

APOLLO

Functions

HL

Mathematics: analysis and approaches

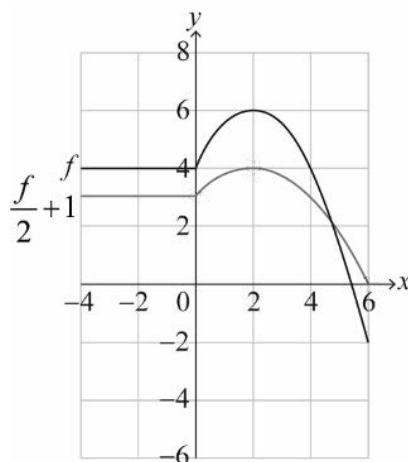
First assessment 2021

68 questions | mark schemes

1. (a) (i) $f(2) = 6$
 (ii) $(f \circ f)(2) = -2$

A1**A1****[2 marks]**

(b)

**M1A1A1**

Note: Award **M1** for an attempt to apply any vertical stretch or vertical translation, **A1** for a correct horizontal line segment between -4 and 0 (located roughly at $y = 3$), **A1** for a correct concave down parabola including max point at $(2, 4)$ and for correct end points at $(0, 3)$ and $(6, 0)$ (within circles). Points do not need to be labelled.

[3 marks]**Total [5 marks]**

$$7. \quad \alpha + \beta + \alpha + \beta = k \quad (\mathbf{A1})$$

$$\alpha + \beta = \frac{k}{2}$$

$$\alpha\beta(\alpha + \beta) = -3k \quad (\mathbf{A1})$$

$$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right) = -3k \quad \left(-\frac{k^3}{8} = -3k\right) \quad \mathbf{M1}$$

$$\text{attempting to solve } -\frac{k^3}{8} + 3k = 0 \text{ (or equivalent) for } k \quad (\mathbf{M1})$$

$$k = 2\sqrt{6} \quad (= \sqrt{24}) \quad (k > 0) \quad \mathbf{A1}$$

Note: Award **A0** for $k = \pm 2\sqrt{6} \quad (\pm \sqrt{24})$.

[5 marks]

11. (a) attempt to solve $4x^2 - 1 = 0$ e.g. by factorising $4x^2 - 1$ (M1)
 $p = \frac{1}{2}, q = -\frac{1}{2}$ or vice versa A1

[2 marks]

- (b) attempt to use quotient rule or product rule (M1)

EITHER

$$f'(x) = \frac{3(4x^2 - 1) - 8x(3x + 2)}{(4x^2 - 1)^2} \left(= \frac{-12x^2 - 16x - 3}{(4x^2 - 1)^2} \right) \quad \text{A1A1}$$

Note: Award **A1** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x) = -8x(3x + 2)(4x^2 - 1)^{-2} + 3(4x^2 - 1)^{-1} \quad \text{A1A1}$$

Note: Award **A1** for each term.

[3 marks]

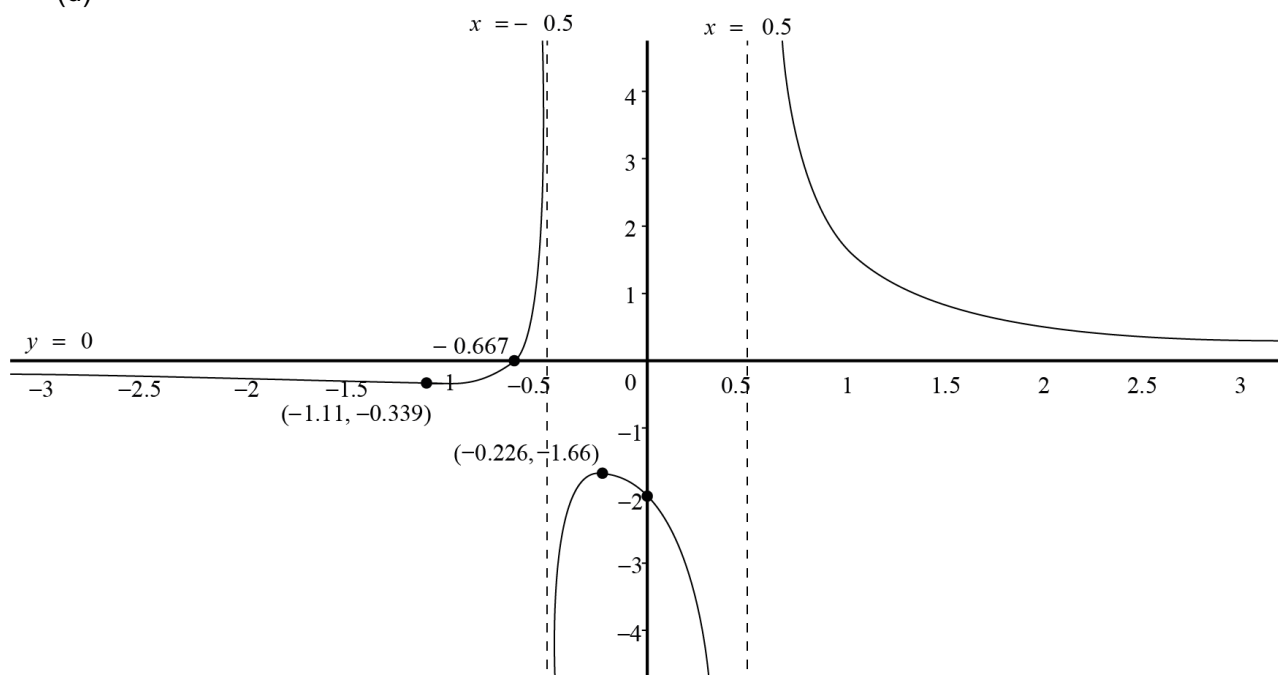
- (c) attempt to find the local min point on $y = f'(x)$ OR solve $f''(x) = 0$ (M1)
 $x = -1.60$ A1

[2 marks]

continued...

Question 11 continued

(d)

**A1A1A1A1A1**

Note: Award **A1** for both vertical asymptotes with their equations, award **A1** for horizontal asymptote with equation, award **A1** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts. If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum **A0A1A0A1A1**.

[5 marks]*continued...*

Question 11 continued

e) $x = -\frac{2}{3} (= -0.667)$ **A1**

(oblique asymptote has) gradient $\frac{4}{3} (= 1.33)$ **(A1)**

appropriate method to find complete equation of oblique asymptote **M1**

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x + 2 \overline{) 4x^2 + 0x - 1} \end{array}$$

$$4x^2 + \frac{8}{3}x$$

$$\hline -\frac{8}{3}x - 1$$

$$-\frac{8}{3}x - \frac{16}{9}$$

$$\hline \frac{7}{9}$$

$y = \frac{4}{3}x - \frac{8}{9} (= 1.33x - 0.889)$ **A1**

Note: Do not award the final A1 if the answer is not given as an equation.
--

[4 marks]

f) attempting to find at least one critical value ($x = -0.568729\dots, x = 1.31872\dots$) **(M1)**

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ **A1A1A1**

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]**Total [20 marks]**

4. (a) attempting to find the vertex (M1)

$$x=1 \text{ OR } y=-5 \text{ OR } f(x)=6(x-1)^2-5$$

$$\text{range is } y \geq -5 \quad \text{A1}$$

[2 marks]

- (b) **METHOD 1**

$$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \quad (= -(6(x-1)^2 - 5) + c) \quad \text{(A1)}$$

EITHER

$$\text{relating to the range of } f \text{ OR attempting to find } g(-5) \quad \text{(M1)}$$

$$5 + c \leq 0 \quad \text{(A1)}$$

OR

$$\text{attempting to find the discriminant of } (g \circ f)(x) \quad \text{(M1)}$$

$$144 + 24(c-1) \leq 0 \quad (120 + 24c \leq 0) \quad \text{(A1)}$$

THEN

$$c \leq -5 \quad \text{A1}$$

[4 marks]

METHOD 2

$$\text{vertical reflection followed by vertical shift} \quad \text{(M1)}$$

$$\text{new vertex is } (1, 5+c) \quad \text{(A1)}$$

$$5 + c \leq 0 \quad \text{(A1)}$$

$$c \leq -5 \quad \text{A1}$$

[4 marks]

Total [6 marks]

5. (a) $100 = A_0 e^0$ **A1**
 $A_0 = 100$ **AG**

[1 mark]

(b) correct substitution of values into exponential equation **(M1)**

$$50 = 100e^{-5730k} \quad \text{OR} \quad e^{-5730k} = \frac{1}{2}$$

EITHER

$$-5730k = \ln \frac{1}{2} \quad \text{A1}$$

$$\ln \frac{1}{2} = -\ln 2 \quad \text{OR} \quad -\ln \frac{1}{2} = \ln 2 \quad \text{A1}$$

OR

$$e^{5730k} = 2 \quad \text{A1}$$

$$5730k = \ln 2 \quad \text{A1}$$

THEN

$$k = \frac{\ln 2}{5730} \quad \text{AG}$$

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if 25 % of the carbon-14 has decayed, 75% remains ie, 75 units remain

$$75 = 100e^{-\frac{\ln 2}{5730}t} \quad \text{(A1)}$$

EITHER

using an appropriate graph to attempt to solve for t **(M1)**

OR

manipulating logs to attempt to solve for t **(M1)**

$$\ln 0.75 = -\frac{\ln 2}{5730}t$$

$$t = 2378.164...$$

THEN

$$t = 2380 \text{ (years) (correct to the nearest 10 years)} \quad \text{A1}$$

[3 marks]**Total [7 marks]**

1

12. (a) **EITHER**

$$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x) \quad \textbf{R1}$$

OR

a sketch graph of $y = f(x)$ with line symmetry in the y -axis indicated **R1**

THEN

so $f(x)$ is an even function **AG**

[1 mark]

(b) as $x \rightarrow \pm\infty$, $f(x) \rightarrow \arcsin 1 \left(\rightarrow \frac{\pi}{2} \right)$ **A1**

so the horizontal asymptote is $y = \frac{\pi}{2}$ **A1**

[2 marks]

continued...

Question 12 continued

- (c) (i) attempting to use the quotient rule to find $\frac{d}{dx} \left(\frac{x^2-1}{x^2+1} \right)$ **M1**

$$\frac{d}{dx} \left(\frac{x^2-1}{x^2+1} \right) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} \left(= \frac{4x}{(x^2+1)^2} \right) \quad \textbf{A1}$$

- attempting to use the chain rule to find $\frac{d}{dx} \left(\arcsin \left(\frac{x^2-1}{x^2+1} \right) \right)$ **M1**

$$\text{let } u = \frac{x^2-1}{x^2+1} \text{ and so } y = \arcsin u \text{ and } \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}} \times \frac{4x}{(x^2+1)^2} \quad \textbf{M1}$$

$$= \frac{4x}{\sqrt{(x^2+1)^2 - (x^2-1)^2}} \times \frac{1}{(x^2+1)} \quad \textbf{A1}$$

$$= \frac{4x}{\sqrt{4x^2}} \times \frac{1}{(x^2+1)} \quad \textbf{A1}$$

$$= \frac{2x}{\sqrt{x^2} (x^2+1)} \quad \textbf{AG}$$

(ii) $f'(x) = \frac{2x}{|x|(x^2+1)}$

EITHER

for $x < 0$, $|x| = -x$ **(A1)**

so $f'(x) = -\frac{2}{x^2+1}$ **A1**

OR

$|x| > 0$ and $x^2+1 > 0$ **A1**

$2x < 0$, $x < 0$ **A1**

THEN

$f'(x) < 0$ **R1**

Note: Award **R1** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x < 0$ **AG**

Note: Do not accept a graphical solution.

[9 marks]

Question 12 continued

$$(d) \quad x = \arcsin\left(\frac{y^2 - 1}{y^2 + 1}\right) \quad \mathbf{M1}$$

$$\sin x = \frac{y^2 - 1}{y^2 + 1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1 \quad \mathbf{A1}$$

$$y^2 = \frac{1 + \sin x}{1 - \sin x} \quad \mathbf{A1}$$

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) $\mathbf{R1}$

Note: The **R1** is dependent on the above **A1**.

$$\text{so } (g^{-1}(x)) = \sqrt{\frac{1 + \sin x}{1 - \sin x}} \quad \mathbf{A1}$$

Note: The final **A1** is not dependent on **R1** mark.

[5 marks]

$$(e) \quad \text{domain is } -\frac{\pi}{2} \leq x < \frac{\pi}{2} \quad \mathbf{A1}$$

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

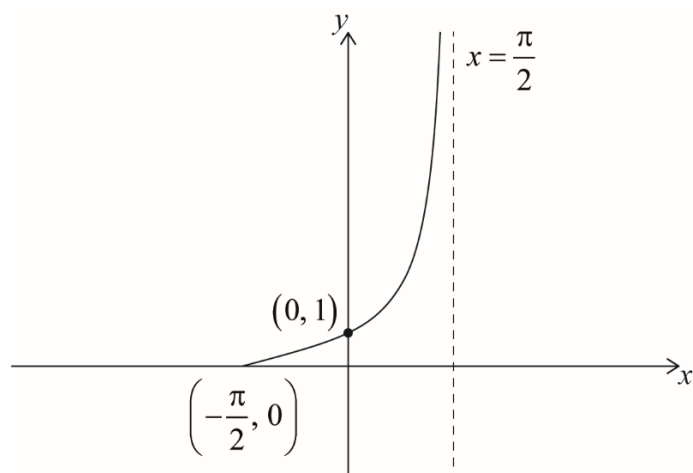
Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

continued...

Question 12 continued

(f)

**A1A1A1**

Note: **A1** for correct domain and correct range and y -intercept at $y = 1$

A1 for asymptotic behaviour $x \rightarrow \frac{\pi}{2}$

A1 for $x = \frac{\pi}{2}$

Coordinates are not required.

Do not accept $x = 1.57$ or other inexact values.

[3 marks]**Total [21 marks]**

- 2.** (a) (i) $x = 3$ **A1**
- (ii) $y = -2$ **A1**

[2 marks]

- (b) (i) $(-2, 0)$ (accept $x = -2$) **A1**

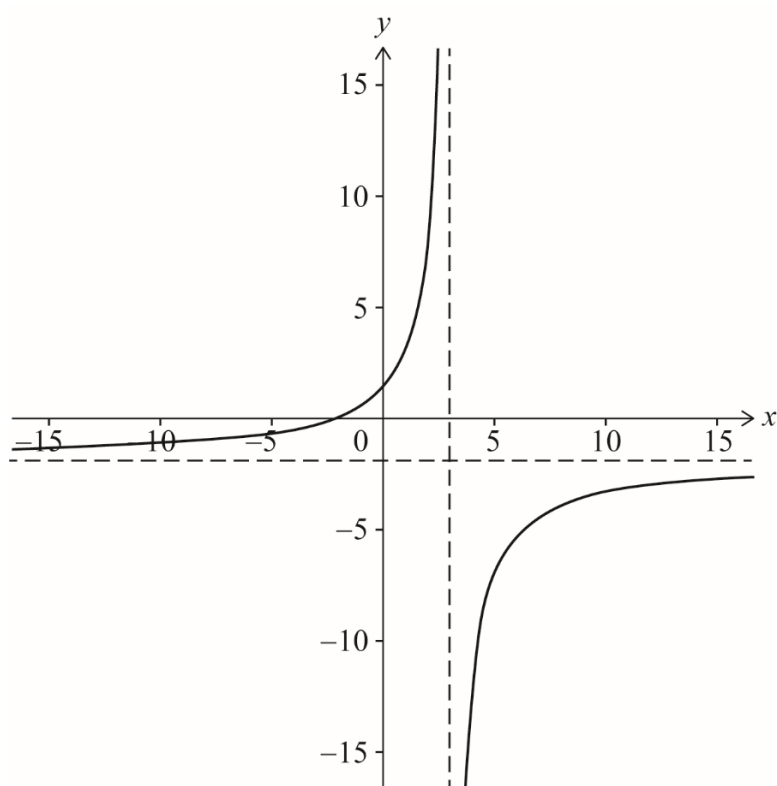
- (ii) $\left(0, \frac{4}{3}\right)$ (accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) **A1**

[2 marks]

continued...

Question 2 continued.

(c)



A1

Note: Award **A1** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]
continue...

Question 2 continued.

(d) **METHOD 1**

$$(g(x))y = \frac{ax+4}{3-x}$$

attempt to find x in terms of y

(M1)

OR exchange x and y and attempt to find y in terms of x

$$3y - xy = ax + 4$$

A1

$$ax + xy = 3y - 4$$

$$x(a + y) = 3y - 4$$

$$x = \frac{3y - 4}{y + a}$$

$$g^{-1}(x) = \frac{3x - 4}{x + a}$$

A1

Note: Condone use of $y =$

$$g(x) \equiv g^{-1}(x)$$

$$\frac{ax+4}{3-x} \equiv \frac{3x-4}{x+a}$$

$$\Rightarrow a = -3$$

A1

[4 marks]
continue...

Question 2 continued.

METHOD 2

$$g(x) = \frac{ax+4}{3-x}$$

attempt to find an expression for $g(g(x))$ and equate to x (M1)

$$gg(x) = \frac{a\left(\frac{ax+4}{3-x}\right) + 4}{3 - \left(\frac{ax+4}{3-x}\right)} = x \quad \text{A1}$$

$$\frac{a(ax+4) + 4(3-x)}{(9-3x) - (ax+4)} = x$$

$$\frac{a(ax+4) + 4(3-x)}{5 - (3+a)x} = x$$

$$a(ax+4) + 4(3-x) = x(5 - (3+a)x) \quad \text{A1}$$

equating coefficients of x^2 (or similar)

$$a = -3 \quad \text{A1}$$

[4 marks]

Total [9 marks]

3. attempt to use change the base

(M1)

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3 (4x^3)$$

attempt to use the power rule

(M1)

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3 (4x^3)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$

(M1)

$$\log_3 \sqrt{x} = \log_3 (4\sqrt{2}x^3)$$

Note: The **M** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen.

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

$$x^5 = \frac{1}{32}$$

(A1)

$$x = \frac{1}{2}$$

A1

[5 marks]

7. (a) attempt to use discriminant $b^2 - 4ac (> 0)$ **M1**

$$(2p)^2 - 4(3p)(1-p) (> 0)$$

$$16p^2 - 12p (> 0) \quad \textbf{(A1)}$$

$$p(4p - 3) (> 0)$$

attempt to find critical values $\left(p = 0, p = \frac{3}{4} \right)$ **M1**

recognition that discriminant > 0 **(M1)**

$$p < 0 \text{ or } p > \frac{3}{4} \quad \textbf{A1}$$

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) $p = 4 \Rightarrow 12x^2 + 8x - 3 = 0$

valid attempt to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (or equivalent) **M1**

$$x = \frac{-8 \pm \sqrt{208}}{24}$$

$$x = \frac{-2 \pm \sqrt{13}}{6}$$

$$a = -2 \quad \textbf{A1}$$

[2 marks]

Total [7 marks]

10. (a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$
 $(-3, 0)$ and $(4, 0)$

(M1)**A1**(ii) $\left(0, \frac{4}{5}\right)$ **A1****[3 marks]**(b) $x = \frac{15}{2}$ **A1**

Note: Award **A0** for $x \neq \frac{15}{2}$.

Award **A1** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark](c) **METHOD 1**

$$(ax + b)(2x - 15) \equiv x^2 - x - 12$$

attempts to expand $(ax + b)(2x - 15)$

(M1)

$$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$$

$$a = \frac{1}{2}$$

A1

equates coefficients of x

(M1)

$$-1 = -\frac{15}{2} + 2b$$

$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4}\right)$$

METHOD 2attempts division on $\frac{x^2 - x - 12}{2x - 15}$ **M1**

$$\frac{x}{2} + \frac{13}{4} + \dots$$

M1

$$a = \frac{1}{2}$$

A1

$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4} \right)$$

METHOD 3

$$a = \frac{1}{2}$$

A1

$$\frac{x^2 - x - 12}{2x - 15} \equiv \frac{x}{2} + b + \frac{c}{2x - 15}$$

M1

$$x^2 - x - 12 \equiv \frac{(2x - 15)x}{2} + (2x - 15)b + c$$

equates coefficients of x :**(M1)**

$$-1 = -\frac{15}{2} + 2b$$

$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4} \right)$$

METHOD 4attempts division on $\frac{x^2 - x - 12}{2x - 15}$ **M1**

$$\frac{x^2 - x - 12}{2x - 15} = \frac{x}{2} + \frac{\frac{13x}{2} - 12}{2x - 15}$$

$$a = \frac{1}{2}$$

A1

$$\frac{\frac{13x}{2} - 12}{2x - 15} = \frac{13}{4} + \dots$$

M1

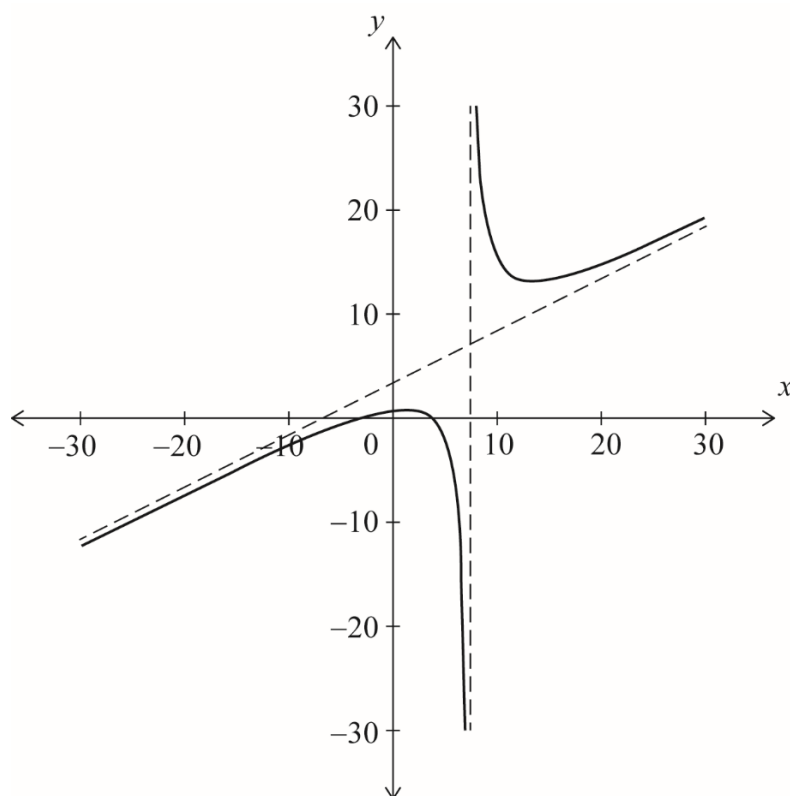
$$b = \frac{13}{4}$$

A1

$$\left(y = \frac{x}{2} + \frac{13}{4} \right)$$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \leq x \leq 30$)

A1

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes

A1

their axes intercepts in approximately the correct positions

A1

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

- (e) (i) attempts to split into partial fractions: **(M1)**

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4) + B(x+3)$$

$$A = 3$$

A1

$$B = -1$$

A1

$$\left(\frac{3}{x+3} - \frac{1}{x-4} \right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4} \right) dx$

attempts to integrate and obtains two terms involving 'ln' **(M1)**

$$= \left[3 \ln|x+3| - \ln|x-4| \right]_0^3$$

A1

$$= 3 \ln 6 - \ln 1 - 3 \ln 3 + \ln 4$$

A1

$$= 3 \ln 2 + \ln 4 \quad (= \ln 8 + \ln 4)$$

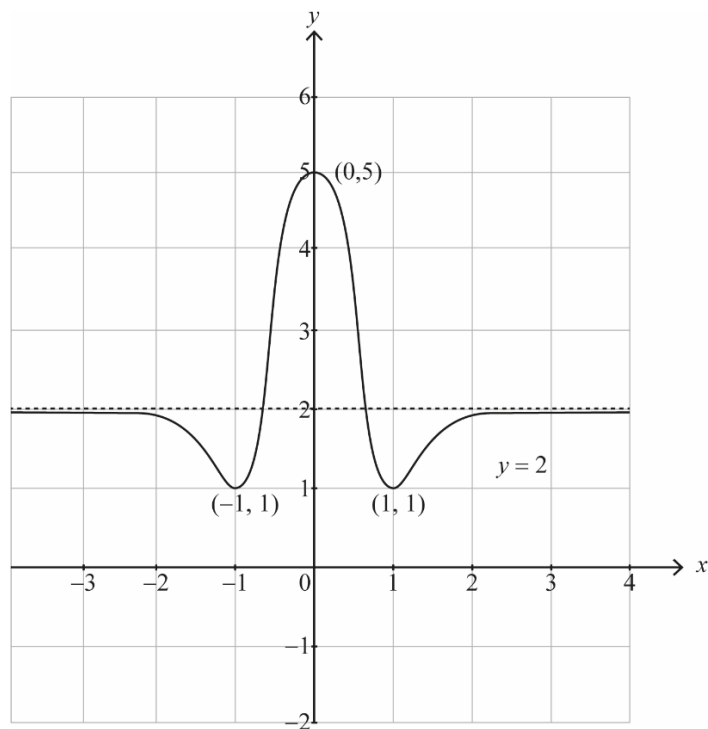
$$= \ln 32 \quad (= 5 \ln 2)$$

A1

Note: The final A1 is dependent on the previous two A marks.

[7 marks]**Total [18 marks]**

4.



no y values below 1

horizontal asymptote at $y = 2$ with curve approaching from below as $x \rightarrow \pm\infty$

$(\pm 1, 1)$ local minima

$(0, 5)$ local maximum

smooth curve and smooth stationary points

A1

A1

A1

A1

A1

Total [5 marks]

- 5.** (a) attempt to form composition **M1**
 correct substitution $g\left(\frac{x+3}{4}\right) = 8\left(\frac{x+3}{4}\right) + 5$ **A1**
 $(g \circ f)(x) = 2x + 11$ **AG**
[2 marks]
- (b) attempt to substitute 4 (seen anywhere) **(M1)**
 correct equation $a = 2 \times 4 + 11$ **(A1)**
 $a = 19$ **A1**
[3 marks]
- Total [5 marks]**

9. (a) attempt to differentiate and set equal to zero

M1

$$f'(x) = 2e^{2x} - 6e^x = 2e^x(e^x - 3) = 0$$

A1

minimum at $x = \ln 3$

$$a = \ln 3$$

A1**[3 marks]**

- (b) **Note:** Interchanging x and y can be done at any stage.

$$y = (e^x - 3)^2 - 4$$

(M1)

$$e^x - 3 = \pm\sqrt{y+4}$$

A1

as $x \leq \ln 3$, $x = \ln(3 - \sqrt{y+4})$

R1

so $f^{-1}(x) = \ln(3 - \sqrt{x+4})$

A1

domain of f^{-1} is $x \in \mathbb{R}, -4 \leq x < 5$

A1**[5 marks]****Total [8 marks]**

Section B

9. METHOD 1

sketching the graph of $y = \frac{x^2}{x-3}$ ($y = x + 3 + \frac{9}{x-3}$) **M1**

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 **R1**

consideration of a straight line steeper than the horizontal line joining

$(-3, 0)$ and $(0, 0)$ **M1**

so $m > 0$ **R1**

hence $0 < m \leq 1$ **A1**

METHOD 2

attempting to eliminate y to form a quadratic equation in x **M1**

$$x^2 = m(x^2 - 9)$$

$$\Rightarrow (m-1)x^2 - 9m = 0$$
 A1

EITHER

attempting to solve $-4(m-1)(-9m) < 0$ for m **M1**

OR

attempting to solve $x^2 < 0$ ie $\frac{9m}{m-1} < 0$ ($m \neq 1$) for m **M1**

THEN

$$\Rightarrow 0 < m < 1$$
 A1

a valid reason to explain why $m = 1$ gives no solutions eg if $m = 1$,

$$(m-1)x^2 - 9m = 0 \Rightarrow -9 = 0 \text{ and so } 0 < m \leq 1$$
 R1

Total [5 marks]

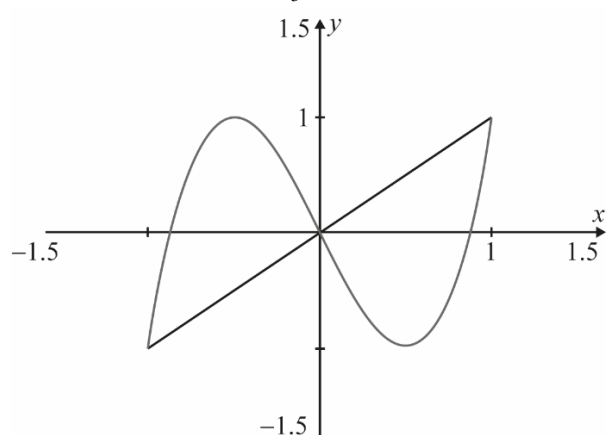
Section B

2. (a) correct graph of $y = f_1(x)$

A1

correct graph of $y = f_3(x)$

A1



[2 marks]

(b) (i) graphical or tabular evidence that n has been systematically varied

M1

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points

(A1)

$\frac{n-1}{2}$ local maximum points

A1

(ii) $\frac{n-1}{2}$ local minimum points

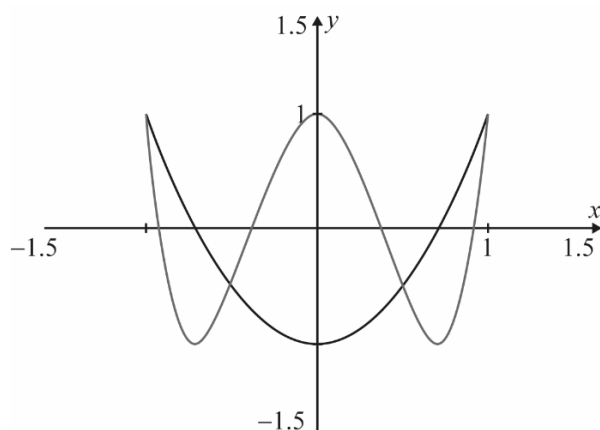
A1

Note: Allow follow through from an incorrect local maximum formula expression.

[4 marks]

continued...

Question 2 continued

(c) correct graph of $y = f_2(x)$ **A1**correct graph of $y = f_4(x)$ **A1****[2 marks]**(d) (i) graphical or tabular evidence that n has been systematically varied**M1**eg $n = 2$, 0 local maximum point and 1 local minimum point $n = 4$, 1 local maximum points and 2 local minimum points $n = 6$, 2 local maximum points and 3 local minimum points**(A1)** $\frac{n-2}{2}$ local maximum points**A1**(ii) $\frac{n}{2}$ local minimum points**A1****[4 marks]**(e) $f_n(x) = \cos(n \arccos(x))$

$$f'_n(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$$

M1A1**Note:** Award **M1** for attempting to use the chain rule.

$$f'_n(x) = 0 \Rightarrow n \sin(n \arccos(x)) = 0$$

M1

$$n \arccos(x) = k\pi \quad (k \in \mathbb{Z}^+)$$

A1

leading to

$$x = \cos \frac{k\pi}{n} \quad (k \in \mathbb{Z}^+ \text{ and } 0 < k < n)$$

AG**[4 marks]**

continued...

Question 2 continued

- (f) $f_2(x) = \cos(2 \arccos x)$
 $= 2(\cos(\arccos x))^2 - 1$ **M1**
 stating that $(\cos(\arccos x)) = x$ **A1**
 so $f_2(x) = 2x^2 - 1$ **AG**
[2 marks]
- (g) $f_{n+1}(x) = \cos((n+1)\arccos x)$
 $= \cos(n \arccos x + \arccos x)$ **A1**
 use of $\cos(A+B) = \cos A \cos B - \sin A \sin B$ leading to **M1**
 $= \cos(n \arccos x) \cos(\arccos x) - \sin(n \arccos x) \sin(\arccos x)$ **AG**
[2 marks]
- (h) (i) $f_{n-1}(x) = \cos((n-1)\arccos x)$ **A1**
 $= \cos(n \arccos x) \cos(\arccos x) + \sin(n \arccos x) \sin(\arccos x)$ **M1**
 $f_{n+1}(x) + f_{n-1}(x) = 2 \cos(n \arccos x) \cos(\arccos x)$ **A1**
 $= 2xf_n(x)$ **AG**
- (ii) $f_3(x) = 2xf_2(x) - f_1(x)$ **(M1)**
 $= 2x(2x^2 - 1) - x$
 $= 4x^3 - 3x$ **A1**
[5 marks]
- Total [25 marks]**
-

5. (a) translation (shift) by $\frac{3\pi}{2}$ to the right OR positive horizontal direction by $\frac{3\pi}{2}$ **A1**
 translation (shift) by q upwards OR positive vertical direction by q **A1**

Note: accept translation by $\begin{pmatrix} \frac{3\pi}{2} \\ q \end{pmatrix}$

Do not accept 'move' for translation/shift.

[2 marks]

(b) **METHOD 1**

minimum of $4\sin\left(x - \frac{3\pi}{2}\right)$ is -4 (may be seen in sketch) **(M1)**

$$-4 + 2.5 + q \geq 7$$

$q \geq 8.5$ (accept $q = 8.5$) **A1**

substituting $x = 0$ and their $q (= 8.5)$ to find r **(M1)**

$$(r =) 4\sin\left(\frac{-3\pi}{2}\right) + 2.5 + 8.5$$

$$4 + 2.5 + 8.5$$
 (A1)

smallest value of r is 15 **A1**

continued...

*Question 5 continued***METHOD 2**

substituting $x=0$ to find an expression (for r) in terms of q **(M1)**

$$(g(0)=r=) \quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+q$$

$$(r=) \quad 6.5+q \quad \text{A1}$$

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 **(M1)**

$$-4+2.5+q \geq 7$$

$$-4+2.5+(r-6.5) \geq 7 \quad (\text{accept } =) \quad \text{A1}$$

smallest value of r is 15 **A1**

METHOD 3

$$4\sin\left(x-\frac{3\pi}{2}\right)+2.5+q=4\cos x+2.5+q \quad \text{A1}$$

y -intercept of $4\cos x+2.5+q$ is a maximum **(M1)**

amplitude of $g(x)$ is 4 **(A1)**

attempt to find least maximum **(M1)**

$$r=2 \times 4+7$$

smallest value of r is 15 **A1**

[5 marks]

Total [7 marks]

3.

(a)

(i)

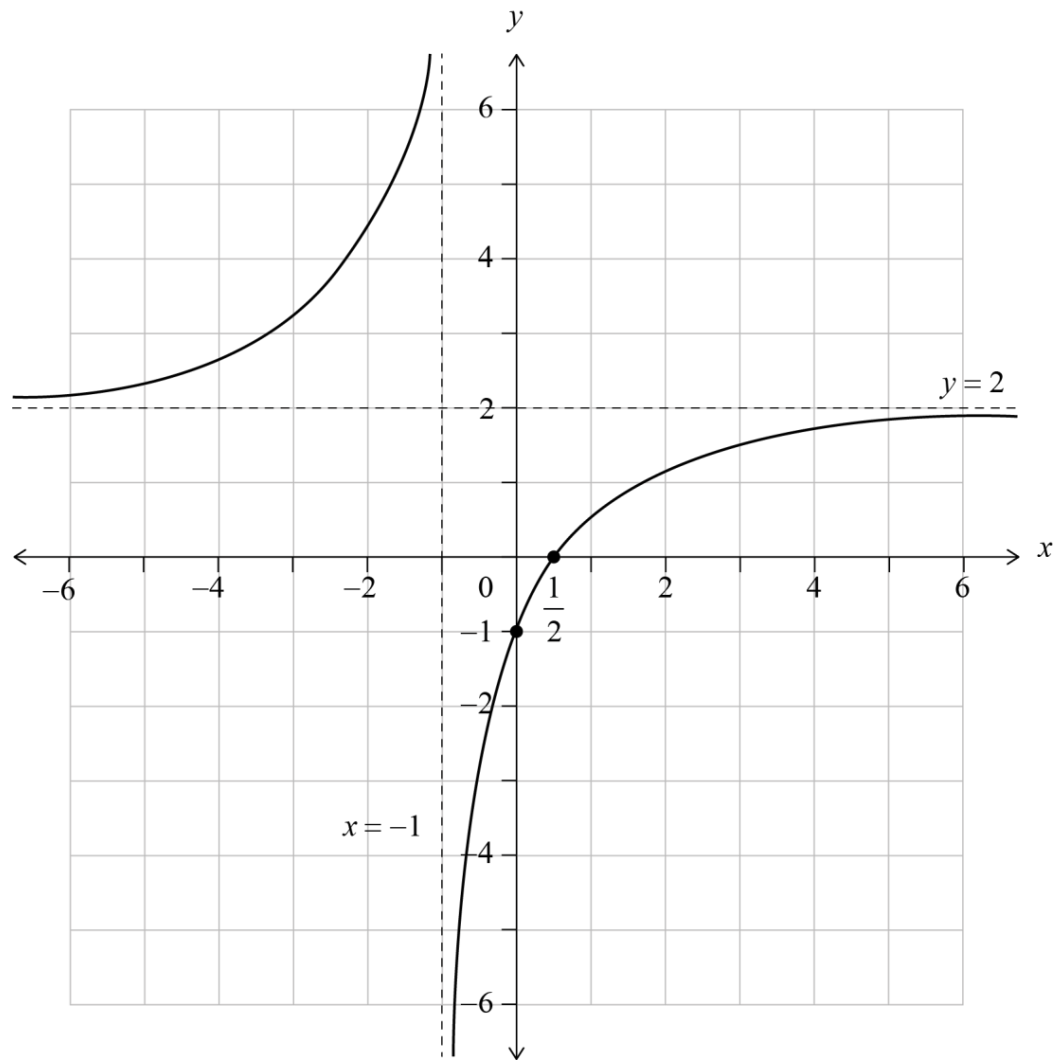
$x = -1$

A1
- (ii)

$y = 2$

A1
- [2 marks]

(b)



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown

A1

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$

A1A1**[3 marks]***continued...*

Question 3 continued

(c) $x > \frac{1}{2}$

A1

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2}, \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark](d) **EITHER**

attempts to sketch $y = \frac{2|x| - 1}{|x| + 1}$

(M1)**OR**

attempts to solve $2|x| - 1 = 0$

(M1)

Note: Award the **(M1)** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

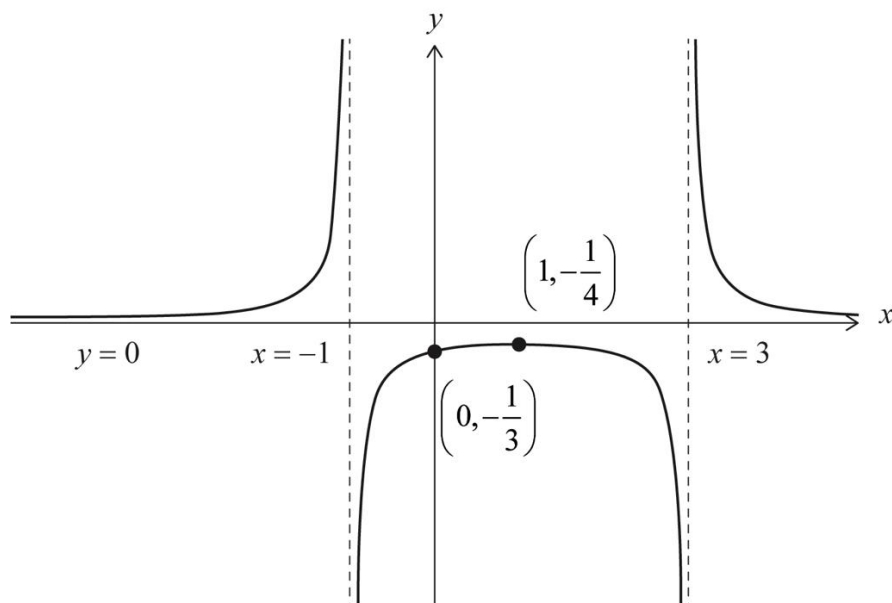
$x < -\frac{1}{2} \text{ or } x > \frac{1}{2}$

A1

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]**Total [8 marks]**

11. (a)



y- intercept $\left(0, -\frac{1}{3}\right)$

A1

Note: Accept an indication of $-\frac{1}{3}$ on the y -axis.

vertical asymptotes $x = -1$ and $x = 3$

A1

horizontal asymptote $y = 0$

A1

uses a valid method to find the x - coordinate of the local maximum point

(M1)

Note: For example, uses the axis of symmetry or attempts to solve $f'(x) = 0$.

local maximum point $\left(1, -\frac{1}{4}\right)$

A1

Note: Award **(M1)A0** for a local maximum point at $x = 1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other

A1**[6 marks]***continued...*

Question 11 continued

(b) (i) $x = \frac{1}{y^2 - 2y - 3}$ **M1**

Note: Award **M1** for interchanging x and y (this can be done at a later stage).

EITHER

attempts to complete the square **M1**

$$y^2 - 2y - 3 = (y - 1)^2 - 4$$
A1

$$x = \frac{1}{(y - 1)^2 - 4}$$

$$(y - 1)^2 - 4 = \frac{1}{x} \left((y - 1)^2 = 4 + \frac{1}{x} \right)$$
A1

$$y - 1 = \pm \sqrt{4 + \frac{1}{x}} \left(= \pm \sqrt{\frac{4x + 1}{x}} \right)$$

OR

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for y **M1**

$$y = \frac{-(-2x) \pm \sqrt{(-2x)^2 + 4x(3x + 1)}}{2x}$$
A1

Note: Award **A1** even if $-$ (in $\pm$) is missing

$$= \frac{2x \pm \sqrt{16x^2 + 4x}}{2x}$$
A1

continued...

Question 11 continued

THEN

$$= 1 \pm \frac{\sqrt{4x^2 + x}}{x}$$

A1

$$y > 3 \text{ and hence } y = 1 - \frac{\sqrt{4x^2 + x}}{x} \text{ is rejected}$$

R1

Note: Award **R1** for concluding that the expression for y must have the '+' sign.

The **R1** may be awarded earlier for using the condition $x > 3$.

$$y = 1 + \frac{\sqrt{4x^2 + x}}{x}$$

$$g^{-1}(x) = 1 + \frac{\sqrt{4x^2 + x}}{x}$$

AG

(ii) domain of g^{-1} is $x > 0$

A1

[7 marks]

continued...

Question 11 continued

(c) attempts to find $(h \circ g)(a)$ (M1)

$$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right) \quad \left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2 - 2a - 3)}\right) \right) \quad (\text{A1})$$

$$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4} \quad \left(\arctan\left(\frac{1}{2(a^2 - 2a - 3)}\right) = \frac{\pi}{4} \right)$$

attempts to solve for $g(a)$ M1

$$\Rightarrow g(a) = 2 \quad \left(\frac{1}{(a^2 - 2a - 3)} = 2 \right)$$

EITHER

$$\Rightarrow a = g^{-1}(2) \quad (\text{A1})$$

attempts to find their $g^{-1}(2)$ M1

$$a = 1 + \frac{\sqrt{4(2)^2 + 2}}{2} \quad (\text{A1})$$

Note: Award all available marks to this stage if x is used instead of a .**OR**

$$\Rightarrow 2a^2 - 4a - 7 = 0 \quad (\text{A1})$$

attempts to solve their quadratic equation M1

$$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4} \quad \left(= \frac{4 \pm \sqrt{72}}{4} \right) \quad (\text{A1})$$

Note: Award all available marks to this stage if x is used instead of a .**THEN**

$$a = 1 + \frac{3}{2}\sqrt{2} \quad (\text{as } a > 3) \quad (\text{A1})$$

$$(p = 1, q = 3, r = 2)$$

Note: Award A1 for $a = 1 + \frac{1}{2}\sqrt{18}$ ($p = 1, q = 1, r = 18$).

[7 marks]

Total [20 marks]

6. (a) attempt to replace x with $-x$

M1

$$f(-x) = 2^{-x} - \frac{1}{2^{-x}}$$

EITHER

$$= \frac{1}{2^x} - 2^x = -f(x)$$

A1

OR

$$= -\left(2^x - \frac{1}{2^x}\right) (= -f(x))$$

A1

Note: Award **M1A0** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180° about the origin **or** the graph is invariant after a reflection in the y -axis and then in the x -axis (or vice versa).

so f is an odd function

AG

[2 marks]

- (b) attempt to find at least one intersection point

(M1)

$$x = -1.26686..., x = 0.177935..., x = 3.06167...$$

$$x = -1.27, x = 0.178, x = 3.06$$

$$-1.27 \leq x < -1,$$

A1

$$0.178 \leq x < 3,$$

A1

$$x \geq 3.06$$

A1

[4 marks]

Total [6 marks]

8. (a) product of roots $= \frac{2k+9}{k}$

A1

[1 mark]

(b) recognition that the product of the roots will be negative

(M1)

$$\frac{2k+9}{k} < 0$$

critical values $k = 0, -\frac{9}{2}$ seen

(A1)

$$-\frac{9}{2} < k < 0$$

A1

[3 marks]

Total [4 marks]

2. (a) attempt to expand $(x - \alpha)(x - \beta)(x - \gamma)$ **M1**
- $$= (x^2 - (\alpha + \beta)x + \alpha\beta)(x - \gamma) \quad \text{OR} \quad = (x - \alpha)(x^2 - (\beta + \gamma)x + \beta\gamma)$$
- A1**
- $$(x^3 + px^2 + qx + r) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$$
- A1**
- comparing coefficients:
- $$p = -(\alpha + \beta + \gamma)$$
- AG**
- $$q = (\alpha\beta + \beta\gamma + \gamma\alpha)$$
- AG**
- $$r = -\alpha\beta\gamma$$
- AG**

Note: For candidates who do not include the **AG** lines award full marks.

[3 marks]

- (b) (i) $p^2 - 2q = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ **(A1)**
- attempt to expand $(\alpha + \beta + \gamma)^2$ **(M1)**
- $$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \text{ or equivalent}$$
- A1**
- $$= \alpha^2 + \beta^2 + \gamma^2$$
- AG**

Note: Accept equivalent working from RHS to LHS.

[3 marks]

continued...

Question 2 continued(ii) **EITHER**attempt to expand $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$ (M1)

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha) \quad \text{A1}$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 2(p^2 - 2q) - 2q \text{ or equivalent} \quad \text{A1}$$

$$= 2p^2 - 6q \quad \text{AG}$$

ORattempt to write $2p^2 - 6q$ in terms of α, β, γ (M1)

$$= 2(p^2 - 2q) - 2q$$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \quad \text{A1}$$

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) + (\beta^2 + \gamma^2 - 2\beta\gamma) + (\gamma^2 + \alpha^2 - 2\gamma\alpha) \quad \text{A1}$$

$$= (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \quad \text{AG}$$

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]*continued...*

Question 2 continued

(c) $p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$$\Rightarrow (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 < 0$$

A1

if all roots were real $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 \geq 0$

R1**Note:** Condone strict inequality in the **R1** line.**Note:** Do not award **A0R1**. $\Rightarrow$ roots cannot all be real**AG****[2 marks]**

(d) $p^2 = (-7)^2 = 49$ and $3q = 51$

A1so $p^2 < 3q \Rightarrow$ the equation has at least one complex root**R1****Note:** Allow equivalent comparisons; e.g. checking $2p^2 < 6q$ **[2 marks]**

(e) (i) use of GDC (eg graphs or tables)

(M1)

$q = 12$

A1**[2 marks]**

(ii) complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).

OR

a cubic curve always crosses the x -axis at at least one point.**R1****[1 mark]**

continued...

Question 2 continued

- (f) (i) attempt to expand $(\alpha + \beta + \gamma + \delta)^2$ (M1)

$$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

(A1)

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$(\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 =) p^2 - 2q$$

A1

[3 marks]

- (ii) $p^2 < 2q$ OR $p^2 - 2q < 0$ A1

Note: Allow **FT** on their result from part (f)(i).

[1 mark]

- (g) $4 < 6$ OR $2^2 - 2 \times 3 < 0$ R1

hence there is at least one complex root. AG

Note: Allow **FT** from part (f)(ii) for the **R** mark provided numerical reasoning is seen.

[1 mark]

continued...

Question 2 continued

(h) (i) $(p^2 > 2q)$ ($81 > 2 \times 24$) (so) nothing can be deduced

R1

Note: Do not allow **FT** for the **R** mark.

[1 mark]

(ii) -1

A1**[1 mark]**

(iii) attempt to express as a product of a linear and cubic factor

M1

$$(x+1)(x^3 - 10x^2 + 34x - 12)$$

A1A1

Note: Award **A1** for each factor. Award at most **A1A0** if not written as a product.

since for the cubic, $p^2 < 3q$ ($100 < 102$)

R1

there is at least one complex root

AG**[4 marks]****[Total: 27 marks]**

8. (a)

$$a = \frac{\pi}{2}$$

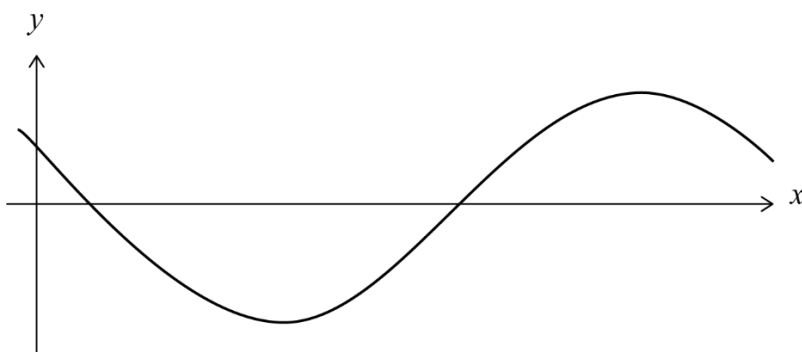
A2

Note: For sinusoidal graph through the origin seen with incorrect a , or use of horizontal line test with incorrect a , award **A1A0**

(b) $a = \pi$

A1

(c)



sketch showing sinusoidal shape decreasing as it crosses the y-axis
(below or above the origin)

(A1)

$$a = k - \pi$$

A1

[5 marks]

^ ^

11. (a) (i) 5^3 (A1)
 $= 125$ A1

(ii) ${}^5P_3 = 5 \times 4 \times 3$ (A1)
 $= 60$ A1

[4 marks]

(b) (i) **METHOD 1**

$$x^2 + 3x + 2 = (x+1)(x+2) \quad (\text{A1})$$

correct use of factor theorem for at least one of their factors (M1)

$$P(-1) = 0 \text{ or } P(-2) = 0$$

attempt to find two equations in a, b and c (M1)

$$(-1)^3 + a(-1)^2 + b(-1) + c = 0 (\Rightarrow -1 + a - b + c = 0)$$

$$(-2)^3 + a(-2)^2 + b(-2) + c = 0$$

$$-8 + 4a - 2b + c = 0 \text{ and } -1 + a - b + c = 0 \quad \text{A1}$$

attempt to combine their two equations in $-8 + 4a - 2b + c = 0$ to eliminate c (M1)

$$b = 3a - 7 \quad \text{A1}$$

Note: Award at most A1M1M1A0M1A0 for $b = -3a - 7$ from $P(1) = P(2) = 0$

continued...

Question 11 continued

METHOD 2

$$P(x) = x^3 + ax^2 + bx + c = (x^2 + 3x + 2)(x + d) \quad (M1)$$

$$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d \quad (A1)$$

attempt to compare coefficients of x^2 and x (M1)

$$a = 3 + d \text{ and } b = 2 + 3d \quad A1$$

attempt to eliminate d (M1)

$$\Rightarrow b = 3a - 7 \quad A1$$

METHOD 3

attempt to divide $x^3 + ax^2 + bx + c$ by $x^2 + 3x + 2$ M1

$$\frac{x^3 + ax^2 + bx + c}{x^2 + 3x + 2} = (x + a - 3) + \frac{(-3a + b + 7)x + (c - 2a + 6)}{x^2 + 3x + 2} \quad A1A1A1$$

Note: Award **A1** for $x + a - 3$, **A1** for $(-3a + b + 7)x$ and **A1** for $c - 2a + 6$

recognition that, if $(x^2 + 3x + 2)$ is a factor of $P(x)$, then $-3a + b + 7 = 0$ (M1)

leading to $b = 3a - 7$ A1

continued...

*Question 11 continued***METHOD 4**

$$x^2 + 3x + 2 = (x+1)(x+2) \quad (\text{A1})$$

attempt to use Vieta's formulae for a cubic with roots -1 , -2 and " p " (M1)

$$(-1) + (-2) + p = -a \Rightarrow p = 3 - a \quad \text{A1}$$

$$(-1)(-2) + (-1)p + (-2)p = b \quad \text{A1}$$

Attempt to eliminate " p " (M1)

$$2 - (3 - a) - 2(3 - a) = b$$

$$b = 3a - 7 \quad \text{A1}$$

Note: Award at most **A1M1A0A0M1A0** for $b = -3a - 7$ from roots 1, 2 and " p "

(ii) METHOD 1

$$a = 1, 2, 5 \text{ lead to invalid values for } b \quad \text{R1}$$

$$a = 3, b = 2 \Rightarrow c = 0 \text{ so not possible} \quad \text{R1}$$

$$\text{so } a = 4, b = 5, c = 2 \text{ is the only solution} \quad \text{AG}$$

METHOD 2

$$c = 2a - 6 \quad \text{R1}$$

$$\text{correctly argues } a = 4 \text{ is the only possibility} \quad \text{R1}$$

$$\text{so } a = 4, b = 5, c = 2 \text{ is the only solution} \quad \text{AG}$$

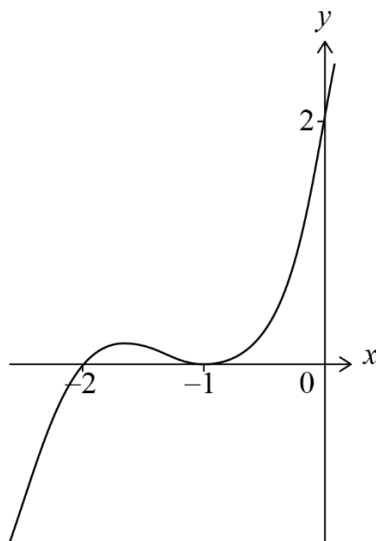
$$\text{(iii) } x^3 + 4x^2 + 5x + 2 = (x^2 + 3x + 2)(x+1)$$

$$= (x+2)(x+1)(x+1) \quad \text{A1}$$

continued...

Question 11 continued

(iv)

positive cubic shape with y -intercept at $(0, 2)$ **A1** x -intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ **A1**local minimum point at $(-1, 0)$ **A1**

Note: Accept answers from an approach based on calculus.

[12 marks]**Total [16 marks]**

4. recognition that initial population is 15000 (seen anywhere) **(A1)**

$$P(0) = 15000 \text{ OR } 0.11 \times 15000 \text{ OR } 0.89 \times 15000$$

population after 11% decrease is $15000 \times 0.89 (=13350)$ **(A1)**

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) **(A1)**

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model **(M1)**

$$15000 \times 0.89 = 15000e^{8k} \text{ OR } 13350 = 15000e^{8k}$$

$$k = \frac{\ln 0.89}{8} (-0.014566) \text{ **(A1)**}$$

substitution of $t = 2041 - 2014 (= 27)$ and their value for k into the model **(M1)**

$$P(27) = 15000e^{-0.0145... \times 27}$$

$$10122.3...$$

$$P(27) = 10100 \text{ (10122)} \text{ **A1**}$$

Total [7 marks]

1. (a) recognizing $f(x)=0$ (M1)
 $x=-1$ A1
[2 marks]

- (b) (i) $x=2$ (must be an equation with x) A1
(ii) $y=\frac{7}{2}$ (must be an equation with y) A1
[2 marks]

- (c) **EITHER**
interchanging x and y (M1)
 $2xy-4x=7y+7$
correct working with y terms on the same side: $2xy-7y=4x+7$ (A1)

- OR**
 $2yx-4y=7x+7$
correct working with x terms on the same side: $2yx-7x=4y+7$ (A1)
interchanging x and y OR making x the subject $x=\frac{4y+7}{2y-7}$ (M1)

- THEN**
 $f^{-1}(x)=\frac{4x+7}{2x-7}$ (or equivalent) $\left(x\neq\frac{7}{2}\right)$ A1

[3 marks]

Total [7 marks]

4. recognition of quadratic in e^x **(M1)**

$$(e^x)^2 - 3e^x + \ln k (= 0) \text{ OR } A^2 - 3A + \ln k (= 0)$$

recognizing discriminant ≥ 0 (seen anywhere) **(M1)**

$$(-3)^2 - 4(1)(\ln k) \text{ OR } 9 - 4 \ln k \quad \textbf{(A1)}$$

$$\ln k \leq \frac{9}{4} \quad \textbf{(A1)}$$

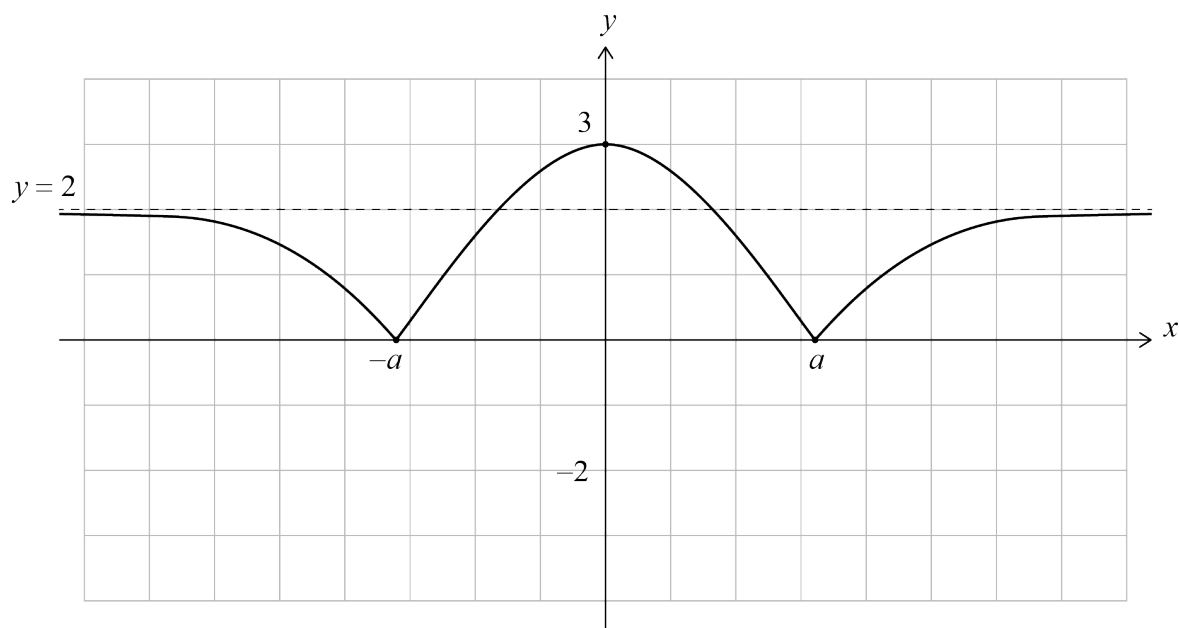
$$e^{9/4} \text{ (seen anywhere)} \quad \textbf{A1}$$

$$0 < k \leq e^{9/4} \quad \textbf{A1}$$

[6 marks]

8. (a) attempt to reflect f in the x OR y axis

(M1)



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:

A1 for correct asymptotic behaviour at $y = 2$ (either side)

A1 for correctly reflected RHS of the graph in the y -axis with smooth maximum at $(0, 3)$.

A1 for labelled x -intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x -intercepts.

[4 marks]

- (b) $k = 0$

A1

$$4 \leq k < 9$$

A2

Note: If final answer incorrect, award **A1** for critical values 4 and 9 seen anywhere.

Exception to FT:

Award a maximum of **A0A2FT** if their graph from (a) is not symmetric about the y -axis.

[3 marks]

Total [7 marks]

2. (a) (i) $x = 2$

A1

(ii) $y = 1$

A1

[2 marks]

(b) (i) $\left(0, \frac{3}{2}\right)$

A1

(ii) $(3, 0)$

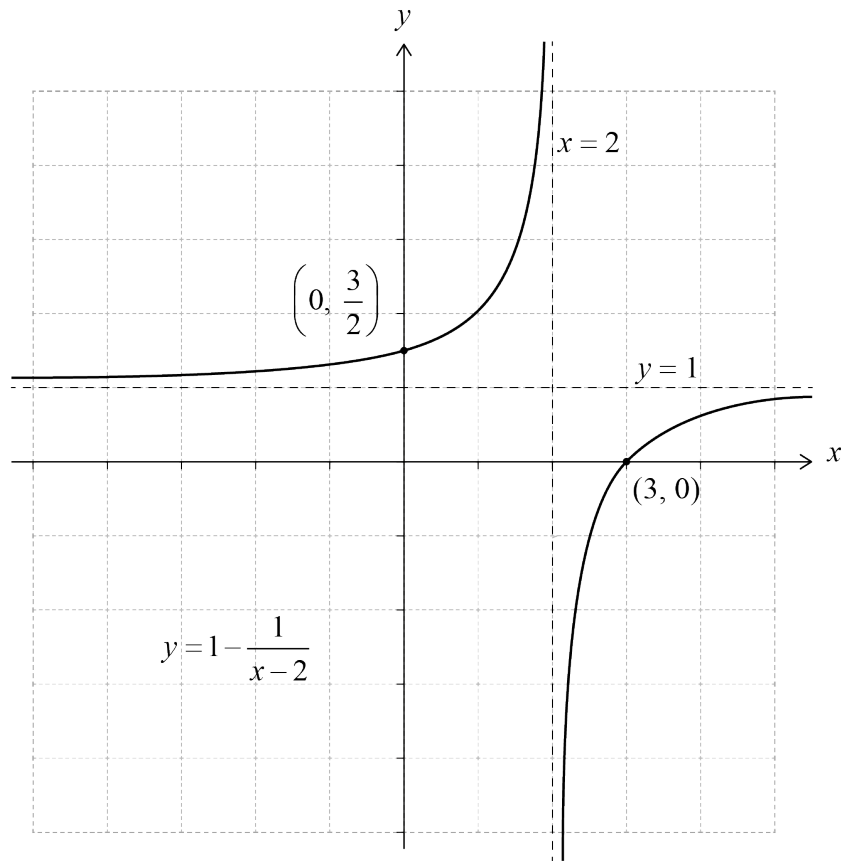
A1

[2 marks]

continued...

Question 2 continued

(c)



two correct branches with correct asymptotic behaviour and intercepts clearly shown

A1

[1 mark]

Total [5 marks]

5. attempts to form $(g \circ f)(x)$ **(M1)**

$$[f(x)]^2 + f(x) + 3 \quad \text{OR} \quad (ax + b)^2 + ax + b + 3$$

$$a^2x^2 + 2abx + b^2 + ax + b + 3 (= 4x^2 - 14x + 15) \quad \textbf{(A1)}$$

equates their corresponding terms to form at least one equation **(M1)**

$$a^2x^2 = 4x^2 \quad \text{OR} \quad a^2 = 4 \quad \text{OR} \quad 2abx + ax = -14x \quad \text{OR} \quad 2ab + a = -14 \quad \text{OR} \\ b^2 + b + 3 = 15$$

$$a = \pm 2 \text{ (seen anywhere)} \quad \textbf{A1}$$

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) **(M1)**

$$f(x) = 2x - 4 \text{ (accept } a = 2 \text{ with } b = -4), f(x) = -2x + 3 \text{ (accept } a = -2 \text{ with } b = 3) \quad \textbf{A1A1}$$

[7 marks]

2. (a) attempt to substitute g into f

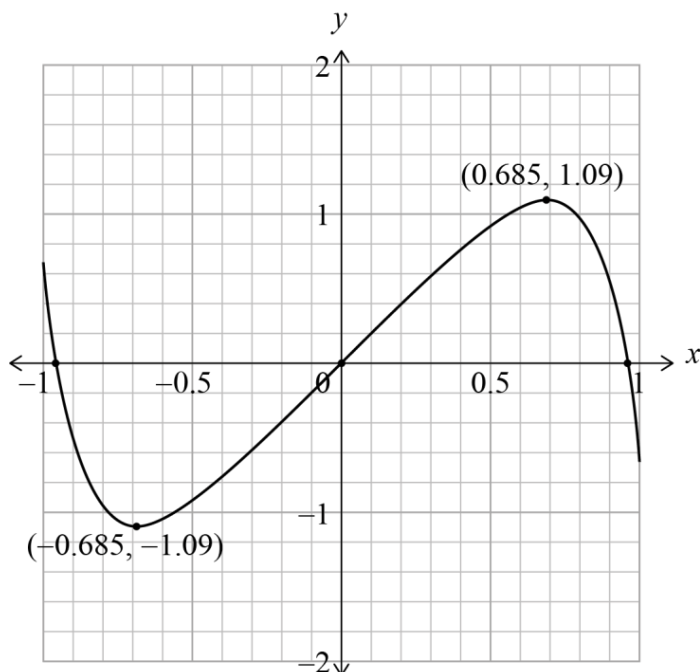
(M1)

$$(f \circ g)(x) = 2 \tan x - \tan^3 x$$

A1

[2 marks]

(b)



A1A1A1

Note: **A1** for approximately correct odd function passing through the origin with a maximum above $y = 1$ and a minimum below $y = -1$.

A1 for endpoints at $x = \pm 1$ and y in the intervals $[0.6, 0.8]$ and $[-0.8, -0.6]$

A1 for maximum in approximately correct position and labelled

$(0.685, 1.09)$ AND minimum in approximately correct position and labelled

$(-0.685, -1.09)$. For approximate position, allow $-0.8 \leq x \leq -0.6$,

$-1.2 \leq y \leq -1$ for minimum and $0.6 \leq x \leq 0.8$, $1 \leq y \leq 1.2$ for maximum. If

the candidate gives the coordinates of extrema below their sketch, only

award this mark if extrema are marked in the correct interval (eg by a dot).

[3 marks]

Total [5 marks]

8. (a) EITHER

attempts to find the y -coordinate of either the local minimum point or the local maximum point

(M1)**OR**

attempts to find the discriminant of $2x - 5 = y(x^2 - 3)$ ($yx^2 - 2x + (5 - 3y) = 0$)

(M1)

$$\Delta = 4 - 4y(5 - 3y) (= 4 - 20y + 12y^2)$$

THEN

$y = 1.43425\dots$ (local min.) and $y = 0.232408\dots$ (local max.)

(A1)(A1)

$$g(x) \leq 0.232 \text{ OR } g(x) \geq 1.43 \quad (g(x) \leq \frac{-\sqrt{13}+5}{6} \text{ OR } g(x) \geq \frac{\sqrt{13}+5}{6})$$

A1

Note: Accept other valid notations such as interval notation.

[4 marks]

(b) $\frac{2|x|-5}{x^2-3} \geq 0$ (since $\cos t < 0$ for $\frac{\pi}{2} < t \leq \pi$)

(R1)

attempts to solve graphically or algebraically

(M1)

$$x \leq -\frac{5}{2} \text{ OR } -\sqrt{3} < x < \sqrt{3} \quad (-1.73 < x < 1.73) \text{ OR } x \geq \frac{5}{2}$$

A1**[3 marks]****Total [7 marks]**

1. (a) attempt to form $(g \circ f)(x)$ (M1)

$$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2) \quad \text{A1}$$

[2 marks]

- (b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ (M1)

$$(2-3)^2 + k^2 = 10 \quad \text{OR} \quad 2^2 - 6(2) + 9 + k^2 = 10$$

$$k^2 = 9 \quad \text{(A1)}$$

$$k = \pm 3 \quad \text{A1}$$

[3 marks]

Total [5 marks]

10. (a) $x = 0$ **A1****[1 mark]**(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ **M1**

attempt to use power rule

(M1) $2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs

(M1)

$$\ln(2x-9) = \ln \frac{x^2}{d} \text{ OR } \ln \frac{x^2}{2x-9} = \ln d \text{ OR } \ln(2x-9)d = \ln x^2$$

$$\frac{x^2}{d} = 2x-9 \text{ OR } \frac{x^2}{2x-9} = d \text{ OR } (2x-9)d = x^2$$

A1

$$x^2 - 2dx + 9d = 0$$

AG(ii) discriminant = $(-2d)^2 - 4 \times 1 \times 9d$ **(A1)**recognizing discriminant > 0 **(M1)**

$$(-2d)^2 - 4 \times 1 \times 9d > 0 \text{ OR } (2d)^2 - 4 \times 9d > 0 \text{ OR } 4d^2 - 36d > 0$$

A1

$$d^2 - 9d > 0$$

AG(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graphOR sign test OR $d^2 > 9d$ **(M1)**

$$d < 0 \text{ or } d > 9, \text{ but } d \in \mathbb{R}^+$$

$$d > 9 \text{ (or }]9, \infty[)$$

A1**[9 marks]**

continued...

Question 10 continued

(c) $x^2 - 20x + 90 (= 0)$

A1

attempting to solve their 3 term quadratic equation

(M1)

$$\left((x-10)^2 - 10 = 0 \right) \text{ or } \left(x = \frac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2} \right)$$

$$x = 10 - \sqrt{10} (= p) \text{ or } x = 10 + \sqrt{10} (= q)$$

(A1)subtracting their values of x **(M1)**

distance $= 2\sqrt{10}$

A1

$$(a = 2, b = 10)$$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.**[5 marks]****Total [15 marks]**

1. (a) attempt to form $(g \circ f)(x)$ (M1)

$$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2) \quad \text{A1}$$

[2 marks]

- (b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ (M1)

$$(2-3)^2 + k^2 = 10 \quad \text{OR} \quad 2^2 - 6(2) + 9 + k^2 = 10$$

$$k^2 = 9 \quad \text{(A1)}$$

$$k = \pm 3 \quad \text{A1}$$

[3 marks]

Total [5 marks]

10. (a) $x = 0$ **A1****[1 mark]**(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ **M1**

attempt to use power rule

(M1) $2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs

(M1)

$$\ln(2x-9) = \ln \frac{x^2}{d} \text{ OR } \ln \frac{x^2}{2x-9} = \ln d \text{ OR } \ln(2x-9)d = \ln x^2$$

$$\frac{x^2}{d} = 2x-9 \text{ OR } \frac{x^2}{2x-9} = d \text{ OR } (2x-9)d = x^2$$

A1

$$x^2 - 2dx + 9d = 0$$

AG(ii) discriminant = $(-2d)^2 - 4 \times 1 \times 9d$ **(A1)**recognizing discriminant > 0 **(M1)**

$$(-2d)^2 - 4 \times 1 \times 9d > 0 \text{ OR } (2d)^2 - 4 \times 9d > 0 \text{ OR } 4d^2 - 36d > 0$$

A1

$$d^2 - 9d > 0$$

AG(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graphOR sign test OR $d^2 > 9d$ **(M1)**

$$d < 0 \text{ or } d > 9, \text{ but } d \in \mathbb{R}^+$$

$$d > 9 \text{ (or }]9, \infty[)$$

A1**[9 marks]**

continued...

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Question 10 continued

(c) $x^2 - 20x + 90 (= 0)$

A1

attempting to solve their 3 term quadratic equation

(M1)

$$\left((x-10)^2 - 10 = 0 \right) \text{ or } \left(x = \frac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2} \right)$$

$$x = 10 - \sqrt{10} (= p) \text{ or } x = 10 + \sqrt{10} (= q)$$

(A1)subtracting their values of x **(M1)**

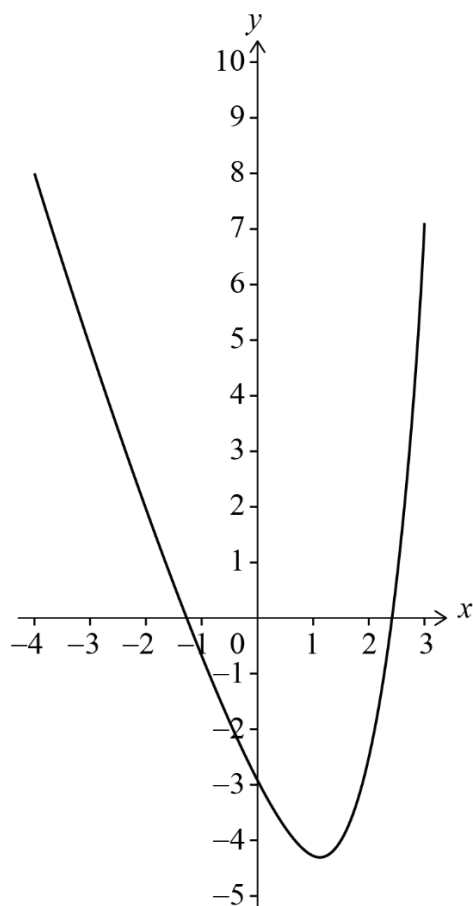
distance $= 2\sqrt{10}$

A1

$$(a = 2, b = 10)$$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.**[5 marks]****Total [15 marks]**

2. (a)

**A1A1A1****Note:** Award marks as follows:

A1 for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$.

A1 for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$.

A1 for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5$, $7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5$, $6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$

A1

$c = -3$ (accept translate/shift 3 (units) down)

A1**[2 marks]****Total [5 marks]**

11. (a) (vertical asymptote equation)
- $x = -3$

A1**Note:** Accept $2x + 6 = 0$ or equivalent.**[1 mark]**

- (b)
- $(2, 0)$
- and
- $(12, 0)$

A1A1**Note:** Award **A1** for $(2, 0)$ and **A1** for $(12, 0)$.Award **A1A0** if only x values are given.**[2 marks]**

- (c)
- METHOD 1**

$$a = \frac{1}{2}$$

A1attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ **(M1)**

$$\frac{x^2 - 14x + 24}{2x + 6}$$

$$= \frac{1}{2}x - \frac{17}{2} \left(+ \frac{\dots}{2x + 6} \right)$$

(A1)

$$b = -\frac{17}{2}$$

A1**Note:** Accept $y = \frac{1}{2}x - \frac{17}{2}$.*continued...*

Question 11 continued

METHOD 2

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6} \quad \text{(A1)}$$

$$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$$

attempt to equate coefficients of x : (M1)

$$-14 = 3 + 2b$$

$$b = -\frac{17}{2} \quad \text{A1}$$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6} \quad \text{(A1)}$$

attempt to find the limit of $f(x) - ax$ as $x \rightarrow \infty$ (M1)

$$b = \lim_{x \rightarrow \infty} \frac{-17x + 24}{2x + 6}$$

$$= -\frac{17}{2} \quad \text{A1}$$

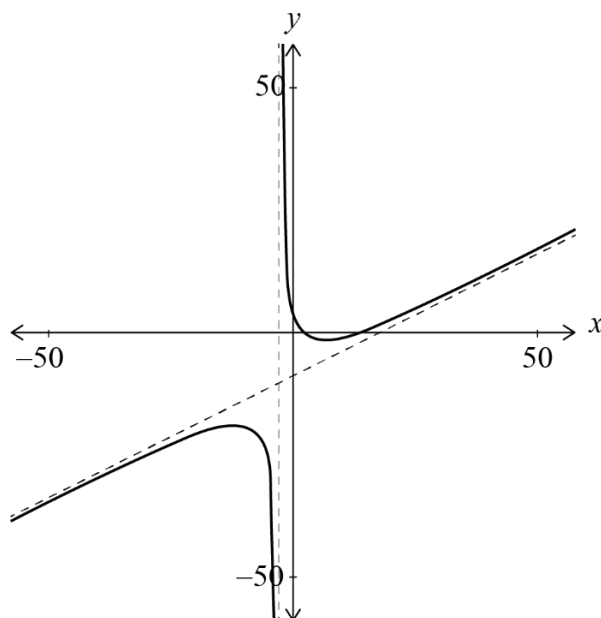
Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

continued...

Question 11 continued

(d)

two branches with approximately correct shape (for $-50 \leq x \leq 50$)**A1****Note:** For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes

A1A1**Note:** Award **A1** for vertical asymptote and behaviour and **A1** for oblique asymptote and behaviour.If only top half of the graph seen only award **A1A0** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions

A1**Note:** Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.**[4 marks]**

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(e) $(-10 - 5\sqrt{3} =) -18.6602....$ OR $(-10 + 5\sqrt{3} =) -1.33974....$ seen anywhere **(A1)**

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$$y \leq -18.7, y \geq -1.34$$

A1A1

Note: Award **A1** for each inequality. Award **A1A0** for strict inequalities in both.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]

(f) $(-10 - 2\sqrt{31} =) -21.1355....$ OR $(-10 + 2\sqrt{31} =) 1.13552....$ seen anywhere **(A1)**

$$x < -21.1, -3 < x < 1.14$$

A1A1A1

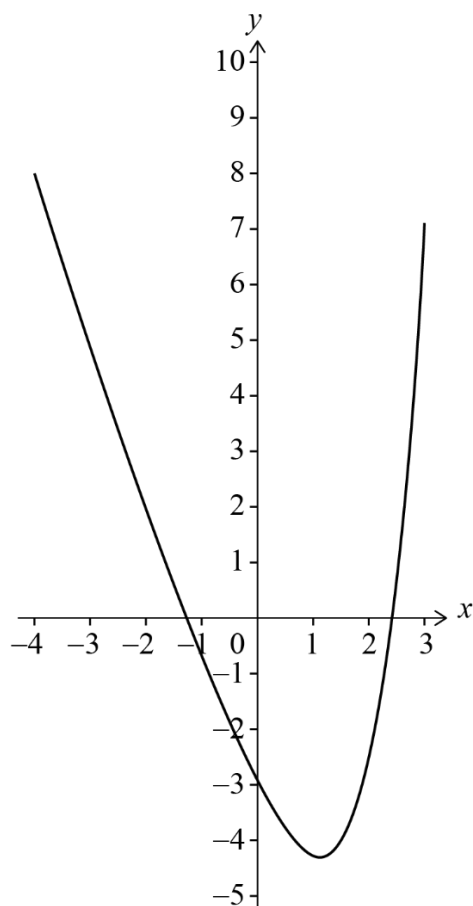
Note: Award **A1** for $x < -21.1$, **A1** for correct endpoints of a single interval -3 and 1.14 and **A1** for $-3 < x < 1.14$.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]**Total [19 marks]**

2. (a)

**A1A1A1****Note:** Award marks as follows:

A1 for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$.

A1 for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$.

A1 for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5$, $7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5$, $6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$

A1

$c = -3$ (accept translate/shift 3 (units) down)

A1**[2 marks]****Total [5 marks]**

11. (a) (vertical asymptote equation)
- $x = -3$

A1**Note:** Accept $2x + 6 = 0$ or equivalent.**[1 mark]**

- (b)
- $(2, 0)$
- and
- $(12, 0)$

A1A1**Note:** Award **A1** for $(2, 0)$ and **A1** for $(12, 0)$.Award **A1A0** if only x values are given.**[2 marks]**

- (c)
- METHOD 1**

$$a = \frac{1}{2}$$

A1attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ **(M1)**

$$\frac{x^2 - 14x + 24}{2x + 6}$$

$$= \frac{1}{2}x - \frac{17}{2} \left(+ \frac{\dots}{2x + 6} \right)$$

(A1)

$$b = -\frac{17}{2}$$

A1**Note:** Accept $y = \frac{1}{2}x - \frac{17}{2}$.*continued...*

Question 11 continued

METHOD 2

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6} \quad \text{(A1)}$$

$$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$$

attempt to equate coefficients of x : (M1)

$$-14 = 3 + 2b$$

$$b = -\frac{17}{2} \quad \text{A1}$$

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$$a = \frac{1}{2} \quad \text{A1}$$

$$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6} \quad \text{(A1)}$$

attempt to find the limit of $f(x) - ax$ as $x \rightarrow \infty$ (M1)

$$b = \lim_{x \rightarrow \infty} \frac{-17x + 24}{2x + 6}$$

$$= -\frac{17}{2} \quad \text{A1}$$

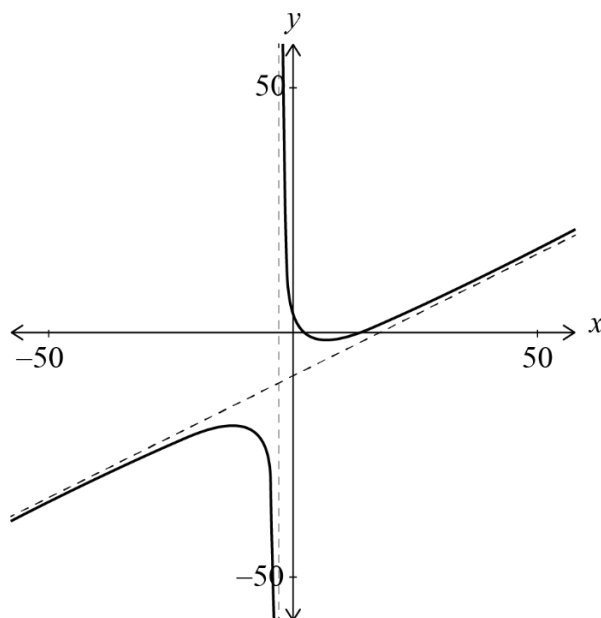
Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

continued...

Question 11 continued

(d)

two branches with approximately correct shape (for $-50 \leq x \leq 50$)**A1****Note:** For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes

A1A1**Note:** Award **A1** for vertical asymptote and behaviour and **A1** for oblique asymptote and behaviour.If only top half of the graph seen only award **A1A0** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions

A1**Note:** Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.**[4 marks]**

– 27 –

8823 – 7112M

(e) $(-10 - 5\sqrt{3} =) -18.6602....$ OR $(-10 + 5\sqrt{3} =) -1.33974....$ seen anywhere **(A1)**

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$$y \leq -18.7, y \geq -1.34$$

A1A1

Note: Award **A1** for each inequality. Award **A1A0** for strict inequalities in both.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]

(f) $(-10 - 2\sqrt{31} =) -21.1355....$ OR $(-10 + 2\sqrt{31} =) 1.13552....$ seen anywhere **(A1)**

$$x < -21.1, -3 < x < 1.14$$

A1A1A1

Note: Award **A1** for $x < -21.1$, **A1** for correct endpoints of a single interval -3 and 1.14 and **A1** for $-3 < x < 1.14$.

Do not award FT from (d).

Accept equivalent set notation.

[4 marks]**Total [19 marks]**

1. (a) varies the value of b with $a = 3$

(M1)

Note: The **(M1)** in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii).
Award **(M1)** for evidence that $b = 0$ case is considered/determined.

$$b = -4, 0$$

A1

[2 marks]

- (b) (i) $b < -4$ or $b > 0$

A1

[1 mark]

- (ii) $-4 < b < 0$

A1

[1 mark]

continued...

Question 1 continued

(c) (i) $b = 0, 4$

A1

[1 mark]

(ii) $b < 0$ or $b > 4$

A1

[1 mark]

(iii) $0 < b < 4$

A1

[1 mark]

continued...

Question 1 continued

- (d) one point of zero gradient is located on either side (of the x -axis) (or equivalent) **A1**

[1 mark]

- (e) **METHOD 1**

$$\frac{dy}{dx} = 3x^2 + 2ax \quad \textbf{(A1)}$$

attempts to solve their $\frac{dy}{dx} = 0$ for x **M1**

$$x(3x + 2a) = 0 \text{ OR } x = \frac{-2a \pm \sqrt{4a^2}}{6} \text{ OR } x + \frac{a}{3} = \pm \frac{a}{3}$$

$$x = -\frac{2}{3}a, 0 \quad \textbf{A1}$$

when $x = 0$, $y = b$ and so $P(0, b)$ is a point of zero gradient **AG**

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ **(M1)**

$$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$$

$$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b \right) \quad \textbf{A1}$$

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient **AG**

[5 marks]*continued...*

Question 1 continued**METHOD 2**

$$\frac{dy}{dx} = 3x^2 + 2ax \quad (\text{A1})$$

substitutes either $x = 0$ or $x = -\frac{2}{3}a$ into their $\frac{dy}{dx}$ **M1**

when $x = 0$, $\frac{dy}{dx} = 0$ and $y = b$ so $P(0, b)$ is a point of zero gradient **AG**

$$\begin{aligned} \frac{dy}{dx} &= 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right) \\ &= \frac{4}{3}a^2 - \frac{4}{3}a^2 (=0) \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2 (=0), = \frac{12}{9}a^2 - \frac{4}{3}a^2 (=0)\right) \end{aligned} \quad \text{A1}$$

and so $\frac{dy}{dx} = 0$ when $x = -\frac{2}{3}a$ **AG**

Note: The following two marks are independent of the first three marks.

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ **(M1)**

$$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$$

$$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right) \quad \text{A1}$$

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient **AG**

[5 marks]

continued...

Question 1 continued

(f) (i) $\frac{d^2y}{dx^2} = 6x + 2a$ **A1**

when $x = 0$, $\frac{d^2y}{dx^2} = 2a$ ($a > 0$) and so (P) is a (local) minimum (point) **R1**

when $x = -\frac{2}{3}a$, $\frac{d^2y}{dx^2} = -2a$ ($a > 0$) and so (Q) is a (local) maximum (point) **R1**

[3 marks]

(ii) (P and Q are) both above (the x -axis) **A1**

Note: Award **A1** if it is made clear that both points are above (the x -axis).
Accept a labelled sketch that clearly shows this information.

[1 mark]*continued...*

Question 1 continued

- (g) (i) (P) is a (local) maximum (point) and is above (the x -axis)

A1**[1 mark]**

- (ii) (Q is below the x -axis when) $\frac{4}{27}a^3 + b < 0$

A1

Note: Award **A1** for an equivalent correct inequality, eg. $\frac{4}{27}a^3 < -b$.

Accept a labelled sketch that clearly shows this information.

[1 mark]*continued...*

Question 1 continued

(h) **METHOD 1**attempts to factorize $4a^3b + 27b^2 (< 0)$ **(M1)** $27b\left(\frac{4}{27}a^3 + b\right)(< 0)$ OR $b(4a^3 + 27b)(< 0)$ **A1** $b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ **A1****Note:** Only award this **A1** if both cases are stated.Award **A1** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).when b and $\frac{4}{27}a^3 + b$ have opposite sign, P and Q are located on either side
(of the x -axis) (or equivalent) **R1****Note:** Accept labelled sketches that clearly show this information.P and Q are located on either side (of the x -axis) if (and only if) the curve has exactly three x -axis intercepts **R1**if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x -axis intercepts **AG****Note:** For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd **A1** line not necessary in reverse method).

continued...

Question 1 continued

METHOD 2

attempts to factorize $4a^3b + 27b^2 (< 0)$ **(M1)**

$27b\left(\frac{4}{27}a^3 + b\right)(< 0)$ OR $b(4a^3 + 27b)(< 0)$ **A1**

either $b > 0$ and $\frac{4}{27}a^3 + b < 0$ or $b < 0$ and $\frac{4}{27}a^3 + b > 0$ **A1**

Note: Only award this **A1** if both cases are stated.

Award **A1** for stating that exactly one of b and $\frac{4}{27}a^3 + b$ is less than zero (or equivalent).

$b > 0$ and $\frac{4}{27}a^3 + b < 0$, $\left(\Rightarrow 0 < b < -\frac{4}{27}a^3\right) \Rightarrow a < 0$ and hence three x -axis intercepts **R1**

$b < 0$ and $\frac{4}{27}a^3 + b > 0$, $\left(\Rightarrow -\frac{4}{27}a^3 < b < 0\right) \Rightarrow a > 0$ and hence three x -axis intercepts **R1**

Note: Accept labelled sketches that clearly show this information.

if $4a^3b + 27b^2 < 0$, then the graph of $y = x^3 + ax^2 + b$ has exactly three x -axis intercepts **AG**

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd **A1** line not necessary in reverse method).

[5 marks]

Total [24 marks]

2. (a) $\log_{10} 1 - \log_{10} a$ OR $\log_{10} a^{-1} = -\log_{10} a$ OR $\log_{10} 10^{-\frac{1}{3}}$ OR $10^x = \frac{1}{10^{\frac{1}{3}}}$ (A1)

$$= -\frac{1}{3}$$

A1

[2 marks]

(b) $\frac{\log_{10} a}{\log_{10} 1000}$ OR $\frac{1}{3} \log_{1000} 10$ OR $\log_{1000} \sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}} = 1000^x \left(= (10^3)^x \right)$ (A1)

$$\frac{\log_{10} a}{3} \text{ OR } \frac{1}{3} \log_{1000} 1000^{\frac{1}{3}} \text{ OR } \log_{1000} 1000^{\frac{1}{9}} \text{ OR } 3x = \frac{1}{3} \quad (A1)$$

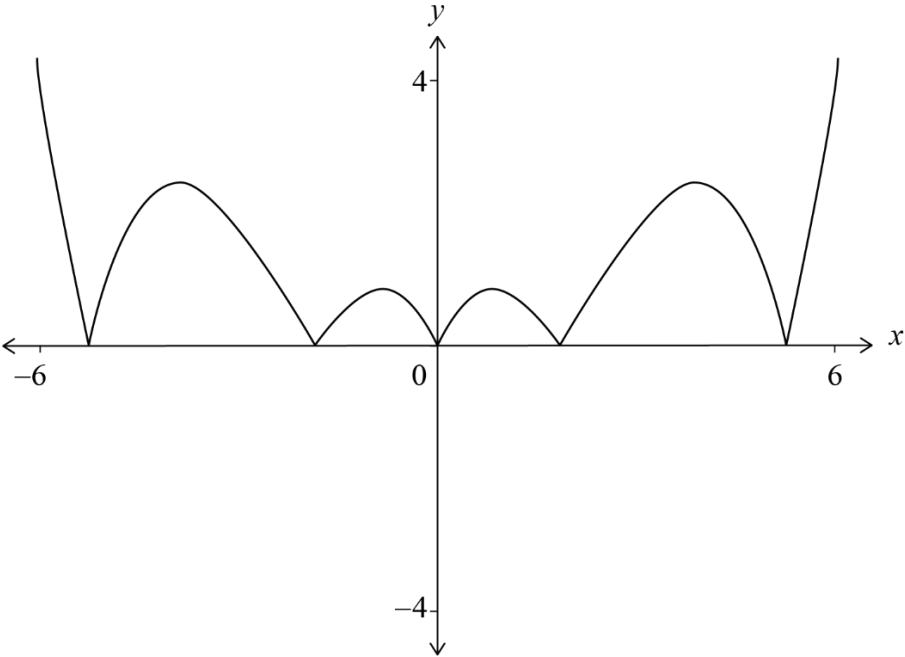
$$= \frac{1}{9}$$

A1

[3 marks]

Total [5 marks]

9. (a)



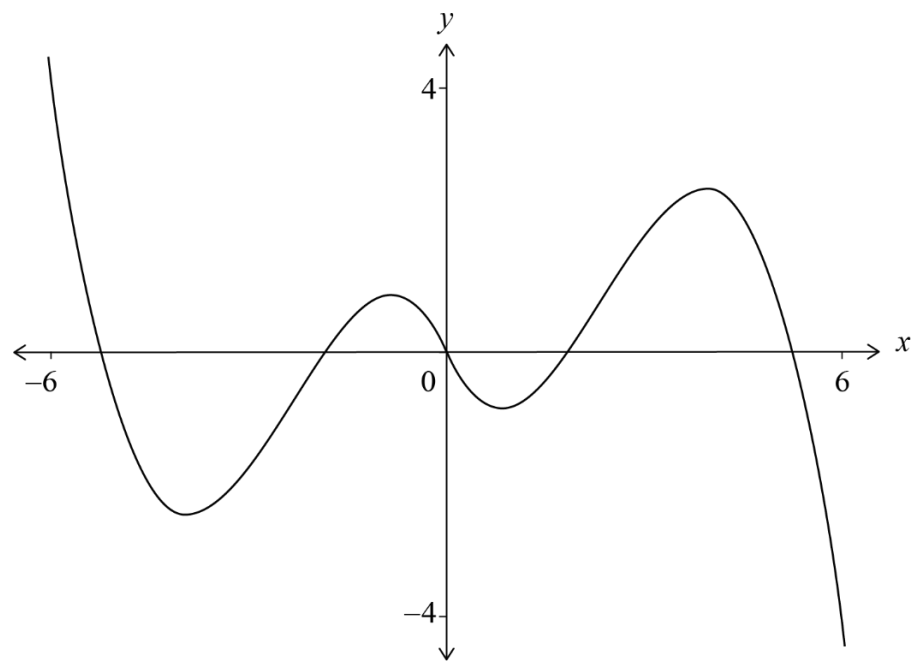
reflection of all negative sections in x -axis **(M1)**
approximately correct graph with sharp points (cusps) at x -intercepts **A1**

Note: Award **A1** only if the heights of the maximum in the middle are lower than the heights of the maximum at the ends.

[2 marks]
continued...

Question 9 continued

(b)



A1A1

Note: Award **A1** for right hand side unchanged and **A1** for rotation 180° about the origin.

[2 marks]

(c) (i) -1.6

A1

(ii) 3.2

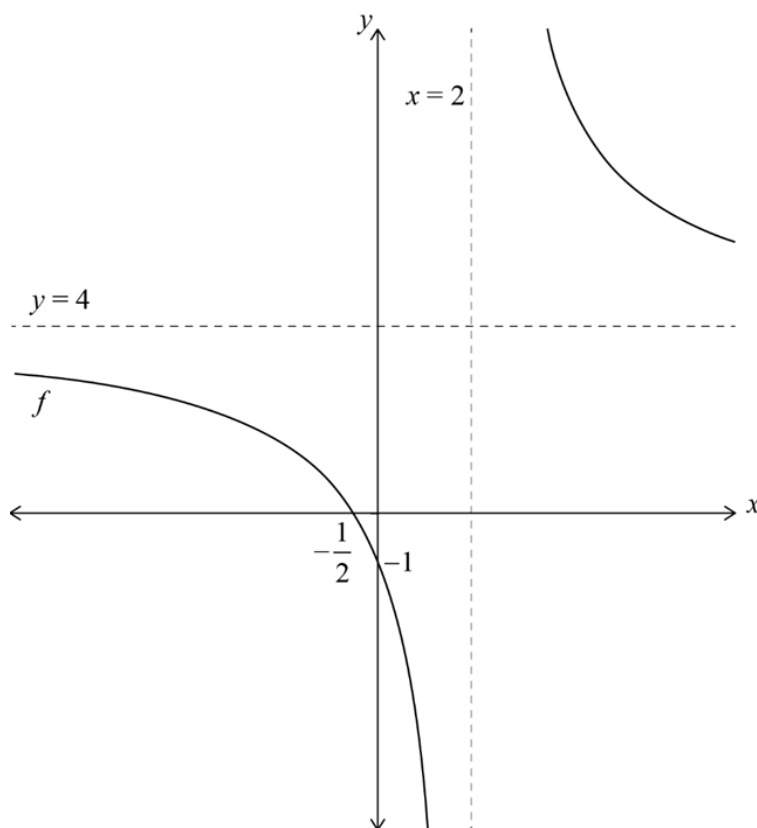
A1

[2 marks]

Total [6 marks]

Section B

10. (a)

vertical asymptote $x = 2$ sketched and labelled with correct equation**A1**horizontal asymptote $y = 4$ sketched and labelled with correct equation**A1**

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x -axis, -1 on y -axis**A1A1**

two branches in correct opposite quadrants with correct asymptotic behaviour

A1**Note:** These marks may be awarded independently.**[5 marks]***continued...*

Question 10 continued

(b) $y \neq 4$ (or equivalent)

A1**[1 mark]**

(c) $2 + \frac{5}{2}$ OR $\left(-\frac{1}{2}\right) + 2 \times \frac{5}{2}$ OR $\frac{-\frac{1}{2} + p}{2} = 2$ OR $-4 = -p + \frac{1}{2}$

A1

$$p = \frac{9}{2}$$

AG**[1 mark]**(d) **METHOD 1**

attempt to substitute both roots to form a quadratic

(M1)**EITHER**

$$\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) \text{ OR } x^2 - \left(-\frac{1}{2} + \frac{9}{2}\right)x + \left(-\frac{1}{2} \times \frac{9}{2}\right)$$

$$= x^2 - 4x - \frac{9}{4}$$

A1A1

$$\left(b = -4, c = -\frac{9}{4}\right)$$

Note: Award **A1** for each correct value. They may be embedded or stated explicitly.**OR**

$$(2x + 1)(2x - 9) = 4\left(x^2 - 4x - \frac{9}{4}\right)$$

$$b = -4, c = -\frac{9}{4}$$

A1A1**Note:** Award **A1** for each correct value. They must be stated explicitly.*continued...*

Question 10 continued

METHOD 2

$$-\frac{b}{2} = 2 \text{ OR } 4 + b = 0 \Rightarrow b = -4 \quad \textbf{A1}$$

attempt to form a valid equation to find c using their b **(M1)**

$$\left(-\frac{1}{2}\right)^2 + -4\left(-\frac{1}{2}\right) + c = 0 \text{ OR } \left(\frac{9}{2}\right)^2 + -4\left(\frac{9}{2}\right) + c = 0$$

$$c = -\frac{9}{4} \quad \textbf{A1}$$

METHOD 3

attempt to form two valid equations in b and c **(M1)**

$$\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + c = 0, \left(\frac{9}{2}\right)^2 + b\left(\frac{9}{2}\right) + c = 0$$

$$b = -4, c = -\frac{9}{4} \quad \textbf{A1A1}$$

METHOD 4

attempt to write $g(x)$ in the form $(x-h)^2 + k$ and substitute for x, h and $g(x)$ **(M1)**

$$\left(-\frac{1}{2} - 2\right)^2 + k = 0 \Rightarrow k = -\frac{25}{4}$$

$$(x-2)^2 - \frac{25}{4}$$

$$= x^2 - 4x - \frac{9}{4} \quad \textbf{A1A1}$$

$$\left(b = -4, c = -\frac{9}{4}\right)$$

Note: Award **A1** for each correct value. They may be embedded or stated explicitly.

[3 marks]

continued...

Question 10 continued

- (e) attempt to substitute
- $x = 2$
- into their
- $g(x)$
- OR

complete the square on their $g(x)$ (may be seen in part (d))**(M1)**

$$y = -\frac{25}{4}$$

A1**[2 marks]**

$$(f) \quad \frac{4x+2}{x-2} = \left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) \text{ OR } \frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$$

attempt to form a cubic equation

(M1)**EITHER**

$$4x+2 = (x-2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) \text{ OR } 4x+2 = \left(x^2 - 4x - \frac{9}{4}\right)(x-2) \text{ OR}$$

$$(x-2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) - 4x - 2 \text{ OR } (x-2)\left(x^2 - 4x - \frac{9}{4}\right) - 4x - 2$$

$$x^3 + \dots + \frac{5}{2}(=0) \text{ OR } 4x^3 + \dots + 10(=0)$$

(A1)(A1)

Note: Award **(A1)** for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

$$\text{product of roots} = \left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right) \text{ OR } \left(\frac{(-1)^3 \times 10}{4}\right)$$

$$= -\frac{5}{2}$$

A1*continued...*

*Question 10 continued***OR**

$$4\left(x + \frac{1}{2}\right) = (x - 2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$$

$$x = -\frac{1}{2} \quad (A1)$$

$$\text{or } 4 = x^2 + \dots + 9 \Rightarrow x^2 + \dots + 5 = 0$$

product of roots of quadratic is 5 (A1)

product is therefore $-\frac{1}{2} \times 5$

$$= -\frac{5}{2} \quad A1$$

[4 marks]**Total [16 marks]**

2. recognising a quadratic in 3^x (M1)

$$3 \times (3^x)^2 + 5 \times 3^x - 2 = 0$$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) (M1)

$$(3 \times 3^x - 1)(3^x + 2) = 0 \text{ OR } 3^x = \frac{-5 \pm \sqrt{25 + 24}}{6} \text{ (or equivalent)} \quad \text{(A1)}$$

$$3^x = \frac{1}{3} \text{ (or } 3^x = -2) \quad \text{(A1)}$$

$$x = -1 \quad \text{A1}$$

Note: Award the final **A1** if candidate's answer includes $x = -1$ and $x = \log_3(-2)$. Award **A0** if other incorrect answers are given.

[5 marks]

✓ ✓ ✓

3. (a) (i) $\left(\frac{9}{2}, \frac{3\sqrt{3}}{2}\right)$ (accept $x = \frac{9}{2}$ and $y = \frac{3\sqrt{3}}{2}$) **A1**

(ii) **METHOD 1**

using $m = \frac{\text{change in } y}{\text{change in } x}$ with their midpoint OR gradient perpendicular to AC

OR $m = \tan 30^\circ$ **(M1)**

$$m = \frac{\sqrt{3}}{3}$$
(A1)

$$y = \frac{\sqrt{3}}{3}x \text{ OR } y - \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}}{3}\left(x - \frac{9}{2}\right) \text{ (must be written as an equation)}$$
A1

METHOD 2

attempt to find the vector equation of the line using a direction and a point on the line

(M1)

direction vector is $\begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ or equivalent/parallel vectors **(A1)**

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \text{ OR } \mathbf{r} = \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \text{ OR } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \text{ (or equivalent)}$$
A1

Note: Vector equation must be in the form $\mathbf{r} =$ or $\begin{pmatrix} x \\ y \end{pmatrix} =$.

Allow equivalent parametric forms such as $x = \frac{9}{2}t$, $y = \frac{3\sqrt{3}}{2}t$.

[4 marks]

continued...

Question 3 continued.

(b) substituting $x = 6$ into their equation **(M1)**

so at B $y = 2\sqrt{3}$ **(A1)**

$$\text{area of triangle OAB} = \frac{1}{2} \times 6 \times 2\sqrt{3} = 6\sqrt{3}$$

area of quadrilateral OABC $= 12\sqrt{3}$ **A1**

[3 marks]

Total [7 marks]

5. (a) $y = \frac{2}{3}$ (must be written as equation with $y =$)

A1**[1 mark]**

(b) (i) 2

A1

(ii) **EITHER**

$$\frac{2(x+3)}{3(x+2)} = mx + 1$$

attempt to expand to obtain a quadratic equation

(M1)

$$2x + 6 = 3mx^2 + 6mx + 3x + 6$$

$$3mx^2 + (6m+1)x = 0 \quad \text{OR} \quad 3mx^2 + 6mx + x = 0$$

A1

recognition that discriminant $\Delta = 0$ for one solution

(M1)

$$(6m+1)^2 = 0$$

continued...

Question 5 continued.

OR

$$\frac{2(x+3)}{3(x+2)} = mx + 1$$

attempt to expand to obtain a quadratic equation **(M1)**

$$2x + 6 = 3mx^2 + 6mx + 3x + 6$$

$$3mx^2 + (6m + 1)x = 0 \quad \text{OR} \quad 3mx^2 + 6mx + x = 0 \quad \textbf{A1}$$

attempt to solve their quadratic for x and equating their solutions **(M1)**

$$x(3mx + 6m + 1) = 0$$

$$x = 0 \quad \text{OR} \quad x = -\frac{6m+1}{3m} (= 0)$$

$$-\frac{6m+1}{3m} = 0$$

OR

attempt to find $f'(x)$ using the quotient rule **(M1)**

$$f'(x) = \frac{2}{3} \left(\frac{(x+2) - (x+3)}{(x+2)^2} \right) = \left(\frac{-2}{3(x+2)^2} \right) \quad \text{OR} \quad \frac{2(3x+6) - 3(2x+6)}{(3x+6)^2} \quad \text{or}$$

equivalent **A1**

recognition that m is the derivative of $f(x)$ at $x = 0$ **(M1)**

THEN

$$\Rightarrow m = -\frac{1}{6} \quad \textbf{A1}$$

continued...

Question 5 continued.

(iii)

Note: In this part, FT may be awarded only for values of m between -1 and 0 .

$$-\frac{1}{6} < m < 0$$

A2

Note: Award **A1** for only $m > -\frac{1}{6}$. Award **A1** for only $m < 0$.

[7 marks]

Total [8 marks]

3. (a) $I = 2 \times 10^{-6} \left(= \frac{1}{500000} \right)$ (units)

A1**[1 mark]**

(b) substitutes their doubled I -value from part (a) into L

(M1)

$$L = 10 \log_{10} (2 \times 10^{-6} \times 10^{12}) (= 63.0102\dots)$$

$$= 63.0 \text{ (decibels)}$$

A1

Note: Accept $60 + 10 \log_{10} 2$ (decibels) as a final answer.
Do not award the final **A1** for $L = 0$ (from $I = 10^{-12}$).

[2 marks]

(c) $115 = 10 \log_{10} (I \times 10^{12})$

(A1)

attempts to solve for I

(M1)

$$I = \frac{10^{11.5}}{10^{12}} \text{ (or equivalent) } (= 0.316227\dots)$$

$$I = 0.316 \text{ (units)}$$

A1

Note: Accept exact final answers such as $10^{-0.5}$ and $\frac{1}{\sqrt{10}}$.

[3 marks]**Total [6 marks]**

6. (a) attempt to find critical values (M1)

$$x = \frac{3}{2}, x = 6 \quad \text{A1}$$

$$\frac{3}{2} < x < 6 \quad \text{A1}$$

Note: Allow equivalent, and/or interval notation.

[3 marks]

(b) $k = \frac{3}{2}$ A1

since we require $2x^2 - 15x + 18 \geq 0$ (and f must be one to one) R1

OR

Does not obey horizontal line test for $x \geq \frac{3}{2}$ R1

[2 marks]

Total [5 marks]

1. (a) (i) $f(0) = -11$

A1

(ii) $-1.80650\dots$

$$f(20) = -1.81 \left(= 11\sqrt{20} - 51 = 22\sqrt{5} - 51 \right)$$

A1

[2 marks]

(b) attempt to find at least one root

(M1)

$$x = 1.72622\dots \text{ and } x = 17.5237\dots$$

$$x = 1.73 \text{ and } x = 17.5$$

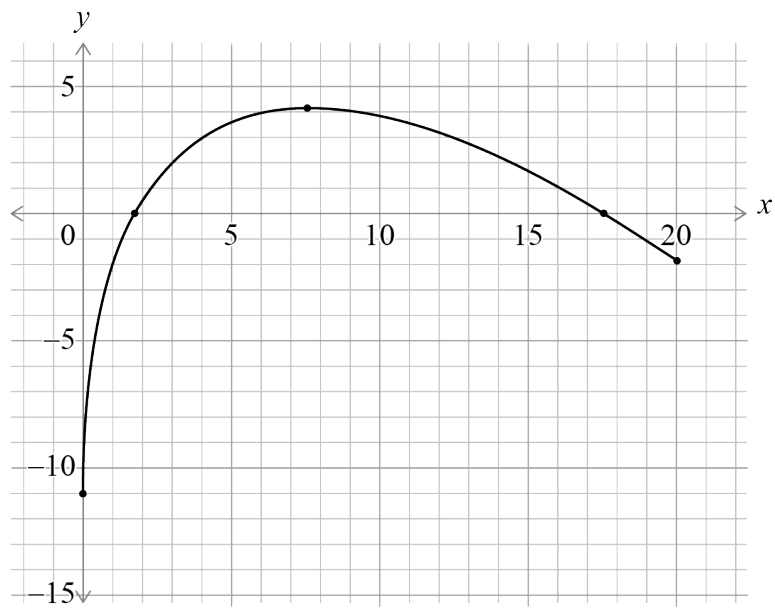
A1

[2 marks]

continued...

Question 1 continued.

(c)



A1A1A1

Note: Follow through from their part (a).

Award **A1** for endpoints at approximately $(0, -11)$ and $(20, -1.8)$. Allow for y -intercept $-12 < y < -10$ and for the right endpoint in the interval $x = 20, -2 < y < -1$.

The following two **A** marks may only be awarded if the approximate shape is correct.

Award **A1** for x -intercepts at approximately $x = 1.7$ and $x = 17.5$ and award **A1** for maximum at approximately $(7.6, 4.1)$. Allow for x -intercepts in the intervals $1 < x < 2$, $17 < x < 18$, and maximum in the intervals $6.5 < x < 8.5$, $3.5 < y < 5$.

[3 marks]

Total [7 marks]

5. METHOD 1

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) **R1**

(discriminant =) $(-r)^2 - 4(4)(r-1)$ $(= r^2 - 16r + 16)$ (seen anywhere) **(A1)**

$1.07179... (= 8 - 4\sqrt{3})$ AND $14.9282... (= 8 + 4\sqrt{3})$ (seen anywhere) **(A1)**

recognition that discriminant of $4x^2 - rx + r - 1$ is less than zero **(M1)**

$1.07 < r < 14.9$ $(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3})$ **A1**

Note: Accept $1.08 \leq r \leq 14.9$.

METHOD 2

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) **R1**

EITHER

minimum when $x = \frac{r}{8} \Rightarrow (y =) 4\left(\frac{r}{8}\right)^2 - r\left(\frac{r}{8}\right) + r - 1$ (> 0) **(A1)**

attempt to solve their inequality for y (must be in terms of r and r^2) **(M1)**

OR

$x < 1 \Rightarrow r > \frac{4x^2 - 1}{x - 1}$ OR $x > 1 \Rightarrow r < \frac{4x^2 - 1}{x - 1}$ **(A1)**

attempt to find local minimum AND local maximum of $r = \frac{4x^2 - 1}{x - 1}$ **(M1)**

THEN

$(r >) 1.07179... (= 8 - 4\sqrt{3})$ AND $(r <) 14.9282... (= 8 + 4\sqrt{3})$ (seen anywhere) **(A1)**

$1.07 < r < 14.9$ $(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3})$ **A1**

Note: Accept $1.08 \leq r \leq 14.9$.

[5 marks]

.....

12. (a) attempt to use quotient rule

M1

$$\frac{dy}{dx} = \frac{(x+k)(4x+6) - (2x^2+6x-3)}{(x+k)^2}$$

A1A1

Note: Award **A1** for $\frac{(x+k)(4x+6)}{(x+k)^2}$, **A1** for $\frac{-(2x^2+6x-3)}{(x+k)^2}$ OR $\frac{-2x^2-6x+3}{(x+k)^2}$

$$= \frac{4x^2 + 4kx + 6x + 6k - 2x^2 - 6x + 3}{(x+k)^2} \text{ or equivalent leading to } \mathbf{AG} \text{ line}$$

A1

$$= \frac{2x^2 + 4kx + 6k + 3}{(x+k)^2}$$

AG

Note: Candidates may use product rule, give marks accordingly.

[4 marks]

- (b) local min or max when $\frac{dy}{dx} = 0$, when $2x^2 + 4kx + 6k + 3 = 0$ has real solutions.

attempt to find the discriminant of numerator

(M1)

$$(4k)^2 - 4 \times 2(6k+3) (= 2k^2 - 6k - 3 < 0)$$

(A1)

attempt to find the critical values of their quadratic equation or inequality

(M1)

$$k = 3.43649... \text{ OR } k = -0.436491... \text{ OR } k = \frac{3 \pm \sqrt{15}}{2}$$

$$k > 3.44 \left(= \frac{3 + \sqrt{15}}{2} \right) \text{ (since } k \text{ is positive)}$$

A1

Note: Accept $k \geq 3.44$.

[4 marks]

- (c) $x = -2$

A1**[1 mark]**

continued...

Question 12 continued.

(d) **METHOD 1**

$$\frac{2x^2 + 6x - 3}{x + 2} = 2x + \dots \quad (\text{A1})$$

$$\text{attempts division on } \frac{2x^2 + 6x - 3}{x + 2} \quad \text{M1}$$

$$\frac{2x^2 + 6x - 3}{x + 2} = 2x + \frac{2x - 3}{x + 2}$$

$$= 2x + 2 + \dots \quad (\text{A1})$$

$$\text{asymptote is } y = 2x + 2 \quad \text{A1}$$

METHOD 2

let asymptote be $y = ax + b$

$$\frac{2x^2 + 6x - 3}{x + 2} = ax + b + \frac{c}{x + 2} \quad \text{M1}$$

$$2x^2 + 6x - 3 = (ax + b)(x + 2) + c$$

equates coefficients of x^2 and x : (M1)

$$a = 2 \quad \text{A1}$$

$$6 = 2a + b$$

$$b = 2 \quad \text{A1}$$

$$(y = 2x + 2)$$

continued...

Question 12 continued.

METHOD 3

$$\frac{2x^2 + 6x - 3}{x + 2} = 2x + \dots$$

A1

$$\frac{2x^2 + 6x - 3}{x + 2} - 2x = \frac{2x - 3}{x + 2}$$

attempt to find the limit as $x \rightarrow \infty$

M1

$$\lim_{x \rightarrow \infty} \frac{2x - 3}{x + 2} = 2$$

A1

$$y = 2x + 2$$

A1

[4 marks]

continued...

Question 12 continued.

(e) **METHOD 1**

EITHER

$$\text{for } k = 2, \frac{dy}{dx} = \frac{2x^2 + 8x + 15}{(x+2)^2} \quad \mathbf{A1}$$

$$\text{attempt to write } \frac{dy}{dx} \text{ in the form } p + \frac{q}{(x+2)^2} \text{ OR to write } \frac{dy}{dx} - 2 \text{ in the form } \frac{q}{(x+2)^2} \quad \mathbf{M1}$$

OR

$$y = 2x + 2 - \frac{7}{x+2} \quad \mathbf{A1}$$

Note: Follow through from their part (d).

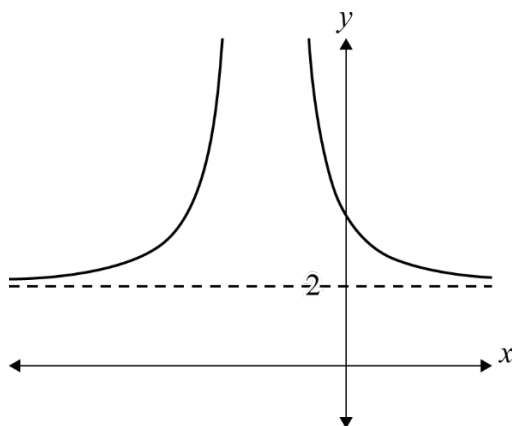
$$\text{attempt to find } \frac{dy}{dx} \text{ in the form } \frac{dy}{dx} = p + \frac{q}{(x+2)^2} \quad \mathbf{M1}$$

THEN

$$\frac{dy}{dx} = 2 + \frac{7}{(x+2)^2} \text{ OR } \frac{dy}{dx} - 2 = \frac{7}{(x+2)^2} \quad \mathbf{A1}$$

$$\text{since } \frac{7}{(x+2)^2} > 0 \text{ for } x \in \mathbb{R}, x \neq -2, \frac{dy}{dx} > 2. \quad \mathbf{R1}$$

Note: Award a maximum of **A1M1A0R0** for a graphical approach which includes a horizontal asymptote at 2, showing that $\frac{dy}{dx} > 2$.



continued...

Question 12 continued.

METHOD 2

$$\text{for } k = 2, \frac{dy}{dx} = \frac{2x^2 + 8x + 15}{(x+2)^2} \quad \mathbf{A1}$$

$$\text{clear assumption that } \frac{2x^2 + 8x + 15}{(x+2)^2} \leq 2 \quad \mathbf{M1}$$

$$(2x^2 + 8x + 15 \leq 2(x+2)^2 \Rightarrow) 15 \leq 8 \quad \mathbf{A1}$$

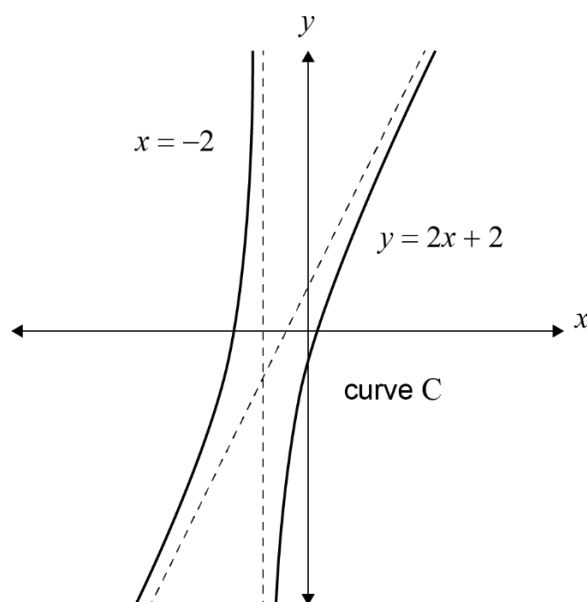
$$\text{contradiction implies that } \frac{dy}{dx} > 2 \quad \mathbf{R1}$$

Note: Award **A1M0A0R0** for solutions which work backwards from

$$\frac{2x^2 + 8x + 15}{(x+2)^2} > 2 \text{ to arrive at } 15 > 8.$$

[4 marks]

(f)



curve C in approx correct place, with negative y -intercept and part of the curve in 4th quadrant

A1

Note: The following **A1** marks should be awarded independently **provided** there is an attempt to draw both branches of $y = f(x)$.

vertical asymptote in approx correct place **A1**

oblique asymptote in approx correct place **A1**

approx correct asymptotic behaviour of C relative to both asymptotes **A1**

[4 marks]**Total [21 marks]**

2. (a) (i) attempts to solve

(M1)**EITHER**

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

OR

$$(x+2)^2 - 3 (= 0)$$

OR

$$(x+2+\sqrt{3})(x+2-\sqrt{3})(= 0)$$

THEN

$$x = -2 \pm \sqrt{3}$$

A1A1

Note: Award **A1** for $-2 + \sqrt{3}$ and **A1** for $-2 - \sqrt{3}$.

[3 marks]*continued...*

Question 2 continued.

(ii) **METHOD 1**

attempts to find their $\frac{1}{-2-\sqrt{3}} \times \frac{-2+\sqrt{3}}{-2+\sqrt{3}}$ (M1)

Note: Only award **(M1)** if the roots are in the form $s \pm \sqrt{t}$.

$$= \frac{-2+\sqrt{3}}{(-2)^2 - (\sqrt{3})^2} \text{ and } (-2)^2 - (\sqrt{3})^2 = 1$$
A1

hence the roots form a reciprocal pair AG

Note: Award as above for attempting to show that $\frac{1}{-2+\sqrt{3}} = -2-\sqrt{3}$.

METHOD 2

considers the product of their roots (M1)

Note: Only award **(M1)** if the roots are in the form $s \pm \sqrt{t}$.

$$(-2+\sqrt{3})(-2-\sqrt{3}) = 1$$
A1

hence the roots form a reciprocal pair AG

METHOD 3

considers $\frac{c}{a} \left(= \frac{1}{1} \right)$ (M1)

the product of roots is 1 A1

hence the roots form a reciprocal pair AG

[2 marks]

continued...

Question 2 continued.

(b) roots are $\pm i \left(= i, \frac{1}{i}, = \operatorname{cis}\left(\pm \frac{\pi}{2}\right), = e^{\pm i \frac{\pi}{2}} \right)$ (A1)

EITHER

$$\frac{1}{-i} \times \frac{i}{i} = i \text{ OR } \frac{1}{-i} = i$$
 A1

Note: Award as above for showing that $\frac{1}{i} = -i$ OR stating that $\frac{1}{i} = -i$.

OR

$$i \times -i = 1 \text{ OR } i \times \frac{1}{i} = 1$$
 A1

OR

the product of roots is 1 A1

THEN

hence the roots form a reciprocal pair AG

[2 marks]

(c) $p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$ (A1)

$$x^2 p\left(\frac{1}{x}\right) = x^2 \left(\frac{a}{x^2} + \frac{b}{x} + a \right)$$
 A1

$$= ax^2 + bx + a$$

$$= p(x)$$
 AG

Note: Award as above for showing that $\frac{p(x)}{x^2} = p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$.

[2 marks]

continued...

Question 2 continued.

(d) **METHOD 1**

$$p(\alpha) = 0 \Rightarrow \alpha^n p\left(\frac{1}{\alpha}\right) = 0$$

A1

EITHER

$$\alpha^n \neq 0 \text{ (as } \alpha \neq 0) \Rightarrow p\left(\frac{1}{\alpha}\right) = 0$$

R1

OR

$$\alpha \neq 0 \text{ (hence } \alpha^n \neq 0) \Rightarrow p\left(\frac{1}{\alpha}\right) = 0$$

R1

THEN

so $\frac{1}{\alpha}$ is also a root

AG

Note: The **R1** is dependent on the **A1**.

[2 marks]

METHOD 2

$$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p\left(\frac{1}{\frac{1}{\alpha}}\right)$$

$$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p(\alpha)$$

A1

for $\alpha \neq 0$ (seen anywhere), $p(\alpha) = 0 \Rightarrow \left(\frac{1}{\alpha}\right)^n p(\alpha) = 0$ and so $p\left(\frac{1}{\alpha}\right) = 0$

R1

so $\frac{1}{\alpha}$ is also a root

AG

Note: The **R1** is dependent on the **A1**.

[2 marks]

continued...

Question 2 continued.

(e) **METHOD 1**

$$p(x) = x^n p\left(\frac{1}{x}\right)$$

$$q(x) = x^m q\left(\frac{1}{x}\right) \quad (\text{A1})$$

attempts to substitute for $p(x)$ and $q(x)$ into $f(x) = p(x)q(x)$ (M1)

$$f(x) = x^n p\left(\frac{1}{x}\right) x^m q\left(\frac{1}{x}\right) \left(= x^{n+m} p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right) \right) \quad \text{A1}$$

$$= x^{n+m} f\left(\frac{1}{x}\right) \text{ (and } n+m \text{ is the degree of } f \text{)} \quad \text{A1}$$

hence f is a palindromic polynomial. AG

METHOD 2

$$p\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}$$

$$q\left(\frac{1}{x}\right) = \frac{q(x)}{x^m} \quad (\text{A1})$$

attempts to substitute for $p\left(\frac{1}{x}\right)$ and $q\left(\frac{1}{x}\right)$ into $f\left(\frac{1}{x}\right) = p\left(\frac{1}{x}\right)q\left(\frac{1}{x}\right)$ (M1)

$$f\left(\frac{1}{x}\right) = \frac{p(x)}{x^n} \frac{q(x)}{x^m} \left(= \frac{p(x)q(x)}{x^{n+m}} \right) \quad \text{A1}$$

$$x^{n+m} f\left(\frac{1}{x}\right) = p(x)q(x) \text{ and } f(x) = p(x)q(x) \text{ (and } n+m \text{ is the degree of } f \text{)} \quad \text{A1}$$

hence f is a palindromic polynomial. AG

[4 marks]

continued...

Question 2 continued.

METHOD 3

for example, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_{n-1} x + a_n$ AND

$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_{m-1} x + b_m$$

A1

attempts to multiply out their $p(x)q(x)$

(M1)

Note: Only award **(M1)** for polynomials of degree n and m respectively with different corresponding coefficients.

$$p(x)q(x) = a_n b_m x^{m+n} + (a_{n-1} b_m + a_n b_{m-1}) x^{m+n-1} + \dots + (a_{n-1} b_m + a_n b_{m-1}) x + a_n b_m$$

A1A1

Note: Award **A1** for the first and second terms and **A1** for the penultimate and last terms.

hence f is a palindromic polynomial

AG

[4 marks]

(f) **METHOD 1**

attempts to expand the product of two quadratic factors

(M1)

EITHER

$$x^4 + ux^3 + vx^3 + uvx^2 + 2x^2 + ux + vx + 1$$

(A1)

OR

$$x^4 + (u+v)x^3 + (uv+2)x^2 + (u+v)x + 1$$

(A1)

THEN

equates coefficients and forms a system of two equations in u and v

(M1)

$$u + v = 2 \text{ and } uv + 2 = -1 \text{ (} uv = -3 \text{)}$$

A1

attempts to solve their system of two equations

(M1)

$$u = -1, v = 3 \text{ OR } (f(x) = (x^2 - x + 1)(x^2 + 3x + 1))$$

A1

Note: Candidates must attempt to form and solve a system of equations in u and v . Hence, award no marks for working backwards from either decimal or exact values of roots given by a GDC.

Award **A0** for $u = -1, v = 3$ and $u = 3, v = -1$ both specified.

continued...

Question 2 continued.

METHOD 2

attempts to divide one of $(x^2 + ux + 1)$ or $(x^2 + vx + 1)$ into $f(x)$ (M1)

EITHER

for example, the quotient is $x^2 + (2 - u)x + u^2 - 2u - 2$ A1

sets their quotient equal to $x^2 + vx + 1$ (M1)

$$x^2 + (2 - u)x + u^2 - 2u - 2 = x^2 + vx + 1$$

$$u^2 - 2u - 2 = 1 \text{ and } 2 - u = v \quad \text{A1}$$

attempts to solve their equations for u and then v (or vice versa) (M1)

OR

for example, the remainder is $u(-u^2 + 2u + 3)x + (-u^2 + 2u + 3)$ A1

sets their remainder equal to zero (M1)

attempts to solve for u (or v) (M1)

$$u = -1, 3 \text{ and } u + v = 2 \quad \text{A1}$$

THEN

$$u = -1, v = 3 \text{ OR } (f(x) = (x^2 - x + 1)(x^2 + 3x + 1)) \quad \text{A1}$$

Note: Award **A0** for $u = -1, v = 3$ and $u = 3, v = -1$ both specified.

continued...

Question 2 continued.

METHOD 3considers $f(1)$, for example **(M1)**

$$(u+2)(v+2)=5$$
 A1

considers $f(2)$, for example **(M1)**

$$(2u+5)(2v+5)=33$$
 A1

attempts to solve their system of two equations **(M1)**

$$u=-1, v=3 \text{ OR } (f(x)=(x^2-x+1)(x^2+3x+1))$$
 A1

Note: Award **A0** for $u=-1, v=3$ and $u=3, v=-1$ both specified.**[6 marks]**(g) attempts to solve $x^4+2x^3-x^2+2x+1=0$ **(M1)****Note:** Award **(M1)** for attempting to solve one of their quadratic equations.

$$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$$
 A1A1

Note: Award **A1** for $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and **A1** for $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$.**[3 marks]**

continued...

Question 2 continued.

(h) **METHOD 1**

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + a_0 (= 0)$

attempts to find $p(-1)$ **(M1)**

$$p(-1) = (-1)^n + a_1(-1)^{n-1} + a_2(-1)^{n-2} + \dots + a_2(-1)^2 + a_1(-1) + 1$$
 (A1)

$$= (-1) + a_1 - a_2 + \dots + a_2 - a_1 + 1$$
 A1

(since n is odd), $p(x)$ has

EITHER

$n+1$ terms so all terms cancel giving $p(-1) = 0$ **R1**

OR

an even number of terms so all terms cancel giving $p(-1) = 0$ **R1**

THEN

hence -1 is always a root **AG**

METHOD 2

considers the product of roots of a polynomial equation of degree n

(M1)

$$\frac{(-1)^n a_0}{a_n}$$

product of roots is $(-1)^n$ (seen anywhere) **A1**

(for n odd) there are $\frac{n-1}{2}$ reciprocal pairs of roots **A1**

product of each reciprocal pair is 1 and $(-1)^n = -1$ (since n is odd) **R1**

hence -1 is always a root **AG**

continued...

Question 2 continued.

METHOD 3

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_2x^2 + a_1x + a_0 (= 0)$

considers $p(x) = x^n p\left(\frac{1}{x}\right)$ with $x = -1$ **(M1)**

$p(-1) = (-1)^n p(-1)$ **A1**

$p(-1) = -p(-1)$ ($2p(-1) = 0$) (since n is odd) **A1**

hence $p(-1) = 0$ **R1**

hence -1 is always a root **AG**

[4 marks]

Total [28 marks]

10. (a) METHOD 1

$$a = 5 \quad (\text{A1})$$

attempt to use roots and symmetry to find h (M1)

$$h = \frac{(-1) + (-3)}{2} \text{ OR half the distance between the roots } \frac{(-1) - (-3)}{2} = 1 \text{ (may be seen on a diagram)}$$

$$h = -2 \text{ (accept } x = -2) \quad (\text{A1})$$

$$f(x) = 5(x - (-2))^2 - 5 (= 5(x + 2)^2 - 5) \quad \text{A1}$$

$$(a = 5, h = -2, k = -5)$$

METHOD 2

$$a = 5 \quad (\text{A1})$$

attempt to expand

$$(x + 1)(x + 3) = x^2 + 4x + 3 \text{ OR } 5(x + 1)(x + 3) = 5x^2 + 20x + 15$$

EITHER

uses their expansion to attempt to complete the square to the form (M1)

$$p(x + q)^2 + r, \text{ where } q \text{ is half the coefficient of their } x \text{ term}$$

$$= (x + 2)^2 - 2^2 + 3 (= (x + 2)^2 - 1) \text{ OR } 5[(x + 2)^2 - 2^2 + 3] (= 5(x + 2)^2 - 5) \quad (\text{A1})$$

OR

uses their expansion to attempt to differentiate and sets equal to zero (M1)

$$\frac{dy}{dx} = 2x + 4 = 0 \text{ OR } \frac{dy}{dx} = 10x + 20 = 0$$

$$h = -2 \text{ (accept } x = -2) \quad (\text{A1})$$

OR

uses their expansion to attempt to find axis of symmetry using $h = \frac{-b}{2a}$ (M1)

$$h = \frac{-4}{2} \text{ OR } h = \frac{-20}{10}$$

$$h = -2 \text{ (accept } x = -2) \quad (\text{A1})$$

THEN

$$f(x) = 5(x - (-2))^2 - 5 (= 5(x + 2)^2 - 5) \quad \text{A1}$$

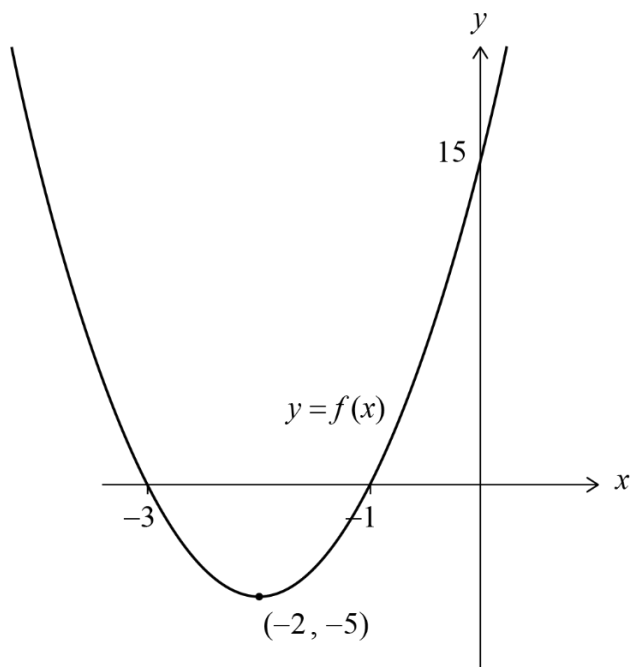
$$(a = 5, h = -2, k = -5)$$

[4 marks]

continued...

Question 10 continued.

(b)

**M1A1A1A1**award **M1** for a roughly symmetric curve which is concave upaward **A1** for x intercepts at -3 and -1 award **A1** for y intercept at 15 award **A1** for vertex at $(-2, -5)$ **[4 marks]**(c) $5(x+2)^2 - 5 \leq 40$ OR $5(x+1)(x+3) \leq 40$ OR $(x+1)(x+3) \leq 8$ leading to $(x+2)^2 \leq 9$ OR $5x^2 + 20x - 25 \leq 0$ OR $x^2 + 4x - 5 \leq 0$ **(A1)**valid attempt to find the critical values for their quadratic inequality **(M1)** $x+2 = \pm 3$ OR $(x+5)(x-1) = 0$ $x = -5, x = 1$ **(A1)** $-5 \leq x \leq 1$ **A1****Note:** Accept $(x \in)[-5, 1]$ or equivalent.**[4 marks]**

continued...

Question 10 continued.

(d) (i) $(f \circ g)(x) = 5(\ln x + 1)(\ln x + 3)$ OR $5(\ln x + 2)^2 - 5$ OR
 $5(\ln x)^2 + 20 \ln x + 15$

A1

(ii) attempt to replace x with $\ln x$ using their solution to part (c)

(M1)

$$-5 \leq \ln x \leq 1$$

$$e^{-5} \leq x \leq e$$

A1

Note: Accept $(x \in) [e^{-5}, e]$ or equivalent.

[3 marks]

(e) $(g \circ f)(x) = \ln(f(x))$

recognition that the domain requires $f(x) > 0$

(M1)

$$(x+1)(x+3) > 0$$

$$x < -3, x > -1$$

A1A1

Note: award **A1** for critical values and **A1** for correct inequalities.

accept $(x \in) (-\infty, -3) \cup (-1, \infty)$ or equivalent.

[3 marks]

Total [18 marks]

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1.

(a) -2 (accept $(0, -2)$)

A1**[1 mark]**

(b) $y = \frac{3}{2}$ (must be an equation)

A1**Note:** Do not accept $\neq$ sign.**[1 mark]**(c) attempt to reflect the graph of f in the x -axis OR one correct value seen**(M1)**both correct values $-\frac{3}{2}$ and 2 (seen anywhere)**A1**

$-\frac{3}{2} < y \leq 2$

A1**[3 marks]****Total [5 marks]**

5.

- (a) recognising $\Delta > 0$ (seen anywhere) **(M1)**

$$\Delta = k^2 - 4(15 - k) \quad (= k^2 + 4k - 60) \quad \text{A1}$$

valid attempt to solve quadratic (in)equality **(M1)**

$$(k - 6)(k + 10) \quad \text{OR} \quad k = \frac{-4 \pm \sqrt{4^2 - 4(-60)}}{2}$$

two correct values -10 and 6 (seen anywhere) **A1**

$$k < -10, \quad k > 6 \quad \text{A1}$$

[5 marks]

- (b) recognising constant or product of roots is positive **(M1)**

$$15 - k > 0 \Rightarrow k < 15$$

$$k < -10, \quad 6 < k < 15 \quad \text{A1}$$

[2 marks]

Total [7 marks]

7. METHOD 1 (product of quadratic factors)

attempt to write as product of two quadratic factors

(M1)

$$x^4 + px^3 - 2x^2 + qx - 3 = (x^2 + 2x + 1)(ax^2 + bx + c)$$

$$= (x^2 + 2x + 1)(x^2 + bx - 3) \text{ OR } (x^2 + 2x + 1)(x^2 - 3)$$

A1

attempt to compare their coefficients

(M1)

$$p = b + 2 \text{ OR } q = b - 6 \text{ OR } -2 = 2b + 1 - 3$$

$$b = 0$$

(A1)

$$p = 2, q = -6$$

A1**METHOD 2** (double root)

$$\text{let } f(x) = x^4 + px^3 - 2x^2 + qx - 3$$

recognition that $f(-1) = 0$ **(M1)**

$$p + q = -4$$

A1recognition that $f'(x) = 0$ at $x = -1$ since it has a double root**(M1)**

$$f'(x) = 4x^3 + 3px^2 - 4x + q$$

$$3p + q = 0$$

A1

$$p = 2, q = -6$$

A1**METHOD 3** (division by $(x+1)$ twice)attempt to use division with $(x+1)$ to find remainder**(M1)**

$$p + q = -4$$

A1attempt to use division once again with $(x+1)$ to find remainder**(M1)**

$$q + 3p = 0$$

A1

$$p = 2, q = -6$$

A1**METHOD 4** (division by $x^2 + 2x + 1$)

attempt to use division to find remainder

(M1)

$$(q + 3p)x + 2p - 4 (= 0) \text{ (or equivalent)}$$

(A1)

equating the coefficients of their remainder to 0

(M1)

$$q + 3p = 0 \text{ and } 2p - 4 = 0 \text{ OR } q - p + 2 = 2(1 - 2p) \text{ and } -3 = 1 - 2p \text{ (or equivalent)}$$

A1

$$p = 2, q = -6$$

A1*continued...*

Question 7 continued.

METHOD 5 (sum and product of roots)

let the four roots be $-1, -1, \alpha$ and β

attempt to set sum of four roots equal to $-p$ OR product of four roots equal to -3

(M1)

Note: Award **M1** for expansion of $(x-1)^2(x-\alpha)(x-\beta)$ leading to a quartic

$$x^4 + (2 - \alpha - \beta)x^3 + (\alpha\beta - 2\alpha - 2\beta + 1)x^2 + (2\alpha\beta - \alpha - \beta)x + \alpha\beta (=0)$$

$$(-1) + (-1) + \alpha + \beta = -p \quad (\Rightarrow \alpha + \beta = -p + 2) \quad \text{and} \quad (-1)(-1)\alpha\beta = -3 \quad (\Rightarrow \alpha\beta = -3)$$

A1

sum of product of pairs and sum of product of roots taken three at a time

$$\alpha\beta - \alpha - \alpha - \beta - \beta + 1 = -2 \quad (\Rightarrow \alpha\beta - 2\alpha - 2\beta = -3) \quad \text{and}$$

$$-\alpha\beta - \alpha\beta + \beta + \alpha = -q \quad (\Rightarrow -2\alpha\beta + \beta + \alpha = -q)$$

A1

Note: Award **A1** for any two correct equations.

attempt to solve *their* four equations to obtain the value of p and q

(M1)

$$\alpha\beta = -3 \quad \text{and} \quad \alpha\beta - 2\alpha - 2\beta = -3 \quad \Rightarrow \alpha + \beta = 0$$

$$p = 2, \quad q = -6$$

A1

[5 marks]

10.

(a) $4^x = 8$ OR $f^{-1}(x) = \log_4 x$ OR $f^{-1}(8) = \log_4 8$ (A1)

attempt to use indices with same base OR change of base of logs (M1)

$$2^{2x} = 2^3, 4^x = 4^{\frac{3}{2}} \text{ OR } f^{-1}(8) = \log_4 4^{\frac{3}{2}} \text{ OR } f^{-1}(8) = \frac{\log_2 8}{\log_2 4}$$

$$f^{-1}(8) = \frac{3}{2} \quad \text{A1}$$

[3 marks]

(b) (i) interchanging x and y (seen anywhere) (M1)

$$x = 1 + \log_2 y \text{ OR } y - 1 = \log_2 x$$

$$x - 1 = \log_2 y \text{ OR } 2^{y-1} = x$$

$$g^{-1}(x) = 2^{x-1} \text{ (or equivalent)} \quad \text{A1}$$

(ii) **METHOD 1**

a horizontal translation/shift by 1 unit to the left (do not accept 'move')

followed by a horizontal stretch/dilation with scale factor $\frac{1}{2}$ (accept horizontal compression by a factor of 2)

one correct transformation A1

two correct transformations in the correct order A1

METHOD 2

horizontal stretch/dilation with scale factor $\frac{1}{2}$ (accept horizontal compression A1

by a factor of 2)

vertical stretch/dilation with scale factor 2 A1

Note: A1s can be awarded independently here as order is not important.

[4 marks]

continued...

Question 10 continued.

- (c) attempt to find composite function (in any order)

M1

$$f(1 + \log_2 x) (= 4^{1 + \log_2 x})$$

$$4 \times 4^{\log_2 x} \text{ OR } 4 \times 2^{2\log_2 x} \text{ OR } 4^{\log_2 2x} \text{ OR } 2^{(2+2\log_2 x)} \text{ OR } 4^{(\log_4 4 + 2\log_4 x)}$$

(A1)

$$= 4 \times 2^{\log_2 x^2} \text{ OR } 2^{\log_2 (2x)^2} \text{ OR } 4^{\log_4 (4x^2)}$$

A1

$$= 4x^2$$

AG**[3 marks]**

$$(d) \quad (i) \quad 2x - 1 + \frac{1}{2x+1} = \frac{(2x-1)(2x+1)+1}{2x+1}$$

(A1)

$$= \frac{4x^2 - 1 + 1}{2x+1}$$

A1

$$2x - 1 + \frac{1}{2x+1} = \frac{4x^2}{2x+1}$$

AG

Note: Accept working from RHS to LHS.

- (ii)
- METHOD 1**

$$\text{enclosed area is } \int_1^3 \left(2x - 1 + \frac{1}{2x+1} \right) dx$$

attempt to integrate (at least one correct term or $\ln(2x+1)$ seen)**(M1)**

$$= x^2 - x + \frac{1}{2} \ln|2x+1| (+c)$$

A1A1

Note: Award **A1** for $x^2 - x$ and **A1** for $\frac{1}{2} \ln|2x+1|$.

$$\text{Accept } \frac{1}{2} \ln(2x+1).$$

substitute correct limits into their integrated expression and subtract

(M1)

$$= \left(9 - 3 + \frac{1}{2} \ln 7 \right) - \left(1 - 1 + \frac{1}{2} \ln 3 \right)$$

$$A = 6 + \frac{1}{2} \ln \frac{7}{3}$$

A1*continued...*

Question 10 continued.

METHOD 2

attempt to use integration by substitution

(M1)

$$\text{let } u = 2x + 1 \text{ OR } u = 2x - 1 \Rightarrow \frac{du}{dx} = 2$$

$$= \frac{1}{2} \int \left(u - 2 + \frac{1}{u} \right) du \text{ OR } \frac{1}{2} \int \frac{(u-1)^2}{u} du \text{ OR } \frac{1}{2} \int \left(u + \frac{1}{u+2} \right) du$$

A1

correct integration

$$= \frac{1}{4} u^2 - u + \frac{1}{2} \ln|u| (+c) \text{ OR } \frac{1}{4} u^2 + \frac{1}{2} \ln|u+2| (+c)$$

A1

substitution of their limits into their integrated expression and subtract

(M1)

$$= \left(\frac{49}{4} - 7 + \frac{1}{2} \ln 7 \right) - \left(\frac{9}{4} - 3 + \frac{1}{2} \ln 3 \right) \text{ OR } \left(\frac{25}{4} + \frac{1}{2} \ln 7 \right) - \left(\frac{1}{4} + \frac{1}{2} \ln 3 \right)$$

$$A = 6 + \frac{1}{2} \ln \frac{7}{3} \text{ (or equivalent)}$$

A1

[7 marks]

Total [17 marks]

2. (a) $f(-3) = -1$

A1**[1 mark]**

(b) $-3 \leq x \leq 5$

A1

Note: Award **A1** for answers using interval notation $[-3, 5]$.

[1 mark]

(c) $(f^{-1}(2x-7) = -3 \Rightarrow) 2x-7 = f(-3)$ **OR** $f^{-1}(-1) = -3$

(M1)

$$2x-7 = -1$$

(A1)

$$x = 3$$

A1**[3 marks]****Total [5 marks]**

4. (a) equating $y = mx - 3$ and $y = x^2 - x - 1$ **M1**
- $x^2 - x - 1 - (mx - 3) = 0$ **OR** $x^2 - mx - x + 2 = 0$ **A1**
- $x^2 - (m + 1)x + 2 = 0$ **AG**
- [2 marks]**

(b) **METHOD 1 (discriminant)**

correct substitution into discriminant (do not award if seen only in quadratic formula) **(A1)**

$(-(m + 1))^2 - 4(1)(2)$ **OR** $(m + 1)^2 - 4(1)(2)$ **A1**

discriminant equals 0 (seen anywhere) **A1**

$(m + 1)^2 = 8$ **OR** $m^2 + 2m - 7 = 0$

$(m =) -1 - 2\sqrt{2}, -1 + 2\sqrt{2}$ **A1A1**

METHOD 2 (derivative)

$\frac{dy}{dx} = 2x - 1$ **A1**

substituting $m = 2x - 1$ or $x = \frac{m + 1}{2}$ into **AG** from part (a) **(A1)**

$x^2 - (2x - 1 + 1)x + 2 = 0$ **OR** $x^2 - 2x^2 + 2 = 0$ **OR** $\left(\frac{m + 1}{2}\right)^2 - (m + 1)\frac{m + 1}{2} + 2 = 0$

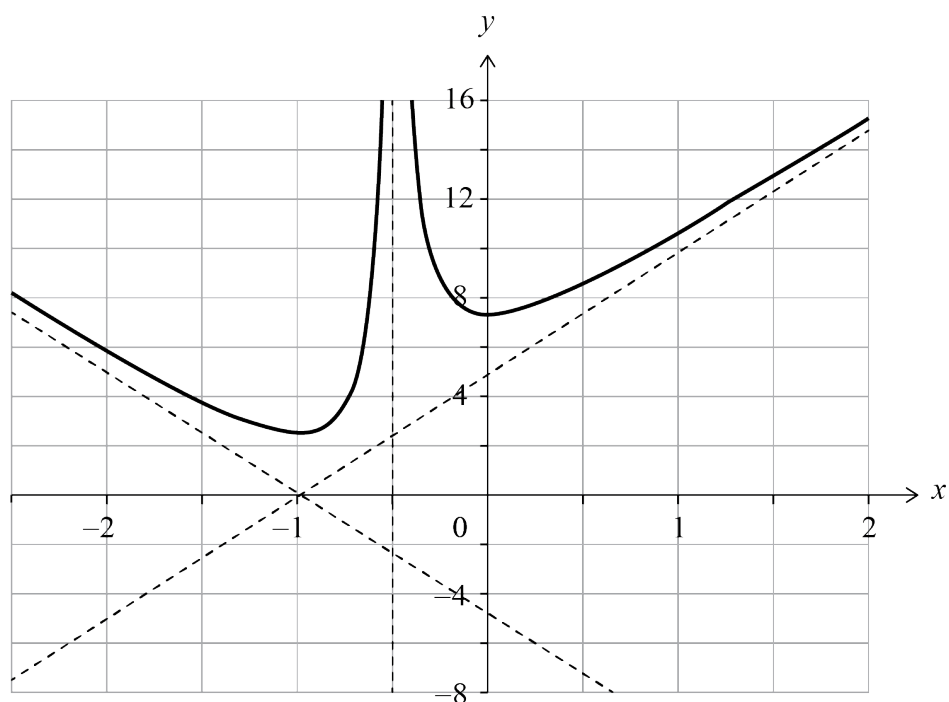
$x = \pm\sqrt{2}$ **OR** $-m^2 - 2m + 7 = 0$ **OR** $m^2 + 2m - 7 = 0$ **A1**

$(m =) 2\sqrt{2} - 1, -2\sqrt{2} - 1$ **A1A1**

[5 marks]

Total [7 marks]

6. (a)



right branch with minimum in approximately correct position **AND** showing asymptotic behaviour to $y = 5x + 5$

A1

asymptotic behaviour on both branches at $x = -\frac{1}{2}$

A1

left branch reflected in x -axis with minimum in approximately correct position

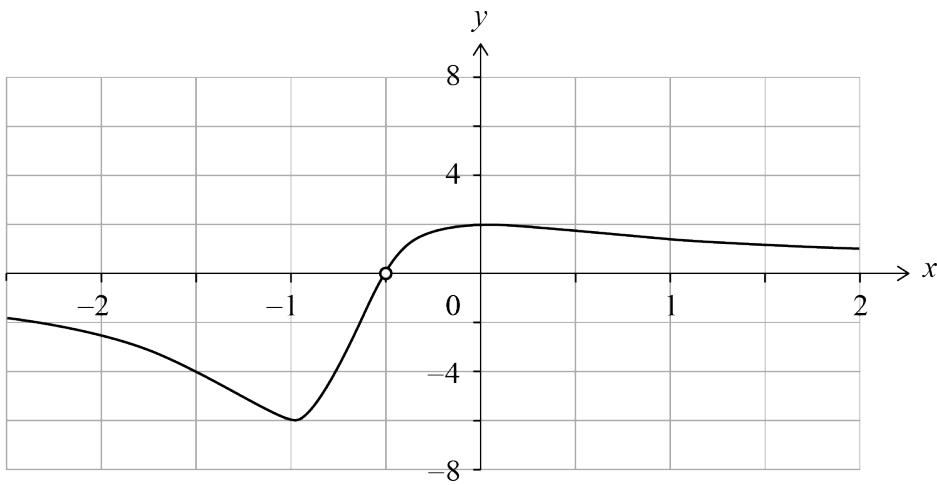
M1

oblique asymptote ($y = -5x - 5$) drawn in approximately correct position

(equation is not required) **AND** left branch with correct asymptotic behaviour

A1**[4 marks]**

(b)



axes intercepts in approximately correct positions **A1**

local extrema in approximately correct positions **A1**

asymptotic behavior to $y = 0$ on both sides **OR** asymptote at $y = 0$ indicated on sketch (by line or label etc. equation not required) **A1**

[3 marks]

Total [7 marks]

2. EITHER

use of the remainder or factor theorem

(M1)

$$f(1) = 0 \text{ OR } f(3) = -2$$

$$f(1) = 2 - 6 + p + q - 2 = 0 \quad (\Rightarrow p + q = 6)$$

(A1)

$$f(3) = 162 - 162 + 9p + 3q - 2 = -2 \quad (\Rightarrow 3p + q = 0)$$

(A1)**OR**

attempt to use synthetic division OR long division to find remainder

(M1)

$$\left(\frac{2x^4 - 6x^3 + px^2 + qx - 2}{x-1} = 2x^3 - 4x^2 + (p-4)x + p+q-4 + \frac{p+q-6}{x-1} \Rightarrow \right) p+q-6=0$$

(A1)

$$\left(\frac{2x^4 - 6x^3 + px^2 + qx - 2}{x-3} = 2x^3 + px + 3p+q + \frac{9p+3q-2}{x-3} \Rightarrow \right) 9p+3q-2=-2$$

(A1)**THEN**solving simultaneously their two equations in p and q **(M1)**

$$p = -3, q = 9$$

A1**[5 marks]**

1. (a) (i) attempts to find an expression for $f^2(x)$ (M1)

Note: For example, award **(M1)** for $f(mx + c)$ seen.

$$= m(mx + c) + c \quad \text{A1}$$

$$= m^2x + cm + c \quad \text{A1}$$

$$f^2(x) = m^2x + c(1 + m) \quad \text{AG}$$

[3 marks]

- (ii) attempts to find an expression for $f^3(x)$ (M1)

Note: For example, award **(M1)** for $f^2(mx + c)$ or $f(m^2x + c(1 + m))$ seen.

EITHER

$$f^3(x) = m^2(mx + c) + c(1 + m) \quad (= m^3x + m^2c + c + cm) \quad \text{A1}$$

OR

$$f^3(x) = m(m^2x + c(1 + m)) + c \quad (= m^3x + mc + m^2c + c) \quad \text{A1}$$

THEN

$$f^3(x) = m^3x + c(1 + m + m^2) \quad \text{AG}$$

[2 marks]

continued...

Question 1 continued

(b) (i) $f^4(x) = m^4x + c(1 + m + m^2 + m^3)$ **A1**

Note: Award **A1** for equivalent forms, for example:

$$f^4(x) = m^4x + c \sum_{i=0}^3 m^i$$

$$f^4(x) = m^3(mx + c) + c(1 + m + m^2)$$

$$f^4(x) = m(m^3x + m^2c + mc + c) + c$$

[1 mark]

(ii) $f^n(x) = m^n x + c(1 + m + (m^2 + \dots + m^{n-1}))$ **A1A1**

Note: Award **A1** for $m^n x$ and **A1** for $c(1 + m + m^2 + \dots + m^{n-1})$ or equivalent,

for example: $c \sum_{i=0}^{n-1} m^i$ or $c \left(\frac{1 - m^n}{1 - m} \right)$

For alternative forms, award **A1** for the correct term in x and **A1** for the correct constant term.

[2 marks]

(iii) **METHOD 1**

attempts to substitute $m = 1$ into their $f^n(x)$ **(M1)**

$$f^n(x) = x + c(1 + 1 + 1 + \dots + 1) \quad \text{A1}$$

$$f^n(x) = x + nc \quad \text{A1}$$

METHOD 2

$$f(x) = x + c \quad \text{A1}$$

attempts to find $f^2(x)$ with their $f(x)$ and $m = 1$ **(M1)**

$$f^2(x) = f(x + c) = x + 2c$$

$$\Rightarrow f^n(x) = x + nc \quad \text{A1}$$

[3marks]

continued...

Question 1 continued

(c) consider $n = 1$

$$f^1(x) = m^1x + c \left(\frac{1-m^1}{1-m} \right) \Rightarrow f(x) = mx + c,$$

R1hence true for $n = 1$ **Note:** The substitution $n = 1$ must be explicit for the **R1** to be awarded.Award **R0** for only considering $n = 2$.Subsequent marks after this **R1** mark are independent of this mark and can be awarded.assume true for $n = k$ ($p(k)$ is true)**M1**

$$f^k(x) = m^kx + c \left(\frac{1-m^k}{1-m} \right) \text{ (for some } k \in \mathbb{Z}^+ \text{)}$$

Note: The assumption of truth must be apparent. Do not award **M1** for statements such as “let $n = k$ ” or “ $n = k$ is true” or “assume $n = k$ is true”. Subsequent marks after this **M1** are independent of this mark and can be awarded.**EITHER**Attempting to find $f(f^k(x))$ **(M1)**

$$f^{k+1}(x) = f(f^k(x))$$

$$f\left(m^kx + c \left(\frac{1-m^k}{1-m} \right)\right) = m\left(m^kx + c \left(\frac{1-m^k}{1-m} \right)\right) + c$$

A1

$$= \left(m^{k+1}x + cm \left(\frac{1-m^k}{1-m} \right) \right) + c$$

ORAttempting to find $f^k(f(x))$ **(M1)**

$$f^{k+1}(x) = f^k(f(x))$$

$$f^k(mx + c) = m^k(mx + c) + c \left(\frac{1-m^k}{1-m} \right)$$

A1

$$= \left(m^{k+1}x + cm^k + c \left(\frac{1-m^k}{1-m} \right) \right)$$

continued...

Question 1 continued

THEN**EITHER**attempts to put c over a common denominator**(M1)**

$$m^{k+1}x + cm \left(\frac{1-m^k}{1-m} \right) + \frac{c(1-m)}{1-m}$$

$$= m^{k+1}x + c \left(\frac{m(1-m^k) + (1-m)}{1-m} \right) \text{ (or equivalent)}$$

A1**OR**attempts to put cm^k over a common denominator**(M1)**

$$m^{k+1}x + \frac{cm^k(1-m)}{1-m} + c \left(\frac{1-m^k}{1-m} \right)$$

$$= m^{k+1}x + c \left(\frac{m^k(1-m) + (1-m^k)}{1-m} \right) \text{ (or equivalent)}$$

A1**OR**recognizes that $\frac{1-m^k}{1-m}$ can be expressed as a geometric series**(M1)**

$$m^{k+1}x + cm(1 + m + m^2 + \dots + m^{k-1}) + c$$

$$= m^{k+1}x + c(1 + m + m^2 + m^3 + \dots + m^k)$$

A1**THEN**

$$= m^{k+1}x + c \left(\frac{1-m^{k+1}}{1-m} \right)$$

A1since true for $n = 1$ and true for $n = k + 1$ if true for $n = k$, hence true for all $n (\in \mathbb{Z}^+)$ **R1**

Note: To obtain the final **R1**, four of the previous seven marks must have been awarded.

The statement 'true for $n = 1$ ' may be seen anywhere in the proof.

[8 marks]

continued...

Question 1 continued

- (d) attempts to apply $m^n \rightarrow 0$ (as $n \rightarrow \infty$) to at least one term in expression (M1)

Note: Award **(M1)** for an equivalent statement/conclusion in words.

$$m^n x \rightarrow 0 \qquad \left(c \left(\frac{1-m^n}{1-m} \right) \rightarrow \right) \frac{c}{1-m} \text{ (as } n \rightarrow \infty \text{)} \qquad \textbf{(A1)A1}$$

Note: Award **(A1)** for the correct term in x and **A1** for the correct constant term.

$$y = \frac{c}{1-m} \qquad \textbf{A1}$$

[4 marks]

(e) (i) $f^n(x) = (-1)^n x + c \left(\frac{1-(-1)^n}{1-(-1)} \right)$ A1

Note: Award **A1** here if the above is seen in part (e)(ii).

Award **A0** for a missing bracket, for example: $f^n(x) = -1^n x + c \left(\frac{1-(-1)^n}{1-(-1)} \right)$.

$$(-1)^n = -1 \text{ (when } n \text{ is odd)} \qquad \textbf{R1}$$

$$\left(f^n(x) = -x + c \left(\frac{1-(-1)}{1-(-1)} \right) \right)$$

Note: The **A1** and **R1** mark can be awarded in either order. **A0R1** is possible.

Award **R0** for arguments based on specific numerical examples.

$$f^n(x) = -x + c \qquad \textbf{AG}$$

[2 marks]

- (ii) evidence that an even n has been considered (M1)

$$\text{for example, } (-1)^2 = 1$$

$$f^n(x) = x \qquad \textbf{A1}$$

Note: Award **(M1)A1** for answer of x (need not be an equation).

[2 marks]

Total [27 marks]

2. (a) attempt to factorize at least one expression (M1)

$$\frac{x}{(x-7)(x-1)} \times \frac{(x-1)(x+1)}{x+1} \quad \text{OR} \quad \frac{x(x^2-1)}{(x-7)(x^2-1)} \quad \text{OR} \quad \frac{x(x^2-1)}{(x-7)(x-1)(x+1)} \quad \text{A1}$$

$$\frac{x}{x-7} \quad \text{AG}$$

[2 marks]

- (b) **METHOD 1** (combines given log terms)

attempt to use the subtraction/quotient property of logs (M1)

$$\log_2 \left(\frac{x(x^2-1)}{(x^2-8x+7)(x+1)} \right)$$

valid attempt to rewrite equation with (or without) logs on both sides (seen anywhere) (M1)

$$2 = \frac{x}{x-7} \quad \text{OR} \quad \log_2 \left(\frac{x}{x-7} \right) = \log_2 2 \quad \text{OR} \quad 2^{\log_2 \left(\frac{x}{x-7} \right)} = 2^1 \quad \text{(A1)}$$

$$x = 14 \quad \text{A1}$$

METHOD 2 (uses log laws with LHS)

attempt to use $1 = \log_2 2$ (seen anywhere) (M1)

attempt to use the subtraction/quotient property of logs (M1)

$$\log_2 \left(\frac{x^3-x}{2} \right) = \log_2 ((x^2-8x+7)(x+1)) \quad \text{OR} \quad \frac{x(x^2-1)}{2} = (x-1)(x-7)(x+1) \quad \text{(A1)}$$

$$\frac{x}{2} = x-7$$

$$x = 14 \quad \text{A1}$$

[4 marks]

Total [6 marks]

4. (a) METHOD 1 x -coordinate of the vertex at A is 2 (A1)attempt to substitute their x -coordinate into equation of line (M1)

$$2(2) - 1 (= 3)$$

$$3 = -(2)^2 + 4(2) + p \quad \text{A1}$$

$$p = -1 \quad \text{AG}$$

METHOD 2 x -coordinate of the vertex at A is 2 (A1)attempt to substitute their x -coordinate into $f(x)$ (M1)

$$f(2) = p + 4$$

$$p + 4 = 2(2) - 1 \quad \text{A1}$$

$$p = -1 \quad \text{AG}$$

METHOD 3attempt to complete the square on $f(x)$ (M1)

$$-(x-2)^2 + p + 4$$

$$x_A = 2 \quad \text{OR} \quad y_A = p + 4 \quad \text{(A1)}$$

$$p + 4 = 2(2) - 1 \quad \text{A1}$$

$$p = -1 \quad \text{AG}$$

Note: Award no marks if $x = 0$ is substituted into $-x^2 + 4x + p = 2x - 1$ to show $p = -1$.**[3 marks]***continued...*

Question 4 continued.

(b) **METHOD 1** (equating $g(x)$ and $2x-1$)

x -coordinate is $\frac{-q}{2}$ (seen anywhere) (A1)

attempt to find intersection points, equating $g(x)$ and $2x-1$ (M1)

$$x^2 + qx - 1 = 2x - 1$$

attempt to substitute their x -coordinate (M1)

$$\left(\frac{-q}{2}\right)^2 + q\left(\frac{-q}{2}\right) - 1 = 2\left(\frac{-q}{2}\right) - 1$$

$$q^2 - 4q = 0$$
 (A1)

$$q = 4 \text{ (since } q \neq 0 \text{)}$$
 A1

METHOD 2 (find coordinates of B in terms of q)

x -coordinate is $\frac{-q}{2}$ (A1)

EITHER

attempt to substitute their x -coordinate into $y = 2x - 1$ (M1)

$$2\left(\frac{-q}{2}\right) - 1 \Rightarrow B\left(\frac{-q}{2}, -q - 1\right)$$

attempt to substitute their coordinates of B into $g(x)$ (M1)

$$-q - 1 = \left(\frac{-q}{2}\right)^2 + q\left(\frac{-q}{2}\right) - 1$$

OR

attempt to substitute their x -coordinate into $g(x)$ (M1)

$$\left(\frac{-q}{2}\right)^2 + q\left(\frac{-q}{2}\right) - 1 \Rightarrow B\left(\frac{-q}{2}, -\frac{q^2}{4} - 1\right)$$

attempt to substitute their coordinates of B into the line (M1)

$$-\frac{q^2}{4} - 1 = 2\left(\frac{-q}{2}\right) - 1$$

continued...

Question 4 continued.

THEN

$$q^2 - 4q = 0 \quad (\text{A1})$$

$$q = 4 \text{ (since } q \neq 0 \text{)} \quad \text{A1}$$

METHOD 3 (completing the square)

attempt to complete the square on $g(x)$ (M1)

$$\left(x + \frac{q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - 1$$

$$x_B = \frac{-q}{2} \quad \text{OR} \quad y_B = -\frac{q^2}{4} - 1 \quad (\text{A1})$$

attempt to substitute their coordinates of B into equation of line (M1)

$$-\frac{q^2}{4} - 1 = 2\left(\frac{-q}{2}\right) - 1$$

$$q^2 - 4q = 0 \quad (\text{A1})$$

$$q = 4 \text{ (since } q \neq 0 \text{)} \quad \text{A1}$$

[5 marks]

Total [8 marks]

3. (a) substitute $P(-10,1)$ or $Q(3,-12)$ into $y = \frac{Ax+B}{x-4}$ (M1)

$$-14 = -10A + B \text{ (or equivalent)}$$

$$12 = 3A + B \text{ (or equivalent)}$$

attempt to solve their equations simultaneously (M1)

$$A = 2, B = 6$$

A1
[3 marks]

(b) EITHER

attempt to express y in the form $P + \frac{Q}{x-4}$ (M1)

OR

attempt to find one or both asymptotes using $x-4=0$ or $y = \frac{a}{c}$ (dividing coefficients) (M1)

THEN

recognizing vertical asymptote at $x = 4$ (may be seen anywhere, including labeled on graph or in part (a)) OR horizontal shift 4 units to the right A1

recognizing horizontal asymptote at $y = 2$ (may be seen anywhere, including labeled on graph or in part (a)) OR vertical shift 2 units up A1

$$\frac{2x+6}{x-4} = 2 + \frac{14}{x-4} \quad (A1)$$

EITHER

horizontal translation 4 units to the right;

vertical stretch scale factor 14 followed by vertical translation 2 up. A1

Note: Horizontal translation may be seen anywhere; y -axis transformations must be seen in the order above.

OR

vertical stretch scale factor 14; followed by translation (through/by) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ A1

Note: In this question, the transformations may be described using terms such as *translate* or *shift*, and *dilate* or *stretch*. Do not accept “move” to describe a transformation.

[5 marks]
Total [8 marks]

4.

Note: Candidates may approach this problem in different ways and may do their steps in many different orders.
The **M** marks are independent and may be awarded in any order. Use the description of each mark to determine when the mark may be awarded.

$$3\log_8 10x - \log_4 x = 1$$

attempt to apply power rule, quotient rule, or product rule (seen anywhere) **(M1)**

attempt to apply change of base rule (seen anywhere) **(M1)**

correct equation with same base in all logarithms **A1**

$$\log_2 10x - \log_2 x^{\frac{1}{2}} = 1 \quad \text{or} \quad \log_2 10 + \log_2 x - \frac{1}{2}\log_2 x = 1 \quad \text{or} \quad \frac{\log_4 (10x)^3}{\log_4 8} - \log_4 x = 1 \quad (\text{or equivalent})$$

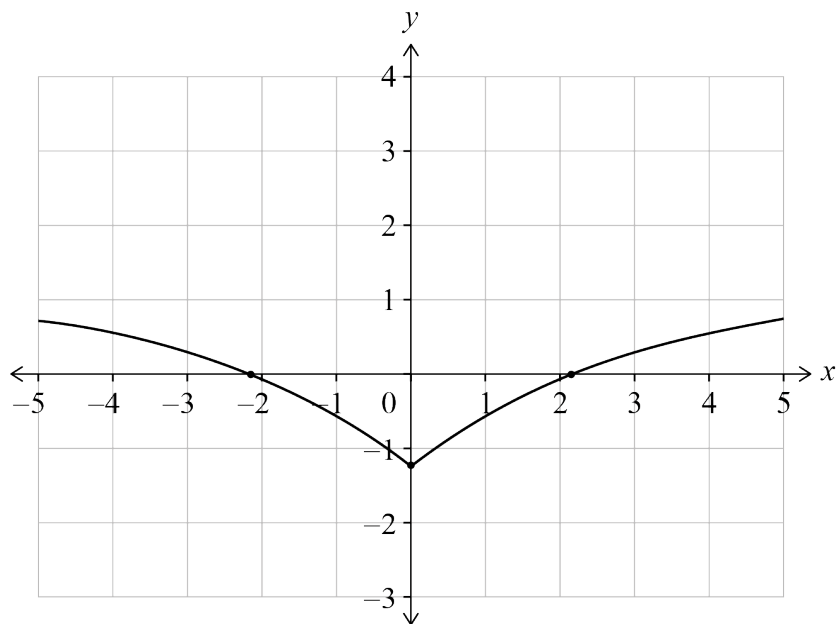
correct equation without logarithms **A1**

$$10\sqrt{x} = 2 \quad \text{or} \quad \sqrt{x} = \frac{2}{10} \left(= \frac{1}{5} \right) \quad (\text{or equivalent})$$

$$x = \frac{1}{25} \quad \text{A1}$$

[5 marks]

5. (a) (i)



attempt to reflect graph in y -axis for $x \geq 0$

M1

intercepts with x -axis and y -axis and clear cusp on y -axis in approximately the correct places

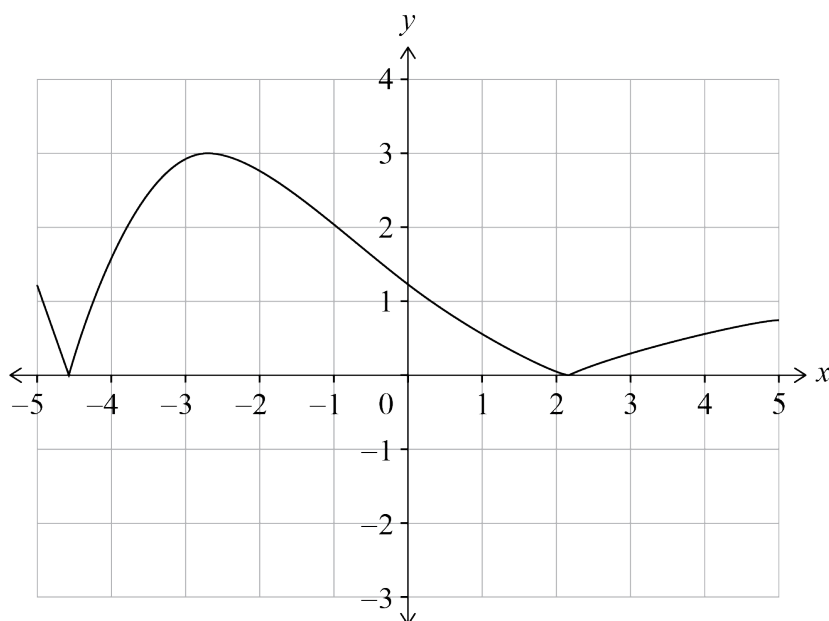
A1

Note: Award **A1** for roots between 2 to 2.5, -2.5 to -2 and y -intercept from -1.5 to -1 .

continued...

Question 5 continued

(ii)

attempt to reflect graph in x -axis for $y \leq 0$ **M1**intercept with y -axis, maximum point and two cusps on x -axis in approximately the correct places**A1**

Note: Award **A1** for cusps on x -axis between 2 to 2.5, -5 to -4 and y -intercept from 1 to 1.5 and maximum point x -coordinate between -3 to -2.5 with $y = 3$.

[4 marks](b) recognition that $|f(x)| \geq (f(x))^2$ when $|f(x)| \leq 1$ **(M1)** $x \geq -4.93$ OR $x \leq 5$ $x \leq -4.27$ OR $x \geq 0.292$ seen**(A1)** $-4.93 \leq x \leq -4.27$, $0.292 \leq x \leq 5$ **A1A1****[4 marks]****Total [8 marks]**

6. attempt to use remainder theorem $P(-1) = 12$ or to divide $P(x)$ by $(x+1)$ (M1)

$$2 - q + r = 12 \quad \text{A1}$$

$$(r - q = 10)$$

EITHER

$$\alpha + \beta = -\frac{q}{2} \text{ and } \alpha\beta = \frac{r}{2} \quad \text{(A1)}$$

attempt to use $(\alpha + \beta)^2 \equiv \alpha^2 + \beta^2 + 2\alpha\beta$ with $\alpha + \beta = -\frac{q}{2}$ and $\alpha\beta = \frac{r}{2}$ (M1)

$$\left(-\frac{q}{2}\right)^2 = \frac{37}{4} + 2\left(\frac{r}{2}\right) \quad \text{(A1)}$$

$$37 = q^2 - 4r$$

attempt to solve two simultaneous equations in q and r by rearranging/eliminating a variable (M1)

$$r = 10 + q \text{ and } r = \frac{q^2 - 37}{4} \text{ OR } q = r - 10 \text{ and } q = \sqrt{37 + 4r} \text{ OR } 37 = q^2 - 4(q + 10) \text{ OR}$$

$$37 = (r - 10)^2 - 4r$$

OR

recognition that α and β are both roots of $P(x)$ and adding the two equations (M1)

$$2\alpha^2 + q\alpha + r = 0 \text{ and } 2\beta^2 + q\beta + r = 0 \text{ then } 2(\alpha^2 + \beta^2) + q(\alpha + \beta) + 2r = 0$$

$$\alpha + \beta = -\frac{q}{2} \quad \text{(A1)}$$

use of $\alpha + \beta = -\frac{q}{2}$ and $\alpha^2 + \beta^2 = \frac{37}{4}$ to form equation in q (M1)

$$2 \times \frac{37}{4} - \frac{1}{2}q^2 + 20 + 2q = 0 \Rightarrow -\frac{1}{2}q^2 + 2q + \frac{77}{2} = 0 \quad \text{(A1)}$$

continued...

Question 6 continued

THEN

$$q = -7, r = 3$$

A1

Note:

Award **A0** if extra values for q and r given without any indication that $q = -7, r = 3$ are chosen as their final answer.

Award a maximum of **(M1)A1(A1)(M1)(A1)(M1)A0** if final answer given as $q = 11, r = 21$

[7 marks]

