

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 8 November 2022 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^n i^q = 1^q + 2^q + 3^q + \dots + n^q \text{ where } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n , for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^n i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^n i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p . [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^n i^2 = a_1n + a_2n^2 + a_3n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1 , a_2 and a_3 . [3]

(iii) Hence, find the values of a_1 , a_2 and a_3 . [2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q .

Consider the function $f(x) = 1 + x + x^2 + \dots + x^n$, $n \in \mathbb{Z}^+$.

(c) Show that $xf'(x) = x + 2x^2 + 3x^3 + \dots + nx^n$. [1]

Let $f_1(x) = xf'(x)$ and consider the following family of functions:

$$f_2(x) = xf_1'(x)$$

$$f_3(x) = xf_2'(x)$$

$$f_4(x) = xf_3'(x)$$

...

$$f_q(x) = xf_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^n i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^n i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \dots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \frac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

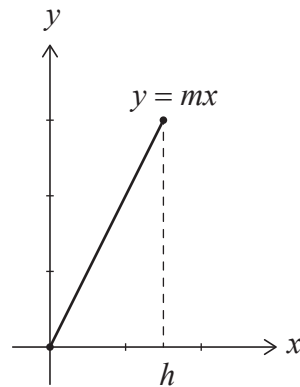
(g) (i) Show that $\lim_{x \rightarrow 1} f_1(x)$ is in indeterminate form. [1]

(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \rightarrow 1} f_1(x) = \frac{1}{2}n(n+1)$. [5]

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \leq x \leq h$ and m, h are positive constants.



When this line is rotated through 360° about the x -axis, a cone is formed with a curved surface area A given by:

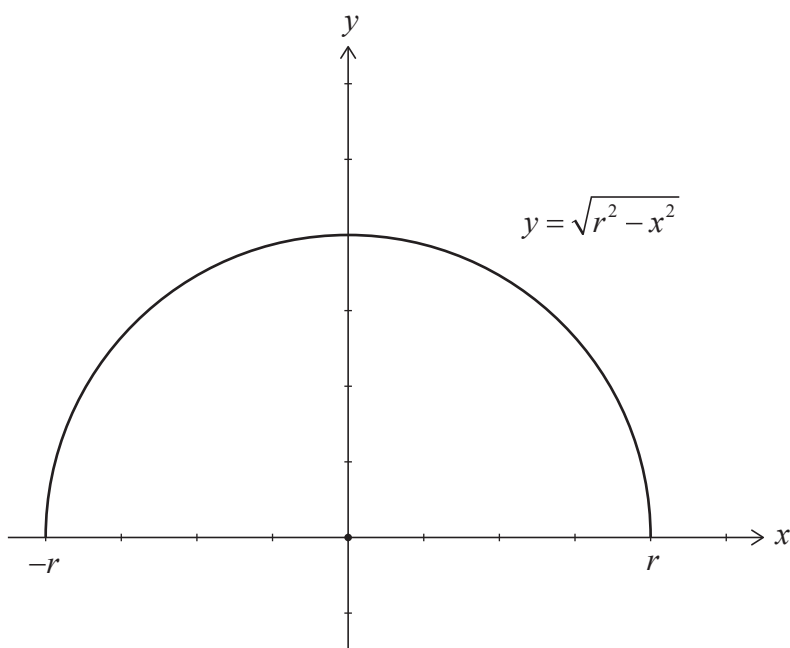
$$A = 2\pi \int_0^h y \sqrt{1 + m^2} \, dx.$$

- (a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\pi$. [2]
- (b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \leq x \leq h$ through 360° about the x -axis.
- (i) Deduce an expression for the radius of this cone r in terms of h and m . [1]
- (ii) Deduce an expression for the slant height l in terms of h and m . [2]
- (iii) Hence, by using the above integral, show that $A = \pi r l$. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the semi-circle, with radius r , defined by $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.



- (c) Find an expression for $\frac{dy}{dx}$. [2]

A differentiable curve $y = f(x)$ is defined for $x_1 \leq x \leq x_2$ and $y \geq 0$. When any such curve is rotated through 360° about the x -axis, the surface formed has an area A given by:

$$A = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

- (d) A sphere is formed by rotating the semi-circle $y = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$ through 360° about the x -axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(This question continues on the following page)

(Question 2 continued)

- (e) Let $f(x) = \sqrt{r^2 - x^2}$ where $-r \leq x \leq r$.

The graph of $y = f(x)$ is transformed to the graph of $y = f(kx)$, $k > 0$. This forms a different curve, called a semi-ellipse.

- (i) Describe this geometric transformation. [2]
- (ii) Write down the x -intercepts of the graph $y = f(kx)$ in terms of r and k . [1]
- (iii) For $y = f(kx)$, find an expression for $\frac{dy}{dx}$ in terms of x , r and k . [2]
- (iv) The semi-ellipse $y = f(kx)$ is rotated 360° about the x -axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A , of the ellipsoid.

Give your answer in the form $2\pi \int_{x_1}^{x_2} \sqrt{p(x)} \, dx$, where $p(x)$ is a polynomial. [4]

- (v) Planet Earth can be modelled as an ellipsoid. In this model:
 - the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
 - the distance from the North Pole to the South Pole is 12 714 km.
 - the diameter of the equator is 12 756 km.

By choosing suitable values for r and k , find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a \times 10^q$ where $1 \leq a < 10$ and $q \in \mathbb{Z}^+$. [4]

References: