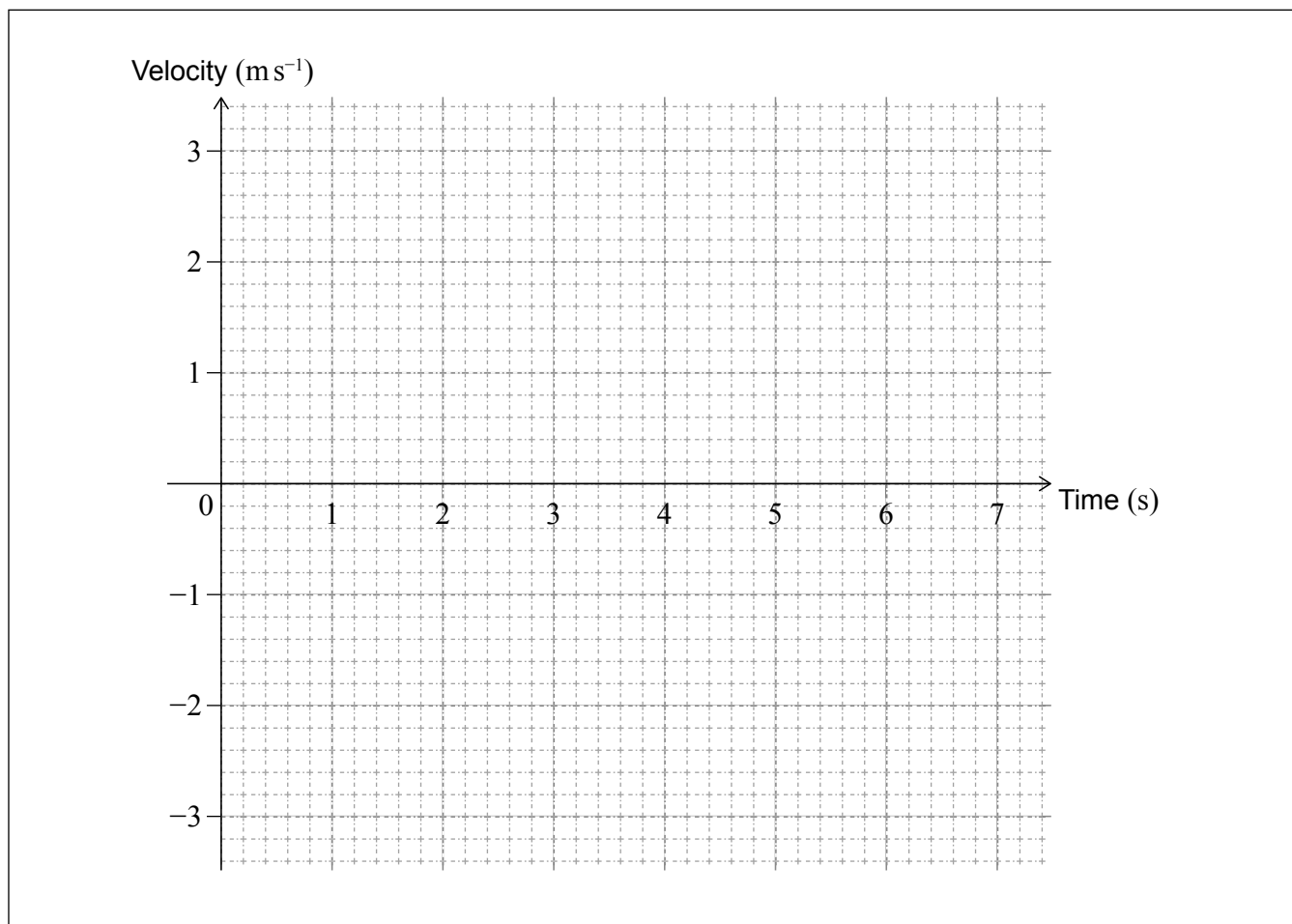
1. InterNations. 2022. *Expatriate Insider 2022*. [online] Available at: <https://www.internations.org/expat-insider/2022/>. Source adapted.

Heliwell, J. F., Huang, H., Wang, S. and Norton, M., 2022. Statistical Appendix for “Happiness, benevolence, and trust during COVID-19 and beyond,” Chapter 2 of *World Happiness Report 2022*. [pdf online] Available at: <https://worldhappiness.report/ed/2022/happiness-benevolence-and-trust-during-covid-19-and-beyond/> [Accessed 13 November 2023]. Source adapted.

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7. [Maximum mark: 8]

Conrad is investigating the motion of a particle. The velocity of the particle, in m s^{-1} , is given by $v(t) = 2\cos t + \sin 2t - 0.2$ where t is the time, in seconds, after the investigation begins.



- (a) On the axes,
- sketch the graph of $v(t)$ for $0 \leq t \leq 6$.
 - label, with coordinates, the point(s) where the particle has zero acceleration. [5]
- (b) (i) Write down an integral expression for the distance the particle travels during the first 6 seconds of the investigation.
- (ii) Hence, find the distance the particle travels during the first 6 seconds of the investigation. [3]

(This question continues on the following page)



(Question 7 continued)

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Turn over

8. [Maximum mark: 7]

Consider the function $f(x) = x^3 - 9x^2 + 23x - 15$ for $0 \leq x \leq 5$.

The graph of $f(x)$ crosses the x -axis when $x = 1$, $x = 5$ and $x = a$.

- (a) Find the value of a . [2]
- (b) Use your graphic display calculator to find the area enclosed by the curve $y = f(x)$ and the x -axis, when $1 \leq x \leq a$. [2]

Another function is defined as $g(x) = -x^2 + 6x - 5$.

The area enclosed by the curve $y = f(x)$ and the x -axis, when $1 \leq x \leq b$, is equal to the area enclosed by the curve $y = g(x)$ and the x -axis, when $1 \leq x \leq b$.

- (c) Find the value of b , where $1 < b < 3$. [3]

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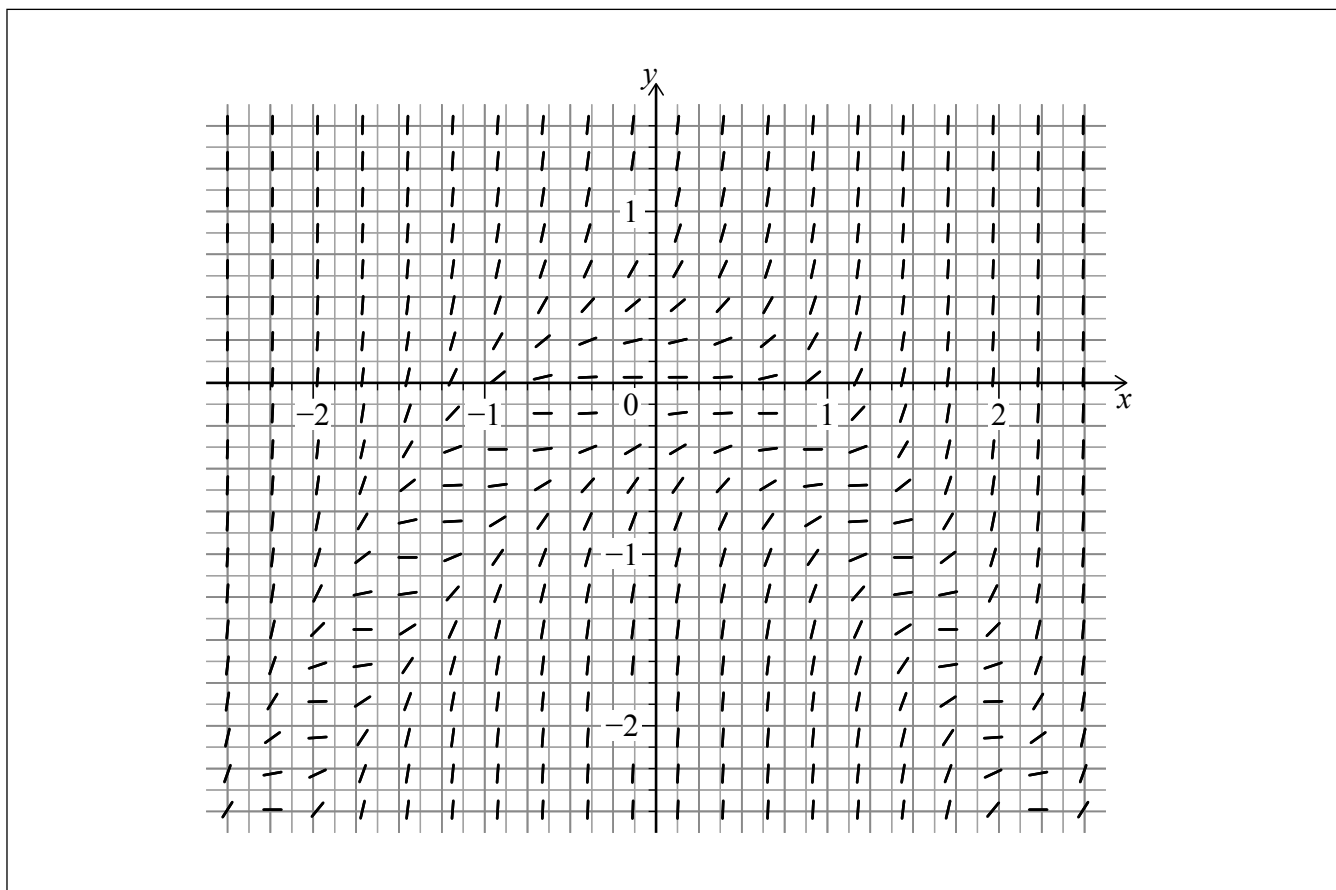
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12. [Maximum mark: 7]

The following graph shows the slope field for the differential equation

$$\frac{dy}{dx} = x^4 + 4x^2y + 4y^2.$$



- (a) On the slope field, sketch the solution to the differential equation that passes through the point $(0, 1)$.

[2]

Some of the solutions to the differential equation have a point at which the gradient is zero. The curve on which all these points lie is given by the equation $y = ax^2$, where a is a constant to be found.

- (b) Sketch this curve on the slope field.

[2]

- (c) Use the differential equation to find the value of a .

[3]

(This question continues on the following page)



(Question 12 continued)

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Turn over

15. [Maximum mark: 8]

Quentin is investigating the motion of a mass hanging on a large spring.

From research, Quentin thinks the displacement, x , in metres, of the mass at time t , in seconds, will satisfy the differential equation

$$9.8 = 2 \frac{d^2x}{dt^2} + 2.4 \frac{dx}{dt} + 8x.$$

Quentin begins his investigation when the mass is at a displacement of 0.5 metres and is moving such that $\frac{dx}{dt} = 9.8$.

- (a) By using Euler's method with $h = 0.1$ to estimate the motion of the mass after being released, find:

- (i) the displacement of the mass after 1 second,
- (ii) the first estimated time the mass reaches $x = 0$.

[5]

Using technology, Quentin correctly measures the displacement of the mass, after 1 second. His software gives him a value of 385 centimetres, which is accurate to the nearest centimetre.

- (b) Find the **maximum possible** percentage error between your estimate from part (a)(i) and the exact displacement after 1 second.

[3]

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[illegible]

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2. fitie, 2017. *Vector illustration of dolphins* [image online] Available at: <https://www.gettyimages.co.uk/detail/illustration/vector-illustration-of-dolphins-royalty-free-illustration/638431854> [Accessed 19 April 2024] Source adapted.



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7. [Maximum mark: 7]

- (a) Find the indefinite integral $\int x e^{-x^2} dx$.

[4]
- (b) Hence find the area bounded by the x -axis, the curve $y = x e^{-x^2}$ and the line $x = k$.
Give your answer in terms of k .

[3]

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9. [Maximum mark: 7]

A climate scientist is modelling an ice sheet as a rectangle.

She believes that the width (x km) is increasing at a constant rate of 10 km per year and the length (y km) is decreasing at a constant rate of 5 km per year.

The time, t , is measured in years, and the area, A , is measured in km^2 .

When $t = 0$ then $x = 75$ and $y = 40$.

- (a) Find $\frac{dA}{dt}$ when $t = 0$.

[3]
- (b) State, with justification, whether the area of the ice sheet is increasing or decreasing when $t = 0$.

[1]
- (c) Find the change in the area of the ice sheet between $t = 0$ and $t = 1$.

[3]

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12. [Maximum mark: 9]

An engineer’s model for an object’s motion is that its acceleration, $\frac{dv}{dt}$, is proportional to $v^{1.5}$, where v is its velocity measured in m s^{-1} .

- (a) Write down a differential equation based on the engineer’s belief.
- [1]

The initial velocity of the object is 4 m s^{-1} and its initial acceleration is -3 m s^{-2} .

- (b) Use the engineer’s model to find an expression for the velocity of the object after t seconds.
- [8]

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15. [Maximum mark: 8]

An astronomer models the shape of a parabolic mirror using the equation $y = x^2$.

- (a) Find the equation of the normal to the mirror at the point $(2, 4)$. [3]

A ray of light comes from an object at coordinates $(0, 10)$ and hits the mirror at the point $(2, 4)$.

- (b) Find the gradient of the ray of light. [2]

- (c) Find the angle between the ray of light and the normal to the mirror. [3]



9. [Maximum mark: 6]

Consider the curve $y = 4x\sqrt{3x-2}$ for $x \geq \frac{2}{3}$.

(a) Find $\frac{dy}{dx}$. [3]

(b) Hence show that the equation of the tangent to the curve at the point (6, 96) is $y = 25x - 54$. [3]



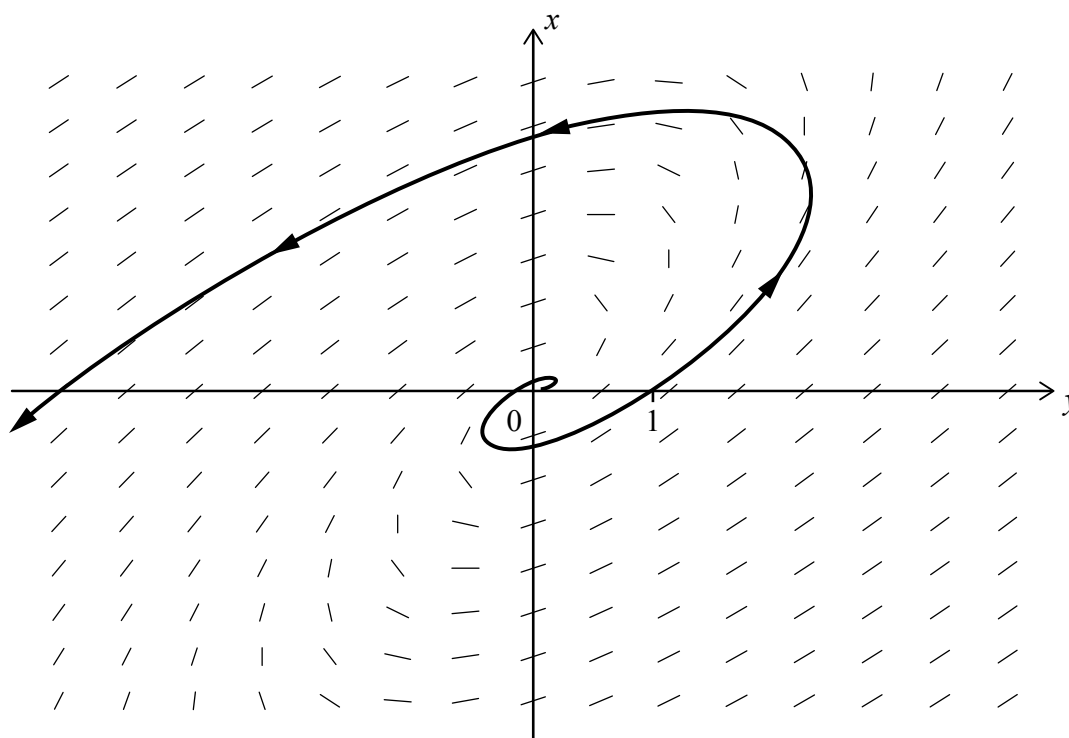
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11. [Maximum mark: 6]

Consider the system given by the coupled differential equations

$$\begin{aligned}\frac{dx}{dt} &= 3x - 2y \\ \frac{dy}{dt} &= 4x - y\end{aligned}$$

The following phase portrait shows the trajectory that passes through the point $(1, 0)$ at time $t = 5$.



- (a) Write down which one of the following could be an eigenvalue for the system.

[1]

- A. 1
- B. $2i$
- C. $1 + 2i$
- D. $-1 + 2i$

- (b) Use Euler's method with a step length of 0.1 to find the approximate values of x and y when $t = 5.5$.

[5]

(This question continues on the following page)



(Question 11 continued)

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Turn over

13. [Maximum mark: 9]

The growth of a fungus on a dish can be modelled by the differential equation

$$\frac{dP}{dt} = 0.2(1 - P), \quad 0 \leq P < 1$$

where P is the proportion of the dish that is covered with fungus and t is the time in days from when the fungus began to form.

- (a) Assuming that $P = 0$ when $t = 0$, show that $P = 1 - e^{-0.2t}$. [7]
- (b) Find the time at which the fungus covers 90% of the dish. [2]

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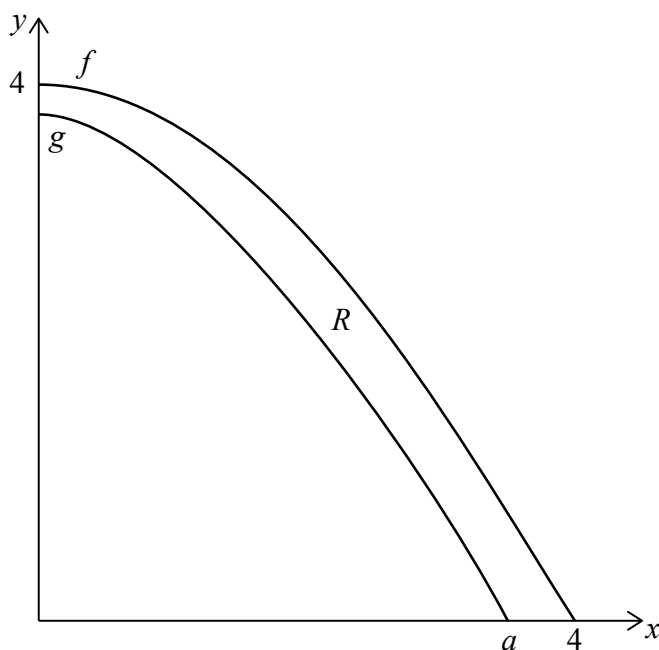
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16. [Maximum mark: 8]

The graph of f is defined by $f(x) = 4\cos\left(\frac{\pi}{8}x\right)$, $0 \leq x \leq 4$.

The graph of $y = g(x)$ is formed by translating the graph of f 0.3 units to the left and 0.2 units down. The graph of g intersects the x -axis at $(a, 0)$ and its domain is then restricted to $0 \leq x \leq a$.

These curves are shown on the following graph where units are in centimetres.



(a) Write down an expression for $g(x)$. [2]

(b) Find the value of a . [1]

A wall-mounted light shade can be modelled by the solid formed when the region between the two curves, R , is rotated through π radians about the x -axis.

(c) (i) Write down an expression for the volume of the solid formed.

(ii) Hence find the volume of material used in the light shade. [5]

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(Question 16 continued)

[illegible]

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3. [Maximum mark: 17]

Ethan and Avery are researching population data for the city of Los Angeles to create a model predicting population values. They collect the following data.

Population data for the city of Los Angeles

Year	Population (thousands)
1900	102
1920	577
1940	1504
1960	2479
1980	2967
2000	3685
2020	3899

Ethan proposes the population can be modelled using quadratic regression to find a function of the form $f(x) = ax^2 + bx + c$, where x is the number of years after 1900.

- (a) Find the equation of Ethan's model. [3]

Ethan finds the coefficient of determination for his model is 0.98843 to five significant figures.

- (b) State whether the coefficient of determination supports Ethan's proposal. Justify your answer. [2]

- (c) Comment on the validity of Ethan's model with reference to one of the parameters in the equation. [1]

- (d) (i) Find the value of $f'(110)$ and interpret this value in context.
 (ii) By considering the population changes in the table, use the value found in part (d)(i) to comment on the validity of Ethan's model. [4]

(This question continues on the following page)

(Question 3 continued)

Avery proposes that the population instead follows a logistic model of the form

$$g(x) = \frac{4000}{1 + 14 e^{-0.05x}}$$

where x is the number of years after 1900.

- (e) State a reason why it may be valid to use Avery's proposal to predict the future population. [1]
- (f) (i) Find $g'(x)$.
- (ii) Hence find the year, according to Avery's model, during which the greatest population growth rate occurred. [6]

Turn over

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6. [Maximum mark: 14]

Rea is investigating how the number of healthy skin cells changes following infection by a virus. At the beginning of the investigation, she inserts 27 000 virus particles into a sample of 1350 healthy skin cells.

Let h be the number of healthy skin cells at time t , where t is the number of hours after the investigation has begun. Let v be the number of virus particles, **in thousands**, at time t .

Rea finds that the rate of change, $\frac{dh}{dt}$, is proportional to $\frac{h}{v}$.

- (a) Given that $\frac{dh}{dt} = -5$ at time $t = 0$, show that $\frac{dh}{dt} = -\frac{h}{10v}$. [3]

Rea models the number of virus particles, in thousands, after t hours as $v = 27 + 0.3t$.

- (b) Find an expression for the number of healthy skin cells at time t . [7]

Rea asks her colleague Artem to attempt the same investigation. Artem's models differ from those of Rea.

Artem models the number of virus particles, in thousands, after t hours as $v = 27 + 0.28t$ and finds that $h = 4380(27 + 0.28t)^{-0.36}$.

- (c) Using **Artem's** models, predict how many hours it will take for the number of virus particles to be at least 100 times the number of healthy skin cells. [4]

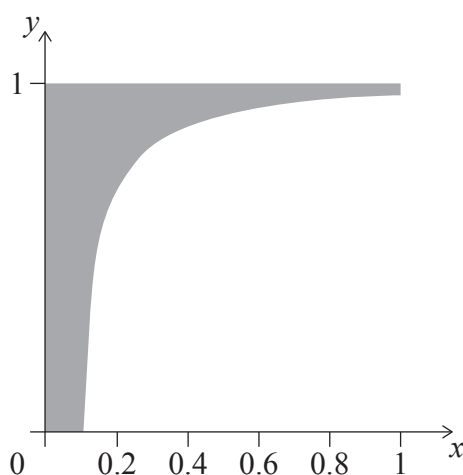
Turn over

7. [Maximum mark: 12]

Bobby is designing a table that will be 1 metre high. The table will have a flat circular top with a radius of 1 metre. To begin the design, he uses the curve of

$$y = \frac{2}{3} \cos^{-1} \left(\frac{1}{10x} \right), \text{ for } 0.1 \leq x \leq 1,$$

to shape the underside of the table, where x and y are measured in metres, as shown.



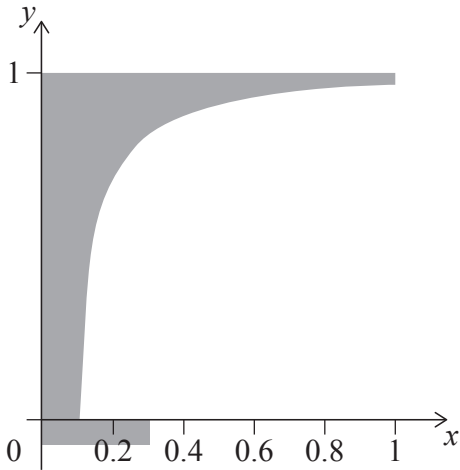
- (a) Find the area of the shaded region.

[4]

(This question continues on the following page)

(Question 7 continued)

To create the base of the table, Bobby adds a rectangle with a height of 0.075 metres and a width of 0.3 metres to the bottom of the figure, as shown.



To finish the design of the table, he rotates the entire shaded region through 2π about the y -axis.

- (b)

Find the three expressions necessary to determine the total volume of this table.

[6]
- (c)

Hence find the total volume of the table.

[2]

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References:

3. Los Angeles Almanac, n.d. *Historical General Population City & County of Los Angeles, 1850 to 2020* [online] Available at: <https://www.laalmanac.com/population/po02.php> [Accessed 4 June 2024]. Source adapted.

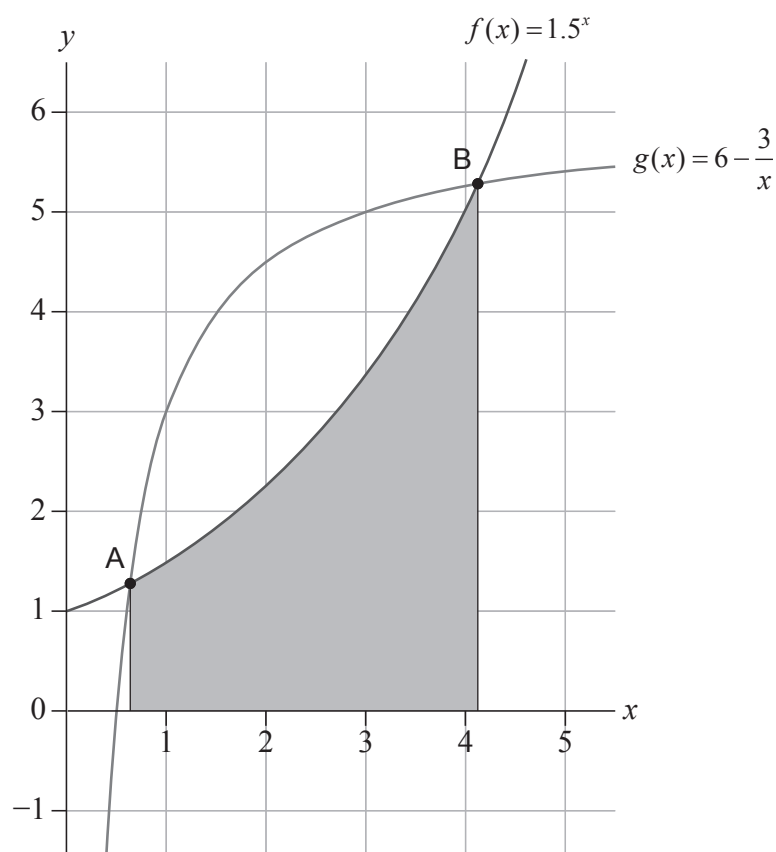
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2. [Maximum mark: 12]

The diagram shows part of the graphs of the functions

$$f(x) = 1.5^x \quad x \geq 0$$

$$g(x) = 6 - \frac{3}{x} \quad x > 0.$$



- (a) Solve $f(x) = g(x)$. [3]
- (b) (i) Write down the integral that represents the area of the shaded region.
- (ii) Calculate the area of this shaded region.
- (iii) Hence, or otherwise, calculate the area of the region enclosed between the curves $y = f(x)$ and $y = g(x)$. [6]

The tangent to the graph of $y = f(x)$ is parallel to the tangent to the graph of $y = g(x)$ at $x = k$.

- (c) Find the value of k . [3]

Turn over

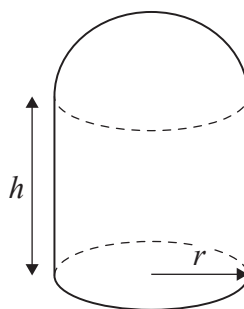
3. [Maximum mark: 16]

Ju Shen designs a plastic cover, in the shape of a cylinder combined with a hemisphere on top, as shown in the diagram.

The plastic used to make the cover forms the curved surface of both the hemisphere and the cylinder; there is no bottom to the cover, however it stands on a flat horizontal surface.

Let the height of the cylinder be h cm and the internal radius of its base be r cm.

diagram not to scale



- (a) Find an expression for the total internal surface area, A cm², of the cover in terms of r and h . [2]
- (b) Show that the total volume, V cm³, enclosed by the cover and the horizontal surface is given by the expression

$$V = \frac{2\pi r^3 + 3\pi r^2 h}{3}. \quad [2]$$

The total volume enclosed by the cover is 10 000 cm³.

- (c) Hence show that $h = \frac{30\,000 - 2\pi r^3}{3\pi r^2}$. [2]

Ju Shen uses the total internal surface area to model the amount of plastic used to construct the cover.

- (d) Show that A is given by the expression

$$A = \frac{2\pi r^2}{3} + \frac{20\,000}{r}. \quad [2]$$

(This question continues on the following page)

(Question 3 continued)

Ju Shen wants to use the minimum amount of plastic in the construction of the cover.

- (e) Find an expression for $\frac{dA}{dr}$. [3]
- (f) Find the value of r and the value of h that minimizes the use of plastic. [4]
- (g) By interpreting your answer to part (f), suggest the best shape for Ju Shen's plastic cover. [1]

Turn over

6. [Maximum mark: 21]

A financial analyst models the change in value of one share, x dollars at time t minutes, after a report is released. She uses the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0.$$

This equation can be written as the coupled differential equations

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -3x - 4y.$$

- (a) Find the general solution for x . [5]

Initially $x = 0$ and $\frac{dx}{dt} = -1$.

- (b) (i) Find an expression for x in terms of t .
 (ii) Sketch x against t in the interval $0 \leq t \leq 4$. [6]

Once the report has been released, the analyst is going to buy some shares and then sell them later.

- (c) (i) Use your graph to find how long after the report is released the analyst should wait to buy the shares in order to maximize her profit.
 (ii) Find the upper limit of the profit the analyst can make per share. [4]

An improved model is written as

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = x \sin t.$$

The same initial conditions as above apply.

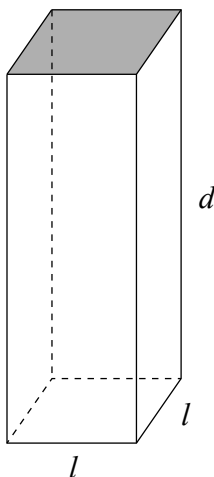
- (d) Use Euler's method with a t -interval of 0.1 to predict the value of x when $t = 1$. [6]

Turn over

1. [Maximum mark: 20]

Kailash manufactures drink containers in the shape of a cuboid. The container has a square top and a square base of length, l cm. Its height, d cm, is three times the length of the base.

diagram not to scale



- (a) Write down an expression for d in terms of l .

[1]

The container can hold 375 cm^3 of drink.

- (b) Find the value of l and d .

[3]

- (c) Calculate the total external surface area of the container.

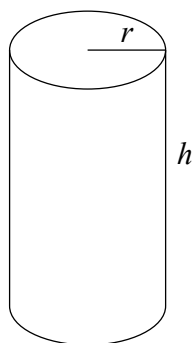
[3]

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(Question 1 continued)

To reduce environmental impact, Kailash is trying to minimize the amount of material needed for the production of the 375 cm^3 container.

He is willing to change the shape to a cylinder with radius $r \text{ cm}$, and height $h \text{ cm}$, as shown below.



The cylindrical container of drink must also hold 375 cm^3 .

- (d) Find an expression for the height, h , of the container in terms of r . [2]

Let the total external surface area be $A \text{ cm}^2$.

- (e) Show that $A = 2\pi r^2 + \frac{750}{r}$. [2]

- (f) Find $\frac{dA}{dr}$. [3]

- (g) Hence or otherwise

- (i) find the value of r that will minimize A .
 (ii) find the minimum value of A needed for the cylinder. [3]

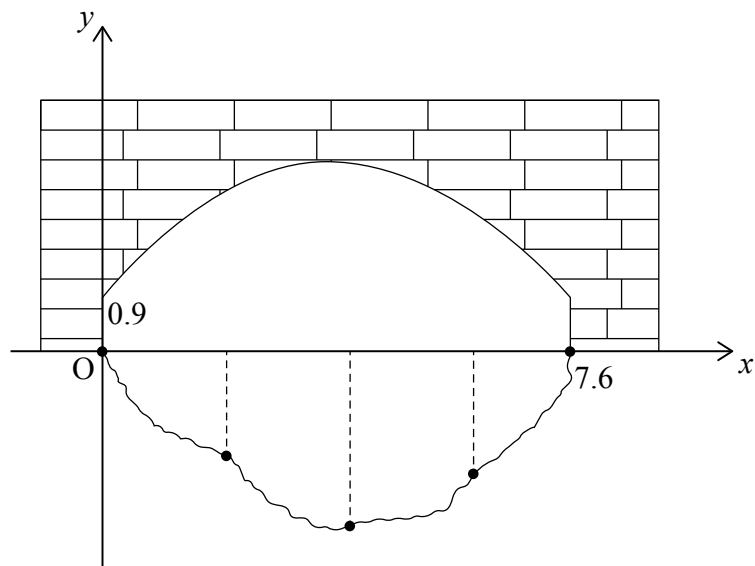
- (h) (i) Find $\frac{d^2A}{dr^2}$.
 (ii) Hence determine whether the graph of A is concave-up or concave-down for $r > 0$. Justify your answer. [3]

Turn over

3. [Maximum mark: 12]

The diagram shows the cross-section of a bridge and a river. A coordinate system has been added with the origin, O , at the point where the bridge meets the water on one side. All units are in metres.

diagram not to scale



A researcher wants to calculate the volume of water that flows under the bridge. To do this he takes measurements of the depth every 1.9 m from O . The depths are shown in the following table.

Horizontal distance from O in metres	0	1.9	3.8	5.7	7.6
Vertical depth of water in metres	0	1.68	2.81	2.32	0

- (a) Use the trapezoidal rule to find the cross-sectional area of the river as it passes under the bridge. [3]

The water flows under the bridge at a rate of 0.3 m s^{-1} .

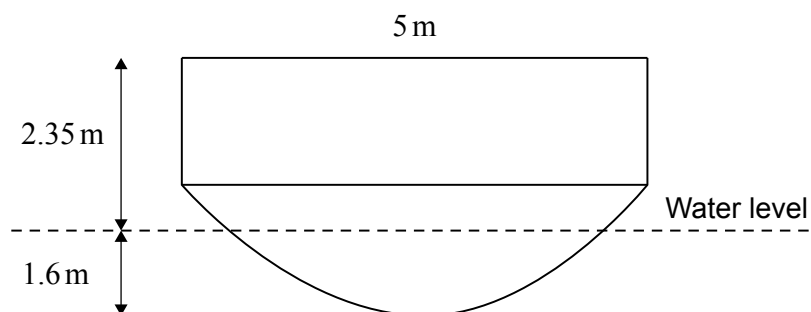
- (b) Find the volume of water that passes under the bridge each second. [2]

(This question continues on the following page)

(Question 3 continued)

A boat is travelling along the river. The cross-section of the boat and the water level is shown in the following diagram.

The top of the boat is parallel to the water level and has a width of 5 m. The height of the boat is 2.35 m above the water level and the lowest part of the boat is 1.6 m below the water level.



The boat is travelling down the centre of the river.

- (c) Find the vertical distance between the lowest part of the boat and the bottom of the river as it passes under the bridge.

[1]

The curved arch of the bridge can be modelled by the equation

$$y = -0.15x^2 + 1.14x + 0.9, \quad 0 \leq x \leq 7.6.$$

- (d) Find the maximum height of the curved arch above the water level.
- (e) Determine whether the top of the boat will be able to pass under the bridge.

[2]

[4]

Turn over

7. [Maximum mark: 8]

Consider the function $B(x) = 20 + \frac{32}{x} + 2x$, where $x \geq 1$, $x \in \mathbb{R}$.

- (a) (i) Find $B'(x)$.

- (ii) Use your answer to part (a)(i) to find the minimum value of $B(x)$. You may assume that $B(x)$ has no local maximum point over the given domain.

[6]

Kim goes to a restaurant that offers a large family banquet. The time, T , in minutes, to prepare the family banquet depends on the number of chefs, n .

To predict the value of T , Kim uses the model $T(n) = 20 + \frac{32}{n} + 2n$, where $n \in \mathbb{Z}^+$.

The restaurant informs Kim that the time taken to prepare the family banquet is less than 40 minutes. There are k chefs preparing the family banquet.

- (b) Find the possible values of k .

[2]

[illegible]

10. [Maximum mark: 5]

The function f is defined by $f(x) = \frac{6}{2x^2 + 5x + 4}$.

(a) Find $f'(x)$.

[2]

(b) Hence or otherwise, find the x -coordinates of the points of inflexion of f .

[3]

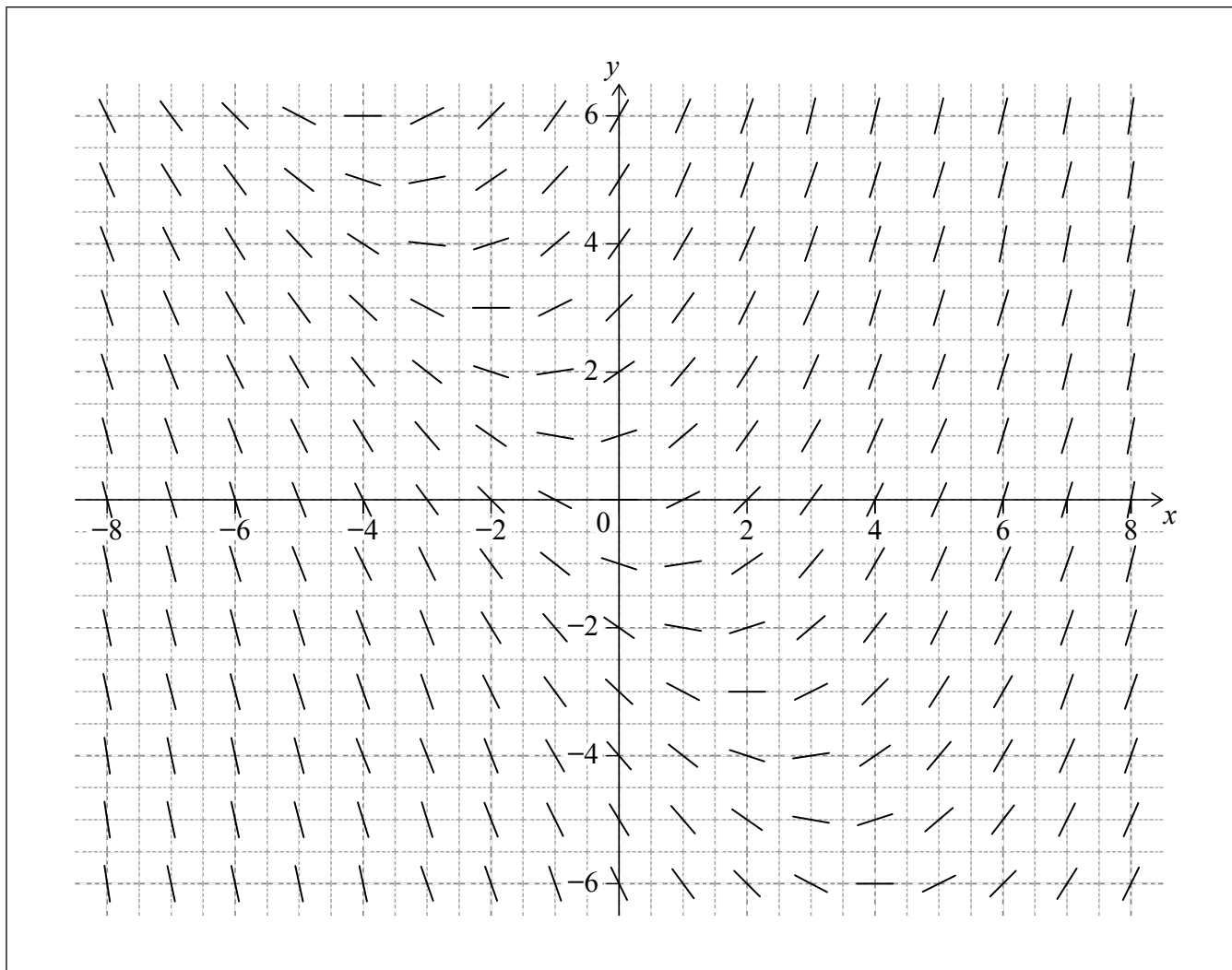
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Turn over

11. [Maximum mark: 4]

Consider the differential equation $\frac{dy}{dx} = \frac{x}{2} + \frac{y}{3}$.

The slope field for the differential equation is shown in the diagram.



All the solutions to $\frac{dy}{dx} = 0$ lie on a straight line.

(a) (i) Find the equation of this line.

(ii) Sketch this line on the slope field.

[2]

One solution to $\frac{dy}{dx} = \frac{x}{2} + \frac{y}{3}$ passes through the point (6, 4).

(b) Sketch this solution on the slope field.

[2]

(This question continues on the following page)



(Question 11 continued)

[illegible]

Turn over

14. [Maximum mark: 6]

Consider the system of coupled differential equations given by

$$\frac{dx}{dt} = 2.2x - 2.6y$$

$$\frac{dy}{dt} = 3.4x - 2.2y.$$

When $t = 0$, $x = 5$ and $y = 2$.

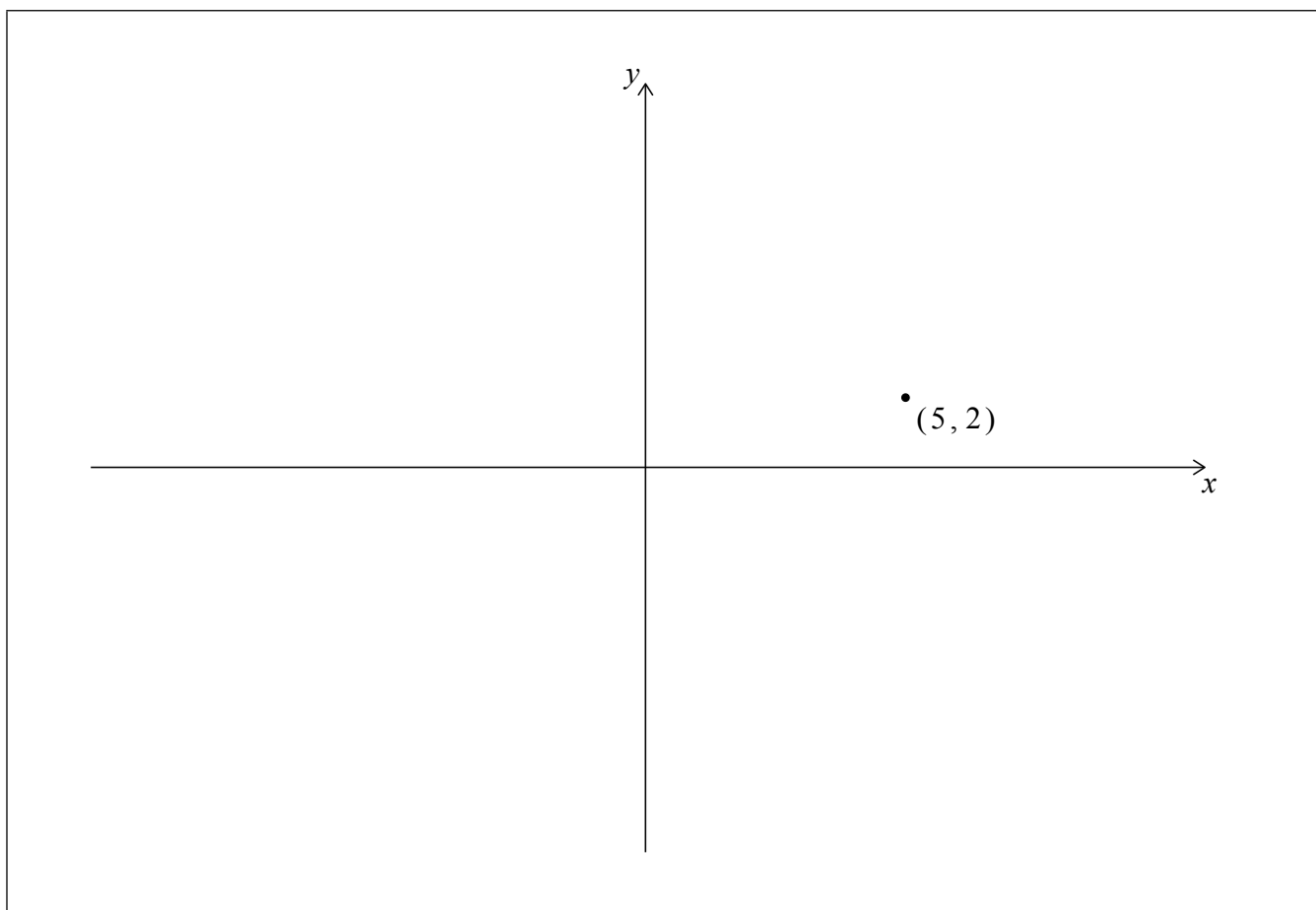
- (a) Find the value of $\frac{dy}{dx}$ at $t = 0$.

[3]

The eigenvalues for this system are $\pm 2i$.

- (b) On the following phase portrait, sketch the trajectory that passes through the point $(5, 2)$. Clearly indicate the direction of this trajectory.

[3]



(This question continues on the following page)



(Question 14 continued)

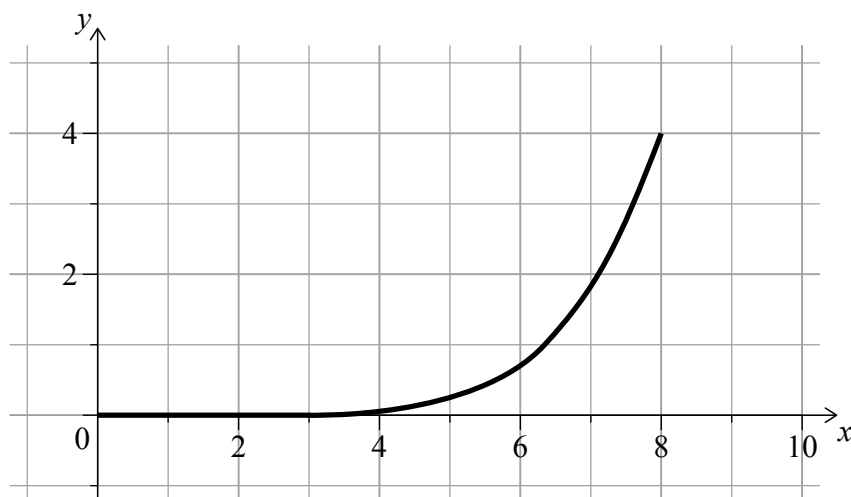
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15. [Maximum mark: 8]

Jun Ho models the cross-section of a bowl, in order to calculate its volume.

His model for part of the cross-section is $y = \frac{x^6}{65536}$, where $0 \leq x \leq 8$, as shown in the following graph. One unit represents one centimetre.



Let R be the region enclosed by this graph, the line $y = 4$ and the line $x = 0$.

He obtains the volume of the bowl by rotating R , 2π radians about the y -axis.

(a) Find the volume of the bowl.

[5]

Jun Ho pours 250 cm^3 of water into the bowl.

(b) Find the depth of the water in the bowl.

[3]

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(Question 15 continued)

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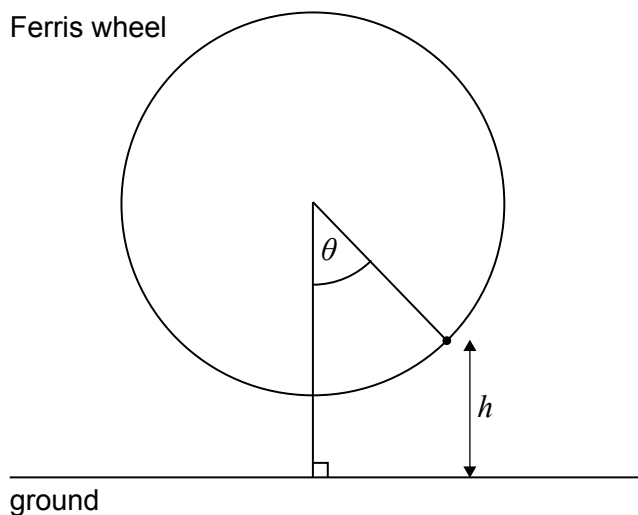
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17. [Maximum mark: 8]

Younsue rides on a Ferris wheel. As the wheel rotates, her height above the ground, h metres, can be modelled in terms of the angle, θ radians, that the wheel has rotated, using $h = 16 - 15 \cos \theta$. The value of θ is measured from Younsue's starting position at the bottom of the wheel.

diagram not to scale

- (a) Write down the radius of the Ferris wheel.

[1]

The speed at which the wheel rotates changes over time and is given by

$$\frac{d\theta}{dt} = \frac{\pi^2}{24} \left| \sin \left(\frac{\pi}{2} t \right) \right|$$

where t is the time, in minutes, after Younsue starts her ride.

- (b) (i) Find the angle rotated by the Ferris wheel between $t = 0$ and $t = 4.5$.

- (ii) Find the value of $\frac{dh}{dt}$, when $t = 4.5$.

[7]

(This question continues on the following page)



(Question 17 continued)

[illegible]

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2. [Maximum mark: 16]

The height of a plant, h cm, depends on the time, t days, after it is planted.

For the first 15 days, the rate of change of the height of the plant is modelled as

$$\frac{dh}{dt} = 1.2 - 0.1t, \text{ where } 0 \leq t \leq 15.$$

- (a) (i) Find the value of $\frac{dh}{dt}$ at $t = 3$.

- (ii) Interpret the meaning of your answer to part (a)(i) in context. [2]

- (b) Use $\frac{dh}{dt}$ to find the value of t when the plant reaches its maximum height. [2]

Two days after it is planted, the height of the plant is 6.5 cm.

- (c) Find an expression for h in terms of t , for $0 \leq t \leq 15$. [5]

- (d) Hence, find the maximum height of the plant. [2]

A second plant was planted at the same time as the first plant. Its height, H cm, is modelled by $H = 2(1.2)^t$, where $0 \leq t \leq 15$.

- (e) Write down the initial height of the second plant. [1]

Each day, the height of the second plant increases by $p\%$.

- (f) Find the value of p . [2]

- (g) Find the value of t when the two plants have the same height. [2]

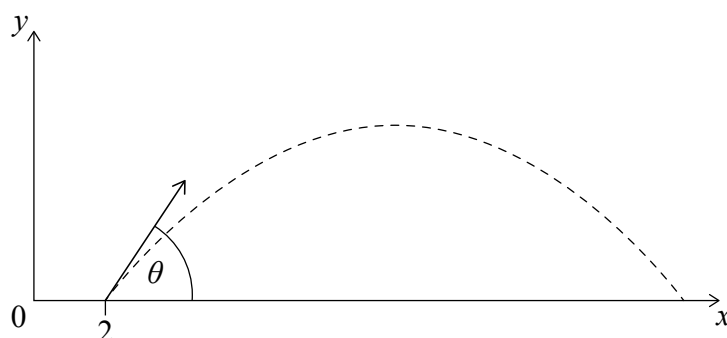
Turn over

6. [Maximum mark: 14]

At an outdoor ballpark for children, a ball is launched from the ground, with initial speed 12 m s^{-1} and initial launch angle θ to the horizontal ground, where $0 < \theta < \frac{\pi}{2}$.

The position of the ball, t seconds after it is launched, is given by $\begin{pmatrix} x \\ y \end{pmatrix}$ where x is the horizontal displacement from an origin and y is the vertical displacement from the ground. Distances are measured in metres.

The ball's initial position is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and its path is shown in the diagram.



The ball can be modelled as a projectile, using $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 12 \cos \theta \\ 12 \sin \theta - 9.8t \end{pmatrix}$.

- (a) Find an expression for x , in terms of θ and t . [3]

It is given that $y = (12 \sin \theta)t - 4.9t^2$.

- (b) When the ball hits the ground, show that $t = \frac{120 \sin \theta}{49}$. [3]

Let x_g be the value of x when the ball hits the ground.

- (c) Find an expression for x_g in terms of θ only. [2]

- (d) Hence, find the value of θ which maximizes the value of x_g . [2]

The model is adapted to account for a horizontal wind with speed 3 m s^{-1} .

The new model for the ball is $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 12 \cos \theta - 3 \\ 12 \sin \theta - 9.8t \end{pmatrix}$.

In the new model, the time taken for the ball to hit the ground remains $t = \frac{120 \sin \theta}{49}$.

- (e) (i) Find an expression for x_g in terms of θ only.
(ii) Hence, find the value of θ which maximizes the value of x_g . [4]

Turn over

2. [Maximum mark: 29]

In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population, x , as a function of t , where t is the time, in years, since the area became protected.

- (a) Initially, the researchers consider using the logistic model

$$x = \frac{L}{1 + Ce^{-kt}}, \text{ where } L, C, k \in \mathbb{R}^+.$$

The researchers decide to let $L = 200$.

- (i) State the assumption being made in assuming $L = 200$. [1]

At $t = 0$, the population of wolves is 40.

- (ii) Find the value of C . [2]

At $t = 5$, the population of wolves is found to have increased to 70.

- (iii) Find the value of k . [2]

- (iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

- (b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$$\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right), \quad a \in \mathbb{R}^+.$$

- (i) Write down the value of $\frac{dx}{dt}$ when $x = 200$. [1]

- (ii) Interpret your answer to part (b)(i) in context. [1]

(This question continues on the following page)

(Question 2 continued)

Consider the function $f(x) = \ln(\ln 200 - \ln x)$, where $0 < x < 200$.

(iii) Show that $f'(x) = \frac{-1}{x \ln\left(\frac{200}{x}\right)}$. [2]

(iv) Hence, use separation of variables to show that the general solution of

$$\frac{dx}{dt} = ax \ln\left(\frac{200}{x}\right), \text{ where } 0 < x < 200,$$

can be written as

$$\ln x = \ln 200 - Ae^{-at},$$

where A is an arbitrary positive constant. [5]

(v) Use the size of the wolf population at $t = 0$ to find the value of A .
Give your answer in the form $A = \ln p$, where $p \in \mathbb{Z}^+$. [2]

(vi) Use the size of the wolf population at $t = 5$, given in part (a), to show that $a = 0.0855$, correct to three significant figures. [2]

(vii) Use the Gompertz model to predict the size of the wolf population at $t = 10$.
Give your answer correct to the nearest whole number. [3]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$$\frac{dx}{dt} = 0.0855x \ln\left(\frac{200}{x}\right) - 0.03x.$$

(d) (i) Use Euler's method, with a step size of 0.5 years and an initial value of $x_0 = 70$ when $t = 5$, to find an estimate for the size of the wolf population when $t = 10$.
Give your answer correct to the nearest whole number. [4]

(ii) Comment on your answer. [1]