APOLLO

Mathematics: analysis and approaches HL Answers

First assessment 2021
Whole questions arranged by syllabus topic.
Original question and scheme crops preserve source diagrams.

APOLLO

Number and algebra

102 questions

2 **METHOD 1 (finding u_1 first, from S_8)**

$4(u_1+8)=8$ ***(A1)***

$u_1=-6$ ***A1***

$u_1+7d=8$ OR $4(2u_1+7d)=8$ (may be seen with their value of u_1) ***(A1)***

attempt to substitute their u_1 ***(M1)***

$d=2$ ***A1***

METHOD 2 (solving simultaneously)

$u_1+7d=8$ ***(A1)***

$4(u_1+8)=8$ OR $4(2u_1+7d)=8$ OR $u_1=-3d$ ***(A1)***

attempt to solve linear or simultaneous equations ***(M1)***

$u_1=-6, d=2$ ***A1A1***

[5 marks]

7. METHOD 1

other two roots are $a-b\text{i}$ and $b-a\text{i}$ ***A1***

sum of roots $=-4$ and product of roots $=400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a+b\text{i}+a-b\text{i}+b+a\text{i}+b-a\text{i}=-4$

$2a+2b=-4\left(\Rightarrow a+b=-2\right)$ ***A1***

$\left(a+b\text{i}\right)\left(a-b\text{i}\right)\ \left(b+a\text{i}\right)\left(b-a\text{i}\right)=400$

$\left(a^2+b^2\right)^2=400$ ***A1***

$a^2+b^2=20$

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

METHOD 2

other two roots are $a-b\text{i}$ and $b-a\text{i}$ ***A1***

$\left(z-\left(a+b\text{i}\right)\right)\left(z-\left(a-b\text{i}\right)\right)\left(z-\left(b+a\text{i}\right)\right)\left(z-\left(b-a\text{i}\right)\right)\left(=0\right)$ ***A1***

$\left((z-a)^2+b^2\right)\left((z-b)^2+a^2\right)\left(=0\right)$

$\left(z^2-2az+a^2+b^2\right)\left(z^2-2bz+b^2+a^2\right)\left(=0\right)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a-2b=4$ and $\left(a^2+b^2\right)^2=400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a=2$ or $a=-4$ ***A1A1***

[8 marks]

9. (a) **METHOD 1**

B has one less pen to select ***(M1)***

EITHER

A and B can be placed in 6×5 ways ***(A1)***

C, D, E have 6 choices each ***(A1)***

OR

A (or B), C, D, E have 6 choices each ***(A1)***

B (or A) has only 5 choices ***(A1)***

THEN

$5\times6^4\left(=6480\right)$ ***A1***

METHOD 2

total number of ways $=6^5$ ***(A1)***

number of ways with Amber and Brownie together $=6^4$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

6^5-6^4

$=5\times6^4\left(=6480\right)$ ***A1***

[4 marks]

(b) **METHOD 1**

total number of ways $=6!\left(=720\right)$ ***(A1)***

number of ways with Amber and Brownie sharing a boundary

$=2\times7\times4!\left(=336\right)$ ***(A1)***

attempt to subtract (may be seen in words) ***(M1)***

$720-336=384$ ***A1***

METHOD 2

case 1: number of ways of placing A in corner pen

$3\times4\times3\times2\times1$

Four corners total no of ways is $4\times\left(3\times4\times3\times2\times1\right)=12\times4!\left(=288\right)$ ***(A1)***

case 2: number of ways of placing A in the middle pen

$2\times4\times3\times2\times1$

two middle pens so $2\times\left(2\times4\times3\times2\times1\right)=4\times4!\left(=96\right)$ ***(A1)***

attempt to add (may be seen in words) ***(M1)***

total no of ways $=288+96$

$=16\times4!\left(=384\right)$ ***A1***

[4 marks]

Total [8 marks]

Section B

1. attempt to subtract squares of integers ***(M1)***

$(n+1)^2 - n^2$

EITHER

correct order of subtraction and correct expansion of $(n+1)^2$, seen anywhere ***A1A1***

$= n^2 + 2n + 1 - n^2 (= 2n+1)$

OR

correct order of subtraction and correct factorization of difference of squares ***A1A1***

$= (n+1-n)(n+1+n)(= 2n+1)$

THEN

$= n + n + 1$ =RHS ***A1***

Note: Do not award final ***A1*** unless all previous working is correct.

which is the sum of n and $n+1$ ***AG***

Note: If expansion and order of subtraction are correct, award full marks for candidates who find the sum of the integers as $2n+1$ and then show that the difference of the squares (subtracted in the correct order) is $2n+1$.

[4 marks]

3. EITHER

attempt to use the binomial expansion of $(x+k)^7$ ***(M1)***

${}^7C_0x^7k^0+{}^7C_1x^6k^1+{}^7C_2x^5k^2+\ldots$ (or ${}^7C_0k^7x^0+{}^7C_1k^5x^1+{}^7C_2k^5x^2+\ldots$)

identifying the correct term ${}^7C_2x^5k^2$ (or ${}^7C_5k^2x^5$) ***(A1)***

OR

attempt to use the general term ${}^7C_rx^rk^{7-r}$ (or ${}^7C_rk^rx^{7-r}$) ***(M1)***

$r=2$ (or $r=5$) ***(A1)***

THEN

${}^7C_2=21$ (or ${}^7C_5=21$) (seen anywhere) ***(A1)***

$21x^5k^2=63x^5$ $(21k^2=63,\ k^2=3)$ ***A1***

$k=\pm\sqrt{3}$ ***A1***

Note: If working shown, award ***M1A1A1A1A0*** for $k=\sqrt{3}$.

[5 marks]

5. METHOD 1

product of a binomial coefficient, a power of 3 (and a power of x^2) seen ***(M1)***
evidence of correct term chosen ***(A1)***

$^{n+1}C_2 \times 3^{n+1-2} \times \left(x^2\right)^2 \left(= \frac{n(n+1)}{2} \times 3^{n-1} \times x^4\right)$ OR $n - r = 1$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

EITHER

$^{n+1}C_2 \times 3^{n-1} = 20412$ ***(A1)***

OR

$^{r+2}C_r \times 3^r = 20412 \Rightarrow r = 6$ ***(A1)***

THEN

$n = 7$ ***A1***

METHOD 2

$3^{n+1}\left(1 + \frac{x^2}{3}\right)^{n+1}$

product of a binomial coefficient and a power of $\frac{x^2}{3}$ OR $\frac{1}{3}$ seen ***(M1)***

evidence of correct term chosen ***(A1)***

$3^{n+1} \times {}^{n+1}C_2 \times \left(\frac{x^2}{3}\right)^2 \left(= 3^{n-1}\frac{n(n+1)}{2}x^4\right)$

equating their coefficient to 20412 or their term to $20412x^4$ ***(M1)***

$3^{n-1} \times \frac{n(n+1)}{2} = 20412$ ***(A1)***

$n = 7$ ***A1***

Total [5 marks]

8. (a) $\left(|zw| = \right)16$ ***A1***

[1 mark]

(b) attempt to find $\arg(z) + \arg(w)$ ***(M1)***

$\arg(zw) = \arg(z) + \arg(w)$

$= \frac{\pi}{5} - \frac{2k\pi}{5}\left(= \frac{(1-2k)\pi}{5} \right)$ ***A1***

[2 marks]

continued…

Question 8 continued

(c) (i) $zw \in \mathrm{Z} \Rightarrow \arg(zw)$ is a multiple of π ***(M1)***

$\Rightarrow 1 - 2k$ is a multiple of 5 ***(M1)***

$k = 3$ ***A1***

(ii) $zw = 16\left(\cos(-\pi) + \mathrm{i}\sin(-\pi)\right)$

-16 ***A1***

[4 marks]

Total [7 marks]

2. (a) attempt to use $u_1 + (n-1)d = 0$ ***(M1)***

$60 - 2.5(k-1) = 0$

$k = 25$ ***A1***

[2 marks]

(b) **METHOD 1**

attempting to express S_n in terms of n ***(M1)***

use of a graph or a table to attempt to find the maximum sum ***(M1)***

$= 750$ ***A1***

METHOD 2

EITHER

recognizing maximum occurs at $n = 25$ ***(M1)***

$S_{25} = \frac{25}{2}(60 + 0),\ S_{25} = \frac{25}{2}(2 \times 60 + 24 \times -2.5)$ ***(A1)***

OR

attempting to calculate S_{24} ***(M1)***

$S_{24} = \frac{24}{2}(2 \times 60 + 23 \times -2.5)$ ***(A1)***

THEN

$= 750$ ***A1***

[3 marks]

Total [5 marks]

7. (a) Jack and Andrea finish in that order (as a unit) so we are considering the arrangement of 7 objects ***(M1***)

$7!\ (=5040)$ ways ***A1***

[2 marks]

(b) **METHOD 1**

the number of ways that Andrea finishes in front of Jack is equal to the number of ways that Jack finishes in front of Andrea ***(M1***)

total number of ways is $8!$ ***(A1)***

$\frac{8!}{2}\ (=20160)$ ways ***A1***

[3 marks]

METHOD 2

the other six runners can finish in $6!\ (=720)$ ways ***(A1)***

when Andrea finishes first, Jack can finish in 7 different positions

when Andrea finishes second, Jack can finish in 6 different positions etc

$7+6+5+4+3+2+1\ (=28)$ ways ***(A1)***

hence there are $(7+6+5+4+3+2+1)\times 6!$ ways

$28\times 6!\ (=20160)$ ways ***A1***

[3 marks]
Total [5 marks]

8. $\dfrac{1+z}{1-z}=\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta}$

attempt to use the complex conjugate of their denominator ***M1***

$=\dfrac{(1+\cos\theta+\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}{(1-\cos\theta-\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}$ ***A1***

$\mathrm{Re}\left(\dfrac{1+z}{1-z}\right)=\dfrac{1-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta}\ \left(=\dfrac{1-\cos^2\theta-\sin^2\theta}{2-2\cos\theta}\right)$ ***M1A1***

Note: Award ***M1*** for expanding the numerator and ***A1*** for a correct numerator. Condone either an incorrect denominator or the absence of a denominator.

using $\cos^2\theta+\sin^2\theta=1$ to simplify the numerator ***(M1)***

$\mathrm{Re}\left(\dfrac{1+z}{1-z}\right)=0$ ***AG***

[5 marks]

2. (a) (i) **METHOD 1**

attempts to expand $(\omega-1)(\omega^2+\omega+1)$ ***(M1)***

$=\omega^3+\omega^2+\omega-\omega^2-\omega-1$ ***A1***

$=\omega^3-1$ ***AG***

[2 marks]

METHOD 2

attempts polynomial division on $\frac{\omega^3-1}{\omega-1}$ ***M1***

$=\omega^2+\omega+1$ ***A1***

so $(\omega-1)(\omega^2+\omega+1)=\omega^3-1$ ***AG***

[2 marks]

(ii) (since ω is a root of $z^3=1)\Rightarrow\omega^3-1=0$ ***R1***

and $\omega\neq1$ ***R1***

$\Rightarrow\omega^2+\omega+1=0$ ***AG***

Note: In part (a), award marks as appropriate where ω has been converted into Cartesian, modulus-argument (polar) or Euler form.

[2 marks]

(b) **METHOD 1**

attempts to find either P_0P_1 or P_0P_2 ***(M1)***

accept any valid method

e.g. $2\sin\frac{\pi}{3}$, $1^2+1^2-2\cos\frac{2\pi}{3}$, $\frac{1}{\sin\frac{\pi}{6}}=\frac{P_0P_1}{\sin\frac{2\pi}{3}}$ from either ΔOP_0P_1 or ΔOP_0P_2

e.g. use of Pythagoras' theorem

e.g. $\left|1-e^{i\frac{2\pi}{3}}\right|$, $\left|1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right|$ by calculating the distance between 2 points

$P_0P_1=\sqrt{3}$ ***A1***

$P_0P_2=\sqrt{3}$ ***A1***

Note: Award a maximum of ***M1A1A0*** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_2 or both.

so $P_0P_1\times P_0P_2=3$ ***AG***

METHOD 2

attempts to find $P_0P_1\times P_0P_2=|1-\omega||1-\omega^2|$ ***(M1)***

$P_0P_1\times P_0P_2=|\omega^3-\omega^2-\omega+1|$ ***A1***

$=|1-(\omega^2+\omega+1)+2|$ and since $\omega^2+\omega+1=0$ ***R1***

so $P_0P_1\times P_0P_2=3$ ***AG***

[3 marks]

(c) **METHOD 1**

$z^4 - 1 = (z-1)(z^3 + z^2 + z + 1)$ ***A1***

(ω is a root hence) $\omega^4 - 1 = 0$ and $\omega \neq 1$ ***R1***

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

Note: Condone the use of ω throughout.

[2 marks]

METHOD 2

considers the sum of roots of $z^4 - 1 = 0$ ***(M1)***

the sum of roots is zero (there is no z^3 term) ***A1***

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

[2 marks]

METHOD 3

substitutes for ω ***(M1)***

e.g. LHS $= e^{i\frac{3\pi}{2}} + e^{\pi i} + e^{i\frac{\pi}{2}} + 1$

$= -i - 1 + i + 1$ ***A1***

Note: This can be demonstrated geometrically or by using vectors.
Accept Cartesian or modulus-argument (polar) form.

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

[2 marks]

METHOD 4

$\omega^3 + \omega^2 + \omega + 1 = \frac{\omega^4 - 1}{\omega - 1}$ ***A1***

$= \frac{0}{\omega - 1} = 0$ as $\omega \neq 1$ ***R1***

$\Rightarrow \omega^3 + \omega^2 + \omega + 1 = 0$ ***AG***

[2 marks]

(d) **METHOD 1**

$P_0P_2 = 2$ ***A1***

attempts to find either P_0P_1 or P_0P_3 ***(M1)***

Note: For example, $P_0P_1 = |1-i|$ and $P_0P_3 = |1+i|$.
Various geometric and trigonometric approaches can be used by candidates.

$P_0P_1 = \sqrt{2}$, $P_0P_3 = \sqrt{2}$ ***A1A1***

Note: Award a maximum of ***A1M1A1A0*** if labels such as P_0P_1 are not clearly shown.
Award full marks if the lengths are shown on a clearly labelled diagram.
Award a maximum of ***A1M1A1A0*** for any decimal approximation seen in the calculation of either P_0P_1 or P_0P_3 or both.

$P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ ***AG***

[4 marks]

METHOD 2

attempts to find $P_0P_1 \times P_0P_2 \times P_0P_3 = |1-\omega||1-\omega^2||1-\omega^3|$ ***M1***

$P_0P_1 \times P_0P_2 \times P_0P_3 = |-\omega^6 + \omega^5 + \omega^4 - \omega^2 - \omega + 1|$ ***A1***

$= |-(-1) + \omega^5 + 1 - (-1) - \omega + 1|$ since $\omega^6 = \omega^2 = -1$ and $\omega^4 = 1$ ***A1***

$= |\omega^5 - \omega + 4|$ and since $\omega^5 = \omega$ ***R1***

so $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ ***AG***

[4 marks]

METHOD 3

$P_0P_2 = 2$ ***A1***

attempts to find $P_0P_1 \times P_0P_3 = |1-\omega||1-\omega^3|$ ***M1***

$P_0P_1 \times P_0P_3 = |\omega^4 - \omega^3 - \omega + 1|$ ***A1***

$= |2-(-\omega)-\omega|$ since $\omega^4 = 1$ and $\omega^3 = -\omega$ ***R1***

so $P_0P_1 \times P_0P_2 \times P_0P_3 = 4$ ***AG***

[4 marks]

(e) $\left(P_0P_1 \times P_0P_2 \times ... \times P_0P_{n-1}\right) = n$ ***A1***

[1 mark]

(f) (i) $P_0P_2 = \left|1-\omega^2\right|,\ P_0P_3 = \left|1-\omega^3\right|$ ***A1A1***

[2 marks]

(ii) $P_0P_{n-1} = \left|1-\omega^{n-1}\right|$ ***A1***

Note: Accept $\left|1-\omega\right|$ from symmetry.

[1 mark]

(g) (i) $z^n - 1 = (z-1)\left(z^{n-1} + z^{n-2} + ... + z + 1\right)$

considers the equation $z^{n-1} + z^{n-2} + ... + z + 1 = 0$ ***(M1)***

the roots are $\omega, \omega^2, ..., \omega^{n-1}$ ***(A1)***

so $(z-\omega)\left(z-\omega^2\right)...\left(z-\omega^{n-1}\right)$ ***A1***

[3 marks]

(ii) **METHOD 1**

substitutes $z = 1$ into $(z-\omega)\left(z-\omega^2\right)...\left(z-\omega^{n-1}\right) \equiv z^{n-1} + z^{n-2} + ... + z + 1$

M1

$(1-\omega)\left(1-\omega^2\right)...\left(1-\omega^{n-1}\right) = n$ ***(A1)***

takes modulus of both sides ***M1***

$\left|(1-\omega)\left(1-\omega^2\right)...\left(1-\omega^{n-1}\right)\right| = |n|$

$|1-\omega|\left|1-\omega^2\right|...\left|1-\omega^{n-1}\right| = n$ ***A1***

so $P_0P_1 \times P_0P_2 \times ... \times P_0P_{n-1} = n$ ***AG***

Note: Award a maximum of ***M1A1FTM1A0*** from part (e).

[4 marks]

METHOD 2

$(1-\omega),(1-\omega^2),...,(1-\omega^{n-1})$ are the roots of $(1-v)^{n-1}+(1-v)^{n-2}+...+(1-v)+1=0$ ***M1***

coefficient of v^{n-1} is $(-1)^{n-1}$ and the coefficient of 1 is n ***A1***

product of the roots is $\frac{(-1)^{n-1}n}{(-1)^{n-1}}=n$ ***A1***

$|1-\omega||1-\omega^2|...|1-\omega^{n-1}|=n$ ***A1***

so $P_0P_1\times P_0P_2\times...\times P_0P_{n-1}=n$ ***AG***

[4 marks]

Total[24 marks]

9. (a) attempt to expand binomial with negative fractional power ***(M1)***

$$\frac{1}{\sqrt{1+ax}} = (1+ax)^{-\frac{1}{2}} = 1 - \frac{ax}{2} + \frac{3a^2x^2}{8} + \ldots$$ ***A1***

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 - \frac{x}{2} - \frac{x^2}{8} + \ldots$$ ***A1***

$$\frac{1}{\sqrt{1+ax}} - \sqrt{1-x} = \frac{(1-a)}{2}x + \left(\frac{3a^2+1}{8}\right)x^2 + \ldots$$

attempt to equate coefficients of x or x^2 ***(M1)***

$$x: \quad \frac{1-a}{2} = 4b;\ x^2: \frac{3a^2+1}{8} = b$$

attempt to solve simultaneously ***(M1)***

$$a = -\frac{1}{3}, b = \frac{1}{6}$$ ***A1***

[6 marks]

(b) $|x| < 1$ ***A1***

[1 mark]

Total [7 marks]

Section B

12. (a) (i) $\left(1+\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}-1\right)$

$=\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$ ***A1***

$=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$=\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}$

$=\mathrm{i}$ ***AG***

Note: Candidates who solve the equation correctly can be awarded the above two marks. The working for part (i) may be seen in part (ii).

(ii) $(z-1)^3=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2\pi k\right)}$ ***(M1)***

$z-1=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+\frac{4\pi k}{6}\right)}$ ***(M1)***

$(k=1)\Rightarrow\omega_2=1+\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ ***A1***

$(k=2)\Rightarrow\omega_3=1+\mathrm{e}^{\mathrm{i}\frac{9\pi}{6}}$ ***A1***

[6 marks]

continue…

Question 12 continued.

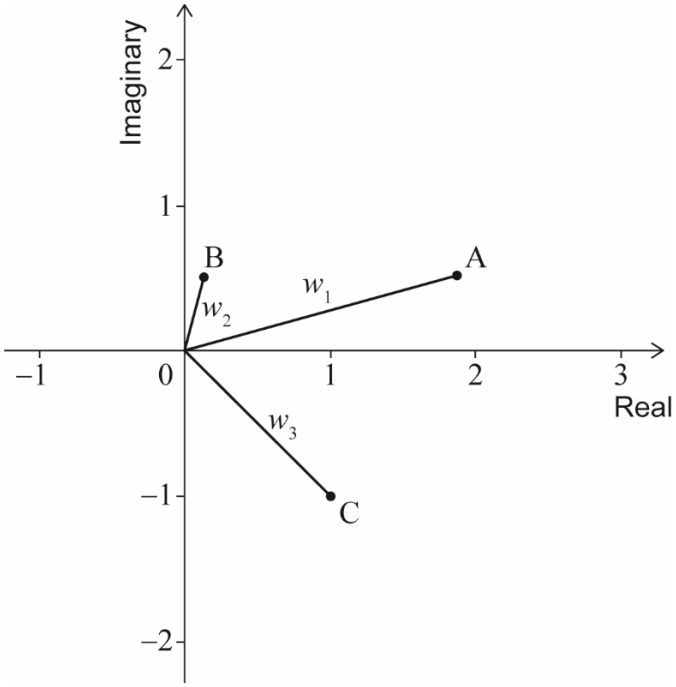
(b) **EITHER**

attempt to express $e^{i\frac{\pi}{6}}$, $e^{i\frac{5\pi}{6}}$, $e^{i\frac{9\pi}{6}}$ in Cartesian form and translate 1 unit in the positive direction of the real axis ***(M1)***

OR

attempt to express w_1, w_2 and w_3 in Cartesian form ***(M1)***

THEN



Note: To award ***A*** marks, it is not necessary to see A,B or C, the w_i , or the solid lines

A1A1A1

[4 marks]

continue…

Question 12 continued.

(c) valid attempt to find $\omega_1 - \omega_3$ (or $\omega_3 - \omega_1$) ***M1***

$$\omega_1 - \omega_3 = \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - (1 - \mathrm{i}) = \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \text{ OR } \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{2}$$

valid attempt to find $\left|\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}\right|$ ***M1***

$$= \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$$\mathrm{AC} = \sqrt{3}$$ ***A1***

[3 marks]

continue…

Question 12 continued.

(d) **METHOD 1**

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(\frac{z-1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$\frac{\alpha-1}{\alpha} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$\alpha - 1 = \alpha\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$

$\alpha - \alpha\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = 1$

$\alpha\left(1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) = 1$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

continue…

Question 12 continued.

METHOD 2

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(1-\frac{1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$1-\frac{1}{z} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

Note: This step to change from z to α may occur at any point in MS.

$1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \frac{1}{\alpha}$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

continue…

Question 12 continued.

METHOD 3

$$\text{LHS}=(z-1)^3=\left(\frac{1}{1-e^{i\frac{\pi}{6}}}-1\right)^3$$

$$=\left(\frac{e^{i\frac{\pi}{6}}}{1-e^{i\frac{\pi}{6}}}\right)^3$$

$$=\frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3}\left(=\frac{i}{\frac{5}{2}-\frac{3\sqrt{3}}{2}+i\left(\frac{3\sqrt{3}}{2}-\frac{5}{2}\right)}\right)$$ ***M1A1***

Note: Award ***M1*** for applying de Moivre's theorem (may be seen in modulus- argument form.)

$$\text{RHS}=\ iz^3=i\left(\frac{1}{1-e^{i\frac{\pi}{6}}}\right)^3$$

$$=\frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3}$$ ***A1***

$$(z-1)^3=iz^3$$ ***AG***

continue…

Question 12 continued.

METHOD 4

$(z-1)^3 = \mathrm{i}z^3$

$z^3 - 3z^2 + 3z - 1 = \mathrm{i}z^3$

$(1-\mathrm{i})z^3 - 3z^2 + 3z - 1 = 0$ ***(M1)***

$$(1-\mathrm{i})\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3 - 3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^2 + 3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right) - 1$$

$$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2 - \left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$$ ***(A1)***

$$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right) - \left(1-3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + 3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} - \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)$$ ***A1***

$= 0$ ***AG***

Note: If the candidate does not interpret their conclusion, award ***(M1)(A1)A0*** as appropriate.

[3 marks]

continue…

Question 12 continued.

(e) **METHOD 1**

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

$$=\frac{2}{2-\sqrt{3}-i}$$ ***A1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{4-2\sqrt{3}+2i}{\left(2-\sqrt{3}\right)^2+1}$$ ***A1***

$$=\frac{4-2\sqrt{3}+2i}{8-4\sqrt{3}}$$ ***A1***

$$=\frac{1}{2}+\frac{1}{4-2\sqrt{3}}i$$

$$\Rightarrow \text{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

continue…

Question 12 continued.

METHOD 2

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{1}{\left(1-\cos\frac{\pi}{6}\right)-i\sin\frac{\pi}{6}}\times\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)^2+\sin^2\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{1-2\cos\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}}$$

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

$$=\frac{1}{2}+\frac{i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

[6 marks]

continue…

Question 12 continued.

METHOD 3

attempt to multiply through by $-\frac{e^{-i\frac{\pi}{12}}}{e^{-i\frac{\pi}{12}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=-\frac{e^{-i\frac{\pi}{12}}}{e^{i\frac{\pi}{12}}-e^{-i\frac{\pi}{12}}}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=-\frac{\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)}{\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}-\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)}$$ ***A1***

$$=-\frac{\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}}{2i\sin\frac{\pi}{12}}$$ ***A1***

$$=\frac{1}{2}-\frac{1}{2i}\cot\frac{\pi}{12}\ \left(=\frac{1}{2}+\frac{1}{2}i\cot\frac{\pi}{12}\right)$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

[6 marks]

continue…

Question 12 continued.

METHOD 4

attempt to multiply through by $\frac{1-e^{-i\frac{\pi}{6}}}{1-e^{-i\frac{\pi}{6}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1-e^{-i\frac{\pi}{6}}}{1-e^{-i\frac{\pi}{6}}-e^{i\frac{\pi}{6}}+1}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=\frac{1-\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

attempt to re-write in Cartesian form ***M1***

$$=\frac{1-\frac{\sqrt{3}}{2}-\frac{1}{2}i}{2-\sqrt{3}}\left(=\frac{2-\sqrt{3}}{2}+i\frac{\frac{1}{2}}{2-\sqrt{3}}\right)$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

Note: Their final imaginary part does not have to be correct in order for the final ***A*** mark to be awarded

[6 marks]

Total [22 marks]

5. (a) $u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ ***(M1)***

$= \frac{14}{24} \left(= \frac{7}{12} = 0.583333\ldots \right)$ ***A1***

[2 marks]

(b) $r = \frac{7}{8} \left(= 0.875 \right)$ ***(A1)***

substituting their values for u_1 and r into $S_\infty = \frac{u_1}{1-r}$ ***(M1)***

$= \frac{14}{3} \left(= 4.66666\ldots \right)$ ***A1***

[3 marks]

(c) attempt to substitute their values into the inequality or formula for S_n ***(M1)***

$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3} \left(\frac{7}{8} \right)^r < 0.001$ OR $S_n = \frac{\frac{7}{12} \left(1 - \left(\frac{7}{8} \right)^n \right)}{\left(1 - \frac{7}{8} \right)}$

attempt to solve their inequality using a table, graph or logarithms
(must be exponential) ***(M1)***

Note: Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$.

correct critical value or at least one correct crossover value ***(A1)***

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036...$ OR $S_\infty - S_{64} = 0.000906...$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683...$ OR $S_\infty - S_{64} - 0.001 = -0.0000931777...$

least value is $n = 64$ ***A1***

[4 marks]

Total [9 marks]

6. (a) **METHOD 1**

$(p+q)^3-3pq(p+q)\equiv p^3+q^3$

attempts to expand $(p+q)^3$ ***M1***

$p^3+3p^2q+3pq^2+q^3$

$(p+q)^3-3pq(p+q)\equiv p^3+3p^2q+3pq^2+q^3-3pq(p+q)$

$\equiv p^3+3p^2q+3pq^2+q^3-3p^2q-3pq^2$ ***A1***

$\equiv p^3+q^3$ ***AG***

Note*:* Condone the use of equals signs throughout.

METHOD 2

$(p+q)^3-3pq(p+q)\equiv p^3+q^3$

attempts to factorise $(p+q)^3-3pq(p+q)$ ***M1***

$\equiv(p+q)\left((p+q)^2-3pq\right)\ \left(\equiv(p+q)\left(p^2-pq+q^2\right)\right)$

$\equiv p^3-p^2q+pq^2+p^2q-pq^2+q^3$ ***A1***

$\equiv p^3+q^3$ ***AG***

Note*:* Condone the use of equals signs throughout.

METHOD 3

$p^3+q^3\equiv(p+q)^3-3pq(p+q)$

attempts to factorise p^3+q^3 ***M1***

$\equiv(p+q)\left(p^2-pq+q^2\right)$

$\equiv(p+q)\left((p+q)^2-3pq\right)$ ***A1***

$\equiv(p+q)^3-3pq(p+q)$ ***AG***

Note*:* Condone the use of the equals sign throughout.

[2 marks]

(b)

Note: Award a maximum of ***A1M0A0A1M0A0*** for $m=-95$ and $n=8$ found by using $\alpha,\beta=\frac{5\pm\sqrt{17}}{4}$ $\left(\alpha,\beta=0.219...,2.28...\right)$.

Condone, as appropriate, solutions that state but clearly do not use the values of α and β.

Special case: Award a maximum of ***A1M1A0A1M0A0*** for $m=-95$ and $n=8$ obtained by solving simultaneously for α and β from product of roots and sum of roots equations.

product of roots of $x^2-\frac{5}{2}x+\frac{1}{2}=0$

$\alpha\beta=\frac{1}{2}$ (seen anywhere) ***A1***

considers $\left(\frac{1}{\alpha^3}\right)\left(\frac{1}{\beta^3}\right)$ by stating $\frac{1}{(\alpha\beta)^3}\left(=n\right)$ ***M1***

Note: Award ***M1*** for attempting to substitute their value of $\alpha\beta$ into $\frac{1}{(\alpha\beta)^3}$.

$\frac{1}{(\alpha\beta)^3}=\frac{1}{\left(\frac{1}{2}\right)^3}$

$n=8$ ***A1***

sum of roots of $x^2-\frac{5}{2}x+\frac{1}{2}=0$

$\alpha+\beta=\frac{5}{2}$ (seen anywhere) ***A1***

considers $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ by stating $\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$ $\left(\left(\frac{\alpha+\beta}{\alpha\beta}\right)^3-\frac{3(\alpha+\beta)}{(\alpha\beta)^2}\right)\left(=-m\right)$ ***M1***

Note: Award ***M1*** for attempting to substitute their values of $\alpha+\beta$ and $\alpha\beta$ into their expression. Award ***M0*** for use of $(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$ only.

$=\frac{\left(\frac{5}{2}\right)^3-\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\frac{1}{8}}$ $\left(=125-30=95\right)$

$m=-95$ ***A1***

$\left(x^2-95x+8=0\right)$

[6 marks]

Total [8 marks]

1. (a) $f'(t) = \frac{e^t - e^{-t}}{2}$ ***A1***

$f''(t) = \frac{e^t + e^{-t}}{2}$ ***A1***

$= f(t)$ ***AG***

[2 marks]

(b) **METHOD 1**

$(f(t))^2 + (g(t))^2$

substituting f and g ***M1***

$= \frac{(e^t + e^{-t})^2 + (e^t - e^{-t})^2}{4}$

$= \frac{(e^t)^2 + 2 + (e^{-t})^2 + (e^t)^2 - 2 + (e^{-t})^2}{4}$ ***(M1)***

$= \frac{(e^t)^2 + (e^{-t})^2}{2} \left(= \frac{e^{2t} + e^{-2t}}{2} \right)$ ***A1***

$= f(2t)$ ***AG***

METHOD 2

$f(2t) = \frac{e^{2t} + e^{-2t}}{2}$

$= \frac{(e^t)^2 + (e^{-t})^2}{2}$ ***M1***

$= \frac{(e^t + e^{-t})^2 + (e^t - e^{-t})^2}{4}$ ***M1A1***

$= (f(t))^2 + (g(t))^2$ ***AG***

[3 marks]

Note: Accept combinations of METHODS 1 & 2 that meet at equivalent expressions.

(c) (i) substituting $e^{iu} = \cos u + i\sin u$ into the expression for f ***(M1)***

obtaining $e^{-iu} = \cos u - i\sin u$ ***(A1)***

$$f(iu) = \frac{\cos u + i\sin u + \cos u - i\sin u}{2}$$

Note: The ***M1*** can be awarded for the use of sine and cosine being odd and even respectively.

$$= \frac{2\cos u}{2}$$

$$= \cos u$$ ***A1***

[3 marks]

(ii) $$g(iu) = \frac{\cos u + i\sin u - \cos u + i\sin u}{2}$$

substituting and attempt to simplify ***(M1)***

$$= \frac{2i\sin u}{2}$$

$$= i\sin u$$ ***A1***

[2 marks]

(d) **METHOD 1**

$$(f(iu))^2 + (g(iu))^2$$

substituting expressions found in part (c) ***(M1)***

$$= \cos^2 u - \sin^2 u \ (= \cos 2u)$$ ***A1***

METHOD 2

$$f(2iu) = \frac{e^{2iu} + e^{-2iu}}{2}$$

$$= \frac{\cos 2u + i\sin 2u + \cos 2u - i\sin 2u}{2}$$ ***M1***

$$= \cos 2u$$ ***A1***

Note: Accept equivalent final answers that have been simplified removing all imaginary parts eg $2\cos^2 u - 1$ etc

[2 marks]

(e) $\left(f(t)\right)^2-\left(g(t)\right)^2=\dfrac{(\mathrm{e}^t+\mathrm{e}^{-t})^2-(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}$ ***M1***

$=\dfrac{(\mathrm{e}^{2t}+\mathrm{e}^{-2t}+2)-(\mathrm{e}^{2t}+\mathrm{e}^{-2t}-2)}{4}$ ***A1***

$=\dfrac{4}{4}=1$ ***A1***

Note: Award ***A1*** for a value of 1 obtained from either LHS or RHS of given expression.

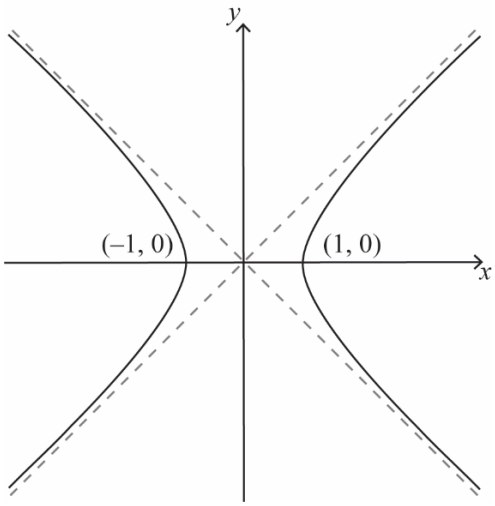
$\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2=\cos^2 u+\sin^2 u$ ***M1***

$=1 \quad \left(\text{hence}\left(f(t)\right)^2-\left(g(t)\right)^2=\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2\right)$ ***AG***

Note: Award full marks for showing that $\left(f(z)\right)^2-\left(g(z)\right)^2=1,\ \forall z\in\mathbb{C}$.

[4 marks]

(f)



A1A1A1A1

Note: Award ***A1*** for correct curves in the upper quadrants, ***A1*** for correct curves in the lower quadrants **,** ***A1*** for correct x-intercepts of $(-1,\ 0)$ and $(1,\ 0)$ (condone $x=-1$ and 1), ***A1*** for $y=x$ and $y=-x$.

[4 marks]

(g) attempt to rotate by 45° in either direction ***(M1)***

Note: Evidence of an attempt to relate to a sketch of $xy = k$ would be sufficient for this ***(M1).***

attempting to rotate a particular point, *eg* $(1, 0)$ ***(M1)***

$(1, 0)$ rotates to $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ (or similar) ***(A1)***

hence $k = \pm\frac{1}{2}$ ***A1A1***

[5 marks]
Total [25 marks]

2. (a) attempting to expand the LHS ***(M1)***

$\text{LHS} = \left(4n^2 - 4n + 1\right) + \left(4n^2 + 4n + 1\right)$ ***A1***

$= 8n^2 + 2 \left(= \text{RHS}\right)$ ***AG***

[2 marks]

(b) ***METHOD 1***

recognition that $2n - 1$ and $2n + 1$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$8n^2 + 2 = 2\left(4n^2 + 1\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

METHOD 2

recognition, *eg* that n and $n + 2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$n^2 + \left(n + 2\right)^2 = 2\left(n^2 + 2n + 2\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

Total [5 marks]

11. (a) attempt to find modulus ***(M1)***

$r = 2\sqrt{3}\left(= \sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3 + \sqrt{3}\mathrm{i} = \sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(= 2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

(b) attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi\mathrm{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi\mathrm{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi\mathrm{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

continued…

Question 11 continued

(c) ***METHOD 1***

attempting to find the total area of (congruent) triangles $\mathrm{UOV}, \mathrm{VOW}$ and UOW ***M1***

$\text{Area} = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$.

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

METHOD 2

$\mathrm{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

$\mathrm{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using $\text{Area} = \frac{1}{2}\times \mathrm{UV}\times \mathrm{VW}\times \sin\alpha$ for example ***M1***

$\text{Area} = \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

(d) $u+v+w=0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{18}\right)+\cos\frac{5\pi}{18}+\mathrm{i}\sin\frac{5\pi}{18}+\cos\frac{17\pi}{18}+\mathrm{i}\sin\frac{17\pi}{18}\right)=0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right)+\cos\frac{5\pi}{18}+\cos\frac{17\pi}{18}\right)=0$

$\cos\left(-\frac{7\pi}{18}\right)=\cos\frac{7\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0$ ***AG***

[4 marks]

Total [18 marks]

2. (a) $\left(1+\frac{5.5}{4\times100}\right)^{\cdot}$ ***(M1)(A1)***

1.056 ***A1***

[3 marks]

continued…

Question 2 continued

(b) ***EITHER***

$2P = P \times \left(1 + \frac{5.5}{100 \times 4}\right)^{4n}$ **OR** $2P = P \times (\text{their } (a))^{m}$ ***(M1)(A1)***

Note: Award ***(M1)*** for substitution into loan payment formula. Award ***(A1)*** for correct substitution.

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I}\% = 5.5$
$\text{P/Y} = 4$
$\text{C/Y} = 4$
$n = 50.756\ldots$ ***(M1)(A1)***

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I}\% = 100\,(\text{their } (a) - 1)$
$\text{P/Y} = 1$
$\text{C/Y} = 1$ ***(M1)(A1)***

THEN

$\Rightarrow 12.7$ years
Laurie will have double the amount she invested during 2032 ***A1***

[3 marks]

Total [6 marks]

8. substituting $w=2\mathrm{i}z$ into $z^{*}-3w=5+5\mathrm{i}$ ***M1***

$z^{*}-6\mathrm{i}z=5+5\mathrm{i}$ ***A1***

let $z=x+y\mathrm{i}$

comparing real and imaginary parts of $(x-y\mathrm{i})-6\mathrm{i}(x+y\mathrm{i})=5+5\mathrm{i}$ ***M1***

to obtain $x+6y=5$ and $-6x-y=5$ ***A1***

attempting to solve for x and y ***M1***

$x=-1$ and $y=1$ and so $z=-1+\mathrm{i}$ ***A1***

hence $w=-2-2\mathrm{i}$ ***A1***

6. EITHER

attempt to obtain the general term of the expansion

$T_{r+1} = {}^nC_r\left(8x^3\right)^{n-r}\left(-\frac{1}{2x}\right)^r$ OR $T_{r+1} = {}^nC_{n-r}\left(8x^3\right)^r\left(-\frac{1}{2x}\right)^{n-r}$ ***(M1)***

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent) ***(M1)***

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - (n-r) = 0$ (or equivalent) ***A1***

r is a multiple of 3 $(r = 3,6,9,...)$ or one correct value of n (seen anywhere) ***(A1)***

$n = 4k, k \in \mathbb{Z}^+$ ***A1***

Note: Accept n is a (positive) multiple of 4 or $n = 4,8,12,...$

Do not accept $n = 4,8,12$

Note: Award full marks for a correct answer using trial and error approach showing $n = 4,8,12,...$ and for recognizing that this pattern continues.

[5 marks]

,

8. Assume that a and b are both odd. ***M1***

Note: Award ***M0*** for statements such as "let a and b be both odd".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m+1$ and $b = 2n+1$ ***A1***

$a^2 + b^2 \equiv (2m+1)^2 + (2n+1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ ***A1***

$= 4\left(m^2 + m + n^2 + n\right) + 2$ ***(A1)***

($4\left(m^2 + m + n^2 + n\right)$ is always divisible by 4) but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2 + b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

Note: Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

9. (a) $z_1 z_2 = (1+b\mathrm{i})\left((1-b^2)-(2b)\mathrm{i}\right)$

$= \left(1-b^2-2\mathrm{i}^2 b^2\right)+\mathrm{i}\left(-2b+b-b^3\right)$ ***M1***

$= \left(1+b^2\right)+\mathrm{i}\left(-b-b^3\right)$ ***A1A1***

Note: Award ***A1*** for $1+b^2$ and ***A1*** for $-b\mathrm{i}-b^3\mathrm{i}$.

[3 marks]

(b) $\arg\left(z_1 z_2\right) = \arctan\left(\dfrac{-b-b^3}{1+b^2}\right) = \dfrac{\pi}{4}$ ***(M1)***

EITHER

$\arctan\left(-b\right) = \dfrac{\pi}{4}$ (since $1+b^2 \neq 0, \text{for } b \in \mathbb{R}$) ***A1***

OR

$-b-b^3 = 1+b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

Total [6 marks]

Section B

10. (a) (i) **EITHER**

attempt to use a ratio from consecutive terms ***M1***

$\frac{p\ln x}{\ln x} = \frac{\frac{1}{3}\ln x}{p\ln x}$ OR $\frac{1}{3}\ln x = (\ln x)r^2$ OR $p\ln x = \ln x\left(\frac{1}{3p}\right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + ...$ and consider the powers of x in geometric sequence.

Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm\frac{1}{\sqrt{3}}$ ***A1***

$p = \pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

continued…

Question 10 continued

(ii) **EITHER**

since, $|p| = \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < 1$ ***R1***

OR

since, $|p| = \frac{1}{\sqrt{3}}$ and $-1 < p < 1$ ***R1***

THEN

$\Rightarrow$ the geometric series converges. ***AG***

Note: Accept r instead of p.
Award ***R0*** if both values of p not considered.

(iii) $\frac{\ln x}{1 - \frac{1}{\sqrt{3}}} \left(= 3 + \sqrt{3}\right)$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x = 3 - \sqrt{3} + \sqrt{3} - 1 \left(\Rightarrow \ln x = 2\right)$ ***A1***

$x = \mathrm{e}^2$ ***A1***

[6 marks]

continued…

Question 10 continued

(b) (i) **METHOD 1**

attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p\ln x - \ln x = \frac{1}{3}\ln x - p\ln x$ OR $\frac{1}{3}\ln x = \ln x + 2\left(p\ln x - \ln x\right)$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + ...$ and consider the powers of x in arithmetic sequence.

Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$.

$2p\ln x = \frac{4}{3}\ln x \quad \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

METHOD 2

attempt to use arithmetic mean $u_2 = \frac{u_1 + u_3}{2}$ ***M1***

$p\ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ ***A1***

$2p\ln x = \frac{4}{3}\ln x \quad \left(\Rightarrow 2p = \frac{4}{3}\right)$ ***A1***

$p = \frac{2}{3}$ ***AG***

continued…

Question 10 continued

METHOD 3

attempt to find difference using u_3 ***M1***

$\frac{1}{3}\ln x = \ln x + 2d \quad \left(\Rightarrow d = -\frac{1}{3}\ln x\right)$

$u_2 = \ln x + \frac{1}{2}\left(\frac{1}{3}\ln x - \ln x\right)$ OR $p\ln x - \ln x = -\frac{1}{3}\ln x$ ***A1***

$p\ln x = \frac{2}{3}\ln x$ ***A1***

$p = \frac{2}{3}$ ***AG***

(ii) $d = -\frac{1}{3}\ln x$ ***A1***

continued…

Question 10 continued

(iii) **METHOD 1**

$S_n = \frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right]$

attempt to substitute into S_n and equate to $\ln\left(\frac{1}{x^3}\right)$ ***(M1)***

$\frac{n}{2}\left[2\ln x + (n-1)\times\left(-\frac{1}{3}\ln x\right)\right] = \ln\left(\frac{1}{x^3}\right)$

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

correct working with S_n (seen anywhere) ***(A1)***

$\frac{n}{2}\left[2\ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x\right]$ OR $n\ln x - \frac{n(n-1)}{6}\ln x$ OR $\frac{n}{2}\left(\ln x + \left(\frac{4-n}{3}\right)\ln x\right)$

correct equation without $\ln x$ ***A1***

$\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$ OR $n - \frac{n(n-1)}{6} = -3$ (or equivalent)

Note: Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to $\frac{n}{2}\left(\frac{7}{3} - \frac{n}{3}\right) = -3$.

attempt to form a quadratic $= 0$ ***(M1)***

$n^2 - 7n - 18 = 0$

attempt to solve their quadratic ***(M1)***

$(n-9)(n+2) = 0$

$n = 9$ ***A1***

continued…

Question 10 continued

METHOD 2

$\ln\left(\frac{1}{x^3}\right) = -\ln x^3 \left(= \ln x^{-3}\right)$ ***(A1)***

$= -3\ln x$ ***(A1)***

listing the first 7 terms of the sequence ***(A1)***

$\ln x + \frac{2}{3}\ln x + \frac{1}{3}\ln x + 0 - \frac{1}{3}\ln x - \frac{2}{3}\ln x - \ln x + ...$

recognizing first 7 terms sum to 0 ***M1***

8th term is $-\frac{4}{3}\ln x$ ***(A1)***

9th term is $-\frac{5}{3}\ln x$ ***(A1)***

sum of 8th and 9th terms $= -3\ln x$ ***(A1)***

$n = 9$ ***A1***

[12 marks]

Total [18 marks]

1. (a) $u_1 = 12$ ***A1***

[1 mark]

(b) $15 - 3n = -33$ ***(A1)***

$n = 16$ ***A1***

[2 marks]

(c) valid approach to find d ***(M1)***

$u_2 - u_1 = 9 - 12$ OR recognize gradient is -3 OR attempts to solve $-33 = 12 + 15d$

$d = -3$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $(n-1)+n+(n+1)$ ***(A1)***

$=3n$ ***A1***

which is always divisible by 3 ***AG***

[2 marks]

(b) $(n-1)^2+n^2+(n+1)^2$ (= $n^2-2n+1+n^2+n^2+2n+1$) ***A1***

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept n^2-1 or n^2+1) ***(M1)***

$=3n^2+2$ ***A1***

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 ***R1***

$3n^2$ is divisible by 3 and so $3n^2+2$ is never divisible by 3

OR the first term is divisible by 3, the second is not

OR $3\left(n^2+\frac{2}{3}\right)$ OR $\frac{3n^2+2}{3}=n^2+\frac{2}{3}$

hence the sum of the squares is never divisible by 3 ***AG***

[4 marks]

Total [6 marks]

5. (a) **EITHER**

recognises the required term (or coefficient) in the expansion ***(M1)***

$bx^5 = {}^7C_2\, x^5 1^2$ OR $b = {}^7C_2$ OR ${}^7C_5$

$b = \frac{7!}{2!5!}\left(= \frac{7!}{2!(7-2)!}\right)$

correct working ***A1***

$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$ OR $\frac{7\times6}{2!}$ OR $\frac{42}{2}$

OR

lists terms from row 7 of Pascal's triangle ***(M1)***

$1, 7, 21, \ldots$ ***A1***

THEN

$b = 21$ ***AG***

[2 marks]

(b) $a = 7$ ***(A1)***

correct equation ***A1***

$21x^5 = \frac{ax^6 + 35x^4}{2}$ OR $21x^5 = \frac{7x^6 + 35x^4}{2}$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve **their** quadratic ***(M1)***

$(x-1)(x-5) = 0$ OR $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$

$x = 1,\ x = 5$ ***A1***

Note: Award final ***A0*** for obtaining $x = 0,\ x = 1,\ x = 5$.

[5 marks]

Total [7 marks]

9. METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.
Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

Note: Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

OR

$1 = 2\left(-\alpha^3 - 3\alpha\right)$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so α cannot be an integer ***R1***

Note: Accept '$\Rightarrow 1$ is even which gives a contradiction'.

continued…

Question 9 continued

OR

$\frac{1}{2} = -\alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ and so α cannot be an integer ***R1***

Note: Accept '$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ which gives a contradiction'.

OR

$\alpha = -\frac{1}{2\left(\alpha^2 + 3\right)}$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2\left(\alpha^2 + 3\right)} \in \mathbb{Z}$ ***R1***

$-\frac{1}{2\left(\alpha^2 + 3\right)}$ is not an integer and so α cannot be an integer ***R1***

Note: Accept $-\frac{1}{2\left(\alpha^2 + 3\right)}$ is not an integer which gives a contradiction'.

THEN

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

continued…

Question 9 continued

METHOD 2

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots…", "let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

let $f(x) = 2x^3 + 6x + 1$ (and $f(\alpha) = 0$)

$f'(x) = 6x^2 + 6 > 0$ for all $x \in \mathbb{R} \Rightarrow f$ is a (strictly) increasing function ***M1A1***

$f(0) = 1$ and $f(-1) = -7$ ***R1***

thus $f(x) = 0$ has only one real root between -1 and 0, which gives a contradiction

(or therefore, contradicting the assumption that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Z}$), ***R1***

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

[5 marks]

Section B

12. (a) $z_2^{*} = r_2 e^{-i\theta}$ ***(A1)***

$z_1 z_2^{*} = r_1 e^{i\alpha} r_2 e^{-i\theta}$ ***A1***

$z_1 z_2^{*} = r_1 r_2 e^{i(\alpha-\theta)}$ ***AG***

Note: Accept working in modulus-argument form

[2 marks]

(b) $\text{Re}\left(z_1 z_2^{*}\right) = r_1 r_2 \cos(\alpha-\theta) \ \ (=0)$ ***A1***

$\alpha - \theta = \arccos 0 \ \ (r_1, r_2 > 0)$

$\alpha - \theta = \frac{\pi}{2}$ (as $0 < \alpha - \theta < \pi$) ***A1***

so $Z_1 O Z_2$ is a right-angled triangle ***AG***

[2 marks]

(c) (i) **EITHER**

$\frac{z_1}{z_2}\left(= \frac{r_1}{r_2} e^{i(\alpha-\theta)}\right) = e^{i\frac{\pi}{3}}$ (since $r_1 = r_2$) ***(M1)***

OR

$z_1 = r_2 e^{i\left(\theta+\frac{\pi}{3}\right)} \left(= r_2 e^{i\theta} e^{i\frac{\pi}{3}}\right)$ ***(M1)***

THEN

$z_1 = z_2 e^{i\frac{\pi}{3}}$ ***A1***

Note: Accept working in either modulus-argument form to obtain $z_1 = z_2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or in Cartesian form to obtain $z_1 = z_2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$.

continued…

Question 12 continued

(ii) substitutes $z_1 = z_2 e^{i\frac{\pi}{3}}$ into $z_1^{\,2} + z_2^{\,2}$ ***M1***

$z_1^{\,2} + z_2^{\,2} = z_2^{\,2} e^{i\frac{2\pi}{3}} + z_2^{\,2} \left(= z_2^{\,2}\left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$ ***A1***

EITHER

$e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$ ***A1***

OR

$z_2^{\,2}\left(e^{i\frac{2\pi}{3}} + 1 \right) = z_2^{\,2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i + 1 \right)$

$= z_2^{\,2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$ ***A1***

THEN

$z_1^{\,2} + z_2^{\,2} = z_2^{\,2} e^{i\frac{\pi}{3}}$

$= z_2\left(z_2 e^{i\frac{\pi}{3}} \right)$ and $z_2 e^{i\frac{\pi}{3}} = z_1$ ***A1***

so $z_1^{\,2} + z_2^{\,2} = z_1 z_2$ ***AG***

Note: For candidates who work on the LHS and RHS separately to show equality, award ***M1A1*** for $z_1^{\,2} + z_2^{\,2} = z_2^{\,2} e^{i\frac{2\pi}{3}} + z_2^{\,2} \left(= z_2^{\,2}\left(e^{i\frac{2\pi}{3}} + 1 \right) \right)$, ***A1*** for $z_1 z_2 = z_2^{\,2} e^{i\frac{\pi}{3}}$ and ***A1*** for $e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$. Accept working in either modulus-argument form or in Cartesian form.

[6 marks]

continued…

Question 12 continued

(d) **METHOD 1**

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ (***A1)***

$a^2 = z_1^{\,2} + z_2^{\,2} + 2z_1 z_2$ ***A1***

$a^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

METHOD 2

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ (***A1)***

$\left(z_1 + z_2\right)^2 = z_1^{\,2} + z_2^{\,2} + 2z_1 z_2$ ***A1***

$\left(z_1 + z_2\right)^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ and $z_1 + z_2 = -a$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

Note: If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.***

so $a^2 - 3b = 0$ ***AG***

[5 marks]

(e) $a^2 - 3 \times 12 = 0$

$a = \pm 6 \ \left(\Rightarrow z^2 \pm 6z + 12 = 0\right)$ ***A1***

for $a = -6$:

$z_1 = 3 + \sqrt{3}\mathrm{i},\ z_2 = 3 - \sqrt{3}\mathrm{i}$ and $\alpha - \theta = -\dfrac{5\pi}{3}$ which does not satisfy $0 < \alpha - \theta < \pi$ ***R1***

for $a = 6$:

$z_1 = -3 - \sqrt{3}\mathrm{i},\ z_2 = -3 + \sqrt{3}\mathrm{i}$ and $\alpha - \theta = \dfrac{\pi}{3}$ ***A1***

so (for $0 < \alpha - \theta < \pi$), only one equilateral triangle can be formed from point O and the two roots of this equation ***AG***

[3 marks]

Total [18 marks]

9. (a) $6 \times 5!$ ***(A1)(A1)***

$= 720$ **(** accept $6!$) ***A1***

[3 marks]

(b) **METHOD 1**

(Peter apart from girls, in an end seat) ${}^{8}P_{4}\left(=1680\right)$ OR

(Peter apart from girls, not in end seat) ${}^{7}P_{4}\left(=840\right)$ ***(A1)***

case 1: Peter at either end

$2 \times {}^{8}P_{4}\left(=3360\right)$ OR $2 \times {}^{8}C_{4} \times 4!\left(=3360\right)$ ***(A1)***

case 2: Peter not at the end

$8 \times {}^{7}P_{4}\left(=6720\right)$ OR $8 \times {}^{7}C_{4} \times 4!\left(=6720\right)$ ***(A1)***

Total number of ways $= 3360 + 6720$

$= 10080$ ***A1***

METHOD 2

(Peter next to girl, in an end seat) $4 \times {}^{8}P_{3}\left(=1344\right)$ OR

(Peter next to one girl, not in end seat) $2 \times 4 \times {}^{7}P_{3}\left(=1680\right)$ OR

(Peter next to two girls, not in end seat) $4 \times 3 \times {}^{7}P_{2}\left(=504\right)$ ***(A1)***

case 1: Peter at either end

$2 \times 4 \times {}^{8}P_{3}\left(=2688\right)$ ***(A1)***

case 2: Peter not at the end

$8\left(2 \times 4 \times {}^{7}P_{3} + 4 \times 3 \times {}^{7}P_{2}\right)\left(=17472\right)$ ***(A1)***

Total number of ways $= {}^{10}P_{5} - \left(2688 + 17472\right)$

$= 10080$ ***A1***

[4 marks]

Total [7 marks]

Section B

9. (a) $9\times9\times8\times7\times6\times5\ \left(=9\times{}^{9}P_{5}\right)$ ***(M1)***

$=136080\ \left(=9\times\dfrac{9!}{4!}\right)$ ***A1***

Note: Award ***M1A0*** for $10\times9\times8\times7\times6\times5\ \left(={}^{10}P_{6}=151200\ =\dfrac{10!}{4!}\right)$.

Note: Award ***M1A0*** for ${}^{9}P_{6}=60480$

[2 marks]

(b) **METHOD 1**

EITHER

every unordered subset of 6 digits from the set of 9 non-zero digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{6}\left(\times1\right)$ ***A1***

THEN

$=84$ ***A1***

METHOD 2

EITHER

removes 3 digits from the set of 9 non-zero digits and these 6 remaining digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{3}\left(\times1\right)$ ***A1***

THEN

$=84$ ***A1***

[2 marks]

Total [4 marks]

Section B

1. (a) (i) $P_3(n) = \frac{(3-2)n^2 - (3-4)n}{2}$ OR $P_3(n) = \frac{n^2 - (-n)}{2}$ ***A1***

$P_3(n) = \frac{n^2 + n}{2}$ ***A1***

Note: Award ***A0A1*** if $P_3(n) = \frac{n^2 + n}{2}$ only is seen.
Do not award any marks for numerical verification.

so for triangular numbers, $P_3(n) = \frac{n(n+1)}{2}$ ***AG***

[2 marks]

(ii) **METHOD 1**

uses a table of values to find a positive integer that satisfies $P_3(n) = 351$ ***(M1)***

for example, a list showing at least 3 consecutive terms $(\ldots\ 325,\ 351,\ 378\ldots)$

Note: Award ***(M1)*** for use of a GDC's numerical solve or graph feature.

$n = 26$ (26th triangular number) ***A1***

Note: Award ***A0*** for $n = -27, 26$. Award ***A0*** if additional solutions besides $n = 26$ are given.

METHOD 2

attempts to solve $\frac{n(n+1)}{2} = 351$ $\left(n^2 + n - 702 = 0\right)$ for n ***(M1)***

$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-702)}}{2}$ OR $(n-26)(n+27) = 0$

$n = 26$ (26th triangular number) ***A1***

Note: Award ***A0*** for $n = -27, 26$. Award ***A0*** if additional solutions besides $n = 26$ are given.

[2 marks]

continued…

Question 1 continued

(b) (i) attempts to form an expression for $P_3(n)+P_3(n+1)$ in terms of n ***M1***

EITHER

$$P_3(n)+P_3(n+1)\equiv\frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}$$

$$\equiv\frac{(n+1)(2n+2)}{2}\ \left(\equiv\frac{2(n+1)(n+1)}{2}\right)$$ ***A1***

OR

$$P_3(n)+P_3(n+1)\equiv\left(\frac{n^2}{2}+\frac{n}{2}\right)+\left(\frac{(n+1)^2}{2}+\frac{n+1}{2}\right)$$

$$\equiv\left(\frac{n^2+n}{2}\right)+\left(\frac{n^2+2n+1+n+1}{2}\right)\ \left(\equiv n^2+2n+1\right)$$ ***A1***

THEN

$$\equiv(n+1)^2$$ ***AG***

[2 marks]

(ii) the sum of the nth and $(n+1)$th triangular numbers is the $(n+1)$th square number ***A1***

[1 mark]

continued…

Question 1 continued

(iii)

X X X X X
O X X X X
O O X X X
O O O X X
O O O O X

A1

Note: Accept equivalent single diagrams, such as the one above, where the 4th and 5th triangular numbers and the 5th square number are clearly shown. Award ***A1*** for a diagram that show $P_3(4)$ (a triangle with 10 dots) and $P_3(5)$ (a triangle with 15 dots) and $P_4(5)$ (a square with 25 dots).

[1 mark]

continued…

Question 1 continued

(c) **METHOD 1**

$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1\ \left(=4n(n+1)+1\right)$ ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$=4n^2+4n+1$

$=(2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

METHOD 2

$8P_3(n)+1=8\left((n+1)^2-P_3(n+1)\right)+1\ \left(=8\left((n+1)^2-\frac{(n+1)(n+2)}{2}\right)+1\right)$ ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$8\left(n^2+2n+1\right)-4\left(n^2+3n+2\right)+1\ \left(=4n^2+4n+1\right)$

$=(2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

continued…

Question 1 continued

METHOD 3

$8P_3(n)+1=8\left(\frac{n(n+1)}{2}\right)+1\ \left(=(An+B)^2\right)$ (where $A,B\in\mathbb{Z}^+$) ***A1***

attempts to expand their expression for $8P_3(n)+1$ ***(M1)***

$4n^2+4n+1\ \left(=A^2n^2+2ABn+B^2\right)$

now equates coefficients and obtains $B=1$ and $A=2$

$=(2n+1)^2$ ***A1***

and $2n+1$ is odd ***AG***

[3 marks]

continued…

Question 1 continued

(d) **EITHER**

$u_1 = 1$ and $d = 3$ ***(A1)***

substitutes their u_1 and their d into $P_5(n) = \frac{n}{2}\left(2u_1 + (n-1)d\right)$ ***M1***

$P_5(n) = \frac{n}{2}\left(2 + 3(n-1)\right)$ $\left(= \frac{n}{2}(2 + 3n - 3)\right)$ ***A1***

OR

$u_1 = 1$ and $u_n = 3n - 2$ ***(A1)***

substitutes their u_1 and their u_n into $P_5(n) = \frac{n}{2}(u_1 + u_n)$ ***M1***

$P_5(n) = \frac{n}{2}(1 + 3n - 2)$ ***A1***

OR

$P_5(n) = \left(3(1) - 2\right) + \left(3(2) - 2\right) + \left(3(3) - 2\right) + ... + 3n - 2$

$P_5(n) = \left(3(1) + 3(2) + 3(3) + ... + 3n\right) - 2n$ $\left(= 3\left(1 + 2 + 3 + ... + n\right) - 2n\right)$ ***(A1)***

substitutes $\frac{n(n+1)}{2}$ into their expression for $P_5(n)$ ***M1***

$P_5(n) = 3\left(\frac{n(n+1)}{2}\right) - 2n$

$P_5(n) = \frac{n}{2}\left(3(n+1) - 4\right)$ ***A1***

OR

attempts to find the arithmetic mean of n terms ***(M1)***

$= \frac{1 + (3n - 2)}{2}$ ***A1***

multiplies the above expression by the number of terms n

$P_5(n) = \frac{n}{2}(1 + 3n - 2)$ ***A1***

THEN

so $P_5(n) = \frac{n(3n-1)}{2}$ ***AG***

[3 marks]

continued…

Question 1 continued

(e) **METHOD 1**

forms a table of $P_3(n)$ values that includes some values for $n > 5$ ***(M1)***

forms a table of $P_5(m)$ values that includes some values for $m > 5$ ***(M1)***

Note: Award ***(M1)*** if at least one $P_3(n)$ value is correct. Award ***(M1)*** if at least one $P_5(m)$ value is correct. Accept as above for $(n^2 + n)$ values and $(3m^2 - m)$ values.

$n = 20$ for triangular numbers ***(A1)***

$m = 12$ for pentagonal numbers ***(A1)***

Note: Award ***(A1)*** if $n = 20$ is seen in or out of a table. Award ***(A1)*** if $m = 12$ is seen in or out of a table. Condone the use of the same parameter for triangular numbers and pentagonal numbers, for example, $n = 20$ for triangular numbers and $n = 12$ for pentagonal numbers.

210 (is a triangular number and a pentagonal number) ***A1***

Note: Award all five marks for 210 seen anywhere with or without working shown.

continued…

Question 1 continued

METHOD 2

EITHER

attempts to express $P_3(n) = P_5(m)$ as a quadratic in n ***(M1)***

$n^2 + n + \left(m - 3m^2\right)(= 0)$ (or equivalent)

attempts to solve their quadratic in n ***(M1)***

$$n = \frac{-1 \pm \sqrt{12m^2 - 4m + 1}}{2} \left(= \frac{-1 \pm \sqrt{1^2 - 4\left(m - 3m^2\right)}}{2} \right)$$

OR

attempts to express $P_3(n) = P_5(m)$ as a quadratic in m ***(M1)***

$3m^2 - m - \left(n^2 + n\right)(= 0)$ (or equivalent)

attempts to solve their quadratic in m ***(M1)***

$$m = \frac{1 \pm \sqrt{12n^2 + 12n + 1}}{6} \left(= \frac{1 \pm \sqrt{(-1)^2 + 12\left(n^2 + n\right)}}{6} \right)$$

THEN

$n = 20$ for triangular numbers ***(A1)***

$m = 12$ for pentagonal numbers ***(A1)***

210 (is a triangular number and a pentagonal number) ***A1***

continued…

Question 1 continued

METHOD 3

$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2}$

let $n = m+k$ $(n > m)$ and so $3m^2 - m = (m+k)(m+k+1)$ ***M1***

$2m^2 - 2(k+1)m - \left(k^2 + k\right) = 0$ ***A1***

attempts to find the discriminant of their quadratic
and recognises that this must be a perfect square ***M1***

$\Delta = 4(k+1)^2 + 8(k^2+k)$

$N^2 = 4(k+1)^2 + 8(k^2+k)$ $\left(= 4(k+1)(3k+1)\right)$

determines that $k = 8$ leading to $2m^2 - 18m - 72 = 0 \Rightarrow m = -3, 12$ and so $m = 12$ ***A1***

210 (is a triangular number and a pentagonal number) ***A1***

METHOD 4

$\frac{n(n+1)}{2} = \frac{m(3m-1)}{2}$

let $m = n-k$ $(m < n)$ and so $n^2 + n = (n-k)\left(3\left(n-k\right)-1\right)$ ***M1***

$2n^2 - 2(3k+1)n + \left(3k^2 + k\right) = 0$ ***A1***

attempts to find the discriminant of their quadratic
and recognises that this must be a perfect square ***M1***

$\Delta = 4(3k+1)^2 - 8(3k^2+k)$

$N^2 = 4(3k+1)^2 - 8(3k^2+k)$ $\left(= 4(k+1)(3k+1)\right)$

determines that $k = 8$ leading to $2n^2 - 50n + 200 = 0 \Rightarrow n = 5, 20$ and so $n = 20$ ***A1***

210 (is a triangular number and a pentagonal number) ***A1***

[5 marks]

continued…

Question 1 continued

(f)

Note: Award a maximum of ***R1M0M0A1M1A1A1R0*** for a 'correct' proof using n and $n+1$.

consider $n=1$: $P_r(1)=1+(1-1)(r-2)=1$ and $P_r(1)=\frac{(r-2)(1^2)-(r-4)(1)}{2}=1$

so true for $n=1$ ***R1***

Note: Accept $P_r(1)=1$ and $P_r(1)=\frac{(r-2)(1^2)-(r-4)(1)}{2}=1$.
Do not accept one-sided considerations such as '$P_r(1)=1$ and so true for $n=1$'.
Subsequent marks after this ***R1*** are independent of this mark can be awarded.

Assume true for $n=k$, *ie*. $P_r(k)=\frac{(r-2)k^2-(r-4)k}{2}$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$". The assumption of truth must be clear.
Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Consider $n=k+1$:

($P_r(k+1)$ can be represented by the sum

$\sum_{m=1}^{k+1}\left(1+\left(m-1\right)\left(r-2\right)\right)=\sum_{m=1}^{k}\left(1+\left(m-1\right)\left(r-2\right)\right)+\left(1+k\left(r-2\right)\right)$ and so

$P_r(k+1)=\frac{(r-2)k^2-(r-4)k}{2}+\left(1+k(r-2)\right)$ $\left(P_r(k+1)=P_r(k)+\left(1+k(r-2)\right)\right)$ ***M1***

$=\frac{(r-2)k^2-(r-4)k+2+2k(r-2)}{2}$ ***A1***

$=\frac{(r-2)(k^2+2k)-(r-4)k+2}{2}$

$=\frac{(r-2)(k^2+2k+1)-(r-2)-(r-4)k+2}{2}$ ***M1***

$=\frac{(r-2)(k+1)^2-(r-4)k-(r-4)}{2}$ ***(A1)***

$=\frac{(r-2)(k+1)^2-(r-4)(k+1)}{2}$ ***A1***

hence true for $n=1$ and $n=k$ true $\Rightarrow n=k+1$ true ***R1***

therefore true for all $n\in\mathbb{Z}^+$

Note: Only award the final ***R1*** if the first five marks have been awarded. Award marks as appropriate for solutions that expand both the LHS and (given) RHS of the equation.

[8 marks]

Total [27 marks]

3. (a)

Note: Award a maximum of ***M1A0A0*** if the candidate manipulates both sides of the equation (such as moving terms from one side to the other).

METHOD 1 (working with LHS)

attempting to expand $\left(a^2-1\right)^2$ (do not accept a^4+1 or a^4-1) ***(M1)***

$\text{LHS} = a^2 + \frac{a^4-2a^2+1}{4}$ or $\frac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$= \frac{a^4+2a^2+1}{4}$ ***A1***

$= \left(\frac{a^2+1}{2}\right)^2 \ (= \text{RHS})$ ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

METHOD 2 (working with RHS)

attempting to expand $\left(a^2+1\right)^2$ ***(M1)***

$\text{RHS} = \frac{a^4+2a^2+1}{4}$

$= \frac{4a^2+a^4-2a^2+1}{4}$ ***A1***

$= a^2 + \frac{a^4-2a^2+1}{4}$ ***A1***

$= a^2 + \left(\frac{a^2-1}{2}\right)^2$ (=LHS) ***AG***

Note: Do not award the final A1 if further working contradicts the AG.

[3 marks]

continued…

Question 3 continued

(b) recognise base and height as a and $\left(\frac{a^2-1}{2}\right)$ (may be seen in diagram) ***(M1)***

correct substitution into triangle area formula ***A1***

$\text{Area} = \frac{a}{2}\left(\frac{a^2-1}{2}\right)$ (or equivalent) $\left(= \frac{a\left(a^2-1\right)}{4} = \frac{a^3-a}{4}\right)$

[2 marks]

Total [5 marks]

5. (a) product of roots $= 80$ ***(A1)***

$3 - \mathrm{i}$ is a root ***(A1)***

attempt to set up an equation involving the product of their four roots and ± 80 ***(M1)***

$(3+\mathrm{i})(3-\mathrm{i})\alpha^3 = 80 \Rightarrow 10\alpha^3 = 80$

$\alpha = 2$ ***A1***

[4 marks]

(b) **METHOD 1**

sum of roots $= -p$ ***(A1)***

$-p = 3+\mathrm{i}+3-\mathrm{i}+2+4$ ***(M1)***

Note: Accept $p = 3+\mathrm{i}+3-\mathrm{i}+2+4$ for ***(M1)***

$p = -12$ ***A1***

METHOD 2

$(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))(z-2)(z-4)$ ***(M1)***

$((z-3)-i)((z-3)+i)(z-2)(z-4)$ ***(A1)***

$(z^2-6z+10)(z^2-6z+8) = z^4-12z^3+....$

$p = -12$ ***A1***

[3 marks]

Total [7 marks]

12. (a) (i) $z_0 = 1 + \mathrm{i}$ ***(A1)***

$\arg(z_0) = \arctan(1) = \frac{\pi}{4} = 45°$ ***A1***

Note: Accept any of these three forms, including an answer marked on an Argand diagram.

(ii) $\arg(z_n) = \arctan\left(\frac{1}{n^2 + n + 1}\right)$ ***A1***

[3 marks]

(b) (i) attempt to use the compound angle formula for tan ***M1***

$$\tan\left(\arctan(a) + \arctan(b)\right) = \frac{\tan\left(\arctan(a)\right) + \tan\left(\arctan(b)\right)}{1 - \tan\left(\arctan(a)\right)\tan\left(\arctan(b)\right)}$$

$= \frac{a + b}{1 - ab}$ ***A1***

$\Rightarrow \arctan(a) + \arctan(b) = \arctan\left(\frac{a + b}{1 - ab}\right)$ ***AG***

(ii) **METHOD 1**

$\arg(w_1) = \arg(z_0 z_1) = \arg(z_0) + \arg(z_1)$ ***M1***

$= \arctan(1) + \arctan\left(\frac{1}{3}\right)$ ***(A1)***

$= \arctan\left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right)$ ***A1***

$= \arctan(2)$ ***AG***

continued…

Question 12 continued

METHOD 2

$w_1 = z_0 z_1 = (1+\mathrm{i})(3+\mathrm{i})$ ***(M1)***

$= 2+4\mathrm{i}$ ***A1***

$\arg(w_1) = \arctan\left(\frac{4}{2}\right)$ or labelled Argand diagram ***A1***

$= \arctan(2)$ ***AG***

[5 marks]

continued…

Question 12 continued

(c) let $n = 0$

$\text{LHS} = \arg(w_0) = \arg(z_0) = \arctan(1)\left(= \frac{\pi}{4}\right)$

$\text{RHS} = \arctan(1)\left(= \frac{\pi}{4}\right)$ so LHS = RHS ***R1***

Note: Award ***R0*** for not starting at $n = 0$, for example by referring to the result in (b) (ii) for $n = 1$. Award subsequent marks.

assume true for $n = k$, (so $\arg(w_k) = \arctan(k+1)$) ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

$\arg(w_{k+1})$

$= \arg(w_k z_{k+1})\left(= \arg(w_k) + \arg(z_{k+1})\right)$ ***(M1)***

$= \arctan(k+1) + \arctan\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$ ***A1***

$= \arctan\left(\frac{(k+1) + \left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)}{1 - (k+1)\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)}\right)$ ***M1***

$= \arctan\left(\frac{(k+1) + \left(\frac{1}{k^2 + 3k + 3}\right)}{1 - (k+1)\left(\frac{1}{k^2 + 3k + 3}\right)}\right)$ ***(A1)***

$= \arctan\left(\frac{(k+1)(k^2 + 3k + 3) + 1}{(k^2 + 3k + 3) - (k+1)}\right)$

continued…

Question 12 continued

$= \arctan\left(\dfrac{k^3 + 4k^2 + 6k + 4}{k^2 + 2k + 2}\right)$ ***A1***

$= \arctan\left(\dfrac{(k+2)\left(k^2 + 2k + 2\right)}{k^2 + 2k + 2}\right)$ ***A1***

$= \arctan(k+2)\left(= \arctan\left((k+1)+1\right)\right)$ ***A1***

since true for $n = 0$, and true for $n = k+1$ if true for $n = k$, the statement is true for all $n \in \mathbb{N}$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** four of the previous marks must have been awarded.

[10 marks]

Total [18 marks]

3. $86.4 = 50r^3$ ***(A1)***

$r = 1.2\left(= \sqrt[3]{\frac{86.4}{50}}\right)$ seen anywhere ***(A1)***

$\frac{50\left(1.2^n - 1\right)}{0.2} > 33500$ OR $250\left(1.2^n - 1\right) = 33500$ ***(A1)***

attempt to solve their geometric S_n inequality or equation ***(M1)***

sketch OR $n > 26.9045, n = 26.9$ OR $S_{26} = 28368.8$ OR $S_{27} = 34092.6$ OR algebraic manipulation involving logarithms

$n = 27$ (accept $n \geq 27$) ***A1***

Total [5 marks]

5.

Note: Do not award any marks if there is clear evidence of adding instead of multiplying, for example ${}^9C_r + (ax)^{9-r} + (1)^r$.

valid approach for expansion (must be the product of a binomial coefficient with $n = 9$ and a power of ax) ***(M1)***

${}^9C_r(ax)^{9-r}(1)^r$ OR ${}^9C_{9-r}(ax)^r(1)^{9-r}$ OR ${}^9C_0(ax)^0(1)^9 + {}^9C_1(ax)^1(1)^8 + \ldots$

recognizing that the term in x^6 is needed ***(M1)***

$\frac{\text{Term in } x^6}{21x^2} = kx^4$ OR $r = 6$ OR $r = 3$ OR $9 - r = 6$

correct term or coefficient in binomial expansion (seen anywhere) ***(A1)***

${}^9C_6(ax)^6(1)^3$ OR ${}^9C_3 a^6x^6$ OR $84\left(a^6x^6\right)(1)$ OR $84a^6$

EITHER

correct term in x^4 or coefficient (may be seen in equation) ***(A1)***

$\frac{{}^9C_6}{21}a^6x^4$ OR $4a^6x^4$ OR $4a^6$

Set their term in x^4 or coefficient of x^4 equal to $\frac{8}{7}a^5x^4$ or $\frac{8}{7}a^5$ (do not accept other powers of *x*) ***(M1)***

$\frac{{}^9C_3}{21}a^6x^4 = \frac{8}{7}a^5x^4$ OR $4a^6 = \frac{8}{7}a^5$

continued…

Question 5 continued

OR

correct term in x^6 or coefficient of x^6 (may be seen in equation) ***(A1)***

$84a^6x^6$ OR $84a^6$

Set their term in x^6 or coefficient of x^6 equal to $24a^5x^6$ or $24a^5$
(do not accept other powers of *x*) ***(M1)***

$84a^6x^6 = 24a^5x^6$ OR $84a = 24$

THEN

$a = \frac{2}{7} \approx 0.286\left(0.285714...\right)$ ***A1***

Note: Award ***A0*** for the final mark for $a = \frac{2}{7}$ and $a = 0$.

Total [6 marks]

.

7. METHOD 1

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

Note: Award ***A1A1*** for $P(3\mathrm{i})=0$ and $P(-3\mathrm{i})=0$ seen in their working.
Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\mathrm{i})=0$ or $P(-3\mathrm{i})=0$ ***(M1)***

$$4m-3m\mathrm{i}+\frac{36}{m}(3\mathrm{i})^2-(3\mathrm{i})^3=0$$

$$4m-3m\mathrm{i}-\frac{36}{m}(-9)+27\mathrm{i}=0$$

attempt to equate the real or imaginary parts ***(M1)***

$27-3m=0$ OR $9\times\frac{36}{m}=4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

Note: Accept sum of three roots set to $-\frac{36}{m}$.
Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$$3\mathrm{i}-3\mathrm{i}+r=\frac{36}{m}\left(\Rightarrow r=\frac{36}{m}\right)$$

substitute their r into product of roots ***(M1)***

$(3\mathrm{i})(-3\mathrm{i})\left(\frac{36}{m}\right)=4m$ OR $\left(z^2+9\right)\left(\frac{36}{m}-z\right)$

$9\times\frac{36}{m}=4m$ OR $\frac{4m}{9}=\frac{36}{m}$

continued…

Question 7 continued

OR

attempt to equate product of three roots to $4m$ ***(M1)***

Note: Accept product of three roots set to $-4m$.
Award ***M0*** for stating product of roots is $\pm 4m$.

$(3\mathrm{i})(-3\mathrm{i}) \times r = 4m \left(\Rightarrow r = \frac{4m}{9} \right)$

substitute their r into sum of roots ***(M1)***

$3\mathrm{i} - 3\mathrm{i} + \frac{4m}{9} = \frac{36}{m}$ OR $\left(z^2 + 9\right)\left(\frac{4m}{9} - z\right)$

$\frac{4m}{9} = \frac{36}{m}$

THEN

$m = 9$ ***(A1)***

third root is 4 ***A1***

[6 marks]

Question 7 continued

METHOD 2

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

recognition that the other factor is $(z+3\mathrm{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(r-z)$ OR $(z-3\mathrm{i})(z+3\mathrm{i})=z^2+9 \Rightarrow P(z)=(z^2+9)\left(\frac{4m}{9}-z\right)$

Note: Accept any attempt at long division of $P(z)$ by z^2+9.
Award ***M0*** for stating other factor is $(z+3\mathrm{i})$ or obtaining z^2+9 with no further working.

attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

$m=9$ ***(A1)***

third root is 4 ***A1***

Note: Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

10. (a) (i) recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

(ii) **METHOD 1**

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

[4 marks]

(b) recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 ***(M1)***

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}(u_1 + 15)$

$u_1 = 5$ ***A1***

[2 marks]

continued…

Question 10 continued

(c) **EITHER**

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}\left(5 + u_n\right)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n-1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

(d) recognition that $v_2 r^2 = v_4$ OR $\left(v_3\right)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3} \quad \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

Total [14 marks]

7. let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\frac{r}{(r+1)!}=1-\frac{1}{(n+1)!}$ for all integers, $n\geq 1$

considering $\mathrm{P}(1)$:

$\mathrm{LHS}=\frac{1}{2}$ and $\mathrm{RHS}=\frac{1}{2}$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *ie*, $\sum_{r=1}^{k}\frac{r}{(r+1)!}=1-\frac{1}{(k+1)!}$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$\sum_{r=1}^{k+1}\frac{r}{(r+1)!}=\sum_{r=1}^{k}\frac{r}{(r+1)!}+\frac{k+1}{((k+1)+1)!}$ ***(M1)***

$=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}$ ***A1***

$=1-\frac{(k+2)-(k+1)}{(k+2)!}$ ***A1***

$=1-\frac{1}{(k+2)!}\left(=1-\frac{1}{((k+1)+1)!}\right)$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true

(for all integers, $n\geq 1$) ***R1***

Note: To obtain the final ***R1***, any four of the previous marks must have been awarded.

[7 marks]

11. (a) **METHOD 1**

$|u| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \left(= \sqrt{1+3}\right)$ ***A1***

$= 2$ ***AG***

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}\left(\sqrt{3}\right)$ OR $\arg u = \pi + \tan^{-1}\left(-\sqrt{3}\right)$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

Note: Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ and so $u = 2e^{i\frac{2\pi}{3}}$ ***AG***

[3 marks]

METHOD 2

reference angle $= \frac{\pi}{3}$ OR $\arg u = \pi - \tan^{-1}\left(\sqrt{3}\right)$ OR $\arg u = \pi + \tan^{-1}\left(-\sqrt{3}\right)$ ***M1***

$= \pi - \frac{\pi}{3}$ ***A1***

Note: Award the above ***M1A1*** for a labelled diagram that convincingly shows that $\arg u = \frac{2\pi}{3}$.

$= \frac{2\pi}{3}$ ***AG***

$r\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

$r = \frac{-1}{\cos\frac{2\pi}{3}} = \frac{-1}{-\frac{1}{2}}$ OR $r = \frac{\sqrt{3}}{\sin\frac{2\pi}{3}} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}}$ ***A1***

$= 2$ and so $u = 2e^{i\frac{2\pi}{3}}$ ***AG***

[3 marks]

continued…

Question 11 continued

(b) (i) $u^n \in \mathbb{R} \Rightarrow \frac{2n\pi}{3} = k\pi \ \ (k \in \mathbb{Z})$ ***(M1)(A1)***

Note: Award ***M1*** for noting that $\sin\frac{2n\pi}{3} = 0$ from $u^n = 2^n\left(\cos\frac{2n\pi}{3} + \mathrm{i}\sin\frac{2n\pi}{3}\right)$.
Award ***(A1)*** for a multiple of 3 considered.

$n = 3$ ***A1***

(ii) substitutes their value (must be a multiple of 3) for n into u^n ***(M1)***

$u^3 = 2^3\cos 2\pi$

$= 8$ ***A1***

[5 marks]

continued…

Question 11 continued

(c) (i) $-1-\sqrt{3}\mathrm{i}$ is a root (by the conjugate root theorem) ***A1***

Note: Accept $2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$.

let $z=c$ be the real root

EITHER

uses sum of roots (equated to ± 5) ***(M1)***

$\left(\left(-1+\sqrt{3}\mathrm{i}\right)+\left(-1-\sqrt{3}\mathrm{i}\right)+c\right)=-5$ ***(A1)***

$-2+c=-5$ ***(A1)***

OR

uses product of roots (equated to ± 12) ***(M1)***

$\left(-1+\sqrt{3}\mathrm{i}\right)\left(-1-\sqrt{3}\mathrm{i}\right)c=-12$ ***(A1)***

$4c=-12$ ***(A1)***

OR

$\left(z-\left(-1+\sqrt{3}\mathrm{i}\right)\right)\left(z-\left(-1-\sqrt{3}\mathrm{i}\right)\right)=z^2+2z+4$ ***(A1)***

compares coefficients *eg* ***(M1)***

$(z-c)\left(z^2+2z+4\right)=z^3+5z^2+10z+12$

$-4c=12$ ***(A1)***

THEN

$c=-3$ (and so $z=-3$ is a root) ***A1***

continued…

Question 11 continued

(ii) **METHOD 1**

compares $z^3+5z^2+10z+12=0$ and $1+5w+10w^2+12w^3=0$

$z=\frac{1}{w}\Rightarrow w=\frac{1}{z}$ ***A2***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\mathrm{i}}\left(=\frac{-1\pm\sqrt{3}\mathrm{i}}{4}\right)$ ***A1A1***

METHOD 2

attempts to factorize into a product of a linear factor and a quadratic factor ***(M1)***

$1+5w+10w^2+12w^3=(3w+1)(4w^2+2w+1)$ ***A1***

$w=-\frac{1}{3},\frac{1}{-1\pm\sqrt{3}\mathrm{i}}\left(=\frac{-1\pm\sqrt{3}\mathrm{i}}{4}\right)$ ***A1A1***

[9 marks]

(d) $(a+b\mathrm{i})^2=2(a-b\mathrm{i})$ ***A1***

attempts to expand and equate real and imaginary parts: ***M1***

$a^2-b^2+2ab\mathrm{i}=2a-2b\mathrm{i}$

$a^2-b^2=2a$ and $2ab=-2b$

attempts to find the value of a or b ***M1***

$2b(a+1)=0$

$b=0\Rightarrow a^2=2a\Rightarrow a=2$ (real root) ***A1***

$a=-1\Rightarrow 1-b^2=-2\Rightarrow b=\pm\sqrt{3}$ (complex roots $-1\pm\sqrt{3}\mathrm{i}$) ***A1***

[5 marks]

Total [22 marks]

5. product of a binomial coefficient, a power of ax^3 and a power of b seen ***(M1)***

evidence of correct term chosen

for $n=8:\ r=2$ (or $r=6$) OR for $n=10:\ r=2$ (or $r=8$) ***(A1)***

correct equations (may include powers of x) ***A1A1***

${}^8C_2a^2b^6=448\ \left(28a^2b^6=448\Rightarrow a^2b^6=16\right)$, ${}^{10}C_2a^2b^8=2880\ \left(45a^2b^8=2880\Rightarrow a^2b^8=64\right)$

attempt to solve their system in a and b algebraically or graphically ***(M1)***

$b=2\ ;\ a=\frac{1}{2}$ ***A1A1***

Note: Award a maximum of ***(M1)(A1)A1A1(M1)A1A0*** for $b=\pm 2$ and/or $a=\pm\frac{1}{2}$.

[7 marks]

6. (a) attempt to use De Moivre's theorem ***(M1)***

$\left(\cos\frac{11\pi}{18}+\mathrm{i}\sin\frac{11\pi}{18}\right)^n=\cos\frac{11\pi n}{18}+\mathrm{i}\sin\frac{11\pi n}{18}\left(=\mathrm{e}^{\frac{11\pi n}{18}\mathrm{i}}\right)$ OR $\cos\left(110^\circ n\right)+\mathrm{i}\sin\left(110^\circ n\right)$

EITHER

attempt to consider imaginary part ***(M1)***

$\sin\frac{11\pi n}{18}=-1$ OR $\sin\left(110^\circ n\right)=-1$

OR

attempt to consider argument of $-\mathrm{i}$ ***(M1)***

$\mathrm{e}^{\frac{11\pi n}{18}\mathrm{i}}=\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$

THEN

$\frac{11\pi n}{18}=\frac{3\pi}{2},\frac{7\pi}{2}\left(,\frac{11\pi}{2}\right)...\left(=\frac{3\pi}{2}+2\pi k,\ k\in\mathbb{Z}\right)$ OR

$110^\circ n=270^\circ,630^\circ\left(,990^\circ\right)...\left(=270^\circ+360^\circ k,\ k\in\mathbb{Z}\right)$ ***(A1)***

$11n=27,63,99,...$

$n=9$ ***A1***

[4 marks]

continued…

Question 6 continued

(b) **EITHER**

$z^{10} = \mathrm{e}^{10\left(\frac{11\pi}{18}\mathrm{i}\right)} \left(= \mathrm{e}^{\frac{55\pi}{9}\mathrm{i}} = \mathrm{e}^{\frac{\pi}{9}\mathrm{i}}\right)$ OR $\arg(z^{10}) = \frac{\pi}{9}$ OR $\arg(z^{10}) = 20^\circ$ ***(A1)***

Note: Accept equivalent arguments given in any interval, in degrees or radians.

recognising that the difference between $\arg(z^{10})$ and $\arg(z)$ is needed ***(M1)***

$\arg(z^{10}) - \arg(z) = \frac{\pi}{9} - \frac{11\pi}{18} = -\frac{\pi}{2}$

OR

recognising that $z^{10} = z^9 \times z$ ***(M1)***

$z^9 = \mathrm{e}^{9\left(\frac{11\pi}{18}\mathrm{i}\right)} \left(= \mathrm{e}^{\frac{11\pi}{2}\mathrm{i}} = \mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}\right)$ OR $\arg(z^9) = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$ OR $\arg(z^9) = 270^\circ$ or -90° ***(A1)***

Note: Accept equivalent arguments given in any interval, in degrees or radians.

THEN

a rotation $\frac{3\pi}{2}$ OR $-\frac{\pi}{2}$ OR equivalent angle about the origin. ***A1***

Note: Accept correct answer given in degrees.

Accept $\frac{\pi}{2}$ clockwise or $\frac{11\pi}{2}$ or $\frac{(4k-1)\pi}{2}$ for $k \in \mathbb{Z}$.

The centre must be stated to gain the final ***A1***.

[3 marks]

Total [7 marks]

9. Assume $p^2 - 8q - 11 = 0$, $(p, q \in \mathbb{Z})$ ***M1***

Note: This ***M1*** is dependent on the assumption of truth (implied by "assume" or "suppose that … is true".)
Subsequent marks should be awarded independently.

EITHER

$p^2 = 8q + 11 (= 2(4q + 5) + 1)$ so p^2 odd $\Rightarrow p$ odd ***R1***

OR

p even $\Rightarrow p^2 - 8q = 11$ even which is a contradiction so p is odd ***R1***

Note: This ***R1*** should be awarded for any valid reason to conclude that p must be odd.

THEN

$p = 2k + 1 (, k \in \mathbb{Z})$ ***M1***

$(2k + 1)^2 = 8q + 11$

$4k^2 + 4k + 1 = 8q + 11$ ***(A1)***

$4k^2 + 4k = 8q + 10$

$2k^2 + 2k = 4q + 5$ or equivalent with one side odd and one side even ***A1***

a contradiction as LHS is even and RHS is odd ***R1***

Note: This ***R1*** is dependent on all previous marks.
Accept correct variations such as work based on $p = 2k - 1$.

therefore, if $p, q \in \mathbb{Z}$ then $p^2 - 8q - 11 \neq 0$ ***AG***

Total [6 marks]

Section B

4. attempt to use the binomial expansion of $(x+h)^8$ ***(M1)***

$^8C_0x^8h^0 + {}^8C_1x^7h^1 + {}^8C_2x^6h^2 + ...$

$a = 8h$ (accept $^8C_1\,h$) ***A1***

$b = 28h^2$ (accept $^8C_2\,h^2$) ***A1***

$d = 70h^4$ (accept $^8C_4\,h^4$) ***A1***

recognition that there is a common ratio between their terms ***(M1)***

$8h \times r = 28h^2$ OR $28h^2 \times r = 70h^4$ OR $8h \times r^2 = 70h^4$

correct equation in terms of h ***(A1)***

$\frac{28h^2}{8h} = \frac{70h^4}{28h^2}$ (or equivalent)

$h = 1.4$ ***A1***

[7 marks]

9. METHOD 1

10 numbers of the form $3n$, 10 numbers of the form $(3n-1)$ and 10 numbers of the form $(3n-2)$ (may be seen anywhere) ***(M1)***

considers one of the following two cases of forming a sum divisible by 3 ***(M1)***

case 1:

chooses 3 numbers of the form $3n$ or chooses 3 numbers of the form $(3n-1)$ or chooses 3 numbers of the form $(3n-2)$

$^{10}C_3 + {}^{10}C_3 + {}^{10}C_3 \ \left(= 3\times {}^{10}C_3 = 3\times 120 = 360\right)$ ways ***A1***

case 2:

chooses 1 number of the form $3n$ and chooses 1 number of the form $(3n-1)$ and chooses 1 number of the form $(3n-2)$

$^{10}C_1 \times {}^{10}C_1 \times {}^{10}C_1 \ \left(= \left({}^{10}C_1\right)^3 = 10^3 = 1000\right)$ ways OR $\frac{{}^{30}C_1 \times {}^{20}C_1 \times {}^{10}C_1}{3!}(=1000)$ ways ***A1***

total number of ways is $3\times {}^{10}C_3 + {}^{10}C_1 \times {}^{10}C_1 \times {}^{10}C_1 \ (=360+1000)$

$=1360$ ***A1***

METHOD 2

total number of ways of choosing 3 numbers (without restriction) is $^{30}C_3 = 4060$ ***A1***

attempts to find the total number of ways of choosing 3 numbers whose sum is not divisible by 3 ***(M1)***

chooses 2 numbers from one group and chooses 1 number from another group

eg chooses 2 numbers of the form $3n$ and chooses 1 number of the form $3n-1$

$3!\times {}^{10}C_2 \times {}^{10}C_1 = 2700$ ***(M1)A1***

Note: Award ***(M1)*** for any integer multiple of $^{10}C_2 \times {}^{10}C_1$.

total number of ways is $4060 - 2700$

$=1360$ ***A1***

[5 marks]

Section B

2. (a) $m = 2$, $c = -1$

$r = \frac{1}{2}$ ***(A1)***

$2, \frac{1}{2}, -1$

EITHER

$d\left(= \frac{1}{2} - 2 = -1 - \frac{1}{2}\right) = -\frac{3}{2}$ ***A1***

OR

this sequence has a common difference of $-\frac{3}{2}$ ***A1***

OR

the (arithmetic) mean of 2 and -1 is $\frac{1}{2}$ ***A1***

THEN

hence $L(x) = 2x - 1$ is an AS-linear function ***AG***

[2 marks]

(b) (i) $\left(L(r) = 0 \Rightarrow\right) mr + c = 0$ ***A1***

$r = -\frac{c}{m}$ ***AG***

Note: Award ***A0*** for numerical verification from $L(x) = 2x - 1$ in part (a).

[1 mark]

continued…

Question 2 continued

(ii) **METHOD 1**

EITHER

attempts to use $(d =) r - m = c - r$ ***(M1)***

Note: Award ***(M1)*** for attempting to use $(d =) m - r = r - c$.

$(d =) -\frac{c}{m} - m = c - \left(-\frac{c}{m}\right)$ ***A1***

Note: Award ***A1*** for $(d =) -\frac{c}{m} - m = \frac{c-m}{2}$.

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $\frac{m+c}{2} = r$ ***(M1)***

$m + c = -\frac{2c}{m}$ ***A1***

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

OR

attempts to use $c = m + 2d$ ***(M1)***

$c = m + 2\left(-\frac{c}{m} - m\right)$ ***A1***

Note: Award ***A1*** for $c = m + 2\left(c - \left(-\frac{c}{m}\right)\right)$.

removes the denominator m from their expression involving m and c ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

continued…

Question 2 continued

OR

attempts to use $r = m + d$ and $c = m + 2d$ $\left(c = m + 2\left(r - m\right)\right)$ ***(M1)***

$m^2 + dm + m + 2d = 0$ (or equivalent) ***A1***

substitutes $d = \frac{c - m}{2}$ into their expression involving m and d ***(M1)***

$m^2 + cm + 2c = 0$ (or equivalent)

THEN

$c\left(m + 2\right) = -m^2 \Rightarrow c = -\frac{m^2}{m + 2}$ ***A1***

Note: Award ***A1*** for a convincing demonstration that $c = -\frac{m^2}{m + 2}$.

so $L\left(x\right) = mx - \frac{m^2}{m + 2}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

METHOD 2

considers $L\left(x\right) = mx - mr$

attempts to use $\left(d =\right) r - m = c - r$ ***(M1)***

Note: Award ***(M1)*** for attempting to use $\left(d =\right) m - r = r - c$.

$\left(d =\right) r - m = -mr - r$ ***A1***

attempts to express r in terms of m ***(M1)***

$2r + mr = m \Rightarrow r = \frac{m}{m + 2}$ ***A1***

so $L\left(x\right) = mx - \frac{m^2}{m + 2}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

[4 marks]

continued…

Question 2 continued

(iii) $m \neq -2 \ (m \neq 0)$ ***A1***

[1 mark]

(c)

attempts to find an integer value of m ***(M1)***

e.g. uses the result that $m+2$ exactly divides 2 OR uses a table OR uses a graph and slider OR uses systematic trial and error

Note: Award ***(M1)*** for solving $m^2 = k(m+2)$ for m or solving $mr - \frac{m^2}{m+2} = 0$ for m or solving $m^2 + cm + 2c = 0$ for m.

$m = -4$ OR $m = -3$ ***(A1)***

$-4, 2, 8$ OR $-3, 3, 9$

$L(x) = -4x + 8, \ L(x) = -3x + 9$ ***A1***

Note: Award ***(M1)(A1)A0*** for $-4x+8$ and $-3x+9$.

[3 marks]

(d) (i) $-\frac{b}{a}$ ***A1***

[1 mark]

(ii) $\frac{c}{a}$ ***A1***

[1 mark]

continued…

Question 2 continued

(e) (i) $b-a$ ***A1***

Note: Award marks as appropriate in parts (e) (ii) and (iii) for use of $r_1 - r_2 = a - b$.

[1 mark]

(ii) attempts to eliminate r_2 ***M1***

$2r_1 = -\frac{b}{a} - (b-a) \Rightarrow 2r_1 = \frac{a^2 - ab - b}{a}$ (or equivalent) ***A1***

Note: Award ***A1*** for a correct alternative form of $\pm r_1$ or $\pm 2r_1$.

so $r_1 = \frac{a^2 - ab - b}{2a}$ ***AG***

Note: Do not accept working backwards from the ***AG***.

[2 marks]

continued…

Question 2 continued

(iii) **METHOD 1**

EITHER

$(r_1 =) \frac{a+b}{2}$ ***(A1)***

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and b ***M1***

$\frac{a+b}{2} = \frac{a^2-ab-b}{2a}$ OR $a+b = \frac{a^2-ab-b}{a}$

OR

$b - r_1 = r_1 - a$ ***(A1)***

attempts to use $b - r_1 = r_1 - a$ with $r_1 = \frac{a^2-ab-b}{2a}$ ***M1***

$b - \left(\frac{a^2-ab-b}{2a}\right) = \frac{a^2-ab-b}{2a} - a$

OR

$(r_1 =) a + d$ ***(A1)***

attempts to use $r_1 = a + d$ with $r_1 = \frac{a^2-ab-b}{2a}$ and $d = \frac{b-a}{2}$ ***M1***

$\frac{a^2-ab-b}{2a} = a + \frac{b-a}{2}$

THEN

$2a^2 + 2ab = 2a^2 - 2ab - 2b$ OR $a^2 + ab = a^2 - ab - b$

$4ab + 2b = 0$ OR $2ab + b = 0$

$2b(2a+1) = 0$ OR $b(2a+1) = 0$ ***A1***

Note: Award ***(A1)M1*** for any valid approach that correctly leads to $2ab + b = 0$ (or equivalent).
Do not accept numerical verification from the ***AG***.

so $b = 0$ or $a = -\frac{1}{2}$ ***AG***

continued…

Question 2 continued

METHOD 2

$(b=)\,a+2d$ OR $(r_1=)\,a+d$ ***(A1)***

attempts to equate two expressions for either r_1 or $2r_1$ in terms of a and d ***M1***

$a+d=\dfrac{a^2-a(a+2d)-(a+2d)}{2a}$ **OR** $2(a+d)=\dfrac{a^2-a(a+2d)-(a+2d)}{a}$

$2a^2+4ad+a+2d=0$

$(2a+1)(a+2d)=0$ ***A1***

Note: Do not accept numerical verification from the ***AG***.

so $b=0$ or $a=-\dfrac{1}{2}$ ***AG***

[3 marks]

continued…

Question 2 continued

(f) **METHOD 1**

$r_1 = \frac{a}{2}$ OR $r_2 = -\frac{a}{2}$ OR $d = -\frac{a}{2}$ ***(A1)***

$c = -a$ ***(A1)***

attempts to find the values of a ***(M1)***

EITHER

the roots of $ax^2 - a = 0$ are ± 1 and $\frac{a}{2} = \pm 1$

OR

substitutes $x = \pm\frac{a}{2}$ into $ax^2 - a = 0$ giving $\frac{a^3}{4} - a = 0$

OR

$\left(r_1 r_2 =\right)\frac{c}{a} = -\frac{a^2}{4} \Rightarrow c = -\frac{a^3}{4}$ and so $-a = -\frac{a^3}{4} \Rightarrow \frac{a^3}{4} - a = 0$

Note: Award ***(M1)*** for attempting to find the values of a from their arithmetic sequence expressed in terms of a.

OR

$c - r_2 = r_2 - b \Rightarrow -\frac{a^3}{4} - \left(-\frac{a}{2}\right) = -\frac{a}{2} \Rightarrow \frac{a^3}{4} - a = 0$

THEN

$a = \pm 2$ ***(A1)***

$\left(r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2\right)$

so $Q(x) = 2x^2 - 2$, $Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $a = 2$ or $a = -2$.

continued…

Question 2 continued

METHOD 2

$r_1 = -d$ OR $r_2 = d$ OR $a = -2d$ ***(A1)***

$c = 2d$ ***(A1)***

attempts to find the values of d ***(M1)***

EITHER

the roots of $-2dx^2 + 2d = 0$ are ± 1

OR

substitutes $x = \pm d$ into $-2dx^2 + 2d = 0$ giving $-2d^3 + 2d = 0$

OR

attempts to use $r_1 r_2 = \frac{c}{a}$ to form $-d^2 = \frac{2d}{-2d}$

Note: Award ***(M1)*** for attempting to find the values of d from their arithmetic sequence expressed in terms of d.

THEN

$d = \pm 1$ ***(A1)***

$(a = \pm 2,\ r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2$, $Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $d = 1$ or $d = -1$.

continued…

Question 2 continued

METHOD 3

$a = 2r_1$ OR $r_2 = -r_1$ OR $d = -r_1$ ***(A1)***

$c = -2r_1$ ***(A1)***

attempts to find the values of r_1 ***(M1)***

EITHER

the roots of $2r_1x^2 - 2r_1 = 0$ are ± 1

OR

substitutes $x = \pm r_1$ into $2r_1x^2 - 2r_1 = 0$ giving $2r_1^3 - 2r_1 = 0$

OR

attempts to use $r_1r_2 = \frac{c}{a}$ to form $-r_1^2 = \frac{-2r_1}{2r_1}$

Note: Award ***(M1)*** for attempting to find the values of r_1 from their arithmetic sequence expressed in terms of r_1.

THEN

$r_1 = \pm 1$ ***(A1)***

$(a = \pm 2,\ r_1 = \pm 1,\ b = 0,\ r_2 = \mp 1,\ c = \mp 2)$

so $Q(x) = 2x^2 - 2$, $Q(x) = -2x^2 + 2$ ***A1***

Note: Award ***A0*** for $2x^2 - 2$ and $-2x^2 + 2$.
Award ***(A1)(A1)(M1)(A0)A0*** for obtaining either $r_1 = 1$ or $r_1 = -1$.

[5 marks]

continued…

Question 2 continued

(g) (i) attempts to express r_1 in terms of b with $a = -\frac{1}{2}$ ***(M1)***

Note: Do not award ***(M1)*** if $a = \frac{1}{2}$ is used.

EITHER

uses $r_1 = \frac{a+b}{2}$

OR

uses $r_1 = \frac{a^2 - ab - b}{2a}$

OR

uses $r_1 - a = b - r_1$

THEN

$r_1 = \frac{2b-1}{4} \left(= \frac{b}{2} - \frac{1}{4}, = \frac{b - \frac{1}{2}}{2} \right)$ ***A1***

[2 marks]

continued…

Question 2 continued

(ii) **METHOD 1**

EITHER

substitutes their expression for r_1 with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$$Q\left(\frac{2b-1}{4}\right)(=0) \Rightarrow -\frac{1}{2}\left(\frac{2b-1}{4}\right)^2 + b\left(\frac{2b-1}{4}\right) + c(=0)$$

OR

$$r_2 = \frac{6b+1}{4}\left(=\frac{3b}{2}+\frac{1}{4}\right)$$

substitutes their expression for r_2 with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$$Q\left(\frac{6b+1}{4}\right)(=0) \Rightarrow -\frac{1}{2}\left(\frac{6b+1}{4}\right)^2 + b\left(\frac{6b+1}{4}\right) + c(=0)$$

THEN

$c = \frac{4b+1}{2}\left(=2b+\frac{1}{2}\right)$ (seen anywhere) ***A1***

$4b^2 + 20b + 5 = 0$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 2

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $Q(x)$ ***(M1)***

$$-\frac{1}{2}\left(x-\left(\frac{2b-1}{4}\right)\right)\left(x-\left(\frac{6b+1}{4}\right)\right)$$

$$-\frac{1}{2}x^2+bx-\frac{3}{8}b^2+\frac{1}{8}b+\frac{1}{32}$$

$c = \frac{4b+1}{2}\left(=2b+\frac{1}{2}\right)$ (seen anywhere) ***A1***

$$2b+\frac{1}{2}=-\frac{3}{8}b^2+\frac{1}{8}b+\frac{1}{32}$$

$$4b^2+20b+5=0$$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5\pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9\pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 3

$r_2 = \frac{6b+1}{4}\left(= \frac{3b}{2} + \frac{1}{4}\right)$

substitutes their expressions for r_1 and r_2 with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ ***(M1)***

$\left(\frac{2b-1}{4}\right)\left(\frac{6b+1}{4}\right) = \frac{c}{-\frac{1}{2}}$ (or equivalent)

EITHER

$c = \frac{4b+1}{2}\left(= 2b + \frac{1}{2}\right)$ (seen anywhere) ***A1***

OR

$-\frac{1}{2}\left(\frac{2b-1}{4}\right)\left(\frac{6b+1}{4}\right) - \left(\frac{6b+1}{4}\right) = \frac{2b-1}{4} - \left(-\frac{1}{2}\right)$ ***A1***

THEN

$4b^2 + 20b + 5 = 0$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 4

attempts to equate two expressions for r_1 with $a = -\frac{1}{2}$ ***(M1)***

$\frac{-b \pm \sqrt{b^2 + 2c}}{-1} = \frac{2b-1}{4} \left(\pm\sqrt{b^2 + 2c} = \frac{2b+1}{4} \right)$

$c = \frac{4b+1}{2} \left(= 2b + \frac{1}{2} \right)$ (seen anywhere) ***A1***

$12b^2 - 4b - 1 + 32\left(2b + \frac{1}{2}\right) = 0 \ \left(4b^2 + 20b + 5 = 0\right)$

attempts to solve their quadratic in b ***(M1)***

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 5

EITHER

$r_1 = d - \frac{1}{2}$

substitutes their expression for r_1 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$Q\left(d - \frac{1}{2}\right)(=0) \Rightarrow -\frac{1}{2}\left(d - \frac{1}{2}\right)^2 + b\left(d - \frac{1}{2}\right) + c(=0)$

OR

$r_2 = 3d - \frac{1}{2}$

substitutes their expression for r_2 in terms of d with $a = -\frac{1}{2}$ into $Q(x)(=0)$ ***(M1)***

$Q\left(3d - \frac{1}{2}\right)(=0) \Rightarrow -\frac{1}{2}\left(3d - \frac{1}{2}\right)^2 + b\left(3d - \frac{1}{2}\right) + c(=0)$

THEN

$b = 2d - \frac{1}{2}$ and $c = 4d - \frac{1}{2}$ (seen anywhere) ***A1***

$4d^2 + 8d - 1 = 0$

attempts to solve their quadratic in d ***(M1)***

$d = \frac{-2 \pm \sqrt{5}}{2}$

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

continued…

Question 2 continued

METHOD 6

$r_1 = d - \frac{1}{2}$ and $r_2 = 3d - \frac{1}{2}$

substitutes their expressions for r_1 and r_2 in terms of d with $a = -\frac{1}{2}$ into $r_1 r_2 = \frac{c}{a}$ ***(M1)***

$\left(d - \frac{1}{2}\right)\left(3d - \frac{1}{2}\right) = \frac{c}{-\frac{1}{2}}$ (or equivalent)

$c = 4d - \frac{1}{2}$ (seen anywhere) ***A1***

$4d^2 + 8d - 1 = 0$

attempts to solve their quadratic in d ***(M1)***

$d = \frac{-2 \pm \sqrt{5}}{2}$

$b = \frac{-5 \pm 2\sqrt{5}}{2}$ ***A1***

substitutes into $c = \frac{4b+1}{2}$

$c = \frac{-9 \pm 4\sqrt{5}}{2}$ ***A1***

Note: Award ***A0A0*** for b and c expressed as decimal values.

Note: Award a maximum of ***(M1)A1(M1)A0A0FT*** for ***FT*** from part (g) (i).

[5 marks]

Total [31 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3$, $q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1$, $d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =) 65 - 40$ OR $(u_5 =) 1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1kx + {}^nC_2k^2x^2 + \ldots$ OR ${}^nC_1k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve (***M1)***

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

Note: The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=\left(5^2\right)5^{2k}-\left(2^3\right)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8\left(5^{2k}-2^{3k}\right)$ OR $25\left(5^{2k}-2^{3k}\right)+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17\left(5^{2k}+8s\right)$ OR $17\left(25s+2^{3k}\right)$ which is divisible by 17 ***A1***

OR

$\left(5^{2k}-2^{3k}\right)\times5^2=5^{2k+2}-25\times2^{3k}=17s\times25$ ***M1***

$=5^{2k+2}-8\times2^{3k}-17\times2^{3k}=17s\times25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times2^{3k}=17s\times25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times2^{3k}=17s\times25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times25+17\times2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17\left(25s+2^{3k}\right)$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if 4 of the previous 6 marks have been awarded

Note: 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

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7. METHOD 1

attempt to substitute solution into given equation ***(M1)***

$(5+q\mathrm{i})^2+\mathrm{i}(5+q\mathrm{i})=-p+25\mathrm{i}$

$25-q^2+10q\mathrm{i}-q+5\mathrm{i}+p-25\mathrm{i}=0$ OR $25-q^2+10q\mathrm{i}-q+5\mathrm{i}=-p+25\mathrm{i}$ ***A1***

$25-q^2+p-q+(10q-20)\mathrm{i}=0$

attempt to equate real or imaginary parts: ***(M1)***

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2$, $p=-19$ ***A1A1***

METHOD 2

$z^2+\mathrm{i}z+p-25\mathrm{i}=0$

sum of roots $=-\mathrm{i}$, product of roots $=p-25\mathrm{i}$ ***M1***

one root is $(5+q\mathrm{i})$ so other root is $(-5-q\mathrm{i}-\mathrm{i})$ ***A1***

product $(5+q\mathrm{i})(-5-q\mathrm{i}-\mathrm{i})=-25-5q\mathrm{i}-5\mathrm{i}-5q\mathrm{i}+q^2+q=p-25\mathrm{i}$

equating real and imaginary parts for product of roots ***(M1)***

Im: $-25=-10q-5$ Re: $p=-25+q^2+q$

$q=2$, $p=-19$ ***A1A1***

[5 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3,\ q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1,\ d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =) 65 - 40$ OR $(u_5 =) 1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1kx + {}^nC_2k^2x^2 + \ldots$ OR ${}^nC_1k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve (***M1)***

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

Note: The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=\left(5^2\right)5^{2k}-\left(2^3\right)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8\left(5^{2k}-2^{3k}\right)$ OR $25\left(5^{2k}-2^{3k}\right)+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17\left(5^{2k}+8s\right)$ OR $17\left(25s+2^{3k}\right)$ which is divisible by 17 ***A1***

OR

$\left(5^{2k}-2^{3k}\right)\times5^2=5^{2k+2}-25\times2^{3k}=17s\times25$ ***M1***

$=5^{2k+2}-8\times2^{3k}-17\times2^{3k}=17s\times25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times2^{3k}=17s\times25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times2^{3k}=17s\times25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times25+17\times2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17\left(25s+2^{3k}\right)$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if 4 of the previous 6 marks have been awarded

Note: 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

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7. METHOD 1

attempt to substitute solution into given equation ***(M1)***

$(5+q\mathrm{i})^2+\mathrm{i}(5+q\mathrm{i})=-p+25\mathrm{i}$

$25-q^2+10q\mathrm{i}-q+5\mathrm{i}+p-25\mathrm{i}=0$ OR $25-q^2+10q\mathrm{i}-q+5\mathrm{i}=-p+25\mathrm{i}$ ***A1***

$25-q^2+p-q+(10q-20)\mathrm{i}=0$

attempt to equate real or imaginary parts: ***(M1)***

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2$, $p=-19$ ***A1A1***

METHOD 2

$z^2+\mathrm{i}z+p-25\mathrm{i}=0$

sum of roots $=-\mathrm{i}$, product of roots $=p-25\mathrm{i}$ ***M1***

one root is $(5+q\mathrm{i})$ so other root is $(-5-q\mathrm{i}-\mathrm{i})$ ***A1***

product $(5+q\mathrm{i})(-5-q\mathrm{i}-\mathrm{i})=-25-5q\mathrm{i}-5\mathrm{i}-5q\mathrm{i}+q^2+q=p-25\mathrm{i}$

equating real and imaginary parts for product of roots ***(M1)***

Im: $-25=-10q-5$ Re: $p=-25+q^2+q$

$q=2$, $p=-19$ ***A1A1***

[5 marks]

7. (a) (i) $(9! =)\ 362880$ ***A1***

Note: Accept 9! or 363000.

(ii) attempt to consider girls as a single object ***(M1)***

$(3! \times 7! =)\ 30240$ ***A1***

Note: Accept 30200.

[3 marks]

(b) **METHOD 1**

recognition of the two different cases for 2 girls and 3 girls ***(M1)***

exactly 2 girls is ${}^6C_3 \times {}^3C_2 = 60$ and exactly 3 girls $({}^3C_3 \times) {}^6C_2 = 15$ ***(A1)***

total $(= 60 + 15) = 75$ ***A1***

METHOD 2

recognition of the three different cases: total choices, 1 girl and no girls ***(M1)***

total choices ${}^9C_5 = 126$, one girl case ${}^3C_1 \times {}^6C_4 = 45$, no girl case ${}^6C_5 = 6$ ***(A1)***

total $(= 126 - 45 - 6) = 75$ ***A1***

[3 marks]

Total [6 marks]

(1)

7. (a) (i) $(9! =)\ 362880$ ***A1***

Note: Accept 9! or 363000.

(ii) attempt to consider girls as a single object ***(M1)***

$(3! \times 7! =)\ 30240$ ***A1***

Note: Accept 30200.

[3 marks]

(b) **METHOD 1**

recognition of the two different cases for 2 girls and 3 girls ***(M1)***

exactly 2 girls is ${}^6C_3 \times {}^3C_2 = 60$ and exactly 3 girls $({}^3C_3 \times) {}^6C_2 = 15$ ***(A1)***

total $(= 60 + 15) = 75$ ***A1***

METHOD 2

recognition of the three different cases: total choices, 1 girl and no girls ***(M1)***

total choices ${}^9C_5 = 126$, one girl case ${}^3C_1 \times {}^6C_4 = 45$, no girl case ${}^6C_5 = 6$ ***(A1)***

total $(= 126 - 45 - 6) = 75$ ***A1***

[3 marks]

Total [6 marks]

(1)

5. (a) $S_n = \frac{10^n - 1}{9}$ ***A1***

$(a = 10, b = 9)$

[1 mark]

(b) **METHOD 1**

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \frac{10-1}{9} + \frac{10^2-1}{9} + \ldots + \frac{10^n - 1}{9}$ ***(A1)***

$= \frac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\frac{9\left(10-1+10^2-1+10^3-1+\ldots+10^n-1\right)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10+10^2+10^3+\ldots+10^n = \frac{10\left(10^n-1\right)}{10-1}$ and $-1-1-1\ldots = -n$ ***A1***

$= \frac{\frac{10\left(10^n-1\right)}{10-1} - n}{9}$ OR $\frac{9\left(\frac{10\left(10^n-1\right)}{10-1}\right) - 9n}{81}$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$= \frac{10\left(10^n-1\right)-9n}{81}$ ***AG***

continued…

Question 5 continued

METHOD 2

attempt to create sum using sigma notation with S_n ***M1***

$\sum_{i=1}^{n} \frac{10^i - 1}{9} \quad \left(= \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1 \right) \right)$

$\sum_{i=1}^{n} 10^i = \frac{10\left(10^n - 1\right)}{9}$ ***A1***

$\sum_{i=1}^{n} 1 = n$ ***A1***

$= \frac{1}{9}\left(\frac{10\left(10^n - 1\right)}{9} - n \right)$ OR $\frac{1}{9}\left(\frac{10\left(10^n - 1\right) - 9n}{9} \right)$ ***A1***

$= \frac{10\left(10^n - 1\right) - 9n}{81}$ ***AG***

continued…

Question 5 continued

METHOD 3

let $\mathrm{P}(n)$ be the proposition that $S_1 + S_2 + S_3 + ... + S_n = \frac{10\left(10^n - 1\right) - 9n}{81}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = S_1 = \frac{10^1 - 1}{9} = 1$ and $\mathrm{RHS} = \frac{10\left(10^1 - 1\right) - 9(1)}{81} = 1$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *i.e.* $S_1 + S_2 + S_3 + ... + S_k = \frac{10\left(10^k - 1\right) - 9k}{81}$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$S_1 + S_2 + S_3 + ... + S_{k+1} = \frac{10\left(10^k - 1\right) - 9k}{81} + \frac{10^{k+1} - 1}{9}$

$= \frac{10^{k+1} - 10 - 9k + 9\left(10^{k+1}\right) - 9}{81}$ ***A1***

$= \frac{10\left(10^{k+1} - 1\right) - 9\left(k + 1\right)}{81}$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true ***R1***

(for all integers $n \geq 1$)

Note: To obtain the final ***R1***, the first ***R1*** and ***A1*** must have been awarded.

[4 marks]

Total [5 marks]

7. Base case $n=1$: LHS $= {}^{1}C_{1}=1$ and RHS $= {}^{2}C_{2}=1$, so true for $n=1$ ***R1***

Note: Award ***R0*** if the value of ${}^{1}C_{1}$ and ${}^{2}C_{2}$ are not evaluated.
Subsequent marks can still be awarded.

assume true for $n=k$ ie $\sum_{r=1}^{k} {}^{r}C_{1} = {}^{k+1}C_{2}$ for some $k \in \mathbb{Z}^{+}$ ***M1***

Note: The assumption of truth must be clear.
Award ***M0*** for statements such as "let $n=k$" or "$n=k$ is true".
Subsequent marks can still be awarded.

consider $n=k+1$

LHS $= \sum_{r=1}^{k+1} {}^{r}C_{1}$

$= \sum_{r=1}^{k} {}^{r}C_{1} + {}^{k+1}C_{1}$ ***(M1)***

$= {}^{k+1}C_{2} + {}^{k+1}C_{1}$ OR $\frac{(k+1)!}{2(k-1)!} + \frac{(k+1)!}{k!}$ ***A1***

EITHER

attempt to cancel factorials and use a common denominator ***M1***

$= \frac{(k+1)k+2(k+1)}{2} \left(= \frac{(k+2)(k+1)}{2} \right)$

OR

attempt to use a common denominator ***M1***

$= \frac{k(k+1)!}{2k!} + \frac{2(k+1)!}{2k!} \left(= \frac{(k+2)(k+1)!}{2k!} \right)$

THEN

$= \frac{(k+2)!}{2!k!} \quad \left(= \frac{(k+2)!}{2!(k+2-2)!} \right)$ ***A1***

$= {}^{k+2}C_{2}$

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n \in \mathbb{Z}^{+}$ ***R1***

Note: Only award the final ***R1*** if 4 of the previous 6 marks have been awarded.

Total *[7 marks]*

Note: Throughout this question, condone presence of any additional terms once the first two correct terms are seen.

$\left(-2\right)^3$

12. (a) attempt to expand the brackets or attempt to find modulus and argument of ϕ ***(M1)***

$(a+b\text{i})^3 = a^3 + 3a^2b\text{i} + 3a(b\text{i})^2 + (b\text{i})^3$ OR $\left(\sqrt{a^2+b^2}\right)^3 \text{cis}\left(3\arctan\left(\frac{b}{a}\right)\right)$

(i) real part is $a^3 - 3ab^2$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

(ii) imaginary part is $3a^2b - b^3$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \sin\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

Note: Award ***(M1)A1A0*** for $\left(a^3 - 3ab^2\right) + \left(3a^2b - b^3\right)\text{i}$ OR

$\left(a^2+b^2\right)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right) + \left(a^2+b^2\right)^{\frac{3}{2}} \text{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

For (ii) condone $\left(3a^2b - b^3\right)\text{i}$ OR $\left(a^2+b^2\right)^{\frac{3}{2}} \text{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

[3 marks]

(b) attempt to substitute $a=1$ and $b=\sqrt{3}$ into their real or imaginary part found in (a) OR to expand the brackets OR to use polar form ***M1***

$(1-9) + \left(3\sqrt{3} - 3\sqrt{3}\right)\text{i}$ OR $\left(2\sqrt{3}\text{i} - 2\right)\left(1+\sqrt{3}\text{i}\right) = 2\sqrt{3}\text{i} - 2 - 6 - 2\sqrt{3}\text{i}$ OR

$\left(2\text{e}^{\frac{\text{i}\pi}{3}}\right)^3 = 8\text{e}^{\text{i}\pi}$ OR $\left(2\text{cis}\left(60^\circ\right)\right)^3 = 8\text{cis}\left(180^\circ\right)$ ***A1***

$= -8$ ***AG***

[2 marks]

(c) $v = -2,\ w = 1 - \sqrt{3}\text{i}$ ***A1A1***

Note: Award ***A1A0*** for $v = 1 - \sqrt{3}\text{i},\ w = -2$ or if the labels v and w are not clearly specified or missing. Candidates may be awarded full ***FT*** marks for subsequent parts.

[2 marks]

continued…

Question 12 continued

(d) **METHOD 1**

triangle UVW has height $h = 3$ and base $b = 2\sqrt{3}$ ***(A1)***

attempt to find area of triangle with their height and base ***(M1)***

area = $\frac{1}{2} \times 2\sqrt{3} \times 3$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 2

triangle UVW has sides of length $\left(\sqrt{3^2 + \left(\sqrt{3}\right)^2} = \right)\sqrt{12}$ ***(A1)***

attempt to find area of equilateral triangle with their side length ***(M1)***

area = $\frac{1}{2}\left(\sqrt{12}\right)^2 \sin\frac{\pi}{3}$ OR $\frac{1}{2}\sqrt{12}(3)$ OR $\left(\sqrt{12}\right)^2 \times \frac{\sqrt{3}}{4}$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 3

triangle UVO has sides of length $\left(\sqrt{1^2 + \left(\sqrt{3}\right)^2} = \right)2$ ***(A1)***

attempt to find area of three isosceles triangles with their side length and angle $\frac{2\pi}{3}$ ***(M1)***

area $= 3\left(\frac{1}{2}(2)^2 \sin\frac{2\pi}{3}\right)$

$= 3\sqrt{3}$ (square units) ***A1***

[3 marks]

continued…

Question 12 continued

(e) attempt to express u, v or w in the form $re^{i\theta}$ and multiply by $e^{\frac{\pi}{4}i}$ ***(M1)***

$\left(u' = 2e^{\frac{\pi}{3}i}e^{\frac{\pi}{4}i} = \right)2e^{\frac{7\pi}{12}i}$ ***A1***

$\left(v' = 2e^{\pi i}e^{\frac{\pi}{4}i} = 2e^{\frac{5\pi}{4}i} = \right)2e^{-\frac{3\pi}{4}i}$ ***A1***

$\left(w' = 2e^{-\frac{\pi}{3}i}e^{\frac{\pi}{4}i} = \right)2e^{-\frac{\pi}{12}i}$ ***A1***

Note: These ***A1*** marks should be awarded independently and in any order.

[4 marks]

(f) **EITHER**

attempt to find one of $(u')^3, (v')^3$ or $(w')^3$ ***(M1)***

$(u')^3 = \left(2e^{\frac{7\pi}{12}i}\right)^3 = 8e^{\frac{7\pi}{4}i} = 8e^{-\frac{\pi}{4}i}$ OR $(v')^3 = \left(2e^{-\frac{3\pi}{4}i}\right)^3 = 8e^{-\frac{\pi}{4}i}$ OR

$(w')^3 = \left(2e^{-\frac{\pi}{12}i}\right)^3 = 8e^{-\frac{\pi}{4}i}$ ***(A1)***

OR

attempt to find product of their three roots u', v' and w' ***(M1)***

$u' \times v' \times w' = (c + di)$

$2e^{\frac{7\pi}{12}i} \times 2e^{-\frac{3\pi}{4}i} \times 2e^{-\frac{\pi}{12}i} = 8e^{-\frac{\pi}{4}i}$ OR $2e^{\frac{7\pi}{12}i} \times 2e^{\frac{5\pi}{4}i} \times 2e^{-\frac{\pi}{12}i} = 8e^{-\frac{\pi}{4}i}$ (or equivalent) ***(A1)***

OR

attempt to find $\left(ze^{\frac{\pi}{4}i}\right)^3$ for any z such that $z^3 = -8$ OR to rotate -8 by $\frac{3\pi}{4}$ ***(M1)***

$\left(\left(ze^{\frac{\pi}{4}i}\right)^3 = z^3e^{\frac{3\pi}{4}i} = \right) -8e^{\frac{3\pi}{4}i}$ OR $8e^{-\frac{\pi}{4}i}$ OR $8\text{cis}(-45^\circ)$ ***(A1)***

continued…

Question 12 continued

THEN

$8e^{-\frac{\pi}{4}i} = 8\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{8}{\sqrt{2}}(1-i)$

$= 4\sqrt{2} - 4\sqrt{2}i$ ***A1***

Note: Accept $c = \frac{8}{\sqrt{2}}, d = -\frac{8}{\sqrt{2}}$.

[3 marks]

(g) **METHOD 1**

attempt to write the arguments of u, v, w, u', v' and w' over a common denominator OR to write the arguments in degrees ***(M1)***

$\frac{-9\pi}{12}, \frac{-4\pi}{12}, \frac{-\pi}{12}, \frac{4\pi}{12}, \frac{7\pi}{12}, \frac{12\pi}{12}$ OR $-135^\circ, -60^\circ, -15^\circ, 60^\circ, 105^\circ, 180^\circ$

THEN

arguments of u, v, w, u', v' and w' differ by $\frac{3\pi}{12}$ and $\frac{5\pi}{12}$ OR 45° and 75°

so arguments of polygon vertices differ by $\frac{\pi}{12}$ or 15° ***(A1)***

$n = 24$ ***A1***

METHOD 2

Let $z = r\,\text{cis}\,\theta \Rightarrow z^n = r^n \text{cis}(n\theta) = r^n \text{cis}(n\theta)$, where θ is the argument of u, v, w, u', v' and w'.

recognition to find $n\theta$ where $n = 6, 12, 18, \ldots$ and $\theta = -\frac{3\pi}{4}, -\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \pi$ ***(M1)***

when $n = 6 \Rightarrow (n\theta =) -\frac{9\pi}{2}, -2\pi, -\frac{\pi}{2}, 2\pi, \frac{7\pi}{2}, 6\pi$

when $n = 12 \Rightarrow (n\theta =) -9\pi, -4\pi, -\pi, 4\pi, 7\pi, 12\pi$ (which is not a multiple of 2π) ***(A1)***

$n = 24$ ***A1***

[3 marks]

Total [20 marks]

7. (a) $\alpha+\beta+\gamma=\frac{7}{2}$ ***A1***

[1 mark]

(b) $p-3\mathrm{i}$ is also a root (seen anywhere) ***A1***

recognition of 5 roots and attempt to sum these roots ***(M1)***

$p+3\mathrm{i}+p-3\mathrm{i}+\frac{7}{2}$

$p+3\mathrm{i}+p-3\mathrm{i}+\frac{7}{2}=\frac{11}{2}$ ***A1***

$p=1$ ***AG***

[3 marks]

(c) (i) attempt to find product of 5 roots and equate to ± 10 ***(M1)***

$(1+3\mathrm{i})(1-3\mathrm{i})\frac{1}{2}\alpha\beta=10$

$\alpha\beta=2$ ***A1***

(ii) $\alpha=1$ and $\beta=2$ ***A1***

[3 marks]

Total [7 marks]

9. (a) $^nC_3$ ***A1***

[1 mark]

(b) **EITHER**

finding the number of ways to assign the students with the two students apart

number of ways to assign two students $^2C_1$ (seen anywhere) ***(A1)***

number of ways to assign others $^{n-2}C_2$ to have one group of 3 (seen anywhere) ***(A1)***

number of ways = $^2C_1 \times {}^{n-2}C_2$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}{}^nC_3 = {}^2C_1 \times {}^{n-2}C_2$ OR $^nC_3 = 2 \times {}^2C_1 \times {}^{n-2}C_2$

valid attempt to eliminate all factorials from their equation ***(M1)***

$\frac{n(n-1)(n-2)}{3\times 2} = 2\times 2\times \frac{(n-2)(n-3)}{2}$ or equivalent with no factorials

$n(n-1) = 12(n-3)$

continued…

Question 9 continued.

OR

finding the number of ways to assign the students with the two students together

number of ways to assign two students and one other to the first group ${}^{n-2}C_1$ (seen anywhere) ***(A1)***

number of ways to assign three other students the first group ${}^{n-2}C_3$ (seen anywhere) ***(A1)***

number of ways $= {}^{n-2}C_1 + {}^{n-2}C_3$

attempt to set up an equation involving either half of their answer to part (a) and their number of ways or their answer to part (a) is twice their number of ways ***M1***

$\frac{1}{2}{}^{n}C_3 = {}^{n-2}C_1 + {}^{n-2}C_3$ OR ${}^{n}C_3 = 2\left({}^{n-2}C_1 + {}^{n-2}C_3\right)$

valid attempt to eliminate all factorials from their equation (***M1)***

$$\frac{n(n-1)(n-2)}{3\times 2} = 2\times(n-2) + 2\times\frac{(n-2)(n-3)(n-4)}{3\times 2}$$

$$n(n-1) = 12 + 2(n-3)(n-4)$$

THEN

$n^2 - 13n + 36 = 0$ ***A1***

$(n-9)(n-4) = 0$

$n = 9$ ***A1***

Note: Do not award the final ***A1*** if additional values of n are given.

[6 marks]

Total [7 marks]

SECTION B

10. (a) attempt to find a difference ***(M1)***

$d = p - a, 2d = q - a, d = q - p$ OR $p = a + d, q = a + 2d, q = p + d$

correct equation ***A1***

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ ***AG***

[2 marks]

(b) attempt to find a ratio ***(M1)***

$r = \frac{s}{a}, r^2 = \frac{t}{a}, \quad r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation ***A1***

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ ***AG***

[2 marks]

continued…

Question 10 continued

(c) **EITHER**

$2p-1=s^2$ (or equivalent) ***A1***

$\left(s^2>0\right) \Rightarrow 2p-1>0$ OR $s=\sqrt{2p-1} \Rightarrow 2p-1>0$ OR $p=\frac{s^2+1}{2}$ (and $s^2>0$) ***R1***

OR

$2p-1=a$ and $s^2=a$ ***A1***

($s^2>0$, so) $a>0 \Rightarrow 2p-1>0$ OR $p=\frac{a+1}{2}$ and $a>0$ ***R1***

$\Rightarrow p>\frac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

continued…

Question 10 continued

(d) (i) $9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

(ii) $9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[4 marks]

(e) (i) attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR $d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2\ln 3$ OR $\ln 1 = 0$ OR $\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

continued…

Question 10 continued.

(ii) **METHOD 1**

attempt to substitute first term and their common difference into S_{10} ***(M1)***

$\frac{10}{2}\left(2(9+\ln 9)+9\left(-4-\ln 3\right)\right)$ OR $\frac{10}{2}\left(2(9+2\ln 3)+9\left(-4-\ln 3\right)\right)$ (or equivalent) ***A1***

$=5\left(-18-5\ln 3\right)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

METHOD 2

$u_{10}=9+\ln 9+9\left(-4-\ln 3\right)\left(=-27+\ln 9-9\ln 3\right)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}\left(2(9+\ln 9)+9\left(-4-\ln 3\right)\right)$ OR $\frac{10}{2}\left(9+\ln 9-27+\ln 9-9\ln 3\right)$ OR

$\frac{10}{2}\left(2(9+2\ln 3)+9\left(-4-\ln 3\right)\right)$ OR $\frac{10}{2}\left(9+\ln 9-27-7\ln 3\right)$ (or equivalent) ***A1***

$=5\left(-18-5\ln 3\right)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90-25\ln 3$ ***AG***

[6 marks]

Total [16 marks]

1. (a) recognition that a 15% loss leaves 85% OR finding 15% and subtracting from original ***(M1)***

0.85×35000 OR $35000-0.15\times 35000$

$=(\$)29750$ ***A1***

Note: Accept $(\$)29800$.

[2 marks]

(b) **EITHER**

29750×0.89^{9} ***(A1)***

OR

$\mathrm{N}=9$
$\mathrm{I}\%=-11$
$\mathrm{PV}=\mp 29750$

(A1)

THEN

value$(FV)=(\$)10423$ ***A1***

Note: For this ***A1*** the answer must be rounded to the nearest dollar. Accept $(\$)10441$ from using 3 sf answer from part (a).

[2 marks]

continued…

Question 1 continued

(c) **METHOD 1**

attempt to solve the inequality (or equation) $29750\times0.89^{n-1}<3500$ OR table of values ***(M1)***

$19.3643...$ OR $(n=19\Rightarrow)\ 3651.80...$ OR $(n=20\Rightarrow)\ 3250.10...$ ***(A1)***

Note: For candidates using $(\$)29800$, $n>19.3787...$, $(n=19\Rightarrow)\ 3657.93...$, $(n=20\Rightarrow)\ 3255.56...$.

$n=20$ ***A1***

[3 marks]

METHOD 2

use of the finance app with $\mathrm{I}\%=-11,\ \mathrm{PV}=\mp29750,\ \mathrm{FV}=\pm3500$

OR $29750\times0.89^{N}<3500$ (condone the use of n or x) ***(M1)***

$(N=)\ 18.3643...$ ***(A1)***

Note: For candidates using $(\$)29800$, $N=18.3787...$

$n=20$ ***A1***

[3 marks]

Total [7 marks]

9. (a) **METHOD 1**

the number of ways Alvin and Bobby can be seated is $2\times 8\left(=16\right)$ ***(A1)***

the number of ways the other children can be seated is $8!\left(=40320\right)$ ***(A1)***

Note: These ***A1*** marks may be awarded independently.

total number of ways is $\left(16\times 8!=\right)645120$ ***A1***

Note: Accept $16\times 8!$ and 645000.

METHOD 2

the number of ways children can be seated in a row of 10 seats is $2\times 9!\left(=725760\right)$ ***(A1)***

the number of ways the children can be seated with Alvin and Bobby in seats 5 and 6 is $2\times 8!\left(=80640\right)$ ***(A1)***

Note: These ***A1*** marks may be awarded independently.

total number of ways is $\left(2\times 9!-2\times 8!=\right)645120$ ***A1***

Note: Accept $16\times 8!$ and 645000.

[3 marks]

(b) **METHOD 1**

attempt to find number of ways that A and B are seated next to each other AND C and D are seated next to each other and subtract from part a) ***(M1)***

Case 1: A and B are sat at the end of a row (8 ways)

$6(2)=12$ ways to seat C and D together

$12\times 6!\left(=8640\right)$ ways ***(A1)***

the total number of ways is $8\times 12\times 6!\left(=69120\right)$

continued…

Question 9 continued

Case 2: A and B are not sat at the end of a row (8 ways)

$5(2)=10$ ways to seat C and D together

$10\times 6!=7200$ ways ***(A1)***

the total number of ways is $8\times 10\times 6!\left(=57600\right)$

total number of ways is $645120-\left(69120+57600\right)$

$=518400$ ***A1***

Note: Accept 518000 or 518280 (from use of 645000).

METHOD 2

attempt to split into cases based on position of A and B and adding all possibilities ***(M1)***

Case 1: A and B are sat at the end of a row (8 ways)

the number of ways C and D can be seated:

with at least one in the same row as A and B $2(1+15)=32$

with both in a different row to A and B $2(6)=12$

the number of ways C and D can be seated is $44\times 6!\left(=31680\right)$ ***(A1)***

the total number of ways is $8\times 31680=253440$

Case 2: A and B are not sat at the end of a row (8 ways)

the number of ways C and D can be seated:

with at least one in the same row as A and B $2(2+15)=34$

with both in a different row to A and B $2(6)=12$

the number of ways C and D can be seated is $46\times 6!\left(=33120\right)$ ***(A1)***

the total number of ways is $8\times 33120=264960$

total number of ways is $253440+264960$

$=518400$ ***A1***

Note: Accept 518000.

[4 marks]

Total [7 marks]

Section B

8. (a) **METHOD 1**

(i) $\arg z = \arctan\left(\frac{\sin 2\theta}{1+\cos 2\theta}\right)$ $\left(\tan(\arg z) = \frac{\sin 2\theta}{1+\cos 2\theta}\right)$ ***A1***

uses $2\sin\theta\cos\theta$ in the numerator and any double angle identity for $\cos 2\theta$ in the denominator ***M1***

$\arg z = \arctan\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right)$ $\left(\tan(\arg z) = \frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right)$

$\Rightarrow \arg z = \arctan(\tan\theta)$ $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ ***A1***

$= \theta$ ***AG***

[3 marks]

(ii) attempts to express $|z|$ in the form $\sqrt{(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2}$ ***(M1)***

$|z| = \sqrt{(1+\cos 2\theta)^2 + \sin^2 2\theta}$

attempts to expand $(1+\cos 2\theta)^2$ and then uses

$\cos^2 2\theta + \sin^2 2\theta = 1$ in an attempt to simplify ***(M1)***

$|z| = \sqrt{2+2\cos 2\theta}$ ***A1***

$|z| = \sqrt{4\cos^2\theta}$ $(= 2|\cos\theta|)$ ***A1***

$= 2\cos\theta$ $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ ***AG***

[4 marks]

METHOD 2 (i) and (ii)

$z = 1 + 2\cos^2\theta - 1 + 2\sin\theta\cos\theta\,\mathrm{i}$ ***M1A1A1***

$z = 2\cos^2\theta + 2\sin\theta\cos\theta\,\mathrm{i}$ ***A1***

attempt to form $z = r\,cis\theta$ ***M1***

$z = 2\cos\theta(\cos\theta + \mathrm{i}\sin\theta)$ ***A1A1***

$\therefore |z| = 2\cos\theta$ and $\arg z = \theta.$ ***AG***

continued…

Question 8 continued.

(b) $2\theta = (2\cos\theta)^3$ ***(A1)***

attempts to solve for θ ***(M1)***

$\theta = 0.913236...$

$\theta = 0.913$ ***A1***

Note: Award all marks for $\theta = 0.913$ found directly without using part (a).

Note: Award ***(A1)(M1)A0*** for $\theta = 3.97^\circ$.

[3 marks]

Total [10 marks]

3. METHOD 1

attempt to expand $(3n+2)^2-(3n-2)^2$ ***M1***

Note: Award ***M0*** for invalid attempts such as $(3n+2)^2=9n^2+4$.

$=9n^2+12n+4-(9n^2-12n+4)$ or equivalent ***A1***

$=24n$ OR $12n+12n$ ***A1***

$=12(2n)$ OR $\frac{24}{12}=2$ OR $\frac{24n}{2}=12n$ OR $12n+12n=12(n+n)$ (or equivalent) ***R1***

so is a multiple of 12 ***AG***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

METHOD 2

use of $a^2-b^2=(a+b)(a-b)$ where $a=3n+2, b=3n-2$ ***M1***

$=(3n+2+3n-2)(3n+2-3n+2)$

$=6n\times 4$ ***A1***

$=24n$ ***A1***

$=12(2n)$ OR $\frac{24n}{12}=2n$ OR $\frac{24n}{2}=12n$ OR $12n+12n=12(n+n)$ (or equivalent) ***R1***

so is a multiple of 12 ***AG***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

continued…

Question 3 continued.

METHOD 3

base case $n=1$: $\left(3(1)+2\right)^2-\left(3(1)-2\right)^2=25-1=24$

so true for $n=1$ ***A1***

assume true for $n=k$ i.e. $\left(3k+2\right)^2-\left(3k-2\right)^2$ is a multiple of 12 ***M1***

consider $n=k+1$:

$\left(3\left(k+1\right)+2\right)^2-\left(3\left(k+1\right)-2\right)^2$

$\left(\left(3k+2\right)+3\right)^2-\left(\left(3k-2\right)+3\right)^2$

$\left(3k+2\right)^2+6\left(3k+2\right)+9-\left(\left(3k-2\right)^2+6\left(3k-2\right)+9\right)$

$\left(3k+2\right)^2-\left(3k-2\right)^2+24$

using the assumption $\left(3k+2\right)^2-\left(3k-2\right)^2=12M$

$12M+24$

$12\left(M+2\right)$

which is a multiple of 12, hence true for $n=k+1$ ***A1***

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Do not award the ***R1*** unless both ***A*** marks have been awarded.

[4 marks]

5. attempt to use $u_n = u_1 + (n-1)d$ or $S_n = \frac{n}{2}\left[2u_1 + (n-1)d\right]$ or $S_n = \frac{n}{2}[u_1 + u_n]$ to

set up at least one equation in u_1 and d ***(M1)***

$16 = u_1 + 9d$ and $100 = \frac{25}{2}\left[2u_1 + 24d\right]$ ***(A1)***

attempt to solve their two linear equations in u_1 and d simultaneously (must eliminate one variable) ***(M1)***

$d = -4 \left(\Rightarrow u_1 = 52\right)$ ***A1***

attempt to solve $u_k = 0$ with their d (or with their d and u_1) ***(M1)***

$\Rightarrow k = 14$ ***A1***

[6 marks]

12. (a) $|16\mathrm{i}| = 16$ and $\arg(16\mathrm{i}) = \frac{\pi}{2}$ ***(A1)***

attempt to use De Moivre's Theorem ***(M1)***

$z_1 = 2\left(\cos\left(\frac{\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)$ ***A1***

attempts to find other solutions using $z = 2\left(\cos\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{8} + \frac{\pi k}{2}\right)\right)$ or equivalent ***(M1)***

$z_2 = 2\left(\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)$ (or any other root) ***A1***

$z_3 = 2\left(\cos\left(\frac{9\pi}{8}\right) + \mathrm{i}\sin\left(\frac{9\pi}{8}\right)\right)$ and $z_4 = 2\left(\cos\left(\frac{13\pi}{8}\right) + \mathrm{i}\sin\left(\frac{13\pi}{8}\right)\right)$ ***A1***

Note: Award a maximum of ***(A1)(M1)A1(M1)A1A0*** for more than four roots or any roots outside the range.

Note: Allow use of r-cis form throughout.

[6 marks]

(b) attempt to evaluate a ratio with their roots eg $\frac{z_2}{z_1}$ ***(M1)***

$\frac{z_2}{z_1} = \frac{2\left(\cos\left(\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(\frac{5\pi}{8}\right)\right)}{2\left(\cos\left(\frac{\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi}{8}\right)\right)}$ or equivalent

$= \left(\cos\left(\frac{\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}\right)\right)$ ***(A1)***

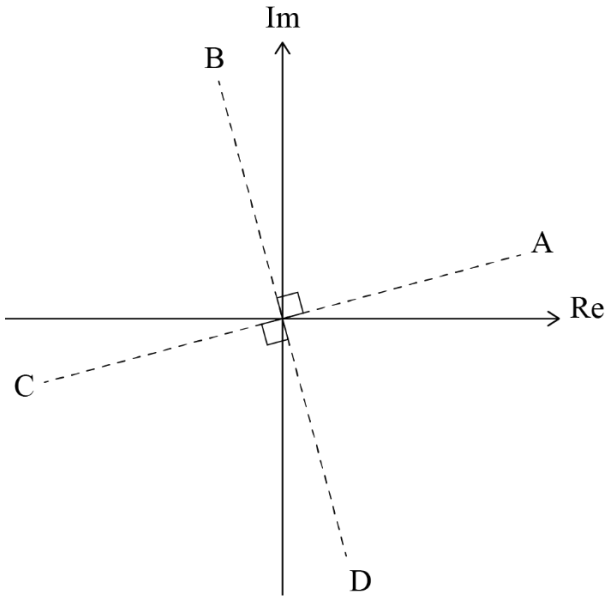
$= \mathrm{i}$ ***A1***

[3 marks]

continued…

Question 12 continued.

(c)



point A in approximately correct place in first quadrant ***A1***

points A, B, C and D approximately the same distance from the origin ***A1***

approximate angular separation of $\frac{\pi}{2}$ ***A1***

Note: Dotted lines not required.

[3 marks]

continued…

Question 12 continued.

(d) **EITHER**

$\left(z_i^*\right)^4 = \left(z_i^4\right)^*$ ***(A1)***

$= \left(16\text{i}\right)^*$ ***(A1)***

OR

$z_1^* = 2\left(\cos\left(-\frac{\pi}{8}\right)+\text{i}\sin\left(-\frac{\pi}{8}\right)\right)$ ***(A1)***

$\left(z_1^*\right)^4 = 2^4\left(\cos\left(-\frac{4\pi}{8}\right)+\text{i}\sin\left(-\frac{4\pi}{8}\right)\right)$ ***(A1)***

OR

$z_1 z_2 z_3 z_4 = -16\text{i}$ ***(A1)***

$\left(z_1 z_2 z_3 z_4\right)^* = \left(-16\text{i}\right)^*$

$z_1^* z_2^* z_3^* z_4^* = 16\text{i}$ ***(A1)***

THEN

$\left(z^4 =\right) -16\text{i}$ ***A1***

$\left(a = 0,\ b = -16\right)$

[3 marks]

(e) $\arg w_1\left(= \frac{\pi}{8}+\frac{\pi}{4}\right) = \frac{3\pi}{8}$ ***A1***

$\text{AB} = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow \text{AA}' = \sqrt{2}$

$\Rightarrow \text{OA}' = \left|w_1\right| = \sqrt{2}$ ***A1***

$\therefore w_1 = \sqrt{2}\text{cis}\frac{3\pi}{8}$

considers when $\arg\left(w_1{}^p\right) \in \mathbb{Z}^+$ (multiple of 2π) ***(M1)***

$\Rightarrow p = 16$ ***A1***

$\Rightarrow q = 8$ ***A1***

[5 marks]

Total [20 marks]

2. EITHER

attempt to form a product of binomial coefficient, a power of $2x$ and a power of -5 seen ***(M1)***

$^9C_3(2x)^6(-5)^3$ OR $^9C_6(2x)^6(-5)^3$ OR $84\times(2x)^6(-5)^3$ ***(A1)(A1)***

Note: Award ***A1*** for $^9C_6$ or $^9C_3$ or 84, ***A1*** for $(2x)^6(-5)^3$.

OR

attempt to use the general term ***(M1)***

$^9C_r(2x)^{9-r}(-5)^r$ and $r=3$ ***(A1)(A1)***

THEN

-672000 (exact) ***A1***

Note: Award ***A0*** for a final answer of $-672000x^6$.

[4 marks]

7. (a) total ways $= 3!\,{}^{12}C_3 \left(= {}^{12}P_3 = 1320\right)$ OR total ways together $= 3!\times 10\left(= 60\right)$ ***(A1)***

attempt to consider the total ways of sitting – total ways of sitting together ***(M1)***

$3!\,{}^{12}C_3 - 3!\times 10$

$= 1260$ ***A1***

[3 marks]

(b) **METHOD 1**

attempt to multiply ways of seating AVP by ways of sitting additional people ***(M1)***

AVP can sit in $3!\times 10\left(= 60\right)$ ways (may be seen in part (a))

other 3 then have $9\times 8\times 7\left(= {}^{9}P_3\right)$ ways to sit ***(A1)***

total ways $= 3!\times 10\times 9\times 8\times 7$

$= 30240$ ***A1***

Note: Award ***(M1)(A0)A0*** for $3!\times 10\times {}^{9}C_3 = 5040$.

METHOD 2

attempt to consider 'AVP' as one item, so 4 'items' in total ***(M1)***

${}^{10}C_4 \times 3!\times 4!\left(= {}^{10}P_4 \times 3!\right)$ ***(A1)***

$= 30240$ ***A1***

[3 marks]

Total [6 marks]

8. (a) **METHOD 1**

suppose $w = x + \mathrm{i}y$

$ww^* = (x + \mathrm{i}y)(x - \mathrm{i}y)$ ***A1***

$= x^2 + y^2$ ***A1***

$= |w|^2$ ***AG***

METHOD 2

suppose $w = r\mathrm{e}^{\mathrm{i}\theta}$

$ww^* = (r\mathrm{e}^{\mathrm{i}\theta})(r\mathrm{e}^{-\mathrm{i}\theta})$ ***A1***

$= r^2$ ***A1***

$= |w|^2$ ***AG***

METHOD 3

suppose $w = r(\cos\theta + \mathrm{i}\sin\theta)$

$ww^* = (r(\cos\theta + \mathrm{i}\sin\theta))(r(\cos\theta - \mathrm{i}\sin\theta))$ ***A1***

$= r^2(\cos^2\theta + \sin^2\theta)(= r^2)$ ***A1***

$= |w|^2$ ***AG***

[2 marks]

(b) **EITHER**

multiplying first equation by w OR multiplying LHS and RHS of both equations ***M1***

$5w^*w = (1 - 2\mathrm{i})z^2w$ OR $5w^*zw = (1 - 2\mathrm{i})z^2(10 + 10\mathrm{i})$

$5(2\sqrt{5})^2 = (1 - 2\mathrm{i})(10 + 10\mathrm{i})z (\Rightarrow 100 = (30 - 10\mathrm{i})z)$ ***A1***

OR

attempt to eliminate w and w^* using $ww^* = 20$ ***M1***

$\left(\frac{(1 - 2\mathrm{i})}{5}z^2\right)\left(\frac{10 + 10\mathrm{i}}{z}\right) = 20$

$\frac{(1 - 2\mathrm{i})(10 + 10\mathrm{i})}{5}z = 20$ ***A1***

continued…

Question 8 continued.

THEN

$z = \dfrac{10}{(1-2\mathrm{i})(1+\mathrm{i})}\left(= \dfrac{10}{3-\mathrm{i}}\right)$ ***(A1)***

$z = 3+\mathrm{i}$ ***A1***

$(a = 3,\ b = 1)$

[4 marks]

Total [6 marks]

9. (a) recognition that $\left|\log_2 c\right| < 1$ ***(M1)***

$0.5 < c < 2, \quad (c \neq 1)$ ***A1A1***

Note: Award ***A1*** for endpoints and ***A1*** for strict inequalities.

[3 marks]

(b) attempt to find $S_\infty = \frac{u_1}{1-r}$ ***(M1)***

$= \frac{5}{1-\log_2(1.5)} \left(= \frac{5}{1-0.58496\ldots} = 12.0471... \right)$ ***(A1)***

attempt to solve their $\left|S_\infty - S_n\right| < 0.1$ ***(M1)***

$\left| \frac{5}{1-\log_2(1.5)} - \sum_{r=0}^{n-1} 5\left(\log_2(1.5)\right)^r \right| < 0.1$ OR $\left| \frac{5}{1-\log_2(1.5)} - \frac{5\left(1-\left(\log_2(1.5)\right)^n\right)}{1-\log_2(1.5)} \right| < 0.1$

Note: Award ***(M1)*** for solving an equality. Condone absence of absolute value signs.

$n = 8.93574...$

$n = 9$ ***A1***

[4 marks]

Total [7 marks]

Section B

1. (a) $T_2 = 1.1(6000) - 500$ OR $T_2 = 6600 - 500$ ***A2***

$= 6100$ ***AG***

[2 marks]

(b) (i) $T_3 = 1.1(6100) - 500$ $\left(T_3 = 1.1\left(1.1(6000) - 500\right) - 500\right)$ OR $T_3 = 6710 - 500$ ***A1***

$= 6210$ ***AG***

[1 mark]

(ii) $T_4 = 1.1(6210) - 500$ OR $T_4 = 6831 - 500$ ***(A1)***

$= 6331$ ***A1***

[2 marks]

continued…

Question 1 continued.

(c) (i) **EITHER**

attempts to eliminate the denominator ***(M1)***

$T_n = 6000(1.1)^{n-1} - 5000\left((1.1)^{n-1} - 1\right)$ $\left(= (6000 - 5000)(1.1)^{n-1} + 5000\right)$ ***A1***

Note: Accept correct equivalent forms, eg. $T_n = 6000(1.1)^{n-1} - 5000(1.1)^{n-1} + 5000$.

OR

attempts to express T_n with a common denominator of 0.1 ***(M1)***

$T_n = \dfrac{600(1.1)^{n-1} - 500\left((1.1)^{n-1} - 1\right)}{0.1}$ $\left(= \dfrac{600(1.1)^{n-1} - 500(1.1)^{n-1} + 500}{0.1}\right)$ ***A1***

Note: Accept correct equivalent forms.

THEN

$T_n = 1000(1.1)^{n-1} + 5000$ ***AG***

[2 marks]

(c) (ii) **EITHER**

attempts to find T_6 using the explicit formula for T_n ***(M1)***

$T_6 = 1000(1.1)^{6-1} + 5000$ OR $T_6 = 6610.51$

OR

attempts to find T_6 using $T_{n+1} = 1.1T_n - 500$ ***(M1)***

$T_4 = 6331$, $T_5 = 6464.1$, $T_6 = 6610.51$

OR

attempts to find T_6 using a finance solver application ***(M1)***

$\mathrm{N} = 5,\ \mathrm{I} = 10\%,\ \mathrm{PV} = -6000$ and $\mathrm{PMT} = 500$

THEN

6611 ***A1***

Note: Accept 6610.

[2 marks]

continued…

Question 1 continued.

(d) (i) **METHOD 1**

attempts to use the explicit formula for D_n to find an expression for $D_{n+1} - D_n$ ***(M1)***

$= 1.1D_n - 750 - \left(-1500(1.1)^{n-1} + 7500\right)$

$= 1.1\left(-1500(1.1)^{n-1} + 7500\right) - 750 - \left(-1500(1.1)^{n-1} + 7500\right)$ ***A1***

EITHER

$= -1650(1.1)^{n-1} + 8250 - 750 + 1500(1.1)^{n-1} - 7500$ ***A1***

OR

$= 1500(1.1)^{n-1} - 1500(1.1)^n$

$= 1500(1.1)^{n-1}(1 - 1.1)\ \left(= 1500(1.1)^{n-1}(-0.1)\right)$ ***A1***

THEN

$D_{n+1} - D_n = -150(1.1)^{n-1}$ ***AG***

METHOD 2

attempts to express $D_{n+1} - D_n$ in terms of D_n ***(M1)***

$D_{n+1} - D_n = 1.1D_n - 750 - D_n$

$= 0.1D_n - 750$ ***(A1)***

$= 0.1\left(-1500(1.1)^{n-1} + 7500\right) - 750\ \left(= -150(1.1)^{n-1} + 750 - 750\right)$ ***A1***

$D_{n+1} - D_n = -150(1.1)^{n-1}$ ***AG***

[3 marks]

continued…

Question 1 continued.

(ii) $D_{n+1} - D_n < 0$ (because $(1.1)^{n-1} > 0$ and so $-150(1.1)^{n-1} < 0$) ***R1***

Note: Award ***R1*** for responses such as:

'The difference is negative'.

$-150(1.1)^{n-1} < 0$

$D_n = D_{n+1} + 150(1.1)^{n-1}$

Accept responses such as:

$D_n - D_{n+1} > 0$ or 'the difference is positive' as long as $D_n - D_{n+1}$ is referred to.

Do not accept arguments based on specific numerical examples.

therefore $D_{n+1} < D_n \Rightarrow D_n > D_{n+1}$ ***AG***

(and hence the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year)

[1 mark]

continued…

Question 1 continued.

(e) **METHOD 1**

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $D_n \leq 0$ ($D_n = 0$ OR $D_n < 0$) ***(M1)***

Note: Award ***(M1)*** for attempting to obtain $-1500(1.1)^{n-1} \leq -7500$ or $-1500(1.1)^{n-1} = -7500$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award ***(M1)***, for considering values of n either side of $D_n = 0$.

EITHER

$17.8863... \left(= \log_{1.1} 5 + 1, = \dfrac{\ln 5}{\ln 1.1} + 1 \right)$ (critical value of n seen anywhere) ***(A2)***

OR

$D_{17} = 607.540...$ ($607.540... < 750$) ***(A2)***

OR

$D_{18} = -81.7054...(< 0)$ ***(A2)***

THEN

trout disappears from the lake during (at the end of) the 17th year ***A1***

Note: Award ***A1*** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

Note: Award as above for $D_{n+1} \leq 0$ ($D_{n+1} = 0$ OR $D_{n+1} < 0$).

continued…

Question 1 continued.

METHOD 2

$S_n = \sum (D_{n+1} - D_n)$ is the total decrease in the number of trout by the end of year n or start of year $n+1$

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $S_n \leq -6000$ ($S_n = -6000$ OR $S_n < -6000$) ***(M1)***

$\frac{-150\left((1.1)^n - 1\right)}{1.1-1}$ $\left(= -1500\left((1.1)^n - 1\right)\right) \leq -6000$

Note: Award ***(M1)*** for attempting to obtain $-1500\left((1.1)^n - 1\right) \leq -6000$ or $-1500\left((1.1)^n - 1\right) = -6000$ (or equivalent) and then attempting to take the log of both sides.

For the use of trial & improvement, award ***(M1)***, for considering values of n either side of $S_n = -6000$.

$n = 16.8863...$ ***(A2)***

trout disappears from the lake during (at the end of) the 17th year ***A1***

Note: Award ***A1*** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

[4 marks]

(f) **METHOD 1**

$C_n = (6000 - 10d)(1.1)^{n-1} + 10d$ ***(A1)***

EITHER

$6000 - 10d = 0$ (recognizes that C_n is independent of n) ***(M1)***

OR

$10d = 6000$ (sets $C_n = 6000$ and compares coefficients) ***(M1)***

THEN

$d = 600$ ***A1***

continued…

Question 1 continued.

METHOD 2

EITHER

$C_{n+1} - C_n \left(= 0\right)$

$6000\left(1.1\right)^n - 10d\left(\left(1.1\right)^n - 1\right) - \left(6000\left(1.1\right)^{n-1} - 10d\left(\left(1.1\right)^{n-1} - 1\right)\right)\left(= 0\right)$ ***(A1)***

OR

$0.1C_n - d\left(= 0\right)$ (similar to part (d) (i) method 2)

$0.1\left(6000\left(1.1\right)^{n-1} - 10d\left(\left(1.1\right)^{n-1} - 1\right)\right) - d\left(= 0\right)$ ***(A1)***

THEN

attempts to expand to eliminate the linear terms in d and then attempts to factorize ***(M1)***

$600\left(1.1\right)^{n-1} - d\left(1.1\right)^{n-1}\left(= 0\right)$ $\left(\left(600 - d\right)\left(1.1\right)^{n-1}\left(= 0\right)\right)$

$d = 600$ ***A1***

METHOD 3

considers $C_{n+1} = 1.1C_n - d$ and $C_1 = 6000$ ***(A1)***

EITHER

$6000 = 1.1 \times 6000 - d$ $\left(C_1 = C_2 = 6000\right)$ ***(A1)***

OR

systematically varies the value of d ***(M1)***

THEN

$d = 600$ ***A1***

continued…

Question 1 continued.

METHOD 4

EITHER

considers a specific $C_n = C_{n+1}$, for example, $C_2 = C_3$ ***(A1)***

$6000(1.1) - 10d(1.1 - 1) = 6000(1.1)^2 - 10d\left((1.1)^2 - 1\right)$

attempts to solve their $C_n = C_{n+1}$ for d ***(M1)***

for example, $1.1d = 660$

OR

considers a specific $C_n = 6000$, for example, $C_2 = 6000$ ***(A1)***

$6000(1.1)^{2-1} - 10d\left((1.1)^{2-1} - 1\right) = 6000$

attempts to solve their $C_n = 6000$ for d ***(M1)***

$6600 - d = 6000$

THEN

$d = 600$ ***A1***

[3 marks]

continued…

Question 1 continued.

(g) $n=1:\ \text{LHS}=u_1$ and $\text{RHS}=\left(u_1r^{1-1}-\dfrac{d\left(r^{1-1}-1\right)}{r-1}\right)=u_1r^0-\dfrac{d\left(r^0-1\right)}{r-1}=u_1$ ***R1***

$\text{LHS}=\text{RHS}$ and so true for $n=1$

Note: Award ***R0*** for considering $n=2$.

Subsequent marks after this ***R1*** are independent of this mark and can be awarded.

assume true for $n=k$ $\left(u_k=u_1r^{k-1}-\dfrac{d\left(r^{k-1}-1\right)}{r-1}\right)$ ***M1***

Note: The assumption of truth must be apparent. Do not award ***M1*** for statements such as "let $n=k$" or "assume that $n=k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

attempts to substitute for u_k in $u_{k+1}=ru_k-d$ ***M1***

$$=r\left(u_1r^{k-1}-\frac{d\left(r^{k-1}-1\right)}{r-1}\right)-d$$

$$=u_1r^k-\frac{dr\left(r^{k-1}-1\right)}{r-1}-\frac{d\left(r-1\right)}{r-1}\quad\left(=u_1r^k-\frac{dr^k-dr}{r-1}-d\right)$$ ***A1***

$$=u_1r^k-\frac{dr^k+dr-dr-d}{r-1}\quad\left(=u_1r^k-\frac{dr^k-d}{r-1}\right)$$ ***A1***

$$=u_1r^k-\frac{d\left(r^k-1\right)}{r-1}$$ ***A1***

since true for $n=1$ and true for $n=k+1$ if true for $n=k$, hence true for all $n\left(\in\mathbb{Z}^+\right)$ ***R1***

Note: To obtain the final ***R1***, four of the previous six marks must have been awarded.

[7 marks]

Total [27 marks]

2. (a) $k = \frac{4}{400}\left(= \frac{1}{100} = 0.01\right)$ ***A1***

[1 mark]

(b) attempt to find binomial coefficients or multiply out brackets ***(M1)***

e.g. Pascal's triangle down to correct row OR $\left(1 + 2x + x^2\right)^2$ OR substitute into binomial expansion

$\left(1 + x\right)^4$

$= 1 + 4x + 6x^2 + 4x^3 + x^4$ ***A1***

[2 marks]

(c) **METHOD 1**

recognition that the expansion can be used with x replaced with k ***(M1)***

$\left(1 + \frac{1}{100}\right)^4$

$= 1 + \frac{4}{100} + \frac{6}{100^2} + ...\left(= 1 + 0.04 + 0.0006 + ...\right)$ ***(A1)***

multiplies by 1000 (seen anywhere) ***(M1)***

$1000\left(1 + \frac{1}{100}\right)^4$

$= 1000 + 40 + 0.6 + ...\left(= 1040.6...\right)$

$= 1041$ (dinar) ***A1***

METHOD 2

attempt to find the value of $\left(1 + k\right)^4$ by hand ***(M1)***

$\left(1.01\right)^4 = \left(1.0201\right)\left(1.01\right)^2 = \left(1.030301\right)\left(1.01\right)$

$= 1.0406...$ ***(A1)***

multiplies by 1000 (seen anywhere) ***(M1)***

$1000\left(1.01\right)^4$

$= 1040.6...$

$= 1041$ (dinar) ***A1***

[4 marks]

Total [7 marks]

5. (a) (i) for a sequence of areas, uses two consecutive terms to find a common ratio OR for sequences of both widths and heights uses two consecutive terms for both sequences to find both common ratios OR recognises that both widths and heights are geometric sequences with common ratio $\frac{3}{2}$ ***M1***

areas form a geometric sequence with first term 20 and common ratio $\frac{45}{20}$ ***A1***

OR area of picture frame F_n is $4\left(\frac{3}{2}\right)^{n-1} \times 5\left(\frac{3}{2}\right)^{n-1}$

area of F_n is $20\left(\frac{9}{4}\right)^{n-1}$ ***AG***

(ii) attempt to find the sum of the areas using $S_n = \frac{u_1\left(r^n - 1\right)}{r-1}$ ***(M1)***

sum of areas $\frac{20\left(\left(\frac{9}{4}\right)^{10} - 1\right)}{\frac{9}{4} - 1}\left(= 16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***(A1)***

mean area $= \frac{1}{10}\left(\frac{20\left(\left(\frac{9}{4}\right)^{10} - 1\right)}{\frac{9}{4} - 1}\right)\left(= \frac{1}{10}\left(16\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)\right)$

$= \frac{16}{10}\left(\left(\frac{9}{4}\right)^{10} - 1\right)\left(= \frac{8}{5}\left(\left(\frac{9}{4}\right)^{10} - 1\right)\right)$ ***A1***

$p = \frac{8}{5},\ a = 10$

[5 marks]

continued…

Question 5 continued.

(b) recognition that median is between 5th and 6th picture frame ***(M1)***

$$\text{median area} = \frac{20\left(\frac{9}{4}\right)^4 + 20\left(\frac{9}{4}\right)^5}{2} \quad \textbf{(A1)}$$

$$= \frac{20\left(\frac{9}{4}\right)^4\left(1+\frac{9}{4}\right)}{2}$$

$$= \frac{65}{2}\left(\frac{9}{4}\right)^4 \quad \textbf{A1}$$

$$q = \frac{65}{2}$$

[3 marks]

Total [8 marks]

$$\left(\quad (0) \right) \qquad \left(\quad (1) \right)$$

7. (a) $\left(3=\right)e^{\ln 3}$ OR $a=\ln 3$ ***A1***

[1 mark]

(b) (i) $z=\frac{1}{3}e^{\mathrm{i}\ln 3}$ OR $\left(\mathrm{Re}(z)\right)=e^{-\ln 3}\cos\left(\ln 3\right)$ ***(A1)***

$\left(\mathrm{Re}(z)\right)=\frac{1}{3}\cos\left(\ln 3\right)$ ***A1***

(ii) $\frac{1}{z}=3e^{-\mathrm{i}\ln 3}\left(=3\left(\cos\left(-\ln 3\right)+\mathrm{i}\sin\left(-\ln 3\right)\right)\right)$ OR $\left(\mathrm{Re}\left(\frac{1}{z}\right)=\right)e^{\ln 3}\cos\left(-\ln 3\right)$ ***(A1)***

$\mathrm{Re}\left(\frac{1}{z}\right)=3\cos\left(-\ln 3\right)$

$=3\cos(\ln 3)$ OR $3\cos\left(\ln\frac{1}{3}\right)$ ***A1***

[4 marks]

Total [5 marks]

8. (a) $1+\log_2 n = \log_2 2 + \log_2 n$

$= \log_2(2n)$ ***A1***

$2n \geq n+1$ OR $\log_2(2n) \geq \log_2(n+1)$ ***R1***

(for $n \in \mathbb{Z}^+$) (since $\log_2 x$ is an increasing function)

$1+\log_2 n \geq \log_2(n+1)$ (for $n \in \mathbb{Z}^+$) ***AG***

Note: Do not award ***A0R1***.

[2 marks]

(b) for $n = 1$

$\log_2 1 = 0$ and $1 > 0$ OR LHS $= 1$, RHS = $\log_2 1 = 0$ OR $\log_2 2 > \log_2 1$ ***R1***

(so true for $n = 1$)

assume true for $n = k$, ie $k > \log_2 k$ ***M1***

Note: Award ***M0*** for statements such as "let $n = k$", "assume $n = k$ is true". The assumption of truth must be clear. "Assume P_k true" is accepted.

The following two marks after this ***M1*** are independent of this mark and can be awarded.

hence

$1+k > 1+\log_2 k$ (using assumption) ***M1***

$\geq \log_2(k+1)$ (using result from part a)) ***A1***

hence if true for $n = k$ then true for $n = k+1$ ***R1***

and as true for $n = 1$, therefore true for all $n \in \mathbb{Z}^+$.

Note: Only award the final **R1** if the first three marks have been awarded.

[5 marks]

Total [7 marks]

9. METHOD 1

assume that $\frac{1}{x(1-x)} < 4$ (for $0 < x < 1$) ***M1***

Note: Award ***M0*** for statements such as "let $\frac{1}{x(1-x)} < 4$".

Award ***M0*** for incorrect use of weak inequality signs such as "assume that $\frac{1}{x(1-x)} \leq 4$".

since $0 < x < 1$ ($\Rightarrow x > 0$ and $1 - x > 0$, therefore) $x(1-x) > 0$ (seen anywhere) ***R1***

Note: Subsequent marks after this ***M1R1*** are independent of these two marks and can only be awarded provided a correct or weak inequality ($<$ or $\leq$) is used at the assumption step.

attempt to form a quadratic inequality in x (using their assumption) ***(M1)***

$1 < 4x(1-x)$ OR $\frac{1-4x(1-x)}{x(1-x)} < 0$

$4x^2 - 4x + 1 < 0$ (or equivalent) ***A1***

EITHER

$(2x-1)^2 < 0$ (or equivalent) ***A1***

a contradiction, since $(2x-1)^2 \geq 0$ (for $0 < x < 1$) ***R1***

OR

a graph of $y = 4x^2 - 4x + 1$ with the vertex touching the x-axis at $x = \frac{1}{2}$ ***A1***

a contradiction, since $4x^2 - 4x + 1 \geq 0$ (for $0 < x < 1$) ***R1***

THEN

hence $\frac{1}{x(1-x)} \geq 4$ for $x \in \mathbb{R}, 0 < x < 1$. ***AG***

Note: Award a maximum of ***M0R1(M1)A1A1R0*** for incorrect use of weak inequality signs.

continued…

Question 9 continued.

METHOD 2

assume that $\frac{1}{x(1-x)} < 4$ (for $0 < x < 1$) ***M1***

Note: Award ***M0*** for statements such as "let $\frac{1}{x(1-x)} < 4$".

Award ***M0*** for incorrect use of weak inequality signs such as "assume that $\frac{1}{x(1-x)} \leq 4$".

Subsequent marks after this ***M1*** are independent and can only be awarded provided a correct or weak inequality ($<$ or $\leq$) is used at the assumption step.

$$\frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x} \Rightarrow 1 \equiv A(1-x) + Bx$$

attempt to equate both coefficients OR substitute two values eg 0 and 1 ***(M1)***

$A = 1$ and $B = 1$ ***A1***

Note: Award ***A1*** for both correct values.

Let $f(x) = \frac{1}{x} + \frac{1}{1-x}$

equating their $f'(x) = 0$ and solve for x OR a concave upward graph for $0 < x < 1$ ***(M1)***

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} = 0 \Rightarrow x^2 - 2x + 1 = x^2 \Rightarrow x = \frac{1}{2}$$

minimum value is $f\left(\frac{1}{2}\right) = 4$ ***A1***

a contradiction, since $4 < 4$ ***R1***

hence $\frac{1}{x(1-x)} \geq 4$ for $x \in \mathbb{R}, 0 < x < 1$. ***AG***

Note: Award a maximum of ***M0(M1)A1(M1)A1R0*** for incorrect use of weak inequality signs.

continued…

Question 9 continued.

METHOD 3

assume that $\frac{1}{x(1-x)} < 4$ (for $0 < x < 1$) ***M1***

Note: Award ***M0*** for statements such as "let $\frac{1}{x(1-x)} < 4$".

Award ***M0*** for incorrect use of weak inequality signs such as "assume that $\frac{1}{x(1-x)} \leq 4$".

Subsequent marks after this ***M1*** are independent and can only be awarded provided a correct or weak inequality ($<$ or $\leq$) is used at the assumption step.

Let $f(x) = \frac{1}{x(1-x)}$

attempt to use product rule to find $f'(x)$ ***(M1)***

$f'(x) = -\frac{1}{x^2(1-x)} + \frac{1}{x(1-x)^2} \left(= \frac{2x-1}{x^2(1-x)^2} \right)$ ***A1***

equating their $f'(x) = 0$ and solve for x ***(M1)***

$x = \frac{1}{2}$

minimum value is $f\left(\frac{1}{2}\right) = 4$ ***A1***

a contradiction, since $4 < 4$ ***R1***

hence $\frac{1}{x(1-x)} \geq 4$ for $x \in \mathbb{R}, 0 < x < 1$. ***AG***

Note: Award a maximum of ***M0(M1)A1(M1)A1R0*** for incorrect use of weak inequality signs.

[6 marks]

Section B

11. (a) attempt to use the binomial expansion with $n = \frac{1}{2}$ ***(M1)***

$= 1 + \left(\frac{1}{2}\right)(5x) + \ldots$

correct substitution into at least three terms ***(A1)***

$= 1 + \left(\frac{1}{2}\right)(5x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(5x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(5x)^3 + \ldots$

$= 1 + \frac{5}{2}x - \frac{25}{8}x^2 + \frac{125}{16}x^3 \ (+\ldots)$ ***A1A1***

Note: Award ***A1*** for $1 + \frac{5}{2}x - \frac{25}{8}x^2$ and ***A1*** for $\frac{125}{16}x^3$.

[4 marks]

(b) at least 3 correct terms in expansion of $(1+qx)^{-1}$ ***A1***

$= 1 + (-1)(qx) + \frac{(-1)(-2)}{2!}(qx)^2 + \ldots \left(= 1 - qx + q^2x^2 + \ldots\right)$

attempt to find product ***(M1)***

$(1+px)\left(1 - qx + q^2x^2 + \ldots\right) = 1 - qx + q^2x^2 + \ldots + px - pqx^2 + \ldots$

$= 1 + px - qx + q^2x^2 - pqx^2 + \ldots \left(= 1 + (p-q)x + \left(q^2 - pq\right)x^2 + \ldots\right)$ ***A1***

Note: Award ***A0*** if the final answer is not in ascending order.

[3 marks]

continued…

Question 11 continued.

(c) equating their coefficients of either x or x^2 ***(M1)***

one correct equation ***A1***

$p-q=\frac{5}{2}$ OR $q^2-pq=-\frac{25}{8}$

valid attempt to solve for q or p ***(M1)***

$-\frac{5}{2}q=-\frac{25}{8}$ OR $q^2-q\left(q+\frac{5}{2}\right)=-\frac{25}{8}$ OR $\frac{15}{4}-q=\frac{5}{2}$ ***A1***

Note: Award full marks for $\frac{15}{4}-q=\frac{5}{2}$ provided a full solution for $p=\frac{15}{4}$ is seen first. Using $p=\frac{15}{4}$ from part (d)'s stem is not sufficient.

$q=\frac{5}{4}$ ***AG***

[4 marks]

(d) (i) attempt to find x ***(M1)***

$1+5x=1.2$

$x=\frac{1}{25}\left(=0.04\right)$ (or equivalent) ***A1***

substituting their value of x with p and q ***(M1)***

$$\frac{1+\frac{15}{4}\left(\frac{1}{25}\right)}{1+\frac{5}{4}\left(\frac{1}{25}\right)}$$

$\frac{115}{105}\ \left(=\frac{23}{21}\right)$ ***A1***

Note: Award ***(M1)A1(M1)A1*** for an answer of $\frac{219}{200}$ or $\frac{2191}{2000}$ using the expansion from part (b) or (a).

continued…

Question 11 continued.

(ii) $\frac{\sqrt{5}}{2} = \sqrt{\frac{5}{4}} = \sqrt{1.25}$

$x = \frac{1}{20}$ ***A1***

recognising $\frac{1}{20} > \frac{1}{25}$ ***R1***

therefore, approximation for $\frac{\sqrt{5}}{2}$ is not as accurate as the approximation for $\sqrt{1.2}$ ***AG***

[6 marks]

Total [17 marks]

12. (a) $z^2 = 2\left(\cos\left(-\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}\right)\right)$ OR $z^2 = 2\left(\cos\left(\frac{4\pi}{3}\right)+\mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right)$ ***A2***

Note: Award ***A1*** for correct modulus or argument seen.

$z^2 = 2\left(\cos\left(-\frac{2\pi}{3}+2k\pi\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}+2k\pi\right)\right)$ OR $z^2 = 2\left(\cos\left(\frac{4\pi}{3}+2k\pi\right)+\mathrm{i}\sin\left(\frac{4\pi}{3}+2k\pi\right)\right)$

attempt to use De Moivre's theorem ***M1***

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{3}+k\pi\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}+k\pi\right)\right)$ OR $z = \sqrt{2}\left(\cos\left(\frac{2\pi}{3}+k\pi\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}+k\pi\right)\right)$

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$; $z = \sqrt{2}\left(\cos\left(\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)$ ***A1***

Note: accept $z = \sqrt{2}\left(\cos\left(\frac{5\pi}{3}\right)+\mathrm{i}\sin\left(\frac{5\pi}{3}\right)\right)$ in place of the 1st term above.

accept $r\,\mathrm{cis}\,\theta$ form throughout.

[4 marks]

(b) (i) $z_1 = \frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\mathrm{i}$ and $z_2 = -\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}$ ***A1A1***

Note: may be seen in part (a).

Award ***A1A0*** for z_1 and z_2 interchanged.

Accept factorised form, e.g. $z_1 = \sqrt{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$.

(ii) $z_3 = \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}$ and $z_4 = -\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\mathrm{i}$ ***A1A1***

Note: Award ***A1A0*** for z_3 and z_4 interchanged.

Accept factorised form, e.g. $z_3 = \sqrt{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$.

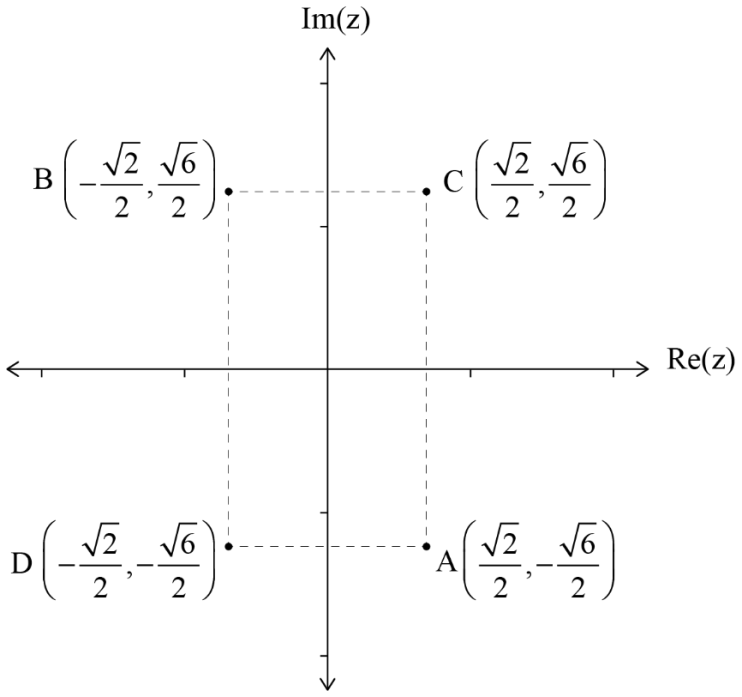
Award ***A1FTA1FT*** for $z_3 = their\left(z_1\right)^*$ with $\mathrm{Re}\left(z_3\right) > 0$ and $z_4 = their\left(z_2\right)^*$.

[4 marks]

continued…

Question 12 continued.

(c) (i) Argand diagram with all four points plotted in approximately correct positions forming a rectangle. ***A2***



Note:	Award ***A1*** for any two points plotted correctly on an argand diagram.
	Award ***A1A0*** if the four points form a square.
	Condone absence of labels, coordinates if they are approximately in correct positions and same distance from the origin. Dotted lines not required.

(ii) **METHOD 1**

attempt to find the length OR the width of rectangle ***(M1)***

$2\times\frac{\sqrt{2}}{2}$ OR $2\times\frac{\sqrt{6}}{2}$

length $=\sqrt{6}$ and width $=\sqrt{2}$

area of rectangle $=\sqrt{12}\left(=2\sqrt{3}\right)$ ***A1***

continued…

Question 12 continued.

METHOD 2

attempt to find the area of triangle ***(M1)***

$\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\sin\frac{2\pi}{3}$ OR $\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\sin\frac{\pi}{3}$

$4\times\left(\frac{1}{2}\times\sqrt{2}\times\sqrt{2}\times\frac{\sqrt{3}}{2}\right)$

area of rectangle $=2\sqrt{3}$ ***A1***

[4 marks]

(d) recognition that $z=\frac{1}{w}$ ***(M1)***

$\left(\frac{1}{w}\right)^4+2\left(\frac{1}{w}\right)^2+4=0$ OR $\frac{1+2w^2+4w^4}{w^4}=0$ ***(A1)***

$4w^4+2w^2+1=0$

$p=4, q=2, r=1$ ***A1***

[3 marks]

(e) (i) **EITHER**

attempt to take conjugate ***M1***

$\frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}\mathrm{i}\right)}\times\frac{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}\right)}{\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}\mathrm{i}\right)}$ OR $\frac{2}{\left(\sqrt{2}-\sqrt{6}\,\mathrm{i}\right)}\times\frac{\left(\sqrt{2}+\sqrt{6}\,\mathrm{i}\right)}{\left(\sqrt{2}+\sqrt{6}\,\mathrm{i}\right)}$

OR

attempt to use De Moivre's theorem ***M1***

$\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$

THEN

$=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\mathrm{i}$ (or equivalent) ***A1***

continued…

Question 12 continued

(ii) recognition of dilation/stretch by a scale factor of $\frac{1}{2}$ for both dimensions ***(M1)***

area of rectangle $= \frac{\sqrt{12}}{4} \left(= \frac{\sqrt{3}}{2} \right)$ ***A1***

Note: Award follow through from their area found in part (c)(ii).

[4 marks]

Total [19 marks]

1. (a) correct application of $\log_a xy = \log_a x + \log_a y$ or $\log_a x^m = m\log_a x$ ***(M1)***

correct expression in terms of $\log_{10} 2$ AND $\log_{10} 3$ (arguments must be 2 and 3) ***(A1)***

$3\log_{10} 2 + \log_{10} 3$ OR $\log_{10} 2 + \log_{10} 2 + \log_{10} 2 + \log_{10} 3$

$3p + q$ ***A1***

[3 marks]

(b) $\left(\log_3 8 = \right) \dfrac{\log_{10} 8}{\log_{10} 3} \left(= \dfrac{3\log_{10} 2}{\log_{10} 3} \right)$ ***(A1)***

$= \dfrac{3p}{q}$ ***A1***

[2 marks]

Total [5 marks]

Total [6 marks]

8. (a) attempt to find modulus **OR** argument of z_1 ***(M1)***

$r = \sqrt{12}\ \left(= 2\sqrt{3}\right)$ or $\theta = -\frac{\pi}{3}$ ***A1***

$z_1 = \sqrt{12}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\left(= 2\sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)$ ***A1***

[3 marks]

(b) $\left(\frac{z_2}{z_1} = \frac{2\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}}{2\sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}} = \right)\mathrm{e}^{\mathrm{i}\frac{7\pi}{6}}\left(= \mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}\right)$ ***(A1)***

recognizing three equivalent arguments ***(M1)***

eg $\frac{z_2}{z_1} = \mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}+2\pi k\right)}$ **OR** $\frac{z_2}{z_1} = \mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}}, \mathrm{e}^{-\frac{17\pi}{6}\mathrm{i}}, \mathrm{e}^{\frac{7\pi}{6}\mathrm{i}}$

attempt to find a root using de Moivre's theorem ***(M1)***

$\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{18}+\frac{2\pi k}{3}\right)}\left(= \mathrm{e}^{\mathrm{i}\left(\frac{(12k-5)\pi}{18}\right)} = \mathrm{e}^{\mathrm{i}\left(\frac{(12k+7)\pi}{18}\right)}\right)$ ***(A1)***

$\mathrm{e}^{-\mathrm{i}\frac{17\pi}{18}}, \mathrm{e}^{-\mathrm{i}\frac{5\pi}{18}}, \mathrm{e}^{\mathrm{i}\frac{7\pi}{18}}$ ***A1***

[5 marks]

Total [8 marks]

10. (a) (i) **METHOD 1**

attempt to equate differences of consecutive terms ***(M1)***

$(3-2k)-(k-5)=(5k+3)-(3-2k)$ **OR** $(k-5)-(3-2k)=(3-2k)-(5k+3)$ ***A1***

$8-3k=7k$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 2 (system of equations)

TWO correct equations involving k and d ***A1***

$k-5+d=3-2k$ OR $3-2k+d=5k+3$ OR $k-5+2d=5k+3$

OR $\frac{3}{2}\left(2\left(k-5\right)+2d\right)=k-5+3-2k+5k+3$ (or equivalent)

valid attempt to solve their system of equations using substitution or elimination ***(M1)***

$\left(d=5.6\right)$

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 3 (in terms of k)

$\frac{3}{2}\left(k-5+5k+3\right)=k-5+3-2k+5k+3$ (or equivalent) ***A1***

combining like terms ***(M1)***

$9k-3=4k+1$ **OR** $5k=4$ (or equivalent)

$(k=)\ \frac{4}{5}$ ***A1***

METHOD 4 (arithmetic mean)

attempt to find mean of u_1 and u_3 ***(M1)***

$\frac{(k-5)+(5k+3)}{2}=3-2k$ ***A1***

$3k-1=3-2k$

$(k=)\ \frac{4}{5}$ ***A1***

(ii) substituting their value of k into expression for u_3 ***(A1)***

$\left(u_3=\right)\ 5\times\frac{4}{5}+3$

$=7$ ***A1***

[5 marks]

continued…

Question 10 continued

(b) (i) substituting $k = 12$ into u_1, u_2 or u_3 ***(M1)***

$(u_1 =)\ 7$ **AND** $(u_2 =) -21$ **AND** $(u_3 =)\ 63$ ***(A1)***

$\left(r =\right) \frac{-21}{7} = \frac{63}{-21}\ (= -3)$ **OR** $(-21)^2 = 7 \times 63$ **OR** $r = -3$ ***R1***

u_1, u_2 and u_3 are in geometric sequence ***AG***

(ii) since $\left|r\right| \geq 1$ (accept $\left|r\right| > 1$ or $r < -1$) ***R1***

hence not convergent ***AG***

[4 marks]

(c) (i) attempts to find ratios, in terms of k, of consecutive terms and equating ***(M1)***

$\frac{(3-2k)}{(k-5)} = \frac{(5k+3)}{(3-2k)}$ **OR** $(3-2k)^2 = (k-5)(5k+3)$ (or equivalent)

$9 - 12k + 4k^2 = 5k^2 - 22k - 15$ ***A1***

Note: Award ***A1*** for correct expansion of all brackets leading to given result.

$k^2 - 10k - 24 = 0$ ***AG***

(ii) recognizing need to factorize, complete the square or substitute into quadratic formula ***(M1)***

$(k+2)(k-12)\left(=0\right)$ **OR** $(k-5)^2 - 49\ (=0)$ **OR** $k = \frac{10 \pm \sqrt{196}}{2}$

$k = -2$ (accept $k = -2$ **and** $k = 12$) ***A1***

substituting their value of k (other than $k = 12$) to find u_1, u_2 or u_3 ***(M1)***

$(u_1 =) -7$ **AND** $(u_2 =)\ 7$ **AND** $(u_3 =) -7$ ***A1***

(iii) $(S_{2m} =)\ 0$ ***A1***

[7 marks]

Total [16 marks]

(2) (4)

12. (a) substituting into modulus formula with either correct real part or correct imaginary part ***(M1)***

$\sqrt{(x-2)^2+(y-1)^2}\ \ (=3)$ ***(A1)***

$x^2-4x+4+y^2-2y+1=9$ ***A1***

$x^2+y^2-4x-2y-4=0$ ***AG***

[3 marks]

(b) **METHOD 1**

recognizing the need to express $\frac{z+p}{z-1}$ in the form $a+\mathrm{i}b$ ***(M1)***

$\frac{z+p}{z-1}\times\frac{z^*-1}{z^*-1}$ **OR** $\frac{z+p}{z-1}\times\frac{(x-y\mathrm{i})-1}{(x-y\mathrm{i})-1}$ **OR** $\frac{(x+p)+\mathrm{i}y}{(x-1)+\mathrm{i}y}\times\frac{(x-1)-\mathrm{i}y}{(x-1)-\mathrm{i}y}$ ***A1***

expanding their $\frac{z+p}{z-1}\times\frac{z^*-1}{z^*-1}$ ***(M1)***

$\frac{zz^*-z+pz^*-p}{zz^*-z-z^*+1}$ **OR** $\frac{x^2+y^2-x-y\mathrm{i}+px-py\mathrm{i}-p}{x^2+y^2-x-y\mathrm{i}-x+y\mathrm{i}+1}$

OR $\frac{(x+p)(x-1)+y^2+((x-1)y-(x+p)y)\mathrm{i}}{(x-1)^2+y^2}$

$\frac{x^2+y^2+(p-1)x-p-(p+1)y\mathrm{i}}{x^2+y^2-2x+1}$ (or equivalent) (seen anywhere) ***A1***

$\left(\Rightarrow \arg\left(\frac{z+p}{z-1}\right)=\right)\ \arctan\left(\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}\right)$ ***(A1)***

equating their $\arg\left(\frac{z+p}{z-1}\right)$ to $\frac{\pi}{4}$ **OR** their expression for $\frac{z+p}{z-1}$ in x and y to 1

OR equating real and imaginary parts ***(M1)***

$\arctan\left(\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}\right)=\frac{\pi}{4}$ **OR** $\frac{-(p+1)y}{x^2+y^2+(p-1)x-p}=1$

$-(p+1)y=x^2+y^2+(p-1)x-p$ ***A1***

$x^2+y^2+(p-1)x+(p+1)y-p=0$ ***AG***

continued…

Question 12 continued

METHOD 2

$\arg\left(\frac{z+p}{z-1}\right) = \arg(z+p) - \arg(z-1)$ ***(M1)***

$= \arg(x+p+y\mathrm{i}) - \arg(x-1+y\mathrm{i})$ ***A1***

$= \arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)$ ***A1***

attempt to use compound angle formula for tan ***(M1)***

$$\tan\left(\arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)\right) = \frac{\tan\left(\arctan\left(\frac{y}{x+p}\right)\right) - \tan\left(\arctan\left(\frac{y}{x-1}\right)\right)}{1 + \tan\left(\arctan\left(\frac{y}{x+p}\right)\right) \times \tan\left(\arctan\left(\frac{y}{x-1}\right)\right)}$$

$$= \frac{\frac{y}{x+p} - \frac{y}{x-1}}{1 + \frac{y}{x+p} \times \frac{y}{x-1}}$$ ***A1***

equating their expression for $\tan\left(\arctan\left(\frac{y}{x+p}\right) - \arctan\left(\frac{y}{x-1}\right)\right)$ to $\tan\frac{\pi}{4}$ ***(M1)***

$$\frac{\frac{y}{x+p} - \frac{y}{x-1}}{1 + \frac{y}{x+p} \times \frac{y}{x-1}} = \tan\frac{\pi}{4} \quad \left(\Rightarrow \frac{(x-1)y - (x+p)y}{(x+p)(x-1) + y^2} = 1\right)$$

$(x+p)(x-1) + y^2 = -y - py$ ***A1***

$x^2 - x + px - p + y^2 = -y - py$

$x^2 + y^2 + (p-1)x + (p+1)y - p = 0$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) $\left(\arg\left(\frac{z+4}{z-1}\right)=\frac{\pi}{4}\Rightarrow\right)\ x^2+y^2+3x+5y-4=0$ (seen anywhere) ***A1***

solving simultaneously their equations in x and y ***(M1)***

$7x+7y=0\ \left(\Rightarrow x=-y\right)$ (or equivalent) ***A1***

substituting to obtain an equation in either x or y ***(M1)***

$x^2-x-2=0$ **OR** $y^2+y-2=0$

solving for either x or y ***(M1)***

$x=-1$ and $x=2$ **OR** $y=-2$ and $y=1$ ***A1***

$(z_1=)\ -1+\mathrm{i}$ **AND** $(z_2=)\ 2-2\mathrm{i}$ (or equivalent) ***A1***

Note: Accept $x=-1,\ y=1$ **AND** $x=2,\ y=-2$.

[7 marks]

(d) let z_1, z_2, z_3 and z_4 be roots of $z^4+ax^3+bz^2+cz+d=0$

EITHER

$a=-\left(z_1+z_2+z_3+z_4\right)$ ***(A1)***

OR

expanding $\left(z-z_1\right)\left(z-z_2\right)\left(z-z_3\right)\left(z-z_4\right)$ to obtain term in z^3

$-\left(z_1+z_2\right)z^3-\left(z_3+z_4\right)z^3$ ***(A1)***

THEN

recognizing that z_3 and z_4 are conjugates of z_1 and z_2 (seen anywhere) ***(M1)***

roots are $-1+\mathrm{i},\ -1-\mathrm{i},\ 2-2\mathrm{i},\ 2+2\mathrm{i}$ ***A1***

$a=-\left((-1+\mathrm{i})+(-1-\mathrm{i})+(2-2\mathrm{i})+(2+2\mathrm{i})\right)$

$=-2$ ***A1***

[4 marks]

Total [21 marks]

8. (a) $\mathrm{Im}\left(\mathrm{e}^{2ti}\left(\mathrm{e}^{i}+\mathrm{e}^{3i}\right)\right)$

$=\mathrm{Im}\left(\mathrm{e}^{(2t+1)i}+\mathrm{e}^{(2t+3)i}\right)$ ***A1***

Note: This ***A1*** is for clearly showing that the powers are added.

Accept alternative notation for the step of adding the arguments e.g. $(\cos(2t)+i\sin(2t))(\cos(1)+i\sin(1))=\cos(2t+1)+i\sin(2t+1)$

$=\sin\left(2t+1\right)+\sin\left(2t+3\right)$ ***A1***

$=h(t)$ ***AG***

Note: Accept argument in reverse

[2 marks]

(b) $\mathrm{e}^{i}+\mathrm{e}^{3i}=1.08060...\mathrm{e}^{2i}$

$1.08\mathrm{e}^{2i}$ ***A1A1***

Note: Award ***A1*** for modulus ***A1*** for argument

$r=1.08\left(=2\cos(1)\right)\ ,\ \theta=2$

[2 marks]

(c) **METHOD 1**

attempt to use their answers to part (a) and (b) to write $h(t)$ as the imaginary part of a number in the form $r\mathrm{e}^{i\theta}$ ***(M1)***

$h(t)=\mathrm{Im}\left(\mathrm{e}^{2ti}\left(1.08060...\mathrm{e}^{2i}\right)\right)$

$=\mathrm{Im}\left(1.08060...\mathrm{e}^{(2t+2)i}\right)$ ***(A1)***

$h(t)=1.08060...\sin(2t+2)$

$=1.08\sin(2t+2)\left(=2\cos(1)\sin(2t+2)\right)$ ***A1***

METHOD 2

Considering the graph of $h(t)$

amplitude $=1.08060...$

$p=1.08\left(=2\cos(1)\right)$ ***(A1)***

attempt to consider the horizontal shift of the graph of h ***(M1)***

first negative zero of graph is -1

$h(t)=1.08060...\sin(2(t+1))\left(=1.08060...\sin(2t+2)\right)$

$=1.08\sin(2t+2)\left(=2\cos(1)\sin(2t+2)\right)$ ***A1***

[3 marks]

Total [7 marks]

10. (a) 15 ***A1***

[1 mark]

(b) attempt to add 11 cards onto a stack with 3 rows OR attempt to consider all 4 rows ***(M1)***

valid diagram with 4 rows OR $t_4 = 15 + 11$ OR $t_4 = 2 + 5 + 8 + 11$

$= 26$ ***A1***

[2 marks]

(c) **METHOD 1**

recognition that t_n is a sum of an arithmetic sequence ***(M1)***

$t_n = 2 + 5 + 8 + 11 + ...$

attempt to use formula for the sum of n terms of an arithmetic sequence ***M1***

$t_n = \frac{n}{2}\left(2\left(2\right) + 3(n-1)\right)$ ***A1***

$t_n = \frac{n}{2}\left(3n + 1\right)$ ***AG***

METHOD 2

attempt to split t_n into the total number of stacked and horizontal cards ***(M1)***

stacked $2 + 4 + 6 + ... = \frac{n}{2}\left(4 + 2(n-1)\right)\left(= \frac{n}{2}\left(2n + 2\right)\right)$ ***A1***

horizontal $0 + 1 + 2 + ... = \frac{n}{2}\left(0 + 1(n-1)\right)\left(= \frac{n}{2}\left(n - 1\right)\right)$ ***A1***

$t_n = \frac{n}{2}\left(4 + 2(n-1)\right) + \frac{n}{2}\left(0 + 1(n-1)\right)\left(= \frac{n}{2}\left(2n + 2\right) + \frac{n}{2}\left(n - 1\right)\right)$

$t_n = \frac{n}{2}\left(3n + 1\right)$ ***AG***

continued…

Question 10 continued.

METHOD 3

recognition that a stack with n rows is made up of complete triangles with the bottom row of horizontal cards removed and that the numbers of complete triangle cards form an arithmetic sequence ***(M1)***

$t_n = (3+6+9+12+...+3n) - n$ OR $t_n = 3(1+2+3+4+...+n) - n$

attempt to use formula for the sum of n terms of an arithmetic sequence ***M1***

$t_n = \frac{n}{2}(2(3)+3(n-1)) - n$ OR $t_n = 3 \times \frac{n}{2}(1+n) - n$ ***A1***

$t_n = \frac{n}{2}(3n+1)$ ***AG***

[3 marks]

(d) attempt to solve $\frac{n(3n+1)}{2} \leq 14(52) (= 728)$ (***M1)***

Note: Accept an attempt to solve an equation for (***M1)***.

$21.8642...$ OR $n = 21, t_n = 672$ and $n = 22, t_n = 737$ ***(A1)***

max number of rows is 21 ***A1***

[3 marks]

continued…

Question 10 continued.

(e) **EITHER**

attempt to solve by listing at least six values of t_n (***M1)***

$2, 7, 15, 26, 40, 57...$

OR

recognition that $\frac{\frac{1}{2}n(3n+1)}{52}$ must be an integer ***(M1)***

$\frac{1}{2}n(3n+1) = 52k \ \left(\text{where } k \text{ is an integer}\right)$

THEN

min number of rows is 13 ***A1***

Note: Award (***M1)A0*** for an answer of 5 packs.

Award ***M0A0*** for any answer resulting from solving $\frac{1}{2}n(3n+1) = 52$.

[2 marks]

continued…

Question 10 continued.

(f) **EITHER**

attempt to use Pythagoras's Theorem or trigonometry to find the height of an equilateral triangle with sides 88mm ***(M1)***

height= $\sqrt{88^2-44^2}$ OR $88\sin 60^\circ$ OR $88\cos 30^\circ$ OR $44\tan 60^\circ$ OR

$\frac{44}{\tan 30^\circ}$ OR $44\sqrt{3}\left(=76.2102...\right)$ ***(A1)***

attempt to solve $44n\sqrt{3}>2000$ OR $\text{their perpendicular height}\times n>2000$ ***(M1)***

Note: Accept an attempt to solve an equation for (***M1)***.

OR

attempt to use trigonometry to find the side of an equilateral triangle with height 2000mm ***(M1)***

side= $\frac{2000}{\sin 60^\circ}$ OR $\frac{2000}{\cos 30^\circ}$ OR $\frac{4000}{\sqrt{3}}\left(=2309.40...\right)$ ***(A1)***

attempt to solve $88n>2309.40...$ OR $88n>\text{their side}$ ***(M1)***

Note: Accept an attempt to solve an equation for (***M1)***.

THEN

$n>26.2431...$

so min number of rows is 27 ***(A1)***

$t_{27}=1107$ ***A1***

[5 marks]

Total [16 marks]

12. (a) attempt to replace x with $-x$ ***M1***

$f_n(-x) = \sum_{r=0}^{n}\left(-2\left(-x\right)^2\right)^r$ OR $-2\left(-x\right)^2 = -2x^2$ (seen anywhere) ***A1***

$f_n(-x) = f_n(x)$ ***A1***

so f_n is even for all values of n ***AG***

[3 marks]

(b) (i) $f_3(x) = 1 - 2x^2 + \left(-2x^2\right)^2 + \left(-2x^2\right)^3$ ***A1***

$= 1 - 2x^2 + 4x^4 - 8x^6$ ***AG***

(ii) $= 1 - 2x^2 + 4x^4 - 8x^6 + 16x^8$ ***A1***

[2 marks]

(c) (i) recognition of geometric series with common ratio $-2x^2$ ***(M1)***

converges for $\left|-2x^2\right| < 1 \left(\Rightarrow x^2 < \frac{1}{2}\right)$ ***(A1)***

$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \left(\Rightarrow -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}\right)$

largest $K = \frac{1}{\sqrt{2}} \left(= \frac{\sqrt{2}}{2}\right)$ ***A1***

(ii) use of formula for S_∞ of a geometric series with first term 1, common ratio $-2x^2$ ***(M1)***

$f(x) = \frac{1}{1-\left(-2x^2\right)}$

$\left(f(x) =\right) \frac{1}{1+2x^2}$ ***A1***

$a = 1,\ b = 2$

[5 marks]

continued…

Question 12 continued.

(d) (i) g is a one-to-one function ***(A1)***

since g is a (strictly) decreasing function OR g has no points of zero gradient (turning points) ***R1***

(ii) attempt to rearrange and swop x and y (at any stage) ***(M1)***

$$x = \frac{1}{1+2y^2} \Rightarrow x + 2xy^2 = 1$$

$$y^2 = \frac{1-x}{2x}$$ ***(A1)***

$$y = \pm\sqrt{\frac{1-x}{2x}}$$

$$g^{-1}(x) = \sqrt{\frac{1-x}{2x}}$$ ***A1***

Note: Award ***A0*** if $g^{-1}(x)$ is missing.

Award **FT** ***A1*** marks only if f is of the correct form $f(x) = \frac{1}{a+bx^2}$

(domain) $\frac{1}{2} < x \leq 1$ ***A1***

Note: This ***A1*** can be awarded independently.

[6 marks]

continued…

Question 12 continued.

(e)

Note: Throughout part (e), do not award ***A1*** marks as **FT** from part (d)

METHOD 1

curves intersect at $x = 0.5897545107...$ ***(A1)***

attempt to add the areas to the left and to the right of the point of intersection ***(M1)***

$$\int_0^{0.589...} g(x)\mathrm{d}x + \int_{0.589...}^{1} g^{-1}(x)\mathrm{d}x \left(= \int_0^{0.589...} \frac{1}{1+2x^2}\mathrm{d}x + \int_{0.589...}^{1} \sqrt{\frac{1-x}{2x}}\mathrm{d}x \right)$$

$= 0.491548... + 0.143738...$ ***(A1)***

Note: Award ***A1*** for one correct value seen, dependent on ***(M1)***

$= 0.635286...$

Area of $R = 0.635$ ***A1***

METHOD 2

curves intersect at $x = 0.5897545107...$ ***(A1)***

attempt to find the area between $y = g(x)$ and $y = x$ to the left of the point of intersection ***(M1)***

$$\int_0^{0.589...} g(x)\mathrm{d}x - \int_0^{0.589...} x\mathrm{d}x = \int_0^{0.589...} \left(\frac{1}{1+2x^2} - x \right)\mathrm{d}x$$

$= 0.317643...$ ***(A1)***

$2\left(0.3176432617...\right) = 0.635286...$

Area of $R = 0.635$ ***A1***

continued…

Question 12 continued.

METHOD 3

curves intersect at $x = 0.5897545107...$ ***A1***

attempt to find the area under $y = g^{-1}(x)$ and $y = x$ to the right of the point of intersection and the area of a square of side $x = 0.589...$ ***(M1)***

$$0.589...^2 + 2\int_{0.589...}^{1} g^{-1}(x)\mathrm{d}x \left(= 0.589...^2 + 2\int_{0.589...}^{1} \sqrt{\frac{1-x}{2x}}\mathrm{d}x \right)$$

$= 0.347810... + 2(0.143738...)$ ***A1***

Note: Award ***A1*** for $0.143..$ seen, dependent on ***(M1)***

$= 0.635286...$

Area of $R = 0.635$ ***A1***

[4 marks]

Total [20 marks]

4. (a) **METHOD 1**

$30000\left(1-\frac{15}{100}\right)^{10}$ OR $30000(0.85)^{10}$ ***(M1)***

$5906.232130...$

€ 5906.23 ***A1***

Note: Award ***(M1)*** for the correct list of values

METHOD 2

$\mathrm{PV} = \pm 30000$

$\mathrm{FV} = \mp 5906.23213...$

$\mathrm{I\%} = -15\%$

$\mathrm{P/Y} = 1$

$\mathrm{C/Y} = 1$

$\mathrm{N} = 10$ ***(M1)***

€ 5906.23 ***A1***

[2 marks]

(b) **METHOD 1**

$r = c - i\left(= 1.5 - 0.8\right)$

$r = 0.7(\%)$ ***(A1)***

attempt to substitute their values into inequality ***(M1)***

$50000\left(1+\frac{0.7}{100}\right)^{n} > 55000$ (accept equality)

correct critical value OR at least one correct crossover value ***(A1)***

$n = 13.6633\ldots$ OR $n = 13, €54746.09$ OR $n = 14, €55129.31$

14 (months) ***A1***

[4 marks]

Note: Award ***(A1)(M1)(A0)A0*** for 164 (months)

continued…

Question 4 continued.

METHOD 2

$\dfrac{1+\dfrac{1.5}{100}}{1+\dfrac{0.8}{100}} = 1.00694...\ \left(r = 0.694...(\%)\right)$ ***(A1)***

attempt to substitute their values into inequality ***(M1)***

$50000(1.00694...)^n > 55000$ (accept equality)

correct critical value OR at least one correct crossover value ***(A1)***

$n = 13.7722\ldots$ OR $n = 13, €54706.84$ OR $n = 14, €55086.75$

14 (months) ***A1***

Note: Award ***(A1)(M1)(A0)A0*** for 164 (months)

[4 marks]

Total [6 marks]

7. (a) attempt to find common ratio ***(M1)***

$80r^3 = 0.74088$

$r = 0.21$ ***(A1)***

$u_2 = 16.8$ ***A1***

[3 marks]

(b) **METHOD 1**

attempt to find common difference ***(M1)***

$u_1 + 10d - u_1 = 16.8 - 80 \ (10d = -63.2)$

$d = -6.32$ ***(A1)***

EITHER

attempt to find the value of n using table or graph of S_n ***(M1)***

$n = 13$

$S_{13} = \frac{13}{2}(2 \times 80 - 6.32 \times 12)$

OR

attempt to find greatest value of S_n using table or graph ***(M1)***

THEN

$= 547.04$ ***A1***

Note: Award ***(M1)(A1)(M1)A0*** for candidates who give maximum at $(13.1582..., 547.119...)$.

[4 marks]

METHOD 2

attempt to find common difference ***(M1)***

$u_1 + 10d - u_1 = 16.8 - 80 \ (10d = -63.2)$

$d = -6.32$ ***(A1)***

attempt to find the value of n using $u_n > 0$ ***(M1)***

$80 + (n-1)(-6.32) > 0$ (accept $=$) $\Rightarrow n < 13.6582...$ OR $n = 13$

$S_{13} = \frac{13}{2}(2 \times 80 - 6.32 \times 12)$

$= 547.04$ ***A1***

[4 marks]

Total [7 marks]

7. $\frac{1}{\sqrt{q-x^2}} = q^{-\frac{1}{2}}\left(1-\frac{x^2}{q}\right)^{-2}$ OR $\frac{1}{\sqrt{q-x^2}} = \left(q-x^2\right)^{-\frac{1}{2}}$ ***(A1)***

substituting into extension of binomial theorem formula to find term in x^6 ***(M1)***

$-\frac{1}{2}\times\left(-\frac{1}{2}-1\right)\times\left(-\frac{1}{2}-2\right)\times\frac{1}{3!}\times\left(-\frac{x^2}{q}\right)^3\left(=\left(\left(-\frac{5}{16}\right)\times\left(-\frac{x^2}{q}\right)^3\right)\right)$ OR $\binom{-\frac{1}{2}}{r}\times q^{-\frac{1}{2}-3}\times\left(-x^2\right)^3$ ***(A1)***

$q^{-\frac{1}{2}}\times\left(-\frac{5}{16}\right)\times\left(-\frac{1}{q}\right)^3\left(=\frac{5}{16}q^{-\frac{7}{2}}\right)$ OR $\frac{5}{16}q^{-\frac{7}{2}}x^6$ ***(A1)***

equating their coefficient to 5120 or their term in x^6 to $5120x^6$ ***(M1)***

$\frac{5}{16}q^{-\frac{7}{2}} = 5120$ OR $\frac{5}{16}q^{-\frac{7}{2}}x^6 = 5120x^6$

$q = 0.0625\left(=\frac{1}{16}\right)$ ***A1***

[6 marks]

8. METHOD 1

EITHER

number of games played by $n-1$ players ${}^{n-1}\mathrm{C}_2 \times 2$ **OR** $(n-1)(n-2)$ ***(A2)***

total number of games is ${}^{n-1}\mathrm{C}_2 \times 2 + 7$ **OR** $(n-1)(n-2)+7$ ***(A1)***

OR

total number of games that should have been played is $2 \times {}^{n}\mathrm{C}_2$ **OR** $n(n-1)$ ***(A1)***

Stephen should have played $2(n-1)$ so he missed $2(n-1)-7$ games ***(A1)***

total number of games played is $n(n-1)-\big(2(n-1)-7\big)$ ***(A1)***

THEN

equating their total number of games to 513 ***(M1)***

$n = 24$ ***A1***

METHOD 2

there are N students other than Stephen

number of games played by N students is $2\big(\big(N-1\big)+\big(N-2\big)+\ldots+2+1\big)\ \big(=N(N-1)\big)$ ***(A1)***

total number of games that are played $N(N-1)+7$ ***(A1)***

equating their total number of games to 513 ***(M1)***

$N = 23$ ***(A1)***

number of students in class $= 24$ ***A1***

[5 marks]

1. (a) attempt to find the common difference ***(M1)***

$12 = 36 + 4d$ OR $\frac{12-36}{5-1}$

$d = -6$ ***(A1)***

attempt to use the term formula for an arithmetic sequence ***(M1)***

$u_{13} = u_1 + 12(-6)$ OR $u_5 + 8(-6)$

$u_{13} = -36$ ***A1***

[4 marks]

(b) **METHOD 1**

attempt to set $S_n = 0$ ***(M1)***

$\frac{n}{2}(2(36) + (n-1)(-6)) = 0$ OR $\frac{n(36+u_n)}{2} = 0 \Rightarrow u_n = -36$

$n = 13$ ***A1***

Note: Condone $n = 0, 13$

METHOD 2

attempt to recognize symmetry ***(M1)***

$36 + 30 + ... + 0 + ... - 30 - 36$

$n = 13$ ***A1***

[2 marks]

Total [6 marks]

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6. METHOD 1 Row Reduction

attempt to reduce a matrix or eliminate one variable ***(M1)***

two correct rows or equivalent equations ***A1A1***

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 2 & 2 & 1 & 14 \\ 1 & 2 & \alpha & \beta \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 2 & 3 & 6 \\ 0 & 2 & 1+\alpha & \beta - 4 \end{bmatrix}$$

correct third row or equivalent equation ***A1***

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & \alpha - 2 & \beta - 10 \end{bmatrix}$$

equating the third row coefficients to zero ***(M1)***

$\alpha = 2$ and $\beta = 10$ ***A1***

METHOD 2 Writing an equation in one variable

attempt to eliminate either x or z using the first two equations (seen anywhere) ***(M1)***

$2(z+4)+2y+z=14$ OR $2x+2y+(x-4)=14$

one correct equation ***A1***

$2y+3z=6$ **OR** $3x+2y=18$

correct substitution into the third equation to create an equation in either x or z ***A2***

$x+18-3x+\alpha(x-4)=\beta$ **OR** $z+4+6-3z+\alpha z=\beta$

OR $(\alpha-2)x=\beta+4\alpha-18$ **OR** $z(\alpha-2)=\beta-10$

attempt to set coefficients equal to zero for infinite solutions ***(M1)***

$\alpha = 2$ and $\beta = 10$ ***A1***

continued…

Question 6 continued.

METHOD 3 Subtracting and comparing equations

attempt to add or subtract two equations ***(M1)***

$x+2y+2z=10$ **OR** $2x+2y+(\alpha-1)z=\beta+4$ **OR** $x+(1-\alpha)z=14-\beta$ ***A2***

attempt to compare coefficients with the third equation ***(M1)***

$\alpha-1=1$, $\beta+4=14$, $1-\alpha=-1$, $14-\beta=4$

$\alpha=2$ and $\beta=10$ ***A1A1***

METHOD 4 Vector approach

attempt to find direction vector of line of intersection using π_1 and π_2 ***(M1)***

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}\times\begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$\left(\vec{d}=\right)\begin{pmatrix}2\\-3\\2\end{pmatrix}$$ ***A1***

normal of π_3 is perpendicular to $\begin{pmatrix}2\\-3\\2\end{pmatrix}$ ***(M1)***

$$\begin{pmatrix}1\\2\\\alpha\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\2\end{pmatrix}=0$$

$\alpha=2$ ***A1***

correct substitution of a point common to the first two planes into the third plane ***A1***

$0+2(9)+2(-4)$, $4+2(3)+2(0)$

$\beta=10$ ***A1***

continued…

Question 6 continued.

METHOD 5 Finding a general point on the line of intersection

attempt to find intersection of π_1 and π_2 ***(M1)***

$x = 4 + z,\ \ y = 3 - \frac{3}{2}z$

$x = 4 + 2t,\ \ y = 3 - 3t,\ \ z = 2t$ **OR** $\mathbf{r} = (4, 3, 0) + t(2, -3, 2)$ ***A1A1***

correct substitution of general point into π_3 ***A1***

$(4 + 2t) + 2(3 - 3t) + \alpha(2t),\quad 10 + t(2\alpha - 4)$

recognizing that the coefficient of t must be zero OR the constant term must be β ***(M1)***

$2\alpha - 4 = 0$ OR $\beta = 10$

$\alpha = 2$ and $\beta = 10$ ***A1***

Total [6 marks]

11. (a) attempt to multiply by conjugate ***M1***

$$\frac{4\mathrm{i}}{1+\mathrm{i}\sqrt{3}}\times\frac{1-\mathrm{i}\sqrt{3}}{1-\mathrm{i}\sqrt{3}}$$

$\frac{4\sqrt{3}+4\mathrm{i}}{4}$, $\mathrm{i}\left(1-\mathrm{i}\sqrt{3}\right)$ ***A1***

$w=\sqrt{3}+\mathrm{i}$ ***AG***

[2 marks]

(b) (i) attempt to find either $|w|$ or $\arg w$ (could be seen in (ii)) ***(M1)***

$|w|=\sqrt{1^2+3}$ OR $\arctan\left(\frac{1}{\sqrt{3}}\right)$ OR labelled diagram

$|w|=2$ ***A1***

[2 marks]

(ii) $\arg w=\frac{\pi}{6}$ (accept 30°) ***A1***

[1 mark]

(c) attempt to find a root using de Moivre's theorem with $n=\frac{1}{2}$ ***(M1)***

$\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{12}}$ ***A1***

attempt to find second root by adding or subtracting π from *their* argument ***(M1)***

$\sqrt{2}\mathrm{e}^{\frac{-11\pi\mathrm{i}}{12}}$ ***A1***

[4 marks]

continued…

Question 11 continued.

(d) (i) **METHOD 1**

attempt to expand and equate to $\sqrt{3}+\mathrm{i}$ ***(M1)***

$a^2-b^2+2ab\mathrm{i}=\sqrt{3}+\mathrm{i}$

at least one correct equation ***A1***

$a^2-b^2=\sqrt{3}$ **OR** $2ab=1$

substituting $b=\frac{1}{2a}$ ***(M1)***

$a^2-\left(\frac{1}{2a}\right)^2=\sqrt{3}$

$4a^4-4\sqrt{3}a^2-1=0$ **OR** $a^4-\sqrt{3}a^2-\frac{1}{4}=0$ ***A1***

attempt to solve quadratic in a^2 ***M1***

$a^2=\frac{4\sqrt{3}\pm\sqrt{48+16}}{8}$ OR $\frac{\sqrt{3}}{2}\pm 1$

since $a^2>0$ (accept $a>0$) ***R1***

$a^2=\frac{\sqrt{3}}{2}+1$ ***AG***

continued…

Question 11 continued.

METHOD 2

$\sqrt{w} = a + bi$ ***(A1)***

$a + bi = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ ***A1***

attempt to equate $\mathrm{Re}\left(\sqrt{w}\right)$ ***M1***

$a = \sqrt{2}\cos\frac{\pi}{12}$

$a^2 = 2\cos^2\frac{\pi}{12}$ ***A1***

attempt to use double angle formula ***(M1)***

$\cos 2\alpha = 2\cos^2\alpha - 1$

$a^2 = 2\left(\frac{\cos\frac{\pi}{6}+1}{2}\right) = \left(\cos\frac{\pi}{6}+1\right)$ ***A1***

$\left(a^2 =\right)\frac{\sqrt{3}}{2}+1$ ***AG***

[6 marks]

(ii) attempt to substitute a^2 into $b^2 = a^2 - \sqrt{3}$ OR $b^2 = \frac{1}{4a^2}$ ***(M1)***

$\left(\frac{\sqrt{3}}{2}+1\right) - \sqrt{3}\,,\ \frac{1}{4\left(\frac{\sqrt{3}}{2}+1\right)}$

$b^2 = 1 - \frac{\sqrt{3}}{2}$ ***A1***

Note: Accept the unrationalized answer of $\frac{1}{2\sqrt{3}+4}$.

[2 marks]

continued…

Question 11 continued.

(e) attempt to set up an equation using an argument of $\frac{\pi}{12}$ ***(M1)***

$\tan\left(\frac{\pi}{12}\right) = \frac{b}{a} = \sqrt{\frac{b^2}{a^2}}$ ***(A1)***

substitution of the given a and their b ***M1***

$\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}}$ **OR** $\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$ ***A1***

$\sqrt{\frac{\left(2-\sqrt{3}\right)^2}{2^2-3}}$

$\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3}$ **,** accept $p = 2,\ q = -1$ ***A1***

[5 marks]

Total [22 marks]

1.

METHOD 1

attempt to use $u_n = u_1 + (n-1)d$ ***(M1)***

$u_1 + 6d = 6$ ***(A1)***

$u_1 + 5d + u_1 + 11d = 24$ ***(A1)***

$2u_1 + 16d = 24 \quad (u_1 + 8d = 12)$

attempt to solve their equations simultaneously ***(M1)***

$u_1 = -12$ **and** $d = 3$ ***A1***

METHOD 2

attempt to express u_6 and u_{12} in terms of the 7th term and d ***(M1)***

$u_6 = 6 - d$ ***(A1)***

$u_{12} = 6 + 5d$ ***(A1)***

setting their sum of u_6 and u_{12} equal to 24 ***(M1)***

$12 + 4d = 24$

$d = 3$ **and** $u_1 = -12$ ***A1***

[5 marks]

11. (a) (i) $w = 2e^{i\frac{\pi}{3}}$ ***A1A1***

Note: Award ***A1*** for $r = 2$ and ***A1*** for $\alpha = \frac{\pi}{3}$.

(ii) attempt to apply De Moivre's theorem: ***M1***

$\left(1+i\sqrt{3}\right)^8 = 2^8 e^{\frac{8\pi i}{3}}$ OR $2^8 \text{cis}\left(\frac{8\pi}{3}\right)$ (or equivalent) ***A1***

$\left(1-i\sqrt{3}\right)^8 = 2^8 e^{-\frac{8\pi i}{3}}$ OR $2^8 \text{cis}\left(-\frac{8\pi}{3}\right)$ (or equivalent) ***A1***

$\left(\left(1+i\sqrt{3}\right)^8 + \left(1-i\sqrt{3}\right)^8 = \right)256\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)+256\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$ OR

$2^8\left(\cos\left(\frac{2\pi}{3}\right)+\cos\left(-\frac{2\pi}{3}\right)\right)$ ***A1***

$= -256$ ***AG***

[6 marks]

(b) recognition that $1 - i\tan\theta = 1 + i\tan(-\theta)$ (***M1***)

$\left(1+i\tan\theta\right)^4 + \left(1-i\tan\theta\right)^4 \equiv \left(1+i\tan\theta\right)^4 + \left(1+i\tan\left(-\theta\right)\right)^4$

$\equiv \frac{\cos 4\theta + i\sin 4\theta}{\cos^4\theta} + \frac{\cos(-4\theta)+i\sin(-4\theta)}{\cos^4(-\theta)}$ (***A1***)

$\equiv \frac{\cos 4\theta + i\sin 4\theta}{\cos^4\theta} + \frac{\cos 4\theta - i\sin 4\theta}{\cos^4\theta}$ ***A1***

$\equiv \frac{2\cos 4\theta}{\cos^4\theta}$ ***AG***

[3 marks]

continued…

Question 11 continued

(c) considering $\cos 4\theta = 0$ ***(M1)***

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ or equivalent ***(A1)***

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ or equivalent ***A1***

hence roots are $z = \mathrm{i}\tan\left(\frac{k\pi}{8}\right)$ where $k = 1,3,5,7$ (or equivalent) ***A1***

Note: For the first two ***A1***'s allow extra correct solutions.
For the final mark award ***A0*** for more than four solutions.

Note: For candidates who use $4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}$, award ***M1A1A0A0.***

[4 marks]

(d) attempt to use binomial expansion ***(M1)***

$(1+z)^4 + (1-z)^4 = (1+4z+6z^2+4z^3+z^4) + (1-4z+6z^2-4z^3+z^4)$

$= 2 + 12z^2 + 2z^4$ ***A1***

attempt to solve (for z^2) $z^4 + 6z^2 + 1 = 0$ using formula or completing the square ***(M1)***

$z^2 = -3 \pm 2\sqrt{2}$ or equivalent ***A1***

substituting $z = \mathrm{i}\tan\left(\frac{\pi}{8}\right)$ ***M1***

$-\tan^2\left(\frac{\pi}{8}\right) = -3 \pm 2\sqrt{2}$ OR $\tan^2\left(\frac{\pi}{8}\right) = 3 \pm 2\sqrt{2}$

$\tan\left(\frac{\pi}{8}\right) = \pm\sqrt{3 \pm 2\sqrt{2}}$ ***A1***

continued…

Question 11 continued

EITHER

$\tan\left(\frac{\pi}{8}\right)$ is the smallest positive root (hence take positive square root and negative sign inside) ***R1***

OR

$0 < \tan\frac{\pi}{8} < \tan\frac{3\pi}{8}$ ***R1***

THEN

$\tan\left(\frac{\pi}{8}\right) = \sqrt{3-2\sqrt{2}}$ ***AG***

[7 marks]

Total [20 marks]

---

2. ${}^{10}P_3$ OR $10\times9\times8$ **OR** ${}^{10}C_3\times3!$ **OR** ${}^{10}C_7\times3!$ $(=720)$ ***(A1)***

correct number of ways the remaining four members can be selected

${}^{7}C_4$ **OR** 35 ***(A1)***

Note: The first two ***A1*** marks are to be awarded independently.

attempt to use the multiplication principle with their values ***(M1)***

720×35

$25\,200$ ***A1***

Total [4 marks]

6. (a) $u_1 = \frac{53}{1000}$

$r = \frac{1}{100}$

$\frac{\frac{53}{1000}}{1-\frac{1}{100}}$ ***(M1)(A1)***

$S_\infty = \frac{53}{990}\left(= 0.0\overline{53}\right)\left(= 0.05\dot{3}\right)$ ***A1***

Note: Award ***(M1)*** for an attempt to use the sum to infinity formula.

[3 marks]

(b) $0.2 + \frac{53}{990}$ ***M1***

$\frac{251}{990}$ ***A1***

[2 marks]

Total [5 marks]

7. (a) attempt to solve using gdc or manual row operations or elimination ***(M1)***

$\begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{17}{4} \\ 0 & 1 & -\frac{3}{2} & \frac{3}{4} \end{pmatrix}$ OR $\begin{matrix} 6 & 18 & -24 & 39 \\ 6 & -2 & 6 & 24 \end{matrix}$ OR $\begin{matrix} 6x+18y-24z=39 \\ 6x-2y+6z=24 \end{matrix}$

$x=-\frac{1}{2}t+\frac{17}{4},\ y=\frac{3}{2}t+\frac{3}{4}, z=t$ $(x=-0.5t+4.25,\ y=1.5t+0.75, z=t)$ OR

$y=t, z=\frac{2}{3}t-\frac{1}{2}, x=\frac{9}{2}-\frac{1}{3}t$ $(y=t, z=0.667t-0.5, x=4.5-0.333t)$ OR

$x=t, y=\frac{27}{2}-3t, z=\frac{17}{2}-2t$ $x=t, y=13.5-3t, z=8.5-2t$ or equivalent ***A1***

[2 marks]

(b) **METHOD 1**

attempt to use GDC or manual row operations or elimination to solve the 3 equations ***(M1)***

$x=29,\ y=-73.5\left(-\frac{147}{2}\right), z=-49.5\left(-\frac{99}{2}\right)$ ***A1***

METHOD 2

attempt to substitute their parametric solution from a) into the third equation with $a=3$ and $k=-3$ and solve to find the value of the parameter ***(M1)***

$3\left(-\frac{1}{2}t+\frac{17}{4}\right)+12\left(\frac{3}{2}t+\frac{3}{4}\right)-16t=-3$

$x=29,\ y=-73.5\left(-\frac{147}{2}\right), z=-49.5\left(-\frac{99}{2}\right)$ ***A1***

[2 marks]

(c) (i) $a=4$ ***A1***

(ii) $k=26$ ***A1***

[2 marks]

Total [6 marks]

8. (a) $15!$ OR $1.31\times10^{12}\left(1.30767....\times10^{12}\right)$ ***A1***

[1 mark]

(b) recognise that the books need to be arranged in separate groups ***(M1)***

$6!(720)$ OR $5!(120)$ OR $4!(24)$ seen

recognise that the three groups can be placed in any order (**M1)**

$3!$ seen

$(6!\times5!\times4!\times3!)=12441600$ ***A1***

Note*:* Accept 12400000 or $6!\times5!\times4!\times3!$.

[3 marks]

(c) recognition of the combination(s) required ***(M1)***

${}^{6}C_{4}(=15)$ OR ${}^{5}C_{4}(=5)$ OR ${}^{4}C_{4}(=1)$ seen

recognition that the sum of 3 different cases is required ***(M1)***

${}^{6}C_{4}+{}^{5}C_{4}+{}^{4}C_{4}=15+5+1$

21 ***A1***

[3 marks]

Total [7 marks]

9. assume $\sqrt{2xy} > \frac{1}{2}(2x+y)$ ***M1***

Note: A statement of assumption of truth is required for example "assume", "presume", "suppose" or "let … be true".

Award **M0** for use of 'let' only or "consider".

attempt to square both sides (which is valid since both x, y are positive) ***M1***

Note: This ***M1*** is independent of the previous method mark.

$2xy > \frac{1}{4}(2x+y)^2$

$8xy > 4x^2 + 4xy + y^2$

$0 > 4x^2 - 4xy + y^2$ OR $0 > x^2 - xy + \frac{y^2}{4}$ ***A1***

$0 > (2x-y)^2$ OR $0 > \left(x - \frac{y}{2}\right)^2$ ***A1***

a square number can never be negative hence contradiction ***R1***

Note: Do not award ***A0R1***

therefore $\sqrt{2xy} \leq \frac{1}{2}(2x+y)$ ***AG***

[5 marks]

Section B

APOLLO

Functions

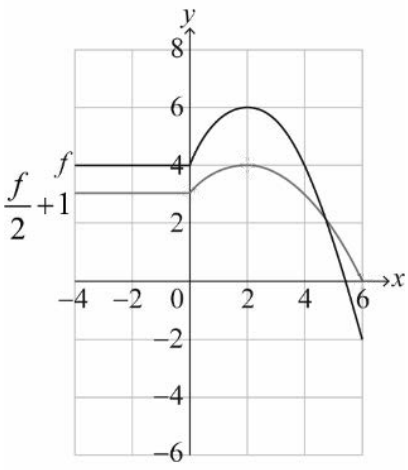
68 questions

1. (a) (i) $f(2)=6$ ***A1***

(ii) $(f\circ f)(2)=-2$ ***A1***

[2 marks]

(b)



M1A1A1

Note: Award ***M1*** for an attempt to apply any vertical stretch or vertical translation,
A1 for a correct horizontal line segment between -4 and 0 (located roughly at $y=3$),
A1 for a correct concave down parabola including max point at (2,4) and for correct end points at (0,3) and (6,0) (within circles). Points do not need to be labelled.

[3 marks]
Total [5 marks]

7. $\alpha+\beta+\alpha+\beta=k$ ***(A1)***

$\alpha+\beta=\frac{k}{2}$

$\alpha\beta(\alpha+\beta)=-3k$ (***A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right)=-3k\ \left(-\frac{k^3}{8}=-3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8}+3k=0$ (or equivalent) for k ***(M1)***

$k=2\sqrt{6}\ \left(=\sqrt{24}\right)(k>0)$ ***A1***

Note: Award ***A0*** for $k=\pm2\sqrt{6}\ \left(\pm\sqrt{24}\right)$.

[5 marks]

11. (a) attempt to solve $4x^2-1=0$ e.g. by factorising $4x^2-1$ ***(M1)***

$p=\frac{1}{2}, q=-\frac{1}{2}$ or vice versa ***A1***

[2 marks]

(b) attempt to use quotient rule or product rule ***(M1)***

EITHER

$$f'(x)=\frac{3\left(4x^2-1\right)-8x\left(3x+2\right)}{\left(4x^2-1\right)^2}\left(=\frac{-12x^2-16x-3}{\left(4x^2-1\right)^2}\right)$$ ***A1A1***

Note: Award ***A1*** for each term in the numerator with correct signs, provided correct denominator is seen.

OR

$$f'(x)=-8x\left(3x+2\right)\left(4x^2-1\right)^{-2}+3\left(4x^2-1\right)^{-1}$$ ***A1A1***

Note: Award ***A1*** for each term.

[3 marks]

(c) attempt to find the local min point on $y=f'(x)$ OR solve $f''(x)=0$ ***(M1)***

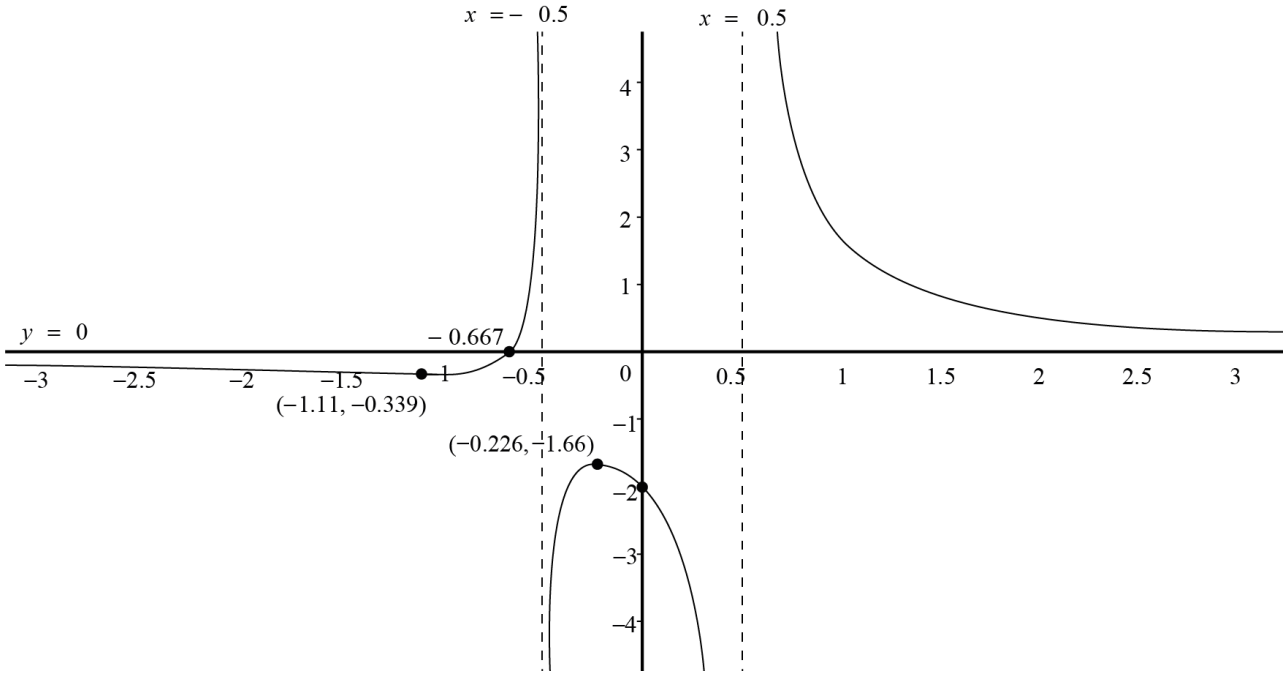
$x=-1.60$ ***A1***

[2 marks]

continued…

Question 11 continued

(d)



A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes with their equations, award ***A1*** for horizontal asymptote with equation, award ***A1*** for each correct branch including asymptotic behaviour, coordinates of minimum and maximum points (may be seen next to the graph) and values of axes intercepts.
If vertical asymptotes are absent (or not vertical) and the branches overlap as a consequence, award maximum ***A0A1A0A1A1.***

[5 marks]

continued…

Question 11 continued

e) $x = -\frac{2}{3}\left(= -0.667\right)$ ***A1***

(oblique asymptote has) gradient $\frac{4}{3}\left(= 1.33\right)$ ***(A1)***

appropriate method to find complete equation of oblique asymptote ***M1***

$$\begin{array}{r} \frac{4}{3}x - \frac{8}{9} \\ 3x+2\overline{)\,4x^2+0x-1} \\ \underline{4x^2+\frac{8}{3}x} \\ -\frac{8}{3}x-1 \\ \underline{-\frac{8}{3}x-\frac{16}{9}} \\ \frac{7}{9} \end{array}$$

$y = \frac{4}{3}x - \frac{8}{9}\left(= 1.33x - 0.889\right)$ ***A1***

Note: Do not award the final ***A1*** if the answer is not given as an equation.

[4 marks]

f) attempting to find at least one critical value $\left(x = -0.568729..., x = 1.31872...\right)$ ***(M1)***

$-\frac{2}{3} < x < -0.569$ OR $-0.5 < x < 0.5$ OR $x > 1.32$ ***A1A1A1***

Note: Only penalize once for use of $\leq$ rather than $<$.

[4 marks]
Total [20 marks]

4. (a) attempting to find the vertex ***(M1)***

$x = 1$ OR $y = -5$ OR $f(x) = 6(x-1)^2 - 5$

range is $y \geq -5$ ***A1***

[2 marks]

(b) **METHOD 1**

$(g \circ f)(x) = -(6x^2 - 12x + 1) + c \ \left(= -\left(6(x-1)^2 - 5\right) + c\right)$ ***(A1)***

EITHER

relating to the range of f OR attempting to find $g(-5)$ ***(M1)***

$5 + c \leq 0$ ***(A1)***

OR

attempting to find the discriminant of $(g \circ f)(x)$ ***(M1)***

$144 + 24(c-1) \leq 0 \ (120 + 24c \leq 0)$ ***(A1)***

THEN

$c \leq -5$ ***A1***

[4 marks]

METHOD 2

vertical reflection followed by vertical shift ***(M1)***

new vertex is $(1, 5 + c)$ ***(A1)***

$5 + c \leq 0$ ***(A1)***

$c \leq -5$ ***A1***

[4 marks]

Total [6 marks]

5. (a) $100 = A_0 e^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

(b) correct substitution of values into exponential equation ***(M1)***

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln\frac{1}{2}$ ***A1***

$\ln\frac{1}{2} = -\ln 2$ OR $-\ln\frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) if $25\,\%$ of the carbon-14 has decayed, $75\,\%$ remains ie, 75 units remain

$75 = 100e^{-\frac{\ln 2}{5730}t}$ ***(A1)***

EITHER

using an appropriate graph to attempt to solve for t ***(M1)***

OR

manipulating logs to attempt to solve for t ***(M1)***

$\ln 0.75 = -\frac{\ln 2}{5730}t$

$t = 2378.164...$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) ***A1***

[3 marks]
Total [7 marks]

12. (a) **EITHER**

$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right) = \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$ ***R1***

OR

a sketch graph of $y = f(x)$ with line symmetry in the y-axis indicated ***R1***

THEN

so $f(x)$ is an even function ***AG***

[1 mark]

(b) as $x \to \pm\infty$, $f(x) \to \arcsin 1 \left(\to \frac{\pi}{2}\right)$ ***A1***

so the horizontal asymptote is $y = \frac{\pi}{2}$ ***A1***

[2 marks]

continued…

Question 12 continued

(c) (i) attempting to use the quotient rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2x\left(x^2+1\right)-2x\left(x^2-1\right)}{\left(x^2+1\right)^2}\left(=\frac{4x}{\left(x^2+1\right)^2}\right)$ ***A1***

attempting to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(\frac{x^2-1}{x^2+1}\right)\right)$ ***M1***

let $u=\frac{x^2-1}{x^2+1}$ and so $y=\arcsin u$ and $\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{\sqrt{1-u^2}}$

$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{\left(x^2+1\right)^2}$ ***M1***

$=\frac{4x}{\sqrt{\left(x^2+1\right)^2-\left(x^2-1\right)^2}}\times\frac{1}{\left(x^2+1\right)}$ ***A1***

$=\frac{4x}{\sqrt{4x^2}}\times\frac{1}{\left(x^2+1\right)}$ ***A1***

$=\frac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ ***AG***

(ii) $f'(x)=\frac{2x}{|x|\left(x^2+1\right)}$

EITHER

for $x<0$, $|x|=-x$ ***(A1)***

so $f'(x)=-\frac{2}{x^2+1}$ ***A1***

OR

$|x|>0$ and $x^2+1>0$ ***A1***

$2x<0,\ x<0$ ***A1***

THEN

$f'(x)<0$ ***R1***

Note: Award ***R1*** for stating that in $f'(x)$, the numerator is negative, and the denominator is positive.

so f is decreasing for $x<0$ ***AG***

Note: Do not accept a graphical solution.

[9 marks]

Question 12 continued

(d) $x = \arcsin\left(\frac{y^2-1}{y^2+1}\right)$ ***M1***

$\sin x = \frac{y^2-1}{y^2+1} \Rightarrow y^2 \sin x + \sin x = y^2 - 1$ ***A1***

$y^2 = \frac{1+\sin x}{1-\sin x}$ ***A1***

domain of g is $x \in \mathbb{R}, x \geq 0$ and so the range of g^{-1} must be $y \in \mathbb{R}, y \geq 0$

hence the positive root is taken (or the negative root is rejected) ***R1***

Note: The ***R1*** is dependent on the above ***A1***.

so $\left(g^{-1}(x) =\right)\sqrt{\frac{1+\sin x}{1-\sin x}}$ ***A1***

Note: The final ***A1*** is not dependent on ***R1*** mark.

[5 marks]

(e) domain is $-\frac{\pi}{2} \leq x < \frac{\pi}{2}$ ***A1***

Note: Accept correct alternative notations, for example, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ or $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

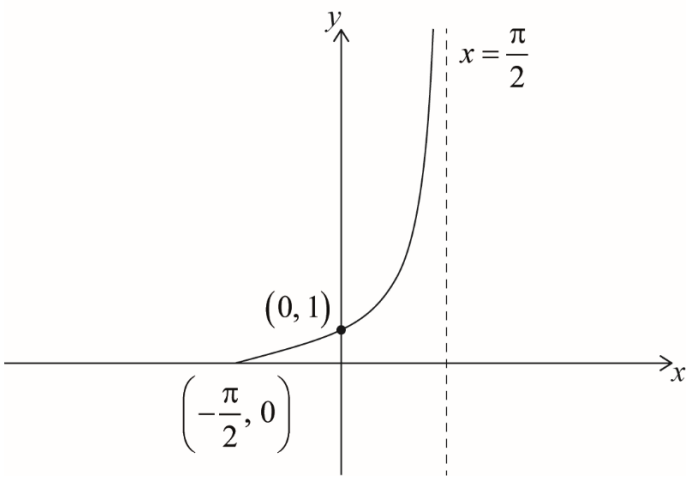
Accept $[-1.57, 1.57[$ if correct to 3 s.f.

[1 mark]

continued…

Question 12 continued

(f)



A1A1A1

Note: ***A1*** for correct domain and correct range and y-intercept at $y = 1$

A1 for asymptotic behaviour $x \to \frac{\pi}{2}$

A1 for $x = \frac{\pi}{2}$

Coordinates are not required.
Do not accept $x = 1.57$ or other inexact values.

[3 marks]
Total [21 marks]

2. (a) (i) $x = 3$ ***A1***

(ii) $y = -2$ ***A1***

[2 marks]

(b) (i) $(-2, 0)$ (accept $x = -2$) ***A1***

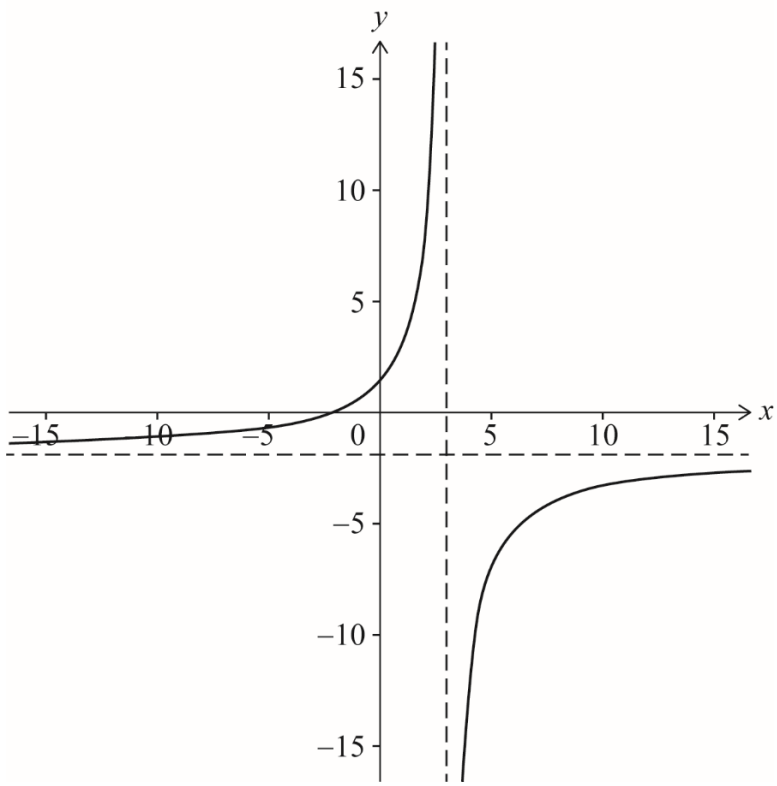
(ii) $\left(0, \frac{4}{3}\right)$ (accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) ***A1***

[2 marks]

continued…

Question 2 continued.

(c)



A1

Note: Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour.

[1 mark]

continue…

Question 2 continued.

(d) **METHOD 1**

$\left(g\left(x\right)=\right)y=\dfrac{ax+4}{3-x}$

attempt to find x in terms of y (***M1)***

OR exchange x and y and attempt to find y in terms of x

$3y-xy=ax+4$ ***A1***

$ax+xy=3y-4$

$x\left(a+y\right)=3y-4$

$x=\dfrac{3y-4}{y+a}$

$g^{-1}\left(x\right)=\dfrac{3x-4}{x+a}$ ***A1***

Note: Condone use of $y=$

$g\left(x\right)\equiv g^{-1}\left(x\right)$

$\dfrac{ax+4}{3-x}\equiv\dfrac{3x-4}{x+a}$

$\Rightarrow a=-3$ ***A1***

[4 marks]

continue…

Question 2 continued.

METHOD 2

$g(x)=\frac{ax+4}{3-x}$

attempt to find an expression for $g(g(x))$ and equate to x **(*M1)***

$gg(x)=\frac{a\left(\frac{ax+4}{3-x}\right)+4}{3-\left(\frac{ax+4}{3-x}\right)}=x$ ***A1***

$\frac{a(ax+4)+4(3-x)}{(9-3x)-(ax+4)}=x$

$\frac{a(ax+4)+4(3-x)}{5-(3+a)x}=x$

$a(ax+4)+4(3-x)=x(5-(3+a)x)$ ***A1***

equating coefficients of x^2 (or similar)

$a=-3$ ***A1***

[4 marks]

Total [9 marks]

3. attempt to use change the base ***(M1)***

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3 \left(4x^3\right)$$

attempt to use the power rule ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3 \left(4x^3\right)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \left(4\sqrt{2}x^3\right)$$

Note: The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen**.**

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

$$x^5 = \frac{1}{32}$$ ***(A1)***

$$x = \frac{1}{2}$$ ***A1***

[5 marks]

7. (a) attempt to use discriminant $b^2 - 4ac(>0)$ ***M1***

$(2p)^2 - 4(3p)(1-p)(>0)$

$16p^2 - 12p(>0)$ ***(A1)***

$p(4p-3)(>0)$

attempt to find critical values $\left(p = 0,\ p = \frac{3}{4}\right)$ ***M1***

recognition that discriminant > 0 ***(M1)***

$p < 0$ or $p > \frac{3}{4}$ ***A1***

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) $p = 4 \Rightarrow 12x^2 + 8x - 3 = 0$

valid attempt to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (or equivalent) ***M1***

$x = \frac{-8 \pm \sqrt{208}}{24}$

$x = \frac{-2 \pm \sqrt{13}}{6}$

$a = -2$ ***A1***

[2 marks]

Total [7 marks]

10. (a) (i)

Note: In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

(ii) $\left(0, \frac{4}{5}\right)$ ***A1***

[3 marks]

(b) $x = \frac{15}{2}$ ***A1***

Note: Award ***A0*** for $x \neq \frac{15}{2}$.

Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

[1 mark]

(c) **METHOD 1**

$(ax + b)(2x - 15) \equiv x^2 - x - 12$

attempts to expand $(ax + b)(2x - 15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of x ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

METHOD 2

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+...$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 3

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of x: ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

METHOD 4

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$

$a=\frac{1}{2}$ ***A1***

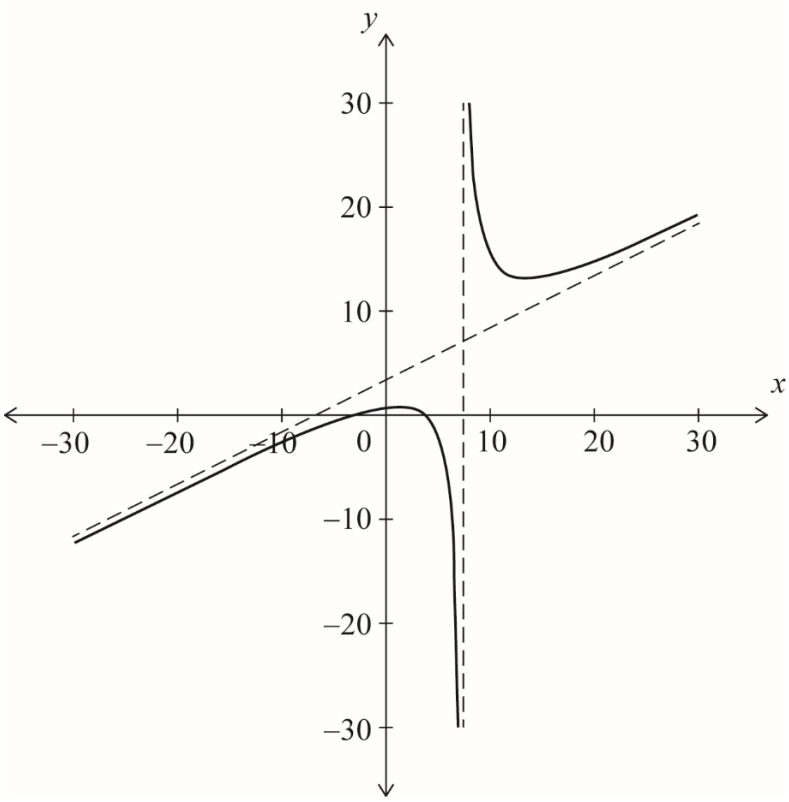
$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+...$ ***M1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

[4 marks]

(d)



two branches with approximately correct shape (for $-30 \leq x \leq 30$) ***A1***

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

[3 marks]

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3}+\frac{B}{x-4}$$

$$2x-15 \equiv A(x-4)+B(x+3)$$

$A=3$ ***A1***

$B=-1$ ***A1***

$$\left(\frac{3}{x+3}-\frac{1}{x-4}\right)$$

(ii) $$\int_0^3 \left(\frac{3}{x+3}-\frac{1}{x-4}\right) \mathrm{d}x$$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$$=\left[3\ln|x+3|-\ln|x-4|\right]_0^3$$ ***A1***

$$=3\ln 6-\ln 1-3\ln 3+\ln 4$$ ***A1***

$$=3\ln 2+\ln 4 \ (=\ln 8+\ln 4)$$

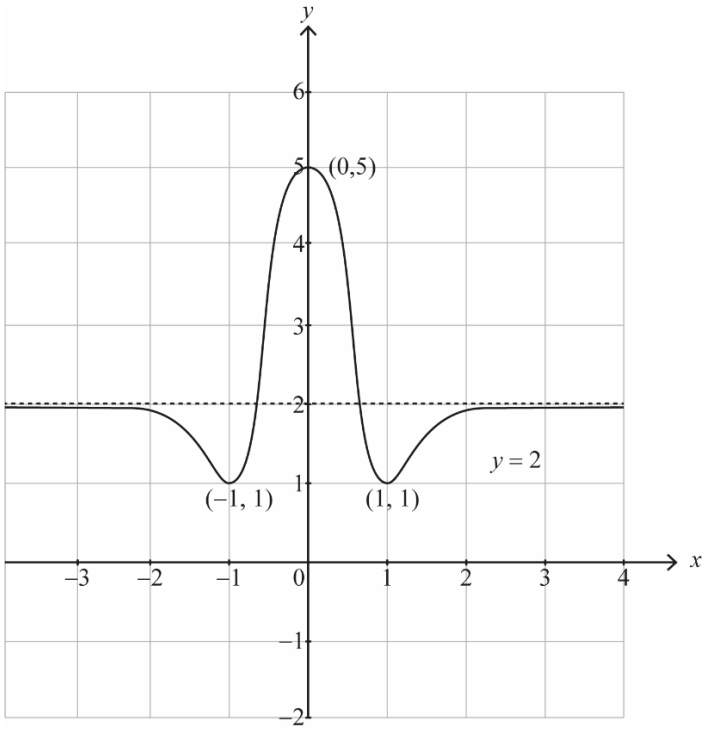
$$=\ln 32 \ (=5\ln 2)$$ ***A1***

Note: The final ***A1*** is dependent on the previous two ***A*** marks.

[7 marks]

Total [18 marks]

4.



no y values below 1 ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ ***A1***

$(\pm 1, 1)$ local minima ***A1***

$(0, 5)$ local maximum ***A1***

smooth curve and smooth stationary points ***A1***

Total [5 marks]

5. (a) attempt to form composition ***M1***

correct substitution $g\left(\frac{x+3}{4}\right)=8\left(\frac{x+3}{4}\right)+5$ ***A1***

$(g\circ f)(x)=2x+11$ ***AG***

[2 marks]

(b) attempt to substitute 4 (seen anywhere) ***(M1)***

correct equation $a=2\times4+11$ ***(A1)***

$a=19$ ***A1***

[3 marks]

Total [5 marks]

9. (a) attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b) **Note:** Interchanging x and y can be done at any stage.

$y = \left(e^{x} - 3\right)^{2} - 4$ ***(M1)***

$e^{x} - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \leq \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \leq x < 5$ ***A1***

[5 marks]

Total [8 marks]

Section B

9. ***METHOD 1***

sketching the graph of $y=\frac{x^2}{x-3}$ $\left(y=x+3+\frac{9}{x-3}\right)$ ***M1***

the (oblique) asymptote has a gradient equal to 1
and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining
$(-3,0)$ and $(0,0)$ ***M1***

so $m>0$ ***R1***

hence $0<m\leq 1$ ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2=m\left(x^2-9\right)$

$\Rightarrow (m-1)x^2-9m=0$ ***A1***

EITHER

attempting to solve $-4(m-1)(-9m)<0$ for m ***M1***

OR

attempting to solve $x^2<0$ *ie* $\frac{9m}{m-1}<0$ $(m\neq 1)$ for m ***M1***

THEN

$\Rightarrow 0<m<1$ ***A1***

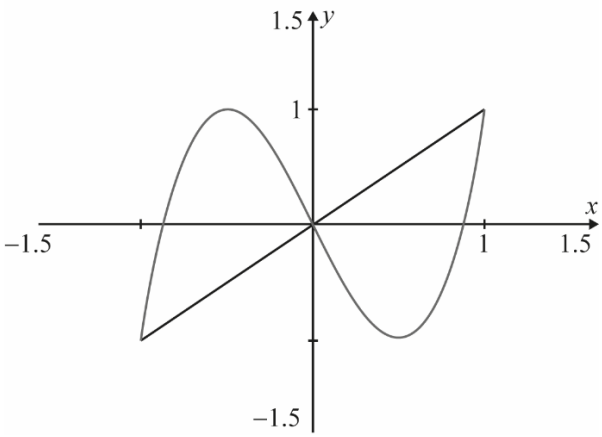
a valid reason to explain why $m=1$ gives no solutions *eg* if $m=1$,
$(m-1)x^2-9m=0\Rightarrow -9=0$ and so $0<m\leq 1$ ***R1***

Total [5 marks]

Section B

2. (a) correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

(b) (i) graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

(ii) $\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

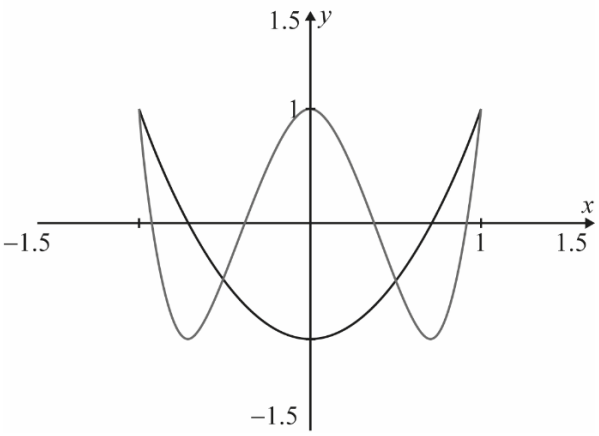
[4 marks]

continued…

Question 2 continued

(c) correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

(d) (i) graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 2$, 0 local maximum point and 1 local minimum point
$n = 4$, 1 local maximum points and 2 local minimum points
$n = 6$, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

(ii) $\frac{n}{2}$ local minimum points ***A1***

[4 marks]

(e) $f_n(x) = \cos\left(n \arccos(x)\right)$

$f_n'(x) = \frac{n \sin\left(n \arccos(x)\right)}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f_n'(x) = 0 \Rightarrow n \sin\left(n \arccos(x)\right) = 0$ ***M1***

$n \arccos(x) = k\pi \ (k \in \mathbb{Z}^+)$ ***A1***

leading to

$x = \cos\frac{k\pi}{n}$ $(k \in \mathbb{Z}^+$ and $0 < k < n)$ ***AG***

[4 marks]

continued…

Question 2 continued

(f) $f_2(x) = \cos(2\arccos x)$

$= 2\left(\cos(\arccos x)\right)^2 - 1$ ***M1***

stating that $\left(\cos(\arccos x)\right) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

(g) $f_{n+1}(x) = \cos\left((n+1)\arccos x\right)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A+B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h) (i) $f_{n-1}(x) = \cos\left((n-1)\arccos x\right)$ ***A1***

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ ***A1***

$= 2xf_n(x)$ ***AG***

(ii) $f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x\left(2x^2 - 1\right) - x$

$= 4x^3 - 3x$ ***A1***

[5 marks]

Total [25 marks]

5. (a) translation (shift) by $\frac{3\pi}{2}$ to the right OR positive horizontal direction by $\frac{3\pi}{2}$ ***A1***

translation (shift) by q upwards OR positive vertical direction by q ***A1***

Note: accept translation by $\begin{pmatrix} \frac{3\pi}{2} \\ q \end{pmatrix}$

Do not accept 'move' for translation/shift.

[2 marks]

(b) **METHOD 1**

minimum of $4\sin\left(x-\frac{3\pi}{2}\right)$ is -4 (may be seen in sketch) ***(M1)***

$-4+2.5+q \geq 7$

$q \geq 8.5$ (accept $q = 8.5$) ***A1***

substituting $x = 0$ and their $q\left(=8.5\right)$ to find r ***(M1)***

$\left(r=\right) \quad 4\sin\left(\frac{-3\pi}{2}\right)+2.5+8.5$

$4+2.5+8.5$ ***(A1)***

smallest value of r is 15 ***A1***

continued…

Question 5 continued

METHOD 2

substituting $x = 0$ to find an expression (for r) in terms of q ***(M1)***

$(g(0) = r =) \quad 4\sin\left(\frac{-3\pi}{2}\right) + 2.5 + q$

$(r =)\ 6.5 + q$ ***A1***

minimum of $4\sin\left(x - \frac{3\pi}{2}\right)$ is -4 ***(M1)***

$-4 + 2.5 + q \geq 7$

$-4 + 2.5 + (r - 6.5) \geq 7$ (accept =) ***(A1)***

smallest value of r is 15 ***A1***

METHOD 3

$4\sin\left(x - \frac{3\pi}{2}\right) + 2.5 + q = 4\cos x + 2.5 + q$ ***A1***

y-intercept of $4\cos x + 2.5 + q$ is a maximum ***(M1)***

amplitude of $g(x)$ is 4 ***(A1)***

attempt to find least maximum ***(M1)***

$r = 2 \times 4 + 7$

smallest value of r is 15 ***A1***

[5 marks]

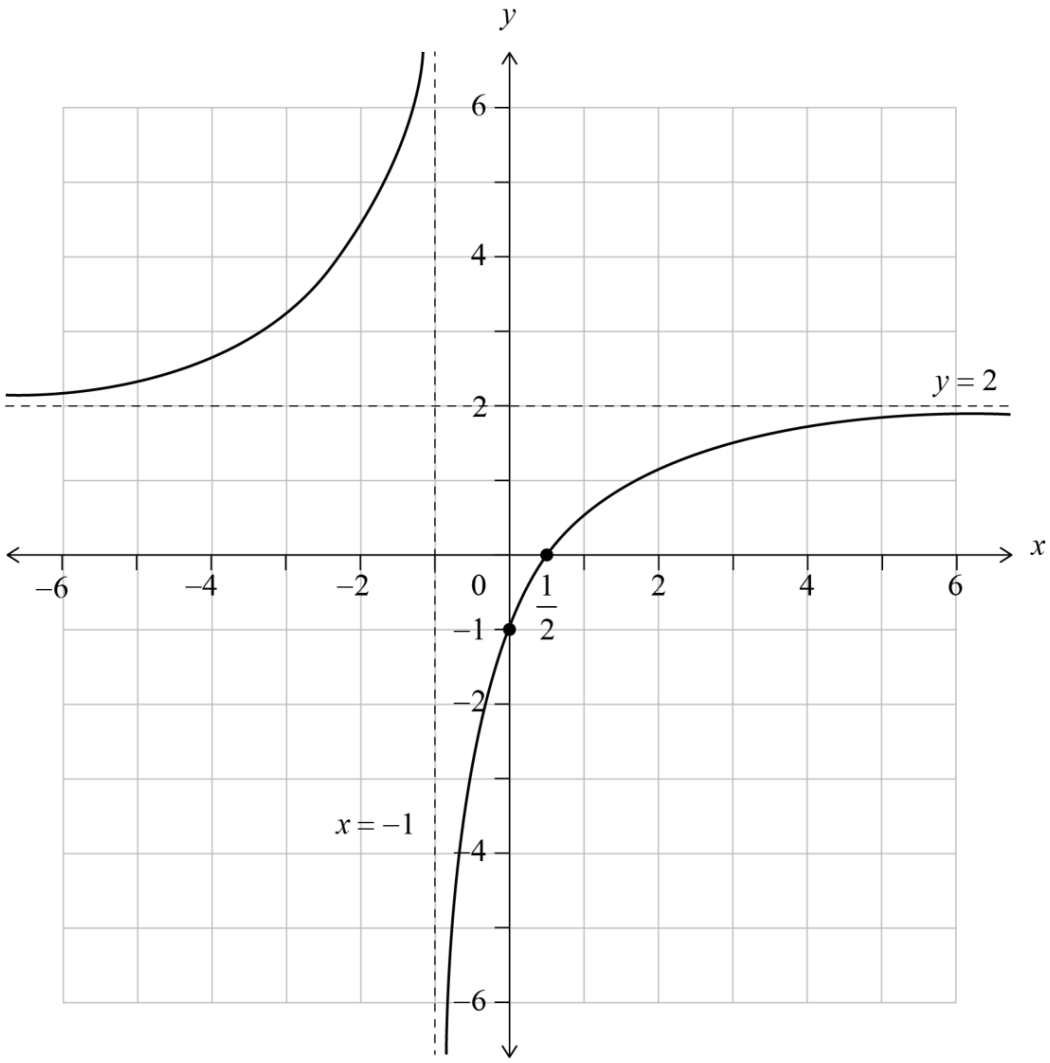
Total [7 marks]

3. (a) (i) $x = -1$ ***A1***

(ii) $y = 2$ ***A1***

[2 marks]

(b)



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$ ***A1A1***

[3 marks]

continued…

Question 3 continued

(c) $x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2}, \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

(d) **EITHER**

attempts to sketch $y = \frac{2|x|-1}{|x|+1}$ ***(M1)***

OR

attempts to solve $2|x|-1=0$ ***(M1)***

Note: Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

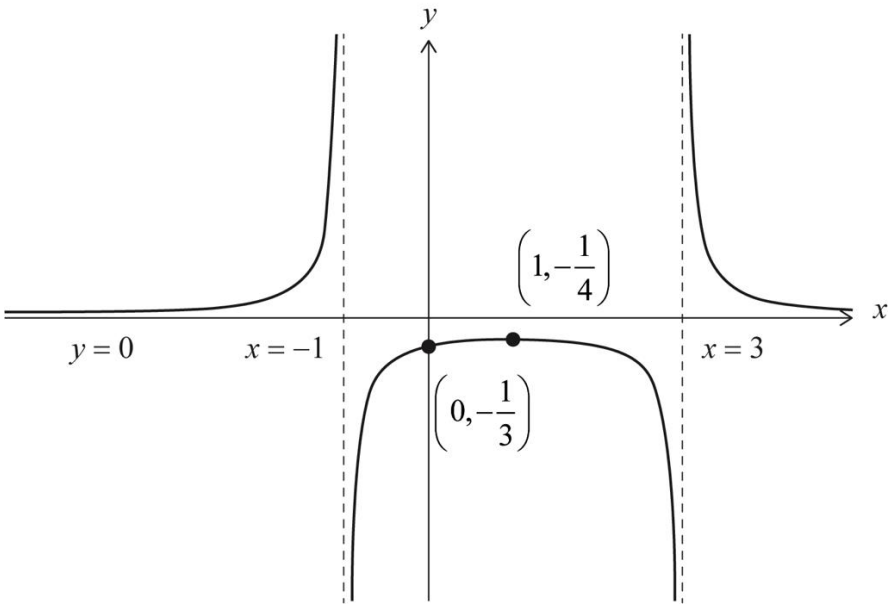
$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]

Total [8 marks]

11. (a)



y- intercept $\left(0,-\frac{1}{3}\right)$ ***A1***

Note: Accept an indication of $-\frac{1}{3}$ on the $y-$axis.

vertical asymptotes $x=-1$ and $x=3$ ***A1***

horizontal asymptote $y=0$ ***A1***

uses a valid method to find the x- coordinate of the local maximum point ***(M1)***

Note: For example, uses the axis of symmetry or attempts to solve $f'(x)=0$.

local maximum point $\left(1,-\frac{1}{4}\right)$ ***A1***

Note: Award ***(M1)A0*** for a local maximum point at $x=1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

[6 marks]

continued…

Question 11 continued

(b) (i) $x = \frac{1}{y^2 - 2y - 3}$ ***M1***

Note: Award ***M1*** for interchanging x and y (this can be done at a later stage).

EITHER

attempts to complete the square ***M1***

$y^2 - 2y - 3 = (y-1)^2 - 4$ ***A1***

$x = \frac{1}{(y-1)^2 - 4}$

$(y-1)^2 - 4 = \frac{1}{x}$ $\left((y-1)^2 = 4 + \frac{1}{x}\right)$ ***A1***

$y - 1 = \pm\sqrt{4 + \frac{1}{x}}$ $\left(= \pm\sqrt{\frac{4x+1}{x}}\right)$

OR

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for y ***M1***

$y = \frac{-(-2x) \pm \sqrt{(-2x)^2 + 4x(3x+1)}}{2x}$ ***A1***

Note: Award ***A1*** even if – (in $\pm$) is missing

$= \frac{2x \pm \sqrt{16x^2 + 4x}}{2x}$ ***A1***

continued…

Question 11 continued

THEN

$= 1 \pm \frac{\sqrt{4x^2 + x}}{x}$ ***A1***

$y > 3$ and hence $y = 1 - \frac{\sqrt{4x^2 + x}}{x}$ is rejected ***R1***

Note: Award ***R1*** for concluding that the expression for y must have the '$+$' sign.
The ***R1*** may be awarded earlier for using the condition $x > 3$.

$y = 1 + \frac{\sqrt{4x^2 + x}}{x}$

$g^{-1}(x) = 1 + \frac{\sqrt{4x^2 + x}}{x}$ ***AG***

(ii) domain of g^{-1} is $x > 0$ ***A1***

[7 marks]

continued…

Question 11 continued

(c) attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right)$ $\left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2-2a-3)}\right)\right)$ ***(A1)***

$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4}$ $\left(\arctan\left(\frac{1}{2(a^2-2a-3)}\right) = \frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a) = 2$ $\left(\frac{1}{(a^2-2a-3)} = 2\right)$

EITHER

$\Rightarrow a = g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a = 1 + \frac{\sqrt{4(2)^2+2}}{2}$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

OR

$\Rightarrow 2a^2 - 4a - 7 = 0$ ***A1***

attempts to solve their quadratic equation ***M1***

$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4}$ $\left(= \frac{4 \pm \sqrt{72}}{4}\right)$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

THEN

$a = 1 + \frac{3}{2}\sqrt{2}$ (as $a > 3$) ***A1***

$(p = 1,\ q = 3,\ r = 2)$

Note: Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}$ $(p = 1,\ q = 1,\ r = 18)$.

[7 marks]

Total [20 marks]

6. (a) attempt to replace x with $-x$ ***M1***

$$f(-x) = 2^{-x} - \frac{1}{2^{-x}}$$

EITHER

$$= \frac{1}{2^x} - 2^x = -f(x)$$ ***A1***

OR

$$= -\left(2^x - \frac{1}{2^x}\right)\left(= -f(x)\right)$$ ***A1***

Note: Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180°about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempt to find at least one intersection point ***(M1)***

$x = -1.26686..., x = 0.177935..., x = 3.06167...$

$x = -1.27, x = 0.178, x = 3.06$

$-1.27 \leq x < -1,$ ***A1***

$0.178 \leq x < 3,$ ***A1***

$x \geq 3.06$ ***A1***

[4 marks]

Total [6 marks]

8. (a) product of roots $= \frac{2k+9}{k}$ ***A1***

[1 mark]

(b) recognition that the product of the roots will be negative ***(M1)***

$\frac{2k+9}{k} < 0$

critical values $k = 0, -\frac{9}{2}$ seen ***(A1)***

$-\frac{9}{2} < k < 0$ ***A1***

[3 marks]

Total [4 marks]

2. (a) attempt to expand $(x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$=\left(x^2-(\alpha+\beta)x+\alpha\beta\right)(x-\gamma)$ **OR** $=(x-\alpha)\left(x^2-(\beta+\gamma)x+\beta\gamma\right)$ ***A1***

$\left(x^3+px^2+qx+r\right)=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$ ***A1***

comparing coefficients:

$p=-(\alpha+\beta+\gamma)$ ***AG***

$q=(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$r=-\alpha\beta\gamma$ ***AG***

Note*:* For candidates who do not include the ***AG*** lines award full marks.

[3 marks]

(b) (i) $p^2-2q=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***(A1)***

attempt to expand $(\alpha+\beta+\gamma)^2$ ***(M1)***

$=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ or equivalent ***A1***

$=\alpha^2+\beta^2+\gamma^2$ ***AG***

Note: Accept equivalent working from RHS to LHS.

[3 marks]

continued…

Question 2 continued

(ii) **EITHER**

attempt to expand $(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***(M1)***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=2\left(p^2-2q\right)-2q$ or equivalent ***A1***

$=2p^2-6q$ ***AG***

OR

attempt to write $2p^2-6q$ in terms of α,β,γ ***(M1)***

$=2\left(p^2-2q\right)-2q$

$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***A1***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***AG***

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

continued…

Question 2 continued

(c) $p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$\Rightarrow (\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 < 0$ ***A1***

if all roots were real $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 \geq 0$ ***R1***

Note: Condone strict inequality in the ***R1*** line.
Note: Do not award ***A0R1.***

$\Rightarrow$ roots cannot all be real ***AG***

[2 marks]

(d) $p^2 = (-7)^2 = 49$ and $3q = 51$ ***A1***

so $p^2 < 3q \Rightarrow$ the equation has at least one complex root ***R1***

Note: Allow equivalent comparisons; e.g. checking $2p^2 < 6q$

[2 marks]

(e) (i) use of GDC (*eg* graphs or tables) ***(M1)***

$q = 12$ ***A1***

[2 marks]

(ii) complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).
OR
a cubic curve always crosses the x-axis at at least one point. ***R1***

[1 mark]

continued…

Question 2 continued

(f) (i) attempt to expand $(\alpha+\beta+\gamma+\delta)^2$ **(*M1*)**

$(\alpha+\beta+\gamma+\delta)^2=\alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

(*A1*)

$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$\left(\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2=\right)p^2-2q$ ***A1***

[3 marks]

(ii) $p^2<2q$ **OR** $p^2-2q<0$ ***A1***

Note: Allow ***FT*** on their result from part (f)(i).

[1 mark]

(g) $4<6$ OR $2^2-2\times3<0$ ***R1***

hence there is at least one complex root. ***AG***

Note: Allow ***FT*** from part (f)(ii) for the ***R*** mark provided numerical reasoning is seen.

[1 mark]

continued…

Question 2 continued

(h) (i) $\left(p^2 > 2q\right)$ $\left(81 > 2\times 24\right)$ (so) nothing can be deduced ***R1***

Note: Do not allow ***FT*** for the ***R*** mark.

[1 mark]

(ii) -1 ***A1***

[1 mark]

(iii) attempt to express as a product of a linear and cubic factor ***M1***

$(x+1)(x^3-10x^2+34x-12)$ ***A1A1***

Note: Award ***A1*** for each factor. Award at most ***A1A0*** if not written as a product.

since for the cubic, $p^2 < 3q$ $(100 < 102)$ ***R1***

there is at least one complex root ***AG***

[4 marks]

[Total: 27 marks]

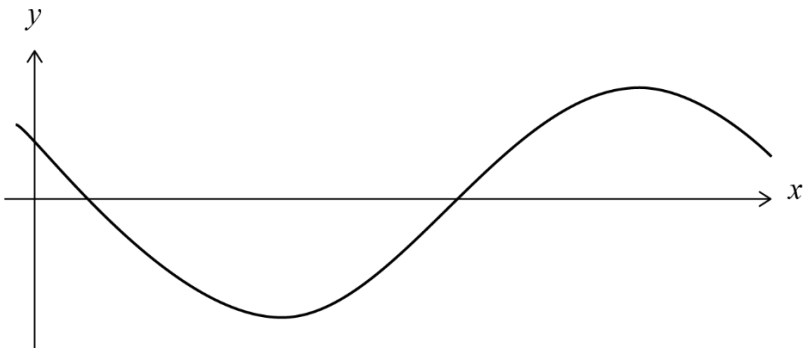
8. (a)

$a = \frac{\pi}{2}$ ***A2***

Note: For sinusoidal graph through the origin seen with incorrect a, or use of horizontal line test with incorrect a, award ***A1A0***

(b) $a = \pi$ ***A1***

(c)



sketch showing sinusoidal shape decreasing as it crosses the y-axis (below or above the origin) ***(A1)***

$a = k - \pi$ ***A1***

[5 marks]

11. (a) (i) 5^3 ***(A1)***

$=125$ ***A1***

(ii) ${}^5P_3 = 5\times 4\times 3$ ***(A1)***

$=60$ ***A1***

[4 marks]

(b) (i) **METHOD 1**

$x^2+3x+2=(x+1)(x+2)$ ***(A1)***

correct use of factor theorem for at least one of their factors ***(M1)***

$P(-1)=0$ or $P(-2)=0$

attempt to find two equations in a, b and c ***(M1)***

$(-1)^3+a(-1)^2+b(-1)+c=0 (\Rightarrow -1+a-b+c=0)$

$(-2)^3+a(-2)^2+b(-2)+c=0$

$-8+4a-2b+c=0$ and $-1+a-b+c=0$ ***A1***

attempt to combine their two equations in $-8+4a-2b+c=0$ to eliminate c ***(M1)***

$b=3a-7$ ***A1***

Note: Award at most ***A1M1M1A0M1A0*** for $b=-3a-7$ from $P(1)=P(2)=0$

continued…

Question 11 continued

METHOD 2

$P(x) = x^3 + ax^2 + bx + c = (x^2 + 3x + 2)(x + d)$ ***(M1)***

$= x^3 + (3 + d)x^2 + (2 + 3d)x + 2d$ ***(A1)***

attempt to compare coefficients of x^2 and x ***(M1)***

$a = 3 + d$ and $b = 2 + 3d$ ***A1***

attempt to eliminate d ***(M1)***

$\Rightarrow b = 3a - 7$ ***A1***

METHOD 3

attempt to divide $x^3 + ax^2 + bx + c$ by $x^2 + 3x + 2$ ***M1***

$\frac{x^3 + ax^2 + bx + c}{x^2 + 3x + 2} = (x + a - 3) + \frac{(-3a + b + 7)x + (c - 2a + 6)}{x^2 + 3x + 2}$ ***A1A1A1***

Note: Award ***A1*** *for* $x + a - 3$, ***A1*** *for* $(-3a + b + 7)x$ and ***A1*** *for* $c - 2a + 6$

recognition that, if $(x^2 + 3x + 2)$ is a factor of $P(x)$, then $-3a + b + 7 = 0$ ***(M1)***

leading to $b = 3a - 7$ ***A1***

continued…

Question 11 continued

METHOD 4

$x^2+3x+2=(x+1)(x+2)$ ***(A1)***

attempt to use Vieta's formulae for a cubic with roots -1, -2 and "p" ***(M1)***

$(-1)+(-2)+p=-a\left(\Rightarrow p=3-a\right)$ ***A1***

$(-1)(-2)+(-1)p+(-2)p=b$ ***A1***

Attempt to eliminate "p" ***(M1)***

$2-(3-a)-2(3-a)=b$

$b=3a-7$ ***A1***

Note: Award at most ***A1M1A0A0M1A0*** for $b=-3a-7$ from roots 1, 2 and "p"

(ii) **METHOD 1**

$a=1,2,5$ lead to invalid values for b ***R1***

$a=3, b=2\Rightarrow c=0$ so not possible ***R1***

so $a=4, b=5, c=2$ is the only solution ***AG***

METHOD 2

$c=2a-6$ ***R1***

correctly argues $a=4$ is the only possibility ***R1***

so $a=4, b=5, c=2$ is the only solution ***AG***

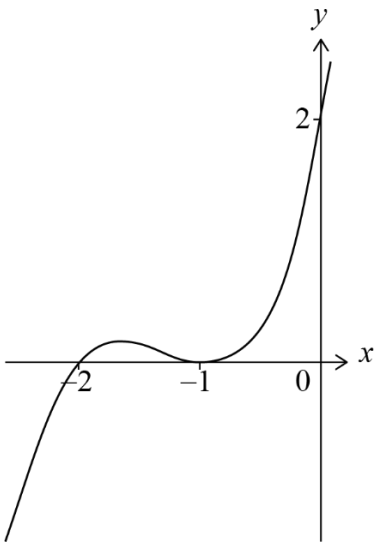
(iii) $x^3+4x^2+5x+2=\left(x^2+3x+2\right)(x+1)$

$=(x+2)(x+1)(x+1)$ ***A1***

continued…

Question 11 continued

(iv)



positive cubic shape with y-intercept at $(0, 2)$ ***A1***

x-intercept at $(-2, 0)$ and local maximum point anywhere between $x = -2$ and $x = -1$ ***A1***

local minimum point at $(-1, 0)$ ***A1***

Note: Accept answers from an approach based on calculus.

[12 marks]

Total [16 marks]

4. recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0) = 15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000 \times 0.89 (= 13350)$ ***(A1)***

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000 \times 0.89 = 15000\mathrm{e}^{8k}$ OR $13350 = 15000\mathrm{e}^{8k}$

$k = \frac{\ln 0.89}{8}$ (-0.014566) ***(A1)***

substitution of $t = 2041 - 2014 (= 27)$ and their value for k into the model ***(M1)***

$P(27) = 15000e^{-0.0145...\times 27}$

$10122.3\ldots$

$P(27) = 10100$ (10122) ***A1***

Total [7 marks]

1. (a) recognizing $f(x)=0$ ***(M1)***

$x=-1$ ***A1***

[2 marks]

(b) (i) $x=2$ (must be an equation with x) ***A1***

(ii) $y=\frac{7}{2}$ (must be an equation with y) ***A1***

[2 marks]

(c) **EITHER**

interchanging x and y ***(M1)***

$2xy-4x=7y+7$

correct working with y terms on the same side: $2xy-7y=4x+7$ ***(A1)***

OR

$2yx-4y=7x+7$

correct working with x terms on the same side: $2yx-7x=4y+7$ ***(A1)***

interchanging x and y OR making x the subject $x=\frac{4y+7}{2y-7}$ ***(M1)***

THEN

$f^{-1}(x)=\frac{4x+7}{2x-7}$ (or equivalent) $\left(x\neq\frac{7}{2}\right)$ ***A1***

[3 marks]

Total [7 marks]

4. recognition of quadratic in e^x ***(M1)***

$\left(e^x\right)^2 - 3e^x + \ln k\,(=0)$ OR $A^2 - 3A + \ln k\,(=0)$

recognizing discriminant ≥ 0 (seen anywhere) (***M1)***

$(-3)^2 - 4(1)(\ln k)$ OR $9 - 4\ln k$ ***(A1)***

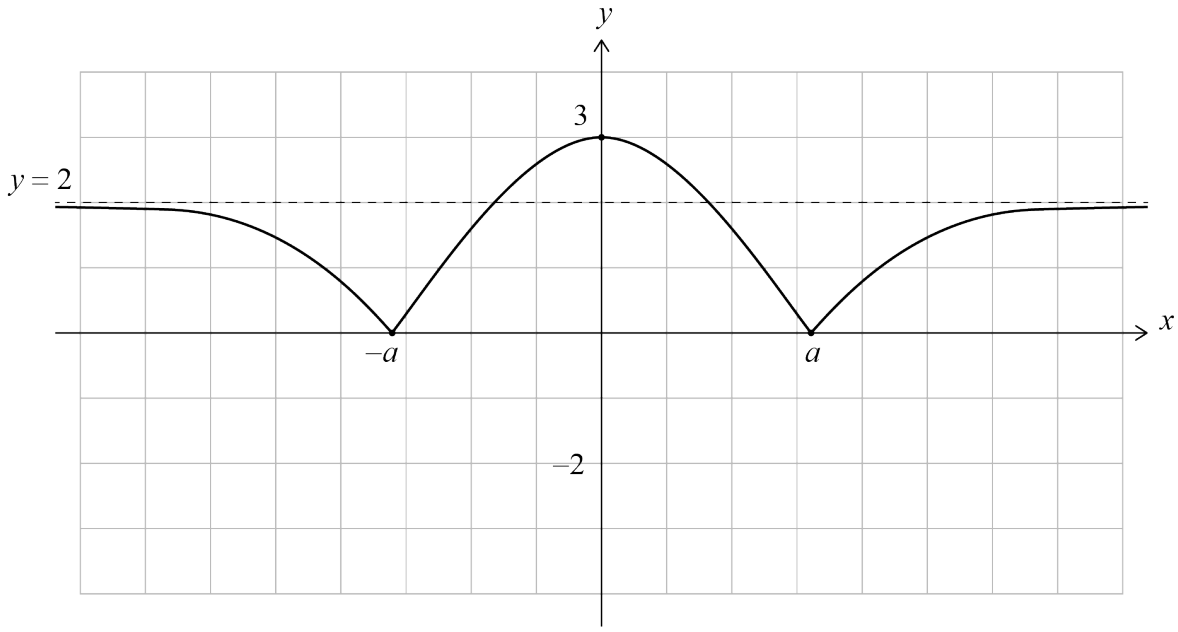
$\ln k \le \frac{9}{4}$ ***(A1)***

$e^{9/4}$ (seen anywhere) ***A1***

$0 < k \le e^{9/4}$ ***A1***

[6 marks]

8. (a) attempt to reflect f in the x OR y axis ***(M1)***



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:
A1 for correct asymptotic behaviour at $y = 2$ (either side)
A1 for correctly reflected RHS of the graph in the *y*-axis with smooth maximum at (0, 3).
A1 for labelled *x*-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

[4 marks]

(b) $k = 0$ ***A1***

$4 \leq k < 9$ ***A2***

Note: If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.
Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the *y*-axis.

[3 marks]

Total [7 marks]

2. (a) (i) $x = 2$ ***A1***

(ii) $y = 1$ ***A1***

[2 marks]

(b) (i) $\left(0, \frac{3}{2}\right)$ ***A1***

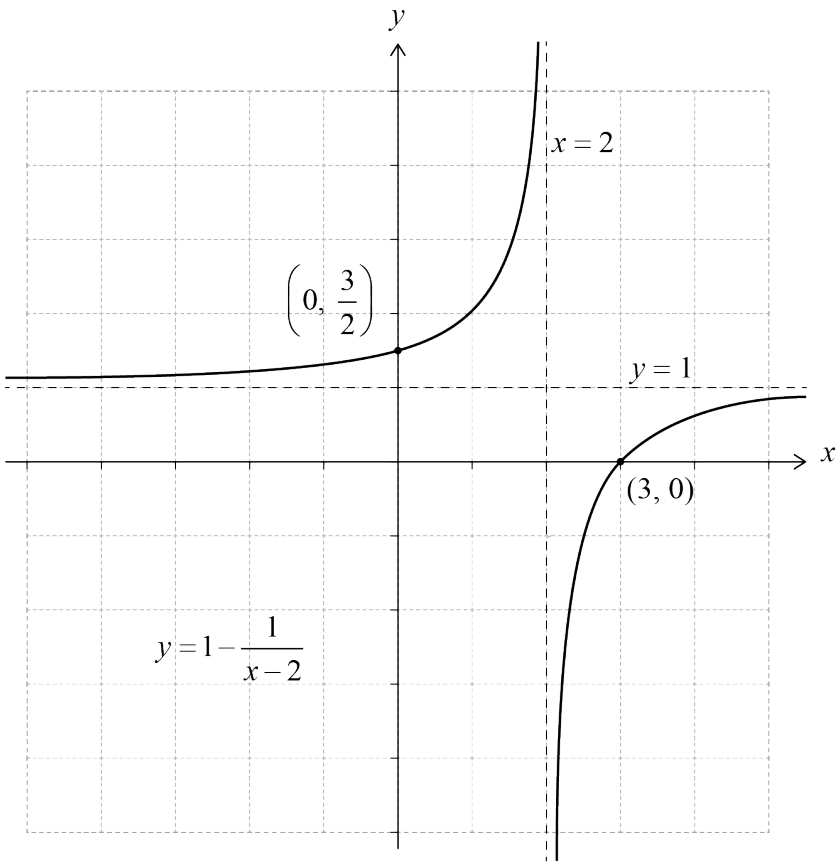
(ii) $(3, 0)$ ***A1***

[2 marks]

continued…

Question 2 continued

(c)



two correct branches with correct asymptotic behaviour and intercepts clearly shown ***A1***

[1 mark]

Total [5 marks]

5. attempts to form $(g \circ f)(x)$ ***(M1)***

$[f(x)]^2 + f(x) + 3$ OR $(ax+b)^2 + ax + b + 3$

$a^2x^2 + 2abx + b^2 + ax + b + 3 (= 4x^2 - 14x + 15)$ ***(A1)***

equates their corresponding terms to form at least one equation ***(M1)***

$a^2x^2 = 4x^2$ OR $a^2 = 4$ OR $2abx + ax = -14x$ OR $2ab + a = -14$ OR $b^2 + b + 3 = 15$

$a = \pm 2$ (seen anywhere) ***A1***

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) ***(M1)***

$f(x) = 2x - 4$ (accept $a = 2$ with $b = -4$), $f(x) = -2x + 3$ (accept $a = -2$ with $b = 3$) ***A1A1***

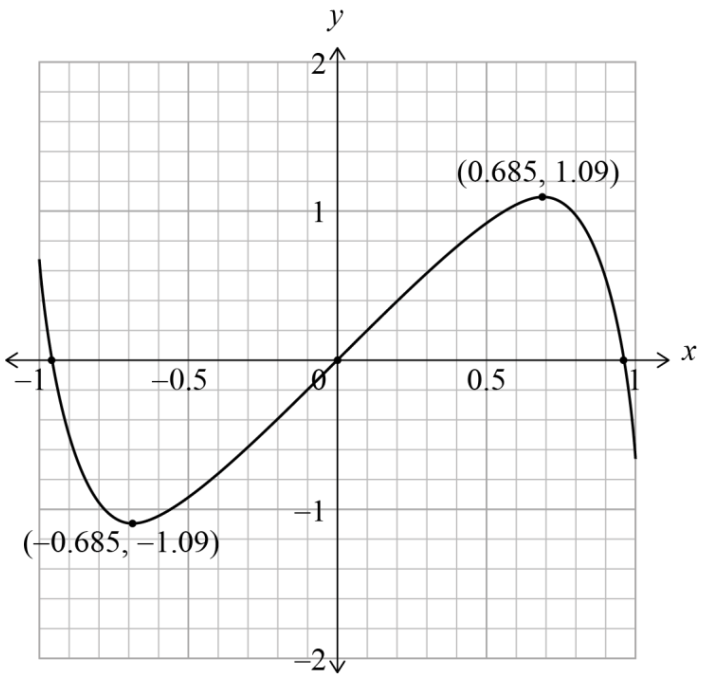
[7 marks]

2. (a) attempt to substitute g into f ***(M1)***

$(f \circ g)(x) = 2\tan x - \tan^3 x$ ***A1***

[2 marks]

(b)



A1A1A1

Note: ***A1*** for approximately correct odd function passing through the origin with a maximum above $y = 1$ and a minimum below $y = -1$.

A1 for endpoints at $x = \pm 1$ and y in the intervals $[0.6, 0.8]$ and $[-0.8, -0.6]$

A1 for maximum in approximately correct position and labelled $(0.685, 1.09)$ AND minimum in approximately correct position and labelled $(-0.685, -1.09)$. For approximate position, allow $-0.8 \le x \le -0.6$, $-1.2 \le y \le -1$ for minimum and $0.6 \le x \le 0.8$, $1 \le y \le 1.2$ for maximum. If the candidate gives the coordinates of extrema below their sketch, only award this mark if extrema are marked in the correct interval (eg by a dot).

[3 marks]

Total [5 marks]

8. (a) **EITHER**

attempts to find the y- coordinate of either the local minimum point or the local maximum point ***(M1)***

OR

attempts to find the discriminant of $2x-5=y\left(x^2-3\right)$ $\left(yx^2-2x+(5-3y)=0\right)$ ***(M1)***

$\Delta=4-4y(5-3y)\left(=4-20y+12y^2\right)$

THEN

$y=1.43425...$ (local min.) and $y=0.232408...$ (local max.) ***(A1)(A1)***

$g(x)\leq 0.232$ OR $g(x)\geq 1.43$ $\left(g(x)\leq\frac{-\sqrt{13}+5}{6}\right.$ OR $\left.g(x)\geq\frac{\sqrt{13}+5}{6}\right)$ ***A1***

Note: Accept other valid notations such as interval notation.

[4 marks]

(b) $\frac{2|x|-5}{x^2-3}\geq 0$ (since $\cos t<0$ for $\frac{\pi}{2}<t\leq\pi$) ***(R1)***

attempts to solve graphically or algebraically ***(M1)***

$x\leq-\frac{5}{2}$ OR $-\sqrt{3}<x<\sqrt{3}$ $(-1.73<x<1.73)$ OR $x\geq\frac{5}{2}$ ***A1***

[3 marks]

Total [7 marks]

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

10. (a) $x = 0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9) = \ln\frac{x^2}{d}$ OR $\ln\frac{x^2}{2x-9} = \ln d$ OR $\ln(2x-9)d = \ln x^2$

$\frac{x^2}{d} = 2x-9$ OR $\frac{x^2}{2x-9} = d$ OR $(2x-9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

(ii) discriminant = $(-2d)^2 - 4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4\times1\times9d > 0$ OR $(2d)^2 - 4\times9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graph
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9,\infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x=)\ \frac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10}\,(= p)$ or $x = 10 + \sqrt{10}\,(= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

$(a = 2, b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

10. (a) $x = 0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9) = 2\ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9) = \ln\frac{x^2}{d}$ OR $\ln\frac{x^2}{2x-9} = \ln d$ OR $\ln(2x-9)d = \ln x^2$

$\frac{x^2}{d} = 2x-9$ OR $\frac{x^2}{2x-9} = d$ OR $(2x-9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

(ii) discriminant = $(-2d)^2 - 4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4\times1\times9d > 0$ OR $(2d)^2 - 4\times9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

(iii) setting $d(d-9) > 0$ OR $d(d-9) = 0$ OR sketch graph
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9,\infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x=)\ \frac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10}\,(= p)$ or $x = 10 + \sqrt{10}\,(= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

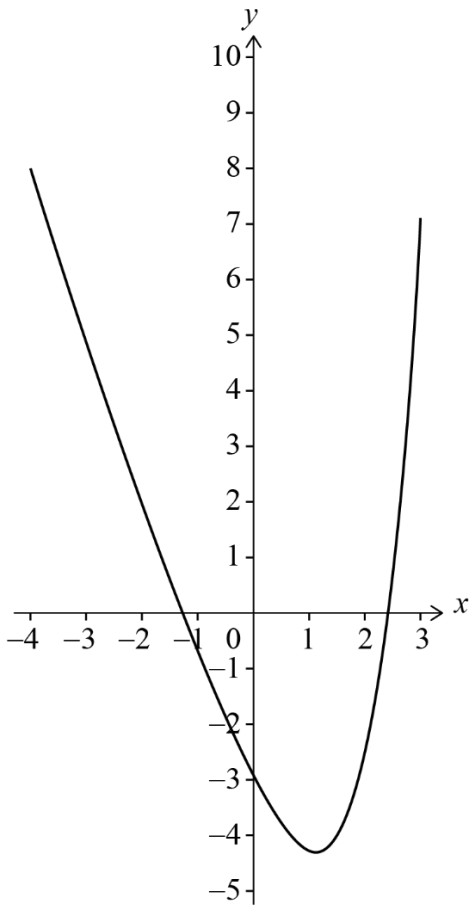
$(a = 2, b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

2. (a)



A1A1A1

Note:	Award marks as follows: ***A1*** for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$. ***A1*** for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$. ***A1*** for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5,\ 7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5,\ 6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$ ***A1***

$c = -3$ (accept translate/shift 3 (units) down) ***A1***

[2 marks]

Total [5 marks]

11. (a) (vertical asymptote equation) $x = -3$ ***A1***

Note: Accept $2x + 6 = 0$ or equivalent.

[1 mark]

(b) $(2, 0)$ and $(12, 0)$ ***A1A1***

Note: Award ***A1*** for $(2, 0)$ and ***A1*** for $(12, 0)$.
Award ***A1A0*** if only x values are given.

[2 marks]

(c) **METHOD 1**

$a = \frac{1}{2}$ ***A1***

attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ ***(M1)***

$\frac{x^2 - 14x + 24}{2x + 6}$

$= \frac{1}{2}x - \frac{17}{2}\left(+\frac{\ldots}{2x + 6}\right)$ ***(A1)***

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

continued…

Question 11 continued

METHOD 2

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6}$ ***(A1)***

$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$

attempt to equate coefficients of x: ***(M1)***

$-14 = 3 + 2b$

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6}$ ***(A1)***

attempt to find the limit of $f(x) - ax$ as $x \to \infty$ ***(M1)***

$b = \lim_{x \to \infty} \frac{-17x + 24}{2x + 6}$

$= -\frac{17}{2}$ ***A1***

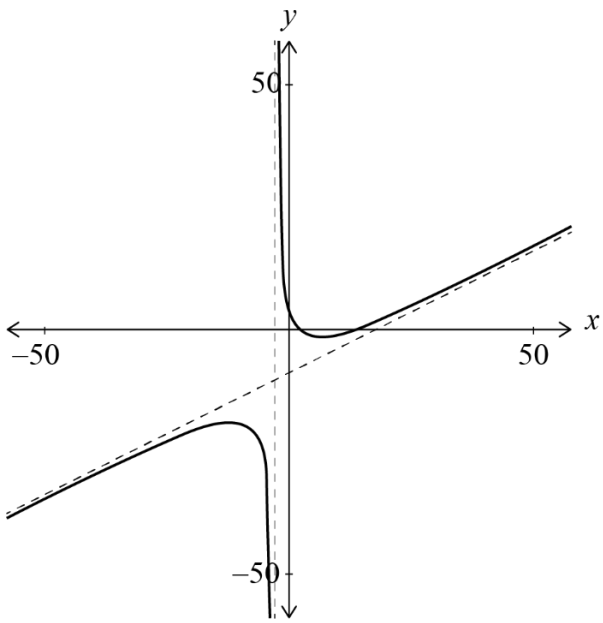
Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

continued…

Question 11 continued

(d)



two branches with approximately correct shape (for $-50 \le x \le 50$) ***A1***

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1A1***

Note: Award ***A1*** for vertical asymptote and behaviour and ***A1*** for oblique asymptote and behaviour.
If only top half of the graph seen only award ***A1A0*** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $\left(-10-5\sqrt{3}=\right)-18.6602....$ OR $\left(-10+5\sqrt{3}=\right)-1.33974....$ seen anywhere ***(A1)***

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$

(M1)

$y \leq -18.7,\ y \geq -1.34$ ***A1A1***

Note: Award ***A1*** for each inequality**.** Award ***A1A0*** for strict inequalities in both.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

(f) $\left(-10-2\sqrt{31}=\right)-21.1355....$ OR $\left(-10+2\sqrt{31}=\right)1.13552....$ seen anywhere ***(A1)***

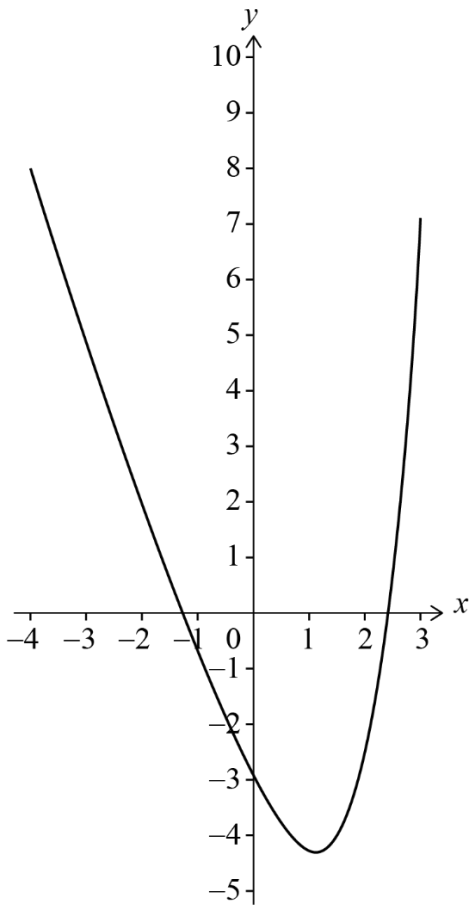
$x<-21.1,\ -3<x<1.14$ ***A1A1A1***

Note: Award ***A1*** for $x<-21.1$, ***A1*** for correct endpoints of a single interval -3 and 1.14 and ***A1*** for $-3<x<1.14$.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

Total [19 marks]

2. (a)



A1A1A1

Note:	Award marks as follows: ***A1*** for approximately correct roots, in the intervals $-2 < x < -1$ and $2 < x < 3$. ***A1*** for y-intercept AND local minimum in approximately correct positions. Allow for y-intercept $-3.5 < y < -2.5$, and for local minimum $0.5 < x < 1.5$, $-5 < y < -4$. ***A1*** for approximately correct endpoints, with the left end in the intervals $-4.5 < x < -3.5,\ 7.5 < y < 8.5$ and the right end in the intervals $2.5 < x < 3.5,\ 6.5 < y < 7.5$

[3 marks]

(b) $k = \frac{1}{2}$ ***A1***

$c = -3$ (accept translate/shift 3 (units) down) ***A1***

[2 marks]

Total [5 marks]

11. (a) (vertical asymptote equation) $x = -3$ ***A1***

Note: Accept $2x + 6 = 0$ or equivalent.

[1 mark]

(b) $(2, 0)$ and $(12, 0)$ ***A1A1***

Note: Award ***A1*** for $(2, 0)$ and ***A1*** for $(12, 0)$.
Award ***A1A0*** if only x values are given.

[2 marks]

(c) **METHOD 1**

$a = \frac{1}{2}$ ***A1***

attempt at 'long division' on $\frac{x^2 - 14x + 24}{2x + 6}$ ***(M1)***

$\frac{x^2 - 14x + 24}{2x + 6}$

$= \frac{1}{2}x - \frac{17}{2}\left(+\frac{\ldots}{2x + 6}\right)$ ***(A1)***

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

continued…

Question 11 continued

METHOD 2

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} \equiv \frac{1}{2}x + b + \frac{c}{2x + 6}$ ***(A1)***

$x^2 - 14x + 24 \equiv \frac{1}{2}x(2x + 6) + b(2x + 6) + c$

attempt to equate coefficients of x: ***(M1)***

$-14 = 3 + 2b$

$b = -\frac{17}{2}$ ***A1***

Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

METHOD 3

$a = \frac{1}{2}$ ***A1***

$\frac{x^2 - 14x + 24}{2x + 6} - \frac{1}{2}x \equiv \frac{-17x + 24}{2x + 6}$ ***(A1)***

attempt to find the limit of $f(x) - ax$ as $x \to \infty$ ***(M1)***

$b = \lim_{x \to \infty} \frac{-17x + 24}{2x + 6}$

$= -\frac{17}{2}$ ***A1***

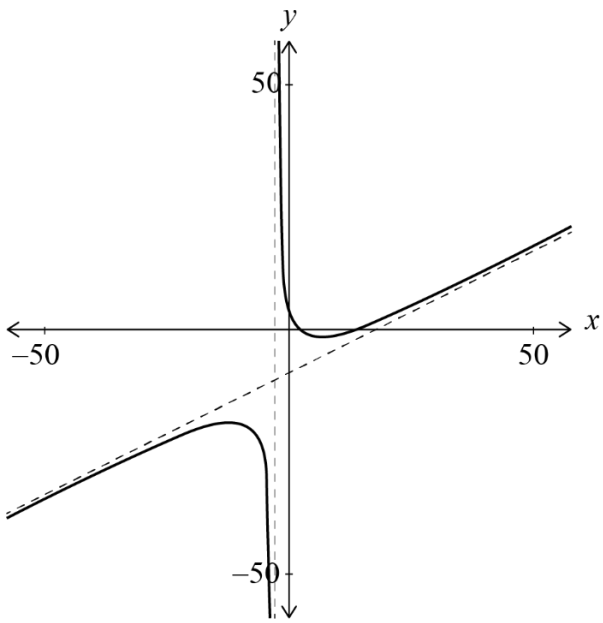
Note: Accept $y = \frac{1}{2}x - \frac{17}{2}$.

[4 marks]

continued…

Question 11 continued

(d)



two branches with approximately correct shape (for $-50 \leq x \leq 50$) ***A1***

Note: For this **A1** the graph must be a function.

their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1A1***

Note: Award ***A1*** for vertical asymptote and behaviour and ***A1*** for oblique asymptote and behaviour.
If only top half of the graph seen only award ***A1A0*** if both asymptotes and behaviour are seen.

their axes intercepts in approximately the correct positions ***A1***

Note: Points of intersection with the axes and the equations of asymptotes do not need to be labelled. Ignore incorrect labels.

[4 marks]

(e) $\left(-10-5\sqrt{3}=\right)-18.6602....$ OR $\left(-10+5\sqrt{3}=\right)-1.33974....$ seen anywhere ***(A1)***

attempt to write the range using at least one value in an interval or an inequality in y or $f(x)$ ***(M1)***

$y \leq -18.7,\ y \geq -1.34$ ***A1A1***

Note: Award ***A1*** for each inequality**.** Award ***A1A0*** for strict inequalities in both.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

(f) $\left(-10-2\sqrt{31}=\right)-21.1355....$ OR $\left(-10+2\sqrt{31}=\right)1.13552....$ seen anywhere ***(A1)***

$x<-21.1,\ -3<x<1.14$ ***A1A1A1***

Note: Award ***A1*** for $x<-21.1$, ***A1*** for correct endpoints of a single interval -3 and 1.14 and ***A1*** for $-3<x<1.14$.
Do not award FT from (d).
Accept equivalent set notation.

[4 marks]

Total [19 marks]

1. (a) varies the value of b with $a = 3$ ***(M1)***

Note: The ***(M1)*** in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii).
Award ***(M1)*** for evidence that $b = 0$ case is considered/determined.

$b = -4, 0$ ***A1***

[2 marks]

(b) (i) $b < -4$ or $b > 0$ ***A1***

[1 mark]

(ii) $-4 < b < 0$ ***A1***

[1 mark]

continued…

Question 1 continued

(c) (i) $b = 0, 4$ ***A1***

[1 mark]

(ii) $b < 0$ or $b > 4$ ***A1***

[1 mark]

(iii) $0 < b < 4$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) one point of zero gradient is located on either side (of the x- axis) (or equivalent) ***A1***

[1 mark]

(e) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 2ax$ ***(A1)***

attempts to solve their $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x ***M1***

$x(3x + 2a)(=0)$ OR $x = \frac{-2a \pm \sqrt{4a^2}}{6}$ OR $x + \frac{a}{3} = \pm\frac{a}{3}$

$x = -\frac{2}{3}a,\ 0$ ***A1***

when $x = 0,\ y = b$ and so $\mathrm{P}(0,\ b)$ is a point of zero gradient ***AG***

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $\mathrm{Q}\left(-\frac{2}{3}a,\ \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

continued…

Question 1 continued

METHOD 2

$\frac{dy}{dx} = 3x^2 + 2ax$ ***(A1)***

substitutes either $x = 0$ or $x = -\frac{2}{3}a$ into their $\frac{dy}{dx}$ ***M1***

when $x = 0$, $\frac{dy}{dx} = 0$ and $y = b$ so $P(0, b)$ is a point of zero gradient ***AG***

$\frac{dy}{dx} = 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right)$

$= \frac{4}{3}a^2 - \frac{4}{3}a^2 \ (= 0) \ \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2(= 0), = \frac{12}{9}a^2 - \frac{4}{3}a^2(= 0)\right)$ ***A1***

and so $\frac{dy}{dx} = 0$ when $x = -\frac{2}{3}a$ ***AG***

Note: The following two marks are independent of the first three marks.

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b \ \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

continued…

Question 1 continued

(f) (i) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 2a$ ***A1***

when $x = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a$ $(a > 0)$ and so (P) is a (local) minimum (point) ***R1***

when $x = -\frac{2}{3}a$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2a$ $(a > 0)$ and so (Q) is a (local) maximum (point) ***R1***

[3 marks]

(ii) (P and Q are) both above (the x- axis) ***A1***

Note: Award ***A1*** if it is made clear that both points are above (the x- axis).
Accept a labelled sketch that clearly shows this information.

[1 mark]

continued…

Question 1 continued

(g) (i) (P) is a (local) maximum (point) and is above (the x- axis) ***A1***

[1 mark]

(ii) (Q is below the x- axis when) $\frac{4}{27}a^3 + b < 0$ ***A1***

Note: Award ***A1*** for an equivalent correct inequality, eg. $\frac{4}{27}a^3 < -b$.
Accept a labelled sketch that clearly shows this information.

[1 mark]

continued…

Question 1 continued

(h) **METHOD 1**

attempts to factorize $4a^3b+27b^2\left(<0\right)$ ***(M1)***

$27b\left(\frac{4}{27}a^3+b\right)\left(<0\right)$ OR $b\left(4a^3+27b\right)\left(<0\right)$ ***A1***

$b>0$ and $\frac{4}{27}a^3+b<0$ or $b<0$ and $\frac{4}{27}a^3+b>0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.
Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3+b$ is less than zero (or equivalent).

when b and $\frac{4}{27}a^3+b$ have opposite sign, P and Q are located on either side (of the x- axis) (or equivalent) ***R1***

Note: Accept labelled sketches that clearly show this information.

P and Q are located on either side (of the x- axis) if (and only if) the curve has exactly three x- axis intercepts ***R1***

if $4a^3b+27b^2<0$, then the graph of $y=x^3+ax^2+b$ has exactly three x- axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

continued…

Question 1 continued

METHOD 2

attempts to factorize $4a^3b+27b^2\ (<0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3+b\right)(<0)$ OR $b\left(4a^3+27b\right)(<0)$ ***A1***

either $b>0$ and $\frac{4}{27}a^3+b<0$ or $b<0$ and $\frac{4}{27}a^3+b>0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.
Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3+b$ is less than zero (or equivalent).

$b>0$ and $\frac{4}{27}a^3+b<0$, $\left(\Rightarrow 0<b<-\frac{4}{27}a^3\right)\Rightarrow a<0$ and hence three x- axis intercepts ***R1***

$b<0$ and $\frac{4}{27}a^3+b>0$, $\left(\Rightarrow -\frac{4}{27}a^3<b<0\right)\Rightarrow a>0$ and hence three x- axis intercepts ***R1***

Note: Accept labelled sketches that clearly show this information.

if $4a^3b+27b^2<0$, then the graph of $y=x^3+ax^2+b$ has exactly three x- axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

[5 marks]

Total [24 marks]

2. (a) $\log_{10} 1 - \log_{10} a$ OR $\log_{10} a^{-1} = -\log_{10} a$ OR $\log_{10} 10^{-\frac{1}{3}}$ OR $10^x = \frac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$= -\frac{1}{3}$ ***A1***

[2 marks]

(b) $\frac{\log_{10} a}{\log_{10} 1000}$ OR $\frac{1}{3}\log_{1000} 10$ OR $\log_{1000} \sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}} = 1000^x \left(= \left(10^3\right)^x\right)$ ***(A1)***

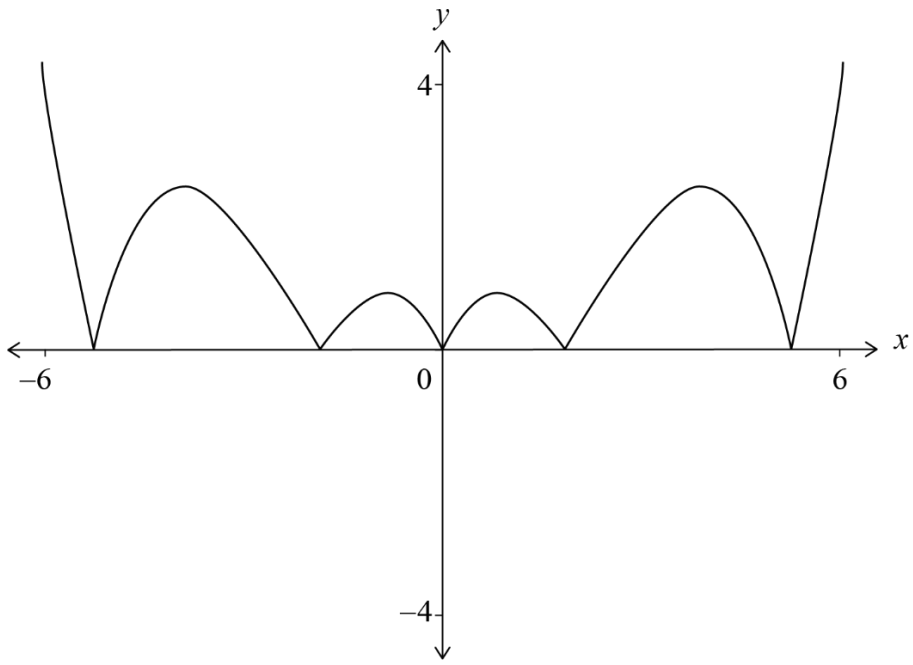
$\frac{\log_{10} a}{3}$ OR $\frac{1}{3}\log_{1000} 1000^{\frac{1}{3}}$ OR $\log_{1000} 1000^{\frac{1}{9}}$ OR $3x = \frac{1}{3}$ ***(A1)***

$= \frac{1}{9}$ ***A1***

[3 marks]

Total [5 marks]

9. (a)



reflection of all negative sections in x-axis ***(M1)***

approximately correct graph with sharp points (cusps) at x-intercepts ***A1***

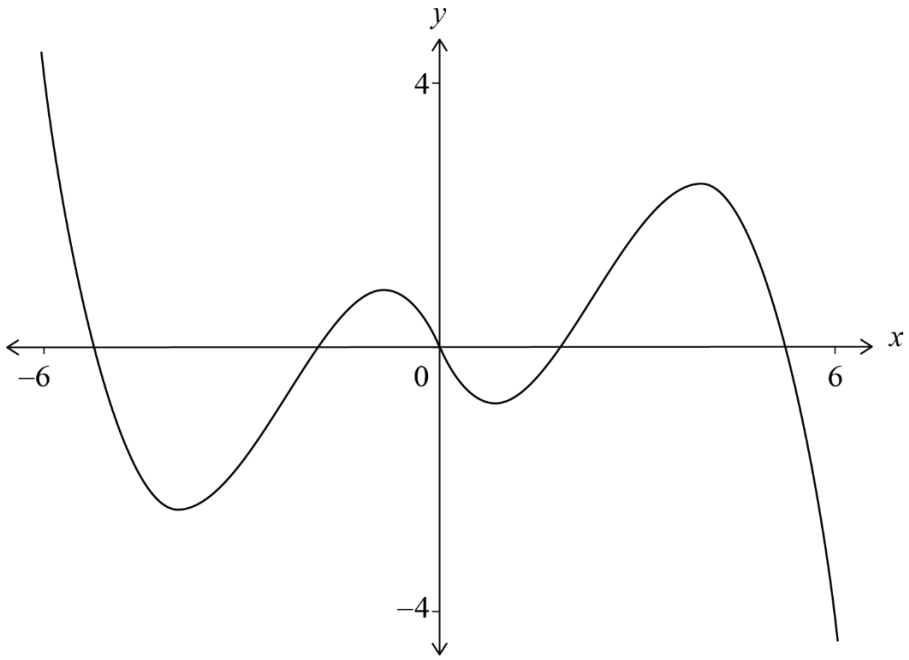
Note:	Award ***A1*** only if the heights of the maximum in the middle are lower than the heights of the maximum at the ends.

[2 marks]

continued…

Question 9 continued

(b)



A1A1

Note: Award ***A1*** for right hand side unchanged and ***A1*** for rotation 180° about the origin.

[2 marks]

(c) (i) -1.6 ***A1***

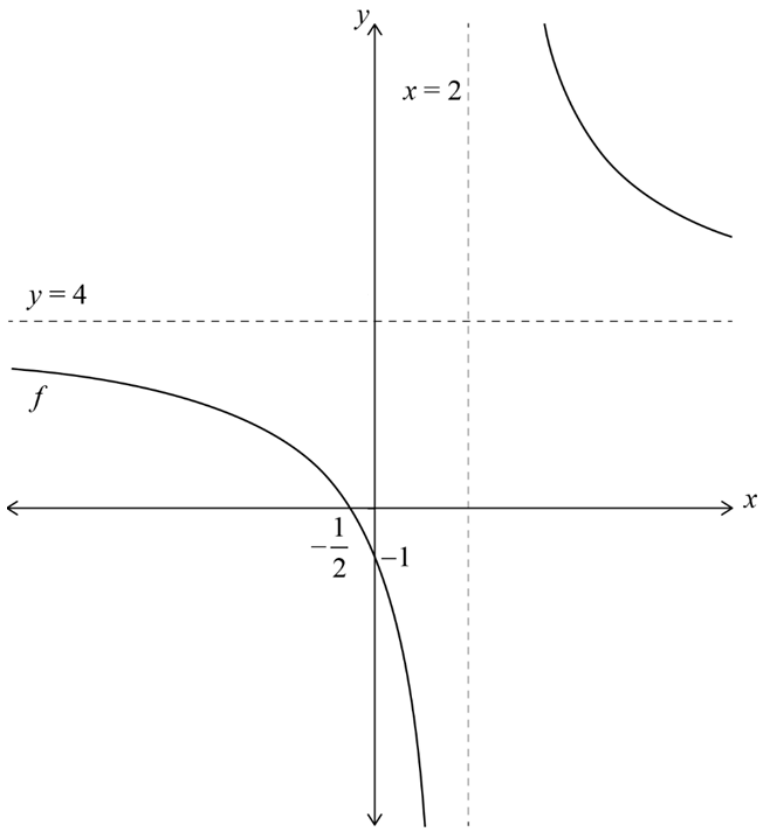
(ii) 3.2 ***A1***

[2 marks]

Total [6 marks]

Section B

10. (a)



vertical asymptote $x = 2$ sketched and labelled with correct equation ***A1***

horizontal asymptote $y = 4$ sketched and labelled with correct equation ***A1***

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x-axis, -1 on y-axis ***A1A1***

two branches in correct opposite quadrants with correct asymptotic behaviour ***A1***

Note:	These marks may be awarded independently.

[5 marks]

continued…

Question 10 continued

(b) $y \neq 4$ (or equivalent) ***A1***

[1 mark]

(c) $2+\frac{5}{2}$ OR $\left(-\frac{1}{2}\right)+2\times\frac{5}{2}$ OR $\frac{-\frac{1}{2}+p}{2}=2$ OR $-4=-p+\frac{1}{2}$ ***A1***

$p=\frac{9}{2}$ ***AG***

[1 mark]

(d) **METHOD 1**

attempt to substitute both roots to form a quadratic ***(M1)***

EITHER

$\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $x^2-\left(-\frac{1}{2}+\frac{9}{2}\right)x+\left(-\frac{1}{2}\times\frac{9}{2}\right)$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

OR

$(2x+1)(2x-9)=4\left(x^2-4x-\frac{9}{4}\right)$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

Note: Award ***A1*** for each correct value. They must be stated explicitly.

continued…

Question 10 continued

METHOD 2

$-\frac{b}{2}=2$ OR $4+b=0 \Rightarrow b=-4$ ***A1***

attempt to form a valid equation to find c using their b ***(M1)***

$\left(-\frac{1}{2}\right)^2+-4\left(-\frac{1}{2}\right)+c=0$ OR $\left(\frac{9}{2}\right)^2+-4\left(\frac{9}{2}\right)+c=0$

$c=-\frac{9}{4}$ ***A1***

METHOD 3

attempt to form two valid equations in b and c ***(M1)***

$\left(-\frac{1}{2}\right)^2+b\left(-\frac{1}{2}\right)+c=0$, $\left(\frac{9}{2}\right)^2+b\left(\frac{9}{2}\right)+c=0$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

METHOD 4

attempt to write $g(x)$ in the form $(x-h)^2+k$ and substitute for x, h and $g(x)$ ***(M1)***

$\left(-\frac{1}{2}-2\right)^2+k=0 \Rightarrow k=-\frac{25}{4}$

$(x-2)^2-\frac{25}{4}$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

[3 marks]

continued…

Question 10 continued

(e) attempt to substitute $x = 2$ into their $g(x)$ OR

complete the square on their $g(x)$ (may be seen in part (d)) ***(M1)***

$y = -\frac{25}{4}$ ***A1***

[2 marks]

(f) $\frac{4x+2}{x-2} = \left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $\frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$

attempt to form a cubic equation ***(M1)***

EITHER

$4x+2 = (x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $4x+2 = \left(x^2-4x-\frac{9}{4}\right)(x-2)$ OR

$(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right) - 4x - 2$ OR $(x-2)\left(x^2-4x-\frac{9}{4}\right) - 4x - 2$

$x^3 + \ldots + \frac{5}{2}(=0)$ OR $4x^3 + \ldots + 10(=0)$ ***(A1)(A1)***

Note: Award ***(A1)*** for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

product of roots = $\left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right)$ OR $\left(\frac{(-1)^3 \times 10}{4}\right)$

$= -\frac{5}{2}$ ***A1***

continued…

Question 10 continued

OR

$4\left(x+\frac{1}{2}\right)=(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$

$x=-\frac{1}{2}$ ***(A1)***

or $4=x^2+\ldots+9\Rightarrow x^2+\ldots+5=0$

product of roots of quadratic is 5 ***(A1)***

product is therefore $-\frac{1}{2}\times 5$

$=-\frac{5}{2}$ ***A1***

[4 marks]

Total [16 marks]

2. recognising a quadratic in 3^x ***(M1)***

$3\times\left(3^x\right)^2+5\times3^x-2=0$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) ***(M1)***

$\left(3\times3^x-1\right)\left(3^x+2\right)=0$ OR $3^x=\dfrac{-5\pm\sqrt{25+24}}{6}$ (or equivalent) ***(A1)***

$3^x=\dfrac{1}{3}$ (or $3^x=-2$) ***(A1)***

$x=-1$ ***A1***

Note: Award the final ***A1*** if candidate's answer includes $x=-1$ and $x=\log_3(-2)$. Award ***A0*** if other incorrect answers are given.

[5 marks]

3. (a) (i) $\left(\frac{9}{2}, \frac{3\sqrt{3}}{2}\right)$ (accept $x = \frac{9}{2}$ and $y = \frac{3\sqrt{3}}{2}$) ***A1***

(ii) **METHOD 1**

using $m = \frac{\text{change in } y}{\text{change in } x}$ with their midpoint OR gradient perpendicular to AC

OR $m = \tan 30^\circ$ ***(M1)***

$m = \frac{\sqrt{3}}{3}$ ***(A1)***

$y = \frac{\sqrt{3}}{3}x$ OR $y - \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}}{3}\left(x - \frac{9}{2}\right)$ (must be written as an equation) ***A1***

METHOD 2

attempt to find the vector equation of the line using a direction and a point on the line ***(M1)***

direction vector is $\begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ or equivalent/parallel vectors ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ OR $\boldsymbol{r} = \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ OR $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{9}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ (or equivalent) ***A1***

Note: Vector equation must be in the form $\boldsymbol{r} =$ or $\begin{pmatrix} x \\ y \end{pmatrix} =$.

Allow equivalent parametric forms such as $x = \frac{9}{2}t$, $y = \frac{3\sqrt{3}}{2}t$.

[4 marks]

continued…

Question 3 continued.

(b) substituting $x = 6$ into their equation ***(M1)***

so at B $y = 2\sqrt{3}$ ***(A1)***

area of triangle $\text{OAB} = \frac{1}{2} \times 6 \times 2\sqrt{3} = 6\sqrt{3}$

area of quadrilateral $\text{OABC} = 12\sqrt{3}$ ***A1***

[3 marks]

Total [7 marks]

5. (a) $y = \frac{2}{3}$ (must be written as equation with $y =$) ***A1***

[1 mark]

(b) (i) 2 ***A1***

(ii) **EITHER**

$$\frac{2(x+3)}{3(x+2)} = mx + 1$$

attempt to expand to obtain a quadratic equation ***(M1)***

$2x + 6 = 3mx^2 + 6mx + 3x + 6$

$3mx^2 + (6m+1)x = 0$ OR $3mx^2 + 6mx + x = 0$ ***A1***

recognition that discriminant $\Delta = 0$ for one solution ***(M1)***

$(6m+1)^2 = 0$

continued…

Question 5 continued.

OR

$$\frac{2(x+3)}{3(x+2)} = mx+1$$

attempt to expand to obtain a quadratic equation ***(M1)***

$$2x+6 = 3mx^2+6mx+3x+6$$

$3mx^2+(6m+1)x=0$ OR $3mx^2+6mx+x=0$ ***A1***

attempt to solve their quadratic for x and equating their solutions ***(M1)***

$$x(3mx+6m+1)=0$$

$x=0$ OR $x=-\frac{6m+1}{3m}(=0)$

$$-\frac{6m+1}{3m}=0$$

OR

attempt to find $f'(x)$ using the quotient rule ***(M1)***

$f'(x)=\frac{2}{3}\left(\frac{(x+2)-(x+3)}{(x+2)^2}\right)=\left(\frac{-2}{3(x+2)^2}\right)$ OR $\frac{2(3x+6)-3(2x+6)}{(3x+6)^2}$ or equivalent ***A1***

recognition that m is the derivative of $f(x)$ at $x=0$ (***M1)***

THEN

$\Rightarrow m=-\frac{1}{6}$ ***A1***

continued…

Question 5 continued.

(iii)

Note: In this part, FT may be awarded only for values of m between -1 and 0.

$$-\frac{1}{6} < m < 0$$ ***A2***

Note: Award ***A1*** for only $m > -\frac{1}{6}$. Award ***A1*** for only $m < 0$.

[7 marks]

Total [8 marks]

3. (a) $I = 2\times10^{-6}\left(=\frac{1}{500000}\right)$ (units) ***A1***

[1 mark]

(b) substitutes their doubled I-value from part (a) into L ***(M1)***

$L = 10\log_{10}\left(2\times10^{-6}\times10^{12}\right)\ \left(=63.0102...\right)$

$=63.0$ (decibels) ***A1***

Note: Accept $60+10\log_{10}2$ (decibels) as a final answer.
Do not award the final ***A1*** for $L=0$ (from $I=10^{-12}$).

[2 marks]

(c) $115 = 10\log_{10}\left(I\times10^{12}\right)$ ***(A1)***

attempts to solve for I ***(M1)***

$I=\frac{10^{11.5}}{10^{12}}$ (or equivalent) $\left(=0.316227...\right)$

$I=0.316$ (units) ***A1***

Note: Accept exact final answers such as $10^{-0.5}$ and $\frac{1}{\sqrt{10}}$.

[3 marks]

Total [6 marks]

6. (a) attempt to find critical values ***(M1)***

$x = \frac{3}{2}, x = 6$ ***(A1)***

$\frac{3}{2} < x < 6$ ***A1***

Note: Allow equivalent, and/or interval notation.

[3 marks]

(b) $k = \frac{3}{2}$ ***A1***

since we require $2x^2 - 15x + 18 \geq 0$ (and f must be one to one) ***R1***

OR

Does not obey horizontal line test for $x \geq \frac{3}{2}$ ***R1***

[2 marks]

Total [5 marks]

1. (a) (i) $f(0) = -11$ ***A1***

(ii) $-1.80650...$

$f(20) = -1.81 \left(= 11\sqrt{20} - 51 = 22\sqrt{5} - 51\right)$ ***A1***

[2 marks]

(b) attempt to find at least one root ***(M1)***

$x = 1.72622...$ and $x = 17.5237...$

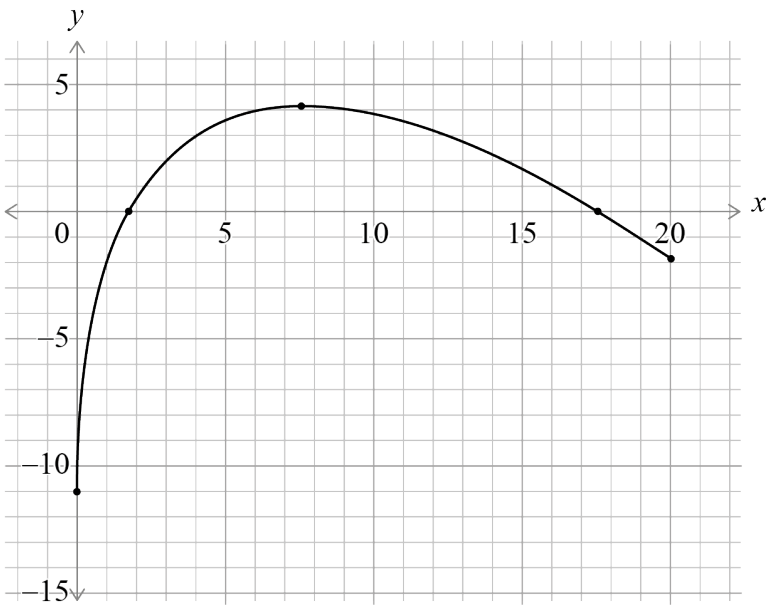
$x = 1.73$ and $x = 17.5$ ***A1***

[2 marks]

continued…

Question 1 continued.

(c)



A1A1A1

Note: Follow through from their part (a).

Award ***A1*** for endpoints at approximately $(0,-11)$ and $(20,-1.8)$. Allow for y-intercept $-12 < y < -10$ and for the right endpoint in the interval $x = 20, -2 < y < -1$.

The following two ***A*** marks may only be awarded if the approximate shape is correct.

Award ***A1*** for x-intercepts at approximately $x = 1.7$ and $x = 17.5$ and award ***A1*** for maximum at approximately $(7.6, 4.1)$. Allow for x-intercepts in the intervals $1 < x < 2$, $17 < x < 18$, and maximum in the intervals $6.5 < x < 8.5$, $3.5 < y < 5$.

[3 marks]

Total [7 marks]

5. METHOD 1

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

(discriminant =) $(-r)^2 - 4(4)(r-1)$ $\left(= r^2 - 16r + 16\right)$ (seen anywhere) ***(A1)***

$1.07179...$ $\left(= 8 - 4\sqrt{3}\right)$ AND $14.9282...$ $\left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

recognition that discriminant of $4x^2 - rx + r - 1$ is less than zero ***(M1)***

$1.07 < r < 14.9$ $(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3})$ ***A1***

Note: Accept $1.08 \leq r \leq 14.9$.

METHOD 2

recognition that $4x^2 - rx + r - 1$ must be greater than zero (seen anywhere) ***R1***

EITHER

minimum when $x = \frac{r}{8} \Rightarrow (y =) 4\left(\frac{r}{8}\right)^2 - r\left(\frac{r}{8}\right) + r - 1$ (> 0) ***(A1)***

attempt to solve their inequality for y (must be in terms of r and r^2) ***(M1)***

OR

$x < 1 \Rightarrow r > \frac{4x^2 - 1}{x - 1}$ OR $x > 1 \Rightarrow r < \frac{4x^2 - 1}{x - 1}$ ***(A1)***

attempt to find local minumum AND local maximum of $r = \frac{4x^2 - 1}{x - 1}$ ***(M1)***

THEN

$(r >) 1.07179...$ $\left(= 8 - 4\sqrt{3}\right)$ AND $(r <) 14.9282...$ $\left(= 8 + 4\sqrt{3}\right)$ (seen anywhere) ***(A1)***

$1.07 < r < 14.9$ $\left(8 - 4\sqrt{3} < r < 8 + 4\sqrt{3}\right)$ ***A1***

Note: Accept $1.08 \leq r \leq 14.9$.

[5 marks]

12. (a) attempt to use quotient rule ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x+k)(4x+6)-(2x^2+6x-3)}{(x+k)^2}$$ ***A1A1***

Note: Award ***A1*** for $=\frac{(x+k)(4x+6)}{(x+k)^2}$, ***A1*** for $\frac{-(2x^2+6x-3)}{(x+k)^2}$ OR $\frac{-2x^2-6x+3}{(x+k)^2}$

$=\frac{4x^2+4kx+6x+6k-2x^2-6x+3}{(x+k)^2}$ or equivalent leading to ***AG*** line ***A1***

$=\frac{2x^2+4kx+6k+3}{(x+k)^2}$ ***AG***

Note: Candidates may use product rule, give marks accordingly.

[4 marks]

(b) local min or max when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, when $2x^2+4kx+6k+3=0$ has real solutions.

attempt to find the discriminant of numerator (***M1)***

$(4k)^2-4\times2(6k+3)\left(=2k^2-6k-3<0\right)$ ***(A1)***

attempt to find the critical values of their quadratic equation or inequality (***M1)***

$k=3.43649...$ OR $k=-0.436491...$ OR $k=\frac{3\pm\sqrt{15}}{2}$

$k>3.44\left(=\frac{3+\sqrt{15}}{2}\right)$ (since k is positive) ***A1***

Note: Accept $k\geq3.44$.

[4 marks]

(c) $x=-2$ ***A1***

[1 mark]

continued…

Question 12 continued.

(d) **METHOD 1**

$\frac{2x^2+6x-3}{x+2} = 2x+...$ ***(A1)***

attempts division on $\frac{2x^2+6x-3}{x+2}$ ***M1***

$\frac{2x^2+6x-3}{x+2} = 2x+\frac{2x-3}{x+2}$

$= 2x+2+...$ ***(A1)***

asymptote is $y = 2x+2$ ***A1***

METHOD 2

let asymptote be $y = ax+b$

$\frac{2x^2+6x-3}{x+2} = ax+b+\frac{c}{x+2}$ ***M1***

$2x^2+6x-3 = (ax+b)(x+2)+c$

equates coefficients of x^2 and x: ***(M1)***

$a = 2$ ***A1***

$6 = 2a+b$

$b = 2$ ***A1***

$(y = 2x+2)$

continued…

Question 12 continued.

METHOD 3

$\frac{2x^2+6x-3}{x+2} = 2x+\ldots$ ***A1***

$\frac{2x^2+6x-3}{x+2} - 2x = \frac{2x-3}{x+2}$

attempt to find the limit as $x \to \infty$ ***M1***

$\lim_{x\to\infty} \frac{2x-3}{x+2} = 2$ ***A1***

$y = 2x+2$ ***A1***

[4 marks]

continued…

Question 12 continued.

(e) **METHOD 1**

EITHER

for $k = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 + 8x + 15}{(x+2)^2}$ ***A1***

attempt to write $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $p + \frac{q}{(x+2)^2}$ OR to write $\frac{\mathrm{d}y}{\mathrm{d}x} - 2$ in the form $\frac{q}{(x+2)^2}$ ***M1***

OR

$y = 2x + 2 - \frac{7}{x+2}$ ***A1***

Note: Follow through from their part (d).

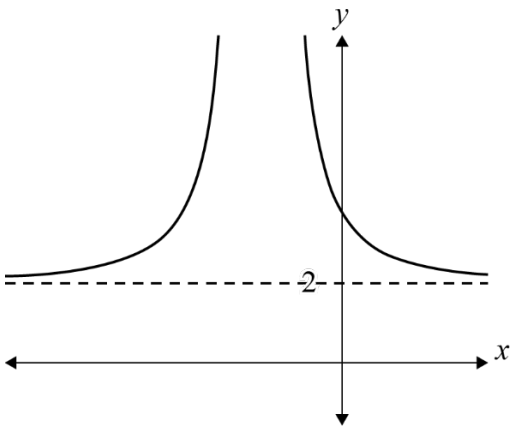
attempt to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = p + \frac{q}{(x+2)^2}$ ***M1***

THEN

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \frac{7}{(x+2)^2}$ OR $\frac{\mathrm{d}y}{\mathrm{d}x} - 2 = \frac{7}{(x+2)^2}$ ***A1***

since $\frac{7}{(x+2)^2} > 0$ for $x \in \mathbb{R}$, $x \neq -2$, $\frac{\mathrm{d}y}{\mathrm{d}x} > 2$. ***R1***

Note: Award a maximum of ***A1M1A0R0*** for a graphical approach which includes a horizontal asymptote at 2, showing that $\frac{\mathrm{d}y}{\mathrm{d}x} > 2$.



continued…

Question 12 continued.

METHOD 2

for $k = 2$, $\frac{dy}{dx} = \frac{2x^2 + 8x + 15}{(x+2)^2}$ ***A1***

clear assumption that $\frac{2x^2 + 8x + 15}{(x+2)^2} \leq 2$ ***M1***

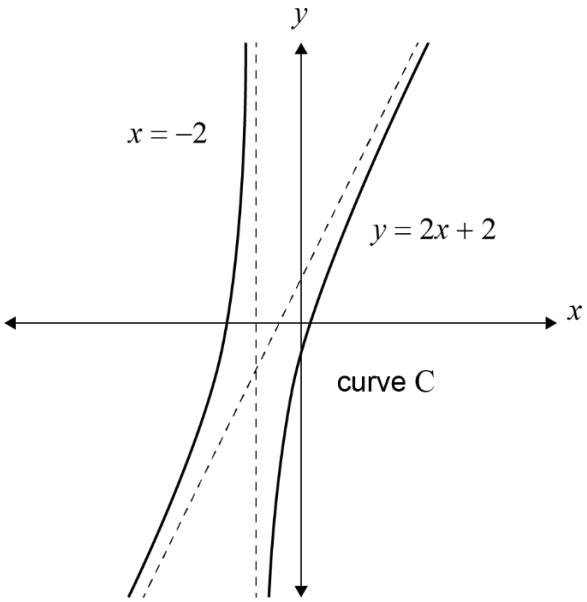
$\left(2x^2 + 8x + 15 \leq 2(x+2)^2 \Rightarrow\right) 15 \leq 8$ ***A1***

contradiction implies that $\frac{dy}{dx} > 2$ ***R1***

Note: Award ***A1M0A0R0*** for solutions which work backwards from $\frac{2x^2 + 8x + 15}{(x+2)^2} > 2$ to arrive at $15 > 8$.

[4 marks]

(f)



curve C in approx correct place, with negative y-intercept and part of the curve in 4th quadrant ***A1***

Note: The following ***A1*** marks should be awarded independently **provided** there is an attempt to draw both branches of $y = f(x)$.

vertical asymptote in approx correct place ***A1***

oblique asymptote in approx correct place ***A1***

approx correct asymptotic behaviour of C relative to both asymptotes ***A1***

[4 marks]

Total [21 marks]

2. (a) (i) attempts to solve ***(M1)***

EITHER

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

OR

$$(x+2)^2 - 3 (= 0)$$

OR

$$\left(x + 2 + \sqrt{3}\right)\left(x + 2 - \sqrt{3}\right)(= 0)$$

THEN

$$x = -2 \pm \sqrt{3}$$ ***A1A1***

Note: Award ***A1*** for $-2 + \sqrt{3}$ and ***A1*** for $-2 - \sqrt{3}$.

[3 marks]

continued…

Question 2 continued.

(ii) **METHOD 1**

attempts to find their $\frac{1}{-2-\sqrt{3}} \times \frac{-2+\sqrt{3}}{-2+\sqrt{3}}$ ***(M1)***

Note: Only award ***(M1)*** if the roots are in the form $s \pm \sqrt{t}$.

$= \frac{-2+\sqrt{3}}{(-2)^2-\left(\sqrt{3}\right)^2}$ and $(-2)^2-\left(\sqrt{3}\right)^2 = 1$ ***A1***

hence the roots form a reciprocal pair ***AG***

Note: Award as above for attempting to show that $\frac{1}{-2+\sqrt{3}} = -2-\sqrt{3}$.

METHOD 2

considers the product of their roots ***(M1)***

Note: Only award ***(M1)*** if the roots are in the form $s \pm \sqrt{t}$.

$\left(-2+\sqrt{3}\right)\left(-2-\sqrt{3}\right) = 1$ ***A1***

hence the roots form a reciprocal pair ***AG***

METHOD 3

considers $\frac{c}{a}\left(=\frac{1}{1}\right)$ ***(M1)***

the product of roots is 1 ***A1***

hence the roots form a reciprocal pair ***AG***

[2 marks]

continued…

Question 2 continued.

(b) roots are $\pm \mathrm{i}\left(=\mathrm{i}, \frac{1}{\mathrm{i}}, =\operatorname{cis}\left(\pm\frac{\pi}{2}\right), =\mathrm{e}^{\pm \mathrm{i}\frac{\pi}{2}}\right)$ ***(A1)***

EITHER

$\frac{1}{-\mathrm{i}} \times \frac{\mathrm{i}}{\mathrm{i}} = \mathrm{i}$ OR $\frac{1}{-\mathrm{i}} = \mathrm{i}$ ***A1***

Note: Award as above for showing that $\frac{1}{\mathrm{i}} = -\mathrm{i}$ OR stating that $\frac{1}{\mathrm{i}} = -\mathrm{i}$.

OR

$\mathrm{i} \times -\mathrm{i} = 1$ OR $\mathrm{i} \times \frac{1}{\mathrm{i}} = 1$ ***A1***

OR

the product of roots is 1 ***A1***

THEN

hence the roots form a reciprocal pair ***AG***

[2 marks]

(c) $p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$ ***(A1)***

$x^2 p\left(\frac{1}{x}\right) = x^2\left(\frac{a}{x^2} + \frac{b}{x} + a\right)$ ***A1***

$= ax^2 + bx + a$

$= p(x)$ ***AG***

Note: Award as above for showing that $\frac{p(x)}{x^2} = p\left(\frac{1}{x}\right) = \frac{a}{x^2} + \frac{b}{x} + a$.

[2 marks]

continued…

Question 2 continued.

(d) **METHOD 1**

$p(\alpha)=0 \Rightarrow \alpha^n p\left(\frac{1}{\alpha}\right)=0$ ***A1***

EITHER

$\alpha^n \neq 0$ (as $\alpha \neq 0$) $\Rightarrow p\left(\frac{1}{\alpha}\right)=0$ ***R1***

OR

$\alpha \neq 0$ (hence $\alpha^n \neq 0$) $\Rightarrow p\left(\frac{1}{\alpha}\right)=0$ ***R1***

THEN

so $\frac{1}{\alpha}$ is also a root ***AG***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

METHOD 2

$p\left(\frac{1}{\alpha}\right)=\left(\frac{1}{\alpha}\right)^n p\left(\frac{1}{\frac{1}{\alpha}}\right)$

$p\left(\frac{1}{\alpha}\right)=\left(\frac{1}{\alpha}\right)^n p(\alpha)$ ***A1***

for $\alpha \neq 0$ (seen anywhere), $p(\alpha)=0 \Rightarrow \left(\frac{1}{\alpha}\right)^n p(\alpha)=0$ and so $p\left(\frac{1}{\alpha}\right)=0$ ***R1***

so $\frac{1}{\alpha}$ is also a root ***AG***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

continued…

Question 2 continued.

(e) **METHOD 1**

$p(x) = x^n p\left(\frac{1}{x}\right)$

$q(x) = x^m q\left(\frac{1}{x}\right)$ ***(A1)***

attempts to substitute for $p(x)$ and $q(x)$ into $f(x) = p(x)q(x)$ ***(M1)***

$f(x) = x^n p\left(\frac{1}{x}\right) x^m q\left(\frac{1}{x}\right) \left(= x^{n+m} p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right)\right)$ ***A1***

$= x^{n+m} f\left(\frac{1}{x}\right)$ (and $n+m$ is the degree of f) ***A1***

hence f is a palindromic polynomial. ***AG***

METHOD 2

$p\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}$

$q\left(\frac{1}{x}\right) = \frac{q(x)}{x^m}$ ***(A1)***

attempts to substitute for $p\left(\frac{1}{x}\right)$ and $q\left(\frac{1}{x}\right)$ into $f\left(\frac{1}{x}\right) = p\left(\frac{1}{x}\right)q\left(\frac{1}{x}\right)$ ***(M1)***

$f\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}\frac{q(x)}{x^m} \left(= \frac{p(x)q(x)}{x^{n+m}}\right)$ ***A1***

$x^{n+m} f\left(\frac{1}{x}\right) = p(x)q(x)$ and $f(x) = p(x)q(x)$ (and $n+m$ is the degree of f) ***A1***

hence f is a palindromic polynomial. ***AG***

[4 marks]

continued…

Question 2 continued.

METHOD 3

for example, $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_{n-1}x + a_n$ AND

$q(x) = b_m x^m + b_{m-1}x^{m-1} + ... + b_{m-1}x + b_m$ ***A1***

attempts to multiply out their $p(x)q(x)$ ***(M1)***

Note: Only award ***(M1)*** for polynomials of degree n and m respectively with different corresponding coefficients.

$p(x)q(x) = a_n b_m x^{m+n} + (a_{n-1}b_m + a_n b_{m-1})x^{m+n-1} + ... + (a_{n-1}b_m + a_n b_{m-1})x + a_n b_m$

A1A1

Note: Award ***A1*** for the first and second terms and ***A1*** for the penultimate and last terms.

hence f is a palindromic polynomial ***AG***

[4 marks]

(f) **METHOD 1**

attempts to expand the product of two quadratic factors ***(M1)***

EITHER

$x^4 + ux^3 + vx^3 + uvx^2 + 2x^2 + ux + vx + 1$ (***A1)***

OR

$x^4 + (u+v)x^3 + (uv+2)x^2 + (u+v)x + 1$ ***(A1)***

THEN

equates coefficients and forms a system of two equations in u and v ***(M1)***

$u + v = 2$ and $uv + 2 = -1$ $(uv = -3)$ ***A1***

attempts to solve their system of two equations ***(M1)***

$u = -1, v = 3$ OR $(f(x) =)(x^2 - x + 1)(x^2 + 3x + 1)$ ***A1***

Note: Candidates must attempt to form and solve a system of equations in u and v. Hence, award no marks for working backwards from either decimal or exact values of roots given by a GDC.

Award ***A0*** for $u = -1, v = 3$ and $u = 3, v = -1$ both specified.

continued…

Question 2 continued.

METHOD 2

attempts to divide one of $\left(x^2+ux+1\right)$ or $\left(x^2+vx+1\right)$ into $f\left(x\right)$ ***(M1)***

EITHER

for example, the quotient is $x^2+\left(2-u\right)x+u^2-2u-2$ ***A1***

sets their quotient equal to x^2+vx+1 ***(M1)***

$x^2+\left(2-u\right)x+u^2-2u-2=x^2+vx+1$

$u^2-2u-2=1$ and $2-u=v$ ***A1***

attempts to solve their equations for u and then v (or vice versa) ***(M1)***

OR

for example, the remainder is $u\left(-u^2+2u+3\right)x+\left(-u^2+2u+3\right)$ ***A1***

sets their remainder equal to zero ***(M1)***

attempts to solve for u (or v) ***(M1)***

$u=-1,3$ and $u+v=2$ ***A1***

THEN

$u=-1,v=3$ OR $\left(f\left(x\right)=\right)\left(x^2-x+1\right)\left(x^2+3x+1\right)$ ***A1***

Note: Award ***A0*** for $u=-1,v=3$ and $u=3,v=-1$ both specified.

continued…

Question 2 continued.

METHOD 3

considers $f(1)$, for example ***(M1)***

$(u+2)(v+2)=5$ ***A1***

considers $f(2)$, for example ***(M1)***

$(2u+5)(2v+5)=33$ ***A1***

attempts to solve their system of two equations ***(M1)***

$u=-1, v=3$ OR $(f(x)=)(x^2-x+1)(x^2+3x+1)$ ***A1***

Note: Award ***A0*** for $u=-1, v=3$ and $u=3, v=-1$ both specified.

[6 marks]

(g) attempts to solve $x^4+2x^3-x^2+2x+1=0$ ***(M1)***

Note: Award ***(M1)*** for attempting to solve one of their quadratic equations.

$\frac{1}{2}\pm\frac{\sqrt{3}}{2}\text{i}, -\frac{3}{2}\pm\frac{\sqrt{5}}{2}$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2}\pm\frac{\sqrt{3}}{2}\text{i}$ and ***A1*** for $-\frac{3}{2}\pm\frac{\sqrt{5}}{2}$.

[3 marks]

continued…

Question 2 continued.

(h) **METHOD 1**

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_2x^2 + a_1x + a_0 (= 0)$

attempts to find $p(-1)$ ***(M1)***

$p(-1) = (-1)^n + a_1(-1)^{n-1} + a_2(-1)^{n-2} + ... + a_2(-1)^2 + a_1(-1) + 1$ ***(A1)***

$= (-1) + a_1 - a_2 + ... + a_2 - a_1 + 1$ ***A1***

(since n is odd), $p(x)$ has

EITHER

$n+1$ terms so all terms cancel giving $p(-1) = 0$ ***R1***

OR

an even number of terms so all terms cancel giving $p(-1) = 0$ ***R1***

THEN

hence -1 is always a root ***AG***

METHOD 2

considers the product of roots of a polynomial equation of degree n ***(M1)***

$\frac{(-1)^n a_0}{a_n}$

product of roots is $(-1)^n$ (seen anywhere) ***A1***

(for n odd) there are $\frac{n-1}{2}$ reciprocal pairs of roots ***A1***

product of each reciprocal pair is 1 and $(-1)^n = -1$ (since n is odd) ***R1***

hence -1 is always a root ***AG***

continued…

Question 2 continued.

METHOD 3

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_2x^2 + a_1x + a_0 (= 0)$

considers $p(x) = x^n p\left(\frac{1}{x}\right)$ with $x = -1$ ***(M1)***

$p(-1) = (-1)^n p(-1)$ ***A1***

$p(-1) = -p(-1)$ $(2p(-1) = 0)$ (since n is odd) ***A1***

hence $p(-1) = 0$ ***R1***

hence -1 is always a root ***AG***

[4 marks]

Total [28 marks]

10. (a) **METHOD 1**

$a = 5$ ***(A1)***

attempt to use roots and symmetry to find h ***(M1)***

$h = \frac{(-1)+(-3)}{2}$ OR half the distance between the roots $\frac{(-1)-(-3)}{2} = 1$ (may be seen on a diagram)

$h = -2$ (accept $x = -2$) ***(A1)***

$f(x) = 5(x-(-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ ***A1***

$(a = 5,\ h = -2,\ k = -5)$

METHOD 2

$a = 5$ ***(A1)***

attempt to expand

$(x+1)(x+3) = x^2 + 4x + 3$ OR $5(x+1)(x+3) = 5x^2 + 20x + 15$

EITHER

uses their expansion to attempt to complete the square to the form ***(M1)***

$p(x+q)^2 + r$,where q is half the coefficient of their x term

$= (x+2)^2 - 2^2 + 3\left(= (x+2)^2 - 1\right)$ OR $5\left[(x+2)^2 - 2^2 + 3\right]\left(= 5(x+2)^2 - 5\right)$ ***(A1)***

OR

uses their expansion to attempt to differentiate and sets equal to zero ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 4 = 0$ OR $\frac{\mathrm{d}y}{\mathrm{d}x} = 10x + 20 = 0$

$h = -2$ (accept $x = -2$) ***(A1)***

OR

uses their expansion to attempt to find axis of symmetry using $h = \frac{-b}{2a}$ ***(M1)***

$h = \frac{-4}{2}$ OR $h = \frac{-20}{10}$

$h = -2$ (accept $x = -2$) ***(A1)***

THEN

$f(x) = 5(x-(-2))^2 - 5\left(= 5(x+2)^2 - 5\right)$ ***A1***

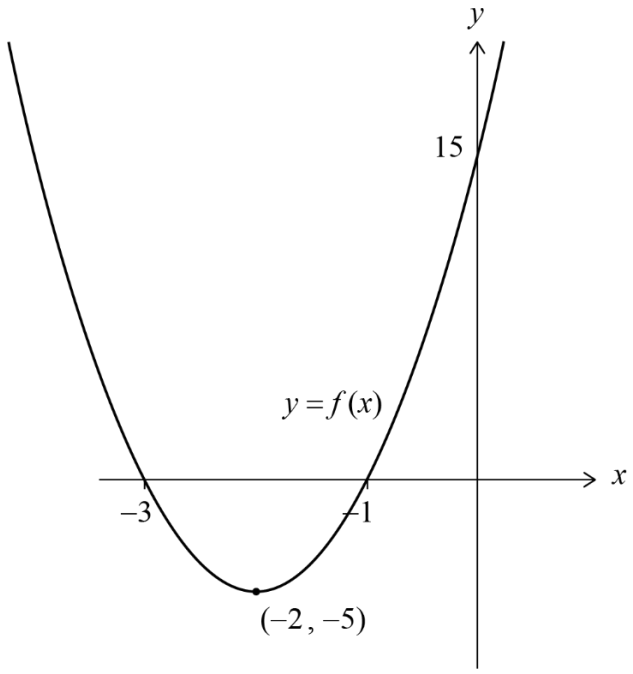
$(a = 5,\ h = -2,\ k = -5)$

[4 marks]

continued…

Question 10 continued.

(b)



M1A1A1A1

award ***M1*** for a roughly symmetric curve which is concave up

award ***A1*** for x intercepts at -3 and -1

award ***A1*** for y intercept at 15

award ***A1*** for vertex at $(-2,-5)$

[4 marks]

(c) $5(x+2)^2-5\leq 40$ OR $5(x+1)(x+3)\leq 40$ OR $(x+1)(x+3)\leq 8$ leading to

$(x+2)^2\leq 9$ OR $5x^2+20x-25\leq 0$ OR $x^2+4x-5\leq 0$ ***(A1)***

valid attempt to find the critical values for their quadratic inequality ***(M1)***

$x+2=\pm 3$ OR $(x+5)(x-1)=0$

$x=-5,\ x=1$ ***(A1)***

$-5\leq x\leq 1$ ***A1***

Note: Accept $(x\in)[-5,1]$ or equivalent.

[4 marks]

continued…

Question 10 continued.

(d) (i) $(f \circ g)(x) = 5(\ln x + 1)(\ln x + 3)$ OR $5(\ln x + 2)^2 - 5$ OR $5(\ln x)^2 + 20\ln x + 15$ ***A1***

(ii) attempt to replace x with $\ln x$ using their solution to part (c) ***(M1)***

$-5 \le \ln x \le 1$

$e^{-5} \le x \le e$ ***A1***

Note: Accept $(x \in)[e^{-5}, e]$ or equivalent.

[3 marks]

(e) $(g \circ f)(x) = \ln(f(x))$

recognition that the domain requires $f(x) > 0$ ***(M1)***

$(x+1)(x+3) > 0$

$x < -3, x > -1$ ***A1A1***

Note: award **A1** for critical values and **A1** for correct inequalities.

accept $(x \in)(-\infty, -3) \cup (-1, \infty)$ or equivalent.

[3 marks]

Total [18 marks]

$\left(1\right)$ $\left(1\right)$

1.

(a) -2 (accept $(0,-2)$) ***A1***

[1 mark]

(b) $y=\frac{3}{2}$ (must be an equation) ***A1***

Note: Do not accept $\neq$ sign.

[1 mark]

(c) attempt to reflect the graph of f in the x-axis OR one correct value seen ***(M1)***

both correct values $-\frac{3}{2}$ and 2 (seen anywhere) ***A1***

$-\frac{3}{2}<y\leq 2$ ***A1***

[3 marks]

Total [5 marks]

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5.

(a) recognising $\Delta > 0$ (seen anywhere) ***(M1)***

$\Delta = k^2 - 4(15-k)$ $\left(= k^2 + 4k - 60\right)$ ***A1***

valid attempt to solve quadratic (in)equality ***(M1)***

$(k-6)(k+10)$ OR $k = \dfrac{-4 \pm \sqrt{4^2 - 4(-60)}}{2}$

two correct values -10 and 6 (seen anywhere) ***A1***

$k < -10,\ k > 6$ ***A1***

[5 marks]

(b) recognising constant or product of roots is positive ***(M1)***

$15 - k > 0 \Rightarrow k < 15$

$k < -10,\ 6 < k < 15$ ***A1***

[2 marks]

Total [7 marks]

7. **METHOD 1** (product of quadratic factors)

attempt to write as product of two quadratic factors ***(M1)***

$x^4 + px^3 - 2x^2 + qx - 3 = (x^2 + 2x + 1)(ax^2 + bx + c)$

$= (x^2 + 2x + 1)(x^2 + bx - 3)$ OR $(x^2 + 2x + 1)(x^2 - 3)$ ***A1***

attempt to compare their coefficients ***(M1)***

$p = b + 2$ OR $q = b - 6$ OR $-2 = 2b + 1 - 3$

$b = 0$ ***(A1)***

$p = 2$, $q = -6$ ***A1***

METHOD 2 (double root)

let $f(x) = x^4 + px^3 - 2x^2 + qx - 3$

recognition that $f(-1) = 0$ ***(M1)***

$p + q = -4$ ***A1***

recognition that $f'(x) = 0$ at $x = -1$ since it has a double root ***(M1)***

$f'(x) = 4x^3 + 3px^2 - 4x + q$

$3p + q = 0$ ***A1***

$p = 2$, $q = -6$ ***A1***

METHOD 3 (division by $(x+1)$ twice)

attempt to use division with $(x+1)$ to find remainder ***(M1)***

$p + q = -4$ ***A1***

attempt to use division once again with $(x+1)$ to find remainder ***(M1)***

$q + 3p = 0$ ***A1***

$p = 2$, $q = -6$ ***A1***

METHOD 4 (division by $x^2 + 2x + 1$)

attempt to use division to find remainder ***(M1)***

$(q + 3p)x + 2p - 4 (= 0)$ (or equivalent) ***(A1)***

equating the coefficients of their remainder to 0 ***(M1)***

$q + 3p = 0$ and $2p - 4 = 0$ OR $q - p + 2 = 2(1 - 2p)$ and $-3 = 1 - 2p$ (or equivalent) ***A1***

$p = 2$, $q = -6$ ***A1***

continued…

Question 7 continued.

METHOD 5 (sum and product of roots)

let the four roots be $-1, -1, \alpha$ and β

attempt to set sum of four roots equal to $-p$ OR product of four roots equal to -3 ***(M1)***

Note: Award ***M1*** for expansion of $(x-1)^2(x-\alpha)(x-\beta)$ leading to a quartic

$$x^4 + (2-\alpha-\beta)x^3 + (\alpha\beta - 2\alpha - 2\beta + 1)x^2 + (2\alpha\beta - \alpha - \beta)x + \alpha\beta \ (=0)$$

$(-1)+(-1)+\alpha+\beta = -p \ (\Rightarrow \alpha+\beta = -p+2)$ and $(-1)(-1)\alpha\beta = -3 \ (\Rightarrow \alpha\beta = -3)$ ***A1***

sum of product of pairs and sum of product of roots taken three at a time

$\alpha\beta - \alpha - \alpha - \beta - \beta + 1 = -2 \ (\Rightarrow \alpha\beta - 2\alpha - 2\beta = -3)$ and

$-\alpha\beta - \alpha\beta + \beta + \alpha = -q \ (\Rightarrow -2\alpha\beta + \beta + \alpha = -q)$ ***A1***

Note: Award ***A1*** for any two correct equations.

attempt to solve *their* four equations to obtain the value of p and q ***(M1)***

$\alpha\beta = -3$ and $\alpha\beta - 2\alpha - 2\beta = -3 \ \Rightarrow \alpha + \beta = 0$

$p = 2$, $q = -6$ ***A1***

[5 marks]

10.

(a) $4^x = 8$ OR $f^{-1}(x) = \log_4 x$ OR $f^{-1}(8) = \log_4 8$ ***(A1)***

attempt to use indices with same base OR change of base of logs ***(M1)***

$2^{2x} = 2^3$, $4^x = 4^{\frac{3}{2}}$ OR $f^{-1}(8) = \log_4 4^{\frac{3}{2}}$ OR $f^{-1}(8) = \dfrac{\log_2 8}{\log_2 4}$

$f^{-1}(8) = \dfrac{3}{2}$ ***A1***

[3 marks]

(b) (i) interchanging x and y (seen anywhere) ***(M1)***

$x = 1 + \log_2 y$ OR $y - 1 = \log_2 x$

$x - 1 = \log_2 y$ OR $2^{y-1} = x$

$g^{-1}(x) = 2^{x-1}$ (or equivalent) ***A1***

(ii) **METHOD 1**

a horizontal translation/shift by 1 unit to the left (do not accept 'move')

followed by a horizontal stretch/dilation with scale factor $\dfrac{1}{2}$ (accept horizontal compression by a factor of 2)

one correct transformation ***A1***

two correct transformations in the correct order ***A1***

METHOD 2

horizontal stretch/dilation with scale factor $\dfrac{1}{2}$ (accept horizontal compression by a factor of 2) ***A1***

vertical stretch/dilation with scale factor 2 ***A1***

Note: ***A1***s can be awarded independently here as order is not important.

[4 marks]

continued…

Question 10 continued.

(c) attempt to find composite function (in any order) ***M1***

$f\left(1+\log_2 x\right)\left(=4^{1+\log_2 x}\right)$

$4\times 4^{\log_2 x}$ OR $4\times 2^{2\log_2 x}$ OR $4^{\log_2 2x}$ OR $2^{(2+2\log_2 x)}$ OR $4^{\left(\log_4 4+2\log_4 x\right)}$ ***(A1)***

$=4\times 2^{\log_2 x^2}$ OR $2^{\log_2 (2x)^2}$ OR $4^{\log_4\left(4x^2\right)}$ ***A1***

$=4x^2$ ***AG***

[3 marks]

(d) (i) $2x-1+\dfrac{1}{2x+1}=\dfrac{(2x-1)(2x+1)+1}{2x+1}$ ***(A1)***

$=\dfrac{4x^2-1+1}{2x+1}$ ***A1***

$2x-1+\dfrac{1}{2x+1}=\dfrac{4x^2}{2x+1}$ ***AG***

Note: Accept working from RHS to LHS.

(ii) **METHOD 1**

enclosed area is $\displaystyle\int_1^3\left(2x-1+\frac{1}{2x+1}\right)\mathrm{d}x$

attempt to integrate (at least one correct term or $\ln\left(2x+1\right)$ seen) ***(M1)***

$=x^2-x+\dfrac{1}{2}\ln\left|2x+1\right|(+c)$ ***A1A1***

Note: Award ***A1*** for x^2-x and ***A1*** for $\dfrac{1}{2}\ln\left|2x+1\right|$.

Accept $\dfrac{1}{2}\ln\left(2x+1\right)$.

substitute correct limits into their integrated expression and subtract ***(M1)***

$=\left(9-3+\dfrac{1}{2}\ln 7\right)-\left(1-1+\dfrac{1}{2}\ln 3\right)$

$A=6+\dfrac{1}{2}\ln\dfrac{7}{3}$ ***A1***

continued…

Question 10 continued.

METHOD 2

attempt to use integration by substitution ***(M1)***

let $u = 2x+1$ OR $u = 2x-1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$

$= \frac{1}{2}\int \left(u - 2 + \frac{1}{u}\right)\mathrm{d}u$ OR $\frac{1}{2}\int \frac{(u-1)^2}{u}\mathrm{d}u$ OR $\frac{1}{2}\int \left(u + \frac{1}{u+2}\right)\mathrm{d}u$ ***A1***

correct integration

$= \frac{1}{4}u^2 - u + \frac{1}{2}\ln|u|(+c)$ OR $\frac{1}{4}u^2 + \frac{1}{2}\ln|u+2|(+c)$ ***A1***

substitution of their limits into their integrated expression and subtract ***(M1)***

$= \left(\frac{49}{4} - 7 + \frac{1}{2}\ln 7\right) - \left(\frac{9}{4} - 3 + \frac{1}{2}\ln 3\right)$ OR $\left(\frac{25}{4} + \frac{1}{2}\ln 7\right) - \left(\frac{1}{4} + \frac{1}{2}\ln 3\right)$

$A = 6 + \frac{1}{2}\ln\frac{7}{3}$ (or equivalent) ***A1***

[7 marks]

Total [17 marks]

2. (a) $f(-3) = -1$ ***A1***

[1 mark]

(b) $-3 \leq x \leq 5$ ***A1***

Note: Award ***A1*** for answers using interval notation $[-3, 5]$.

[1 mark]

(c) $\left(f^{-1}(2x-7) = -3 \Rightarrow\right) 2x - 7 = f(-3)$ **OR** $f^{-1}(-1) = -3$ ***(M1)***

$2x - 7 = -1$ ***(A1)***

$x = 3$ ***A1***

[3 marks]

Total [5 marks]

4. (a) equating $y = mx - 3$ and $y = x^2 - x - 1$ ***M1***

$x^2 - x - 1 - (mx - 3) = 0$ **OR** $x^2 - mx - x + 2 = 0$ ***A1***

$x^2 - (m+1)x + 2 = 0$ ***AG***

[2 marks]

(b) **METHOD 1 (discriminant)**

correct substitution into discriminant (do not award if seen only in quadratic formula) ***(A1)***

$(-(m+1))^2 - 4(1)(2)$ **OR** $(m+1)^2 - 4(1)(2)$ ***A1***

discriminant equals 0 (seen anywhere) ***A1***

$(m+1)^2 = 8$ **OR** $m^2 + 2m - 7 = 0$

$(m =)\ -1 - 2\sqrt{2},\ -1 + 2\sqrt{2}$ ***A1A1***

METHOD 2 (derivative)

$\frac{dy}{dx} = 2x - 1$ ***A1***

substituting $m = 2x - 1$ or $x = \frac{m+1}{2}$ into ***AG*** from part (a) ***(A1)***

$x^2 - (2x - 1 + 1)x + 2 = 0$ **OR** $x^2 - 2x^2 + 2 = 0$ **OR** $\left(\frac{m+1}{2}\right)^2 - (m+1)\frac{m+1}{2} + 2 = 0$

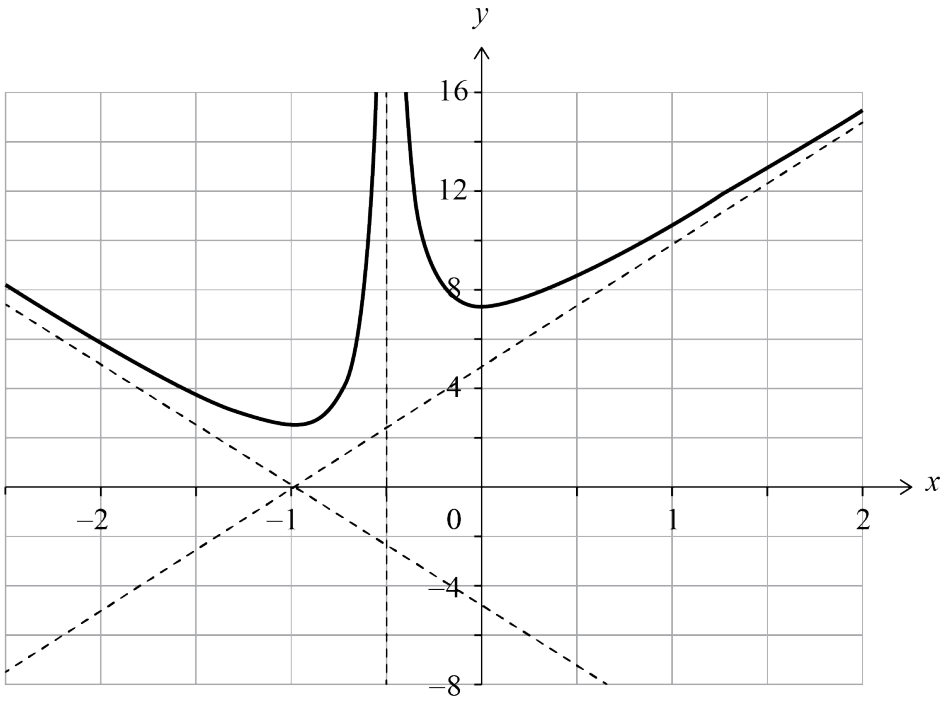
$x = \pm\sqrt{2}$ **OR** $-m^2 - 2m + 7 = 0$ **OR** $m^2 + 2m - 7 = 0$ ***A1***

$(m =)\ 2\sqrt{2} - 1,\ -2\sqrt{2} - 1$ ***A1A1***

[5 marks]

Total [7 marks]

6. (a)



right branch with minimum in approximately correct position **AND** showing asymptotic behaviour to $y = 5x + 5$ ***A1***

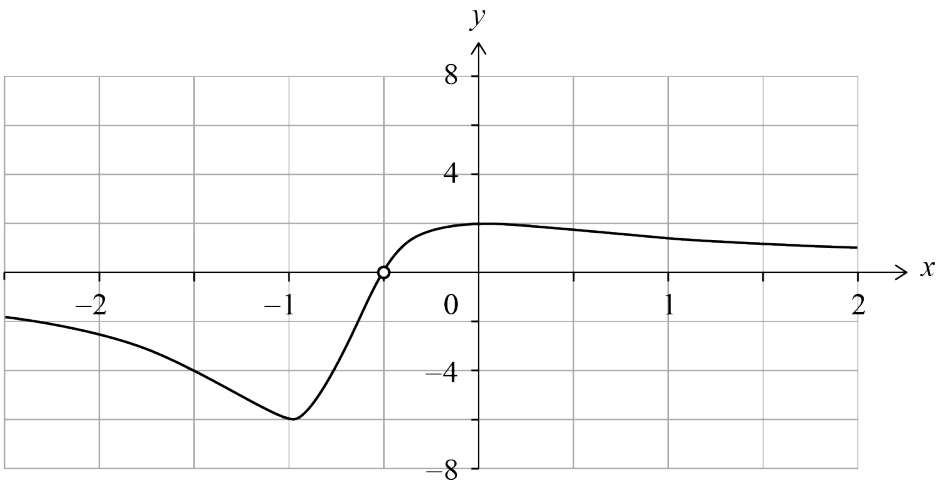
asymptotic behaviour on both branches at $x = -\frac{1}{2}$ ***A1***

left branch reflected in x-axis with minimum in approximately correct position ***M1***

oblique asymptote ($y = -5x - 5$) drawn in approximately correct position (equation is not required) **AND** left branch with correct asymptotic behaviour ***A1***

[4 marks]

(b)



axes intercepts in approximately correct positions ***A1***

local extrema in approximately correct positions ***A1***

asymptotic behavior to $y = 0$ on both sides **OR** asymptote at $y = 0$ indicated on sketch (by line or label etc. equation not required) ***A1***

[3 marks]

Total [7 marks]

2. EITHER

use of the remainder or factor theorem ***(M1)***

$f(1)=0$ **OR** $f(3)=-2$

$f(1)=2-6+p+q-2=0 \ (\Rightarrow p+q=6)$ ***(A1)***

$f(3)=162-162+9p+3q-2=-2 \ (\Rightarrow 3p+q=0)$ ***(A1)***

OR

attempt to use synthetic division OR long division to find remainder ***(M1)***

$\left(\frac{2x^4-6x^3+px^2+qx-2}{x-1}=2x^3-4x^2+(p-4)x+p+q-4+\frac{p+q-6}{x-1}\Rightarrow\right) p+q-6=0$ ***(A1)***

$\left(\frac{2x^4-6x^3+px^2+qx-2}{x-3}=2x^3+px+3p+q+\frac{9p+3q-2}{x-3}\Rightarrow\right) 9p+3q-2=-2$ ***(A1)***

THEN

solving simultaneously their two equations in p and q ***(M1)***

$p=-3, q=9$ ***A1***

[5 marks]

1. (a) (i) attempts to find an expression for $f^2(x)$ ***(M1)***

Note: For example, award ***(M1)*** for $f(mx+c)$ seen.

$= m(mx+c)+c$ ***A1***

$= m^2x+cm+c$ ***A1***

$f^2(x) = m^2x+c(1+m)$ ***AG***

[3 marks]

(ii) attempts to find an expression for $f^3(x)$ ***(M1)***

Note: For example, award ***(M1)*** for $f^2(mx+c)$ or $f\left(m^2x+c(1+m)\right)$ seen.

EITHER

$f^3(x) = m^2(mx+c)+c(1+m) \quad \left(= m^3x+m^2c+c+cm\right)$ ***A1***

OR

$f^3(x) = m\left(m^2x+c(1+m)\right)+c \quad \left(= m^3x+mc+m^2c+c\right)$ ***A1***

THEN

$f^3(x) = m^3x+c\left(1+m+m^2\right)$ ***AG***

[2 marks]

continued…

Question 1 continued

(b) (i) $f^4(x) = m^4x + c\left(1 + m + m^2 + m^3\right)$ ***A1***

Note: Award ***A1*** for equivalent forms, for example:

$f^4(x) = m^4x + c\sum_{i=0}^{3} m^i$

$f^4(x) = m^3\left(mx + c\right) + c\left(1 + m + m^2\right)$

$f^4(x) = m\left(m^3x + m^2c + mc + c\right) + c$

[1 mark]

(ii) $f^n(x) = m^nx + c\left(1 + m + (m^2 +)... + m^{n-1}\right)$ ***A1A1***

Note: Award ***A1*** for m^nx and ***A1*** for $c\left(1 + m + m^2 + ... + m^{n-1}\right)$ or equivalent,

for example: $c\sum_{i=0}^{n-1} m^i$ or $c\left(\frac{1-m^n}{1-m}\right)$

For alternative forms, award ***A1*** for the correct term in x and ***A1*** for the correct constant term.

[2 marks]

(iii) **METHOD 1**

attempts to substitute $m = 1$ into their $f^n(x)$ ***(M1)***

$f^n(x) = x + c(1 + 1 + 1 + ... + 1)$ ***(A1)***

$f^n(x) = x + nc$ ***A1***

METHOD 2

$f(x) = x + c$ ***(A1)***

attempts to find $f^2(x)$ with their $f(x)$ and $m = 1$ ***(M1)***

$f^2(x) = f(x + c) = x + 2c$

$\Rightarrow f^n(x) = x + nc$ ***A1***

[3marks]

continued…

Question 1 continued

(c) consider $n = 1$

$f^1(x) = m^1x + c\left(\frac{1-m^1}{1-m}\right) \Rightarrow f(x) = mx + c$, ***R1***

hence true for $n = 1$

Note: The substitution $n = 1$ must be explicit for the ***R1*** to be awarded.

Award ***R0*** for only considering $n = 2$.

Subsequent marks after this ***R1*** mark are independent of this mark and can be awarded.

assume true for $n = k$ ($p(k)$ is true) ***M1***

$f^k(x) = m^kx + c\left(\frac{1-m^k}{1-m}\right)$ (for some $k \in \mathbb{Z}^+$)

Note: The assumption of truth must be apparent. Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true" or "assume $n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

EITHER

Attempting to find $f\left(f^k(x)\right)$ ***(M1)***

$f^{k+1}(x) = f\left(f^k(x)\right)$

$f\left(m^kx + c\left(\frac{1-m^k}{1-m}\right)\right) = m\left(m^kx + c\left(\frac{1-m^k}{1-m}\right)\right) + c$ ***A1***

$= \left(m^{k+1}x + cm\left(\frac{1-m^k}{1-m}\right)\right) + c$

OR

Attempting to find $f^k\left(f(x)\right)$ ***(M1)***

$f^{k+1}(x) = f^k\left(f(x)\right)$

$f^k(mx + c) = m^k(mx + c) + c\left(\frac{1-m^k}{1-m}\right)$ ***A1***

$= \left(m^{k+1}x + cm^k + c\left(\frac{1-m^k}{1-m}\right)\right)$

continued…

Question 1 continued

THEN

EITHER

attempts to put c over a common denominator ***(M1)***

$$m^{k+1}x + cm\left(\frac{1-m^k}{1-m}\right) + \frac{c(1-m)}{1-m}$$

$$= m^{k+1}x + c\left(\frac{m(1-m^k)+(1-m)}{1-m}\right) \text{ (or equivalent)} \quad \textbf{A1}$$

OR

attempts to put cm^k over a common denominator ***(M1)***

$$m^{k+1}x + \frac{c\,m^k(1-m)}{1-m} + c\left(\frac{1-m^k}{1-m}\right)$$

$$= m^{k+1}x + c\left(\frac{m^k(1-m)+(1-m^k)}{1-m}\right) \text{ (or equivalent)} \quad \textbf{A1}$$

OR

recognizes that $\frac{1-m^k}{1-m}$ can be expressed as a geometric series ***(M1)***

$$m^{k+1}x + cm\left(1+m+m^2+\ldots+m^{k-1}\right)+c$$

$$= m^{k+1}x + c\left(1+m+m^2+m^3+\ldots+m^k\right) \quad \textbf{A1}$$

THEN

$$= m^{k+1}x + c\left(\frac{1-m^{k+1}}{1-m}\right) \quad \textbf{A1}$$

since true for $n=1$ and true for $n=k+1$ if true for $n=k$, hence true for all $n(\in\mathbb{Z}^+)$ ***R1***

Note: To obtain the final ***R1***, four of the previous seven marks must have been awarded.

The statement 'true for $n=1$' may be seen anywhere in the proof.

[8 marks]

continued…

Question 1 continued

(d) attempts to apply $m^n \to 0$ (as $n \to \infty$) to at least one term in expression ***(M1)***

Note: Award ***(M1)*** for an equivalent statement/conclusion in words.

$m^n x \to 0$ $\left(c\left(\frac{1-m^n}{1-m}\right) \to\right) \frac{c}{1-m}$ (as $n \to \infty$) ***(A1)A1***

Note: Award ***(A1)*** for the correct term in x and ***A1*** for the correct constant term.

$y = \frac{c}{1-m}$ ***A1***

[4 marks]

(e) (i) $f^n(x) = (-1)^n x + c\left(\frac{1-(-1)^n}{1-(-1)}\right)$ ***A1***

Note: Award ***A1*** here if the above is seen in part (e)(ii).

Award ***A0*** for a missing bracket, for example: $f^n(x) = -1^n x + c\left(\frac{1-(-1)^n}{1-(-1)}\right)$.

$(-1)^n = -1$ (when n is odd) ***R1***

$\left(f^n(x) = -x + c\left(\frac{1-(-1)}{1-(-1)}\right)\right)$

Note: The ***A1*** and ***R1*** mark can be awarded in either order. ***A0R1*** is possible.

Award ***R0*** for arguments based on specific numerical examples.

$f^n(x) = -x + c$ ***AG***

[2 marks]

(ii) evidence that an even n has been considered ***(M1)***

for example, $(-1)^2 = 1$

$f^n(x) = x$ ***A1***

Note: Award ***(M1)A1*** for answer of x (need not be an equation).

[2 marks]

Total [27 marks]

2. (a) attempt to factorize at least one expression ***(M1)***

$\frac{x}{(x-7)(x-1)} \times \frac{(x-1)(x+1)}{x+1}$ **OR** $\frac{x(x^2-1)}{(x-7)(x^2-1)}$ **OR** $\frac{x(x^2-1)}{(x-7)(x-1)(x+1)}$ ***A1***

$\frac{x}{x-7}$ ***AG***

[2 marks]

(b) **METHOD 1** (combines given log terms)

attempt to use the subtraction/quotient property of logs ***(M1)***

$\log_2\left(\frac{x\left(x^2-1\right)}{\left(x^2-8x+7\right)(x+1)}\right)$

valid attempt to rewrite equation with (or without) logs on both sides (seen anywhere) ***(M1)***

$2=\frac{x}{x-7}$ **OR** $\log_2\left(\frac{x}{x-7}\right)=\log_2 2$ **OR** $2^{\log_2\left(\frac{x}{x-7}\right)}=2^1$ ***(A1)***

$x=14$ ***A1***

METHOD 2 (uses log laws with LHS)

attempt to use $1=\log_2 2$ (seen anywhere) ***(M1)***

attempt to use the subtraction/quotient property of logs ***(M1)***

$\log_2\left(\frac{x^3-x}{2}\right)=\log_2\left(\left(x^2-8x+7\right)(x+1)\right)$ **OR** $\frac{x\left(x^2-1\right)}{2}=(x-1)(x-7)(x+1)$ ***(A1)***

$\frac{x}{2}=x-7$

$x=14$ ***A1***

[4 marks]

Total [6 marks]

4. (a) **METHOD 1**

x-coordinate of the vertex at A is 2 ***(A1)***

attempt to substitute their x-coordinate into equation of line ***(M1)***

$2(2)-1(=3)$

$3=-(2)^2+4(2)+p$ ***A1***

$p=-1$ ***AG***

METHOD 2

x-coordinate of the vertex at A is 2 ***(A1)***

attempt to substitute their x-coordinate into $f(x)$ ***(M1)***

$f(2)=p+4$

$p+4=2(2)-1$ ***A1***

$p=-1$ ***AG***

METHOD 3

attempt to complete the square on $f(x)$ ***(M1)***

$-(x-2)^2+p+4$

$x_{\mathrm{A}}=2$ **OR** $y_{\mathrm{A}}=p+4$ ***(A1)***

$p+4=2(2)-1$ ***A1***

$p=-1$ ***AG***

Note: Award no marks if $x=0$ is substituted into $-x^2+4x+p=2x-1$ to show $p=-1$.

[3 marks]

continued…

Question 4 continued.

(b) **METHOD 1** (equating $g(x)$ and $2x-1$)

x-coordinate is $\frac{-q}{2}$ (seen anywhere) ***(A1)***

attempt to find intersection points, equating $g(x)$ and $2x-1$ ***(M1)***

$x^2+qx-1=2x-1$

attempt to substitute their x-coordinate ***(M1)***

$\left(\frac{-q}{2}\right)^2+q\left(\frac{-q}{2}\right)-1=2\left(-\frac{q}{2}\right)-1$

$q^2-4q=0$ ***(A1)***

$q=4$ (since $q\neq 0$) ***A1***

METHOD 2 (find coordinates of B in terms of q)

x-coordinate is $\frac{-q}{2}$ ***(A1)***

EITHER

attempt to substitute their x-coordinate into $y=2x-1$ ***(M1)***

$2\left(\frac{-q}{2}\right)-1 \Rightarrow \mathrm{B}\left(\frac{-q}{2},-q-1\right)$

attempt to substitute their coordinates of B into $g(x)$ ***(M1)***

$-q-1=\left(\frac{-q}{2}\right)^2+q\left(\frac{-q}{2}\right)-1$

OR

attempt to substitute their x-coordinate into $g(x)$ ***(M1)***

$\left(\frac{-q}{2}\right)^2+q\left(\frac{-q}{2}\right)-1 \Rightarrow \mathrm{B}\left(\frac{-q}{2},-\frac{q^2}{4}-1\right)$

attempt to substitute their coordinates of B into the line ***(M1)***

$-\frac{q^2}{4}-1=2\left(\frac{-q}{2}\right)-1$

continued…

Question 4 continued.

THEN

$q^2 - 4q = 0$ ***(A1)***

$q = 4$ (since $q \neq 0$) ***A1***

METHOD 3 (completing the square)

attempt to complete the square on $g(x)$ ***(M1)***

$\left(x + \frac{q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - 1$

$x_B = \frac{-q}{2}$ **OR** $y_B = -\frac{q^2}{4} - 1$ ***(A1)***

attempt to substitute their coordinates of B into equation of line ***(M1)***

$-\frac{q^2}{4} - 1 = 2\left(\frac{-q}{2}\right) - 1$

$q^2 - 4q = 0$ ***(A1)***

$q = 4$ (since $q \neq 0$) ***A1***

[5 marks]

Total [8 marks]

3. (a) substitute $\mathrm{P}(-10,1)$ or $\mathrm{Q}(3,-12)$ into $y=\frac{Ax+B}{x-4}$ ***(M1)***

$-14=-10A+B$ (or equivalent)

$12=3A+B$ (or equivalent)

attempt to solve their equations simultaneously ***(M1)***

$A=2,\ B=6$ ***A1***

[3 marks]

(b) **EITHER**

attempt to express y in the form $P+\frac{Q}{x-4}$ ***(M1)***

OR

attempt to find one or both asymptotes using $x-4=0$ or $y=\frac{a}{c}$ (dividing coefficients) ***(M1)***

THEN

recognizing vertical asymptote at $x=4$ (may be seen anywhere, including labeled on graph or in part (a)) OR horizontal shift 4 units to the right ***A1***

recognizing horizontal asymptote at $y=2$ (may be seen anywhere, including labeled on graph or in part (a)) OR vertical shift 2 units up ***A1***

$\frac{2x+6}{x-4}=2+\frac{14}{x-4}$ ***(A1)***

EITHER

horizontal translation 4 units to the right;
vertical stretch scale factor 14 followed by vertical translation 2 up. ***A1***

Note: Horizontal translation may be seen anywhere; y -axis transformations must be seen in the order above.

OR

vertical stretch scale factor 14; followed by translation (through/by) $\begin{pmatrix}4\\2\end{pmatrix}$ ***A1***

Note: In this question, the transformations may be described using terms such as *translate* or *shift*, and *dilate* or *stretch*. Do not accept "*move*" to describe a transformation.

[5 marks]
Total [8 marks]

4.

Note: Candidates may approach this problem in different ways and may do their steps in many different orders.
The ***M*** marks are independent and may be awarded in any order. Use the description of each mark to determine when the mark may be awarded.

$3\log_8 10x - \log_4 x = 1$

attempt to apply power rule, quotient rule, or product rule (seen anywhere) ***(M1)***

attempt to apply change of base rule (seen anywhere) ***(M1)***

correct equation with same base in all logarithms ***A1***

$\log_2 10x - \log_2 x^{\frac{1}{2}} = 1$ or $\log_2 10 + \log_2 x - \frac{1}{2}\log_2 x = 1$ or $\frac{\log_4 (10x)^3}{\log_4 8} - \log_4 x = 1$ (or equivalent)

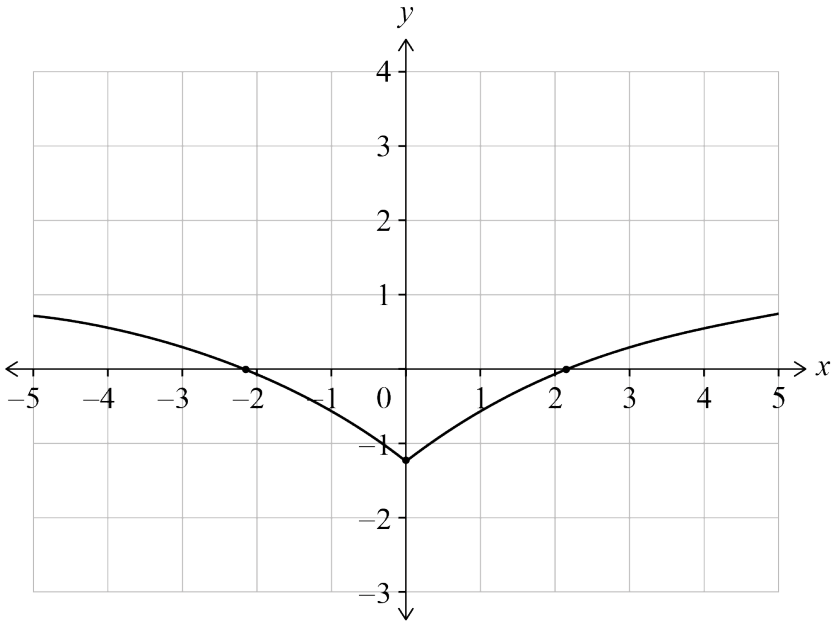
correct equation without logarithms ***A1***

$10\sqrt{x} = 2$ or $\sqrt{x} = \frac{2}{10}\left(= \frac{1}{5}\right)$ (or equivalent)

$x = \frac{1}{25}$ ***A1***

[5 marks]

5. (a) (i)



attempt to reflect graph in y-axis for $x \geq 0$ ***M1***

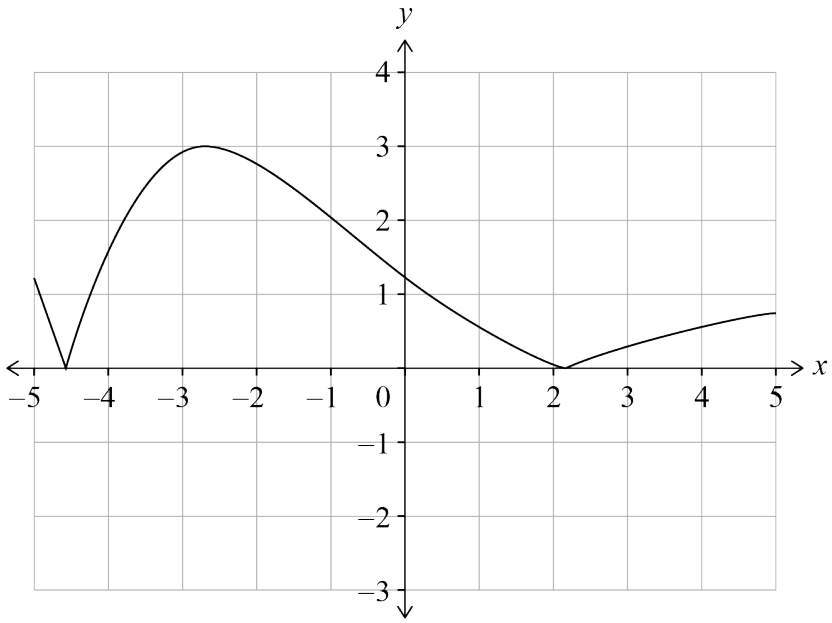
intercepts with x-axis and y-axis and clear cusp on y-axis in approximately the correct places ***A1***

Note: Award ***A1*** for roots between 2 to 2.5, -2.5 to -2 and y-intercept from -1.5 to -1.

continued…

Question 5 continued

(ii)



attempt to reflect graph in x-axis for $y \leq 0$ ***M1***

intercept with y-axis, maximum point and two cusps on x-axis in approximately the correct places ***A1***

Note: Award ***A1*** for cusps on x-axis between 2 to 2.5, -5 to -4 and y-intercept from 1 to 1.5 and maximum point x-coordinate between -3 to -2.5 with $y = 3$.

[4 marks]

(b) recognition that $|f(x)| \geq (f(x))^2$ when $|f(x)| \leq 1$ ***(M1)***

$x \geq -4.93$ OR $x \leq 5$

$x \leq -4.27$ OR $x \geq 0.292$ seen ***(A1)***

$-4.93 \leq x \leq -4.27$, $0.292 \leq x \leq 5$ ***A1A1***

[4 marks]

Total [8 marks]

6. attempt to use remainder theorem $P(-1)=12$ or to divide $P(x)$ by $(x+1)$ ***(M1)***

$2-q+r=12$ ***A1***

$(r-q=10)$

EITHER

$\alpha+\beta=-\frac{q}{2}$ and $\alpha\beta=\frac{r}{2}$ ***(A1)***

attempt to use $(\alpha+\beta)^2\equiv\alpha^2+\beta^2+2\alpha\beta$ with $\alpha+\beta=-\frac{q}{2}$ and $\alpha\beta=\frac{r}{2}$ ***(M1)***

$\left(-\frac{q}{2}\right)^2=\frac{37}{4}+2\left(\frac{r}{2}\right)$ ***(A1)***

$37=q^2-4r$

attempt to solve two simultaneous equations in q and r by rearranging/eliminating a variable ***(M1)***

$r=10+q$ and $r=\frac{q^2-37}{4}$ OR $q=r-10$ and $q=\sqrt{37+4r}$ OR $37=q^2-4(q+10)$ OR $37=(r-10)^2-4r$

OR

recognition that α and β are both roots of $P(x)$ and adding the two equations ***(M1)***

$2\alpha^2+q\alpha+r=0$ and $2\beta^2+q\beta+r=0$ then $2(\alpha^2+\beta^2)+q(\alpha+\beta)+2r=0$

$\alpha+\beta=-\frac{q}{2}$ ***(A1)***

use of $\alpha+\beta=-\frac{q}{2}$ and $\alpha^2+\beta^2=\frac{37}{4}$ to form equation in q ***(M1)***

$2\times\frac{37}{4}-\frac{1}{2}q^2+20+2q=0\Rightarrow-\frac{1}{2}q^2+2q+\frac{77}{2}=0$ ***(A1)***

continued…

Question 6 continued

THEN

$q = -7, r = 3$ ***A1***

Note:

Award ***A0*** if extra values for q and r given without any indication that $q = -7, r = 3$ are chosen as their final answer.

Award a maximum of ***(M1)A1(A1)(M1)(A1)(M1)A0*** if final answer given as $q = 11, r = 21$

[7 marks]

APOLLO

Geometry and trigonometry

97 questions

5. (a)

Note: Do not award the final ***A1*** for proofs which work from both sides to find a common expression other than $2\sin x\cos x - 2\sin^2 x$.

METHOD 1 (LHS to RHS)

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

LHS= $2\sin x\cos x + \cos 2x - 1$ OR

$\sin 2x + 1 - 2\sin^2 x - 1$ OR

$2\sin x\cos x + 1 - 2\sin^2 x - 1$

$= 2\sin x\cos x - 2\sin^2 x$ ***A1***

$\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$ =RHS ***AG***

METHOD 2 (RHS to LHS)

RHS $= 2\sin x\cos x - 2\sin^2 x$

attempt to use double angle formula for $\sin 2x$ or $\cos 2x$ ***M1***

$= \sin 2x + 1 - 2\sin^2 x - 1$ ***A1***

$= \sin 2x + \cos 2x - 1 =$ LHS ***AG***

[2 marks]

(b) attempt to factorise ***M1***

$(\cos x - \sin x)\left(2\sin x + 1\right) = 0$ ***A1***

recognition of $\cos x = \sin x \Rightarrow \dfrac{\sin x}{\cos x} = \tan x = 1$ OR $\sin x = -\dfrac{1}{2}$ ***(M1)***

one correct reference angle seen anywhere, accept degrees ***(A1)***

$\dfrac{\pi}{4}$ OR $\dfrac{\pi}{6}$ (accept $-\dfrac{\pi}{6}, \dfrac{7\pi}{6}$)

Note*:* This ***(M1)(A1)*** is independent of the previous ***M1A1.***

$x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ ***A2***

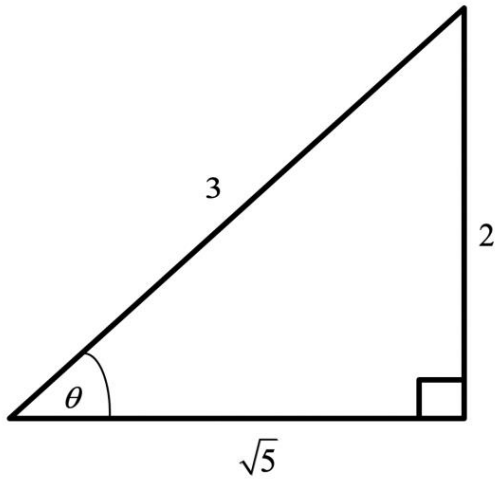
Note: Award ***A1*** for any two correct (radian) answers.
Award ***A1A0*** if additional values given with the four correct (radian) answers. Award ***A1A0*** for four correct answers given in degrees.

[6 marks]

Total [8 marks]

6. METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\text{cosec}\,\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

METHOD 2

Attempt to use $1+\cot^2\theta = \text{cosec}^2\theta$ ***M1***

$1+\cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\text{cosec}\,\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ *(*since $\text{cosec}\,\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

11. (a) (i) $\frac{-1+1}{2} = 0 = 3-3$ ***A1***

the point $(-1, 0, 3)$ lies on L_1. ***AG***

(ii) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$\frac{x+1}{2} = y = 3-z = \lambda \Rightarrow x = 2\lambda - 1,\ y = \lambda,\ z = 3-\lambda$ OR $\frac{x+1}{2} = \frac{y-0}{1} = \frac{z-3}{-1}$

correct direction vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ or equivalent seen in vector form ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $\boldsymbol{r} =$ is omitted.

[4 marks]

(b) attempt to use the scalar product formula ***(M1)***

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} = (\pm)\sqrt{6}\sqrt{a^2+2}\cos 45^\circ$ ***(A1)(A1)***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$2a + 2 = \frac{(\pm)\sqrt{6}\sqrt{a^2+2}\sqrt{2}}{2} \left(\Rightarrow 2a + 2 = (\pm)\sqrt{3}\sqrt{a^2+2}\right)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$4a^2 + 8a + 4 = 3a^2 + 6$

$a^2 + 8a - 2 = 0$ ***A1***

attempt to solve their quadratic ***M1***

$a = \frac{-8 \pm \sqrt{64+8}}{2} = \frac{-8 \pm \sqrt{72}}{2} \left(= -4 \pm 3\sqrt{2}\right)$ ***A1***

[8 marks]

continued…

Question 11 continued

(c) **METHOD 1**

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

Solutions exist unless $a - 2 = 0$

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

$t = \frac{1}{a-2}$ or $\lambda = \frac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\frac{a}{a-2}, 1 + \frac{1}{a-2}, 2 - \frac{1}{a-2}\right)\left(= \left(\frac{a}{a-2}, \frac{a-1}{a-2}, \frac{2a-5}{a-2}\right)\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

METHOD 2

no unique point of intersection implies direction vectors of L_1 and L_2 parallel

$k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks.

attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$$\begin{cases} 2\lambda - 1 = ta \\ \lambda = 1 + t \\ 3 - \lambda = 2 - t \end{cases}$$ ***A1***

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = t(a-2)$ or $2\lambda - 1 = (\lambda - 1)a \Rightarrow a - 1 = \lambda(a-2)$

$t = \frac{1}{a-2}$ or $\lambda = \frac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\frac{a}{a-2}, 1 + \frac{1}{a-2}, 2 - \frac{1}{a-2}\right)\left(= \left(\frac{a}{a-2}, \frac{a-1}{a-2}, \frac{2a-5}{a-2}\right)\right)$ ***A2***

continued…

Question 11 continued

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

[7 marks]
Total [19 marks]

2. (a) attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{4} \left(= \frac{5 \pm 3}{4} \right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1A1***

Total [7 marks]

5. METHOD 1

use of $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ on the LHS ***(M1)***

$|\boldsymbol{a}\times\boldsymbol{b}|^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***A1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\left(1-\cos^2\theta\right)$ ***M1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ OR $=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-\left(|\boldsymbol{a}||\boldsymbol{b}|\cos\theta\right)^2$ ***A1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-\left(\boldsymbol{a}\cdot\boldsymbol{b}\right)^2$ ***AG***

METHOD 2

use of $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ on the RHS ***(M1)***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***A1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\left(1-\cos^2\theta\right)$ ***M1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ OR $=\left(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta\right)^2$ ***A1***

$=|\boldsymbol{a}\times\boldsymbol{b}|^2$ ***AG***

Note: If candidates attempt this question using cartesian vectors, e.g

$\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ **and** $\boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ **,**

award full marks if fully developed solutions are seen.
Otherwise award no marks.

[4 marks]

6. METHOD 1

attempt to use the cosine rule to find the value of x ***(M1)***

$100 = x^2 + 4x^2 - 2(x)(2x)\left(\frac{3}{4}\right)$ ***A1***

$2x^2 = 100$

$x^2 = 50$ OR $x = \sqrt{50}$ $\left(= 5\sqrt{2}\right)$ ***A1***

attempt to find $\sin\hat{C}$ (seen anywhere) ***(M1)***

$\sin^2\hat{C} + \left(\frac{3}{4}\right)^2 = 1$ or $x^2 + 3^2 = 4^2$ OR right triangle with side 3 and hypotenuse 4

$\sin\hat{C} = \frac{\sqrt{7}}{4}$ ***(A1)***

Note: The marks for finding $\sin\hat{C}$ may be awarded independently of the first three marks for finding x.

correct substitution into the area formula using their value of x (or x^2) and their value of $\sin\hat{C}$ ***(M1)***

$A = \frac{1}{2} \times 5\sqrt{2} \times 10\sqrt{2} \times \frac{\sqrt{7}}{4}$ or $A = \frac{1}{2} \times \sqrt{50} \times 2\sqrt{50} \times \frac{\sqrt{7}}{4}$

$A = \frac{25\sqrt{7}}{2}$ ***A1***

continued…

METHOD 2

attempt to find the height, h, of the triangle in terms of x ***(M1)***

$h^2+\left(\frac{3}{4}x\right)^2=x^2$ OR $h^2+\left(\frac{5}{4}x\right)^2=10^2$ OR $h=\frac{\sqrt{7}}{4}x$ ***A1***

equating their expressions for either h^2 or h ***(M1)***

$x^2-\left(\frac{3}{4}x\right)^2=10^2-\left(\frac{5}{4}x\right)^2$ OR $\sqrt{100-\frac{25}{16}x^2}=\frac{\sqrt{7}}{4}x$ (or equivalent) ***A1***

$x^2=50$ OR $x=\sqrt{50}\ \left(=5\sqrt{2}\right)$ ***A1***

correct substitution into the area formula using their value of x (or x^2) ***(M1)***

$A=\frac{1}{2}\times 2\sqrt{50}\times\frac{\sqrt{7}}{4}\sqrt{50}$ OR $A=\frac{1}{2}\left(2\times 5\sqrt{2}\right)\left(\frac{\sqrt{7}}{4}5\sqrt{2}\right)$

$A=\frac{25\sqrt{7}}{2}$ ***A1***

Total [7 marks]

8. (a) **METHOD 1**

setting at least two components of l_1 and l_2 equal ***M1***

$$3+2\lambda = 2+\mu \quad (1)$$
$$2-2\lambda = -\mu \quad (2)$$
$$-1+2\lambda = 4+\mu \quad (3)$$

attempt to solve two of the equations eg. adding (1) and (2) ***M1***

gives a contradiction (no solution), eg $5 = 2$ ***R1***

so l_1 and l_2 do not intersect ***AG***

Note: For an error within the equations award ***M0M1R0.***
Note: The contradiction must be correct to award the ***R1.***

[3 marks]

METHOD 2

l_1 and l_2 are parallel, so l_1 and l_2 are either identical or distinct. ***R1***

Attempt to subtract two position vectors from each line,

e.g. $\begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\0\\4\end{pmatrix} \left(= \begin{pmatrix}1\\2\\-5\end{pmatrix} \right)$ ***M1***

$\begin{pmatrix}3\\2\\-1\end{pmatrix} \neq k \begin{pmatrix}1\\-1\\1\end{pmatrix}$ ***A1***

[3 marks]

(b) **METHOD 1**

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be $(3,2,-1)$ on l_1 and let B be $(2,0,4)$ on l_2

Attempt to find vector $\overrightarrow{\mathrm{AB}} \left(= \begin{pmatrix}-1\\-2\\5\end{pmatrix} \right)$ ***(M1)***

Distance required is $\dfrac{\left|\boldsymbol{v} \times \overrightarrow{\mathrm{AB}}\right|}{|\boldsymbol{v}|}$ ***M1***

continued…

Question 8 continued

$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right|$ ***(A1)***

$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}3\\6\\3\end{pmatrix}\right|$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

METHOD 2

l_1 and l_2 are parallel (as $\begin{pmatrix}2\\-2\\2\end{pmatrix}$ is a multiple of $\begin{pmatrix}1\\-1\\1\end{pmatrix}$)

let A be a fixed point on l_1 eg $(3,2,-1)$ and let B be a general point on l_2 $(2+\mu,-\mu,4+\mu)$

attempt to find vector $\vec{\mathrm{AB}}$ ***(M1)***

$\vec{\mathrm{AB}} = \begin{pmatrix}-1\\-2\\5\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\1\end{pmatrix}\ (\mu\in\mathbb{R})$ ***A1***

$\left|\vec{\mathrm{AB}}\right| = \sqrt{(-1+\mu)^2+(-2-\mu)^2+(5+\mu)^2}\ \left(=\sqrt{3\mu^2+12\mu+30}\right)$ ***M1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\left|\vec{\mathrm{AB}}\right|^2\right)=0\Rightarrow 6\mu+12=0\Rightarrow\mu=-2$ ***A1***

OR

$\left|\vec{\mathrm{AB}}\right| = \sqrt{3(\mu+2)^2+18}$ to obtain $\mu=-2$ ***A1***

THEN

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]

continued…

Question 8 continued

METHOD 3

let A be $(3,2,-1)$ on l_1 and let B be $(2+\mu,-\mu,4+\mu)$ on l_2 ***(M1)***

*(*or let A be $(2,0,4)$ on l_2 and let B be $(3+2\lambda,2-2\lambda,-1+2\lambda)$ on l_1)

$\vec{AB}=\begin{pmatrix}-1\\-2\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}\ (\mu\in\mathbb{R})$ (or $\vec{AB}=\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}$) ***A1***

$\begin{pmatrix}\mu-1\\-\mu-2\\\mu+5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ (or $\begin{pmatrix}2\lambda+1\\-2\lambda+2\\2\lambda-5\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$) ***M1***

$\mu=-2$ or $\lambda=1$ ***A1***

minimum distance is $\sqrt{18}\left(=3\sqrt{2}\right)$ ***A1***

[5 marks]
Total [8 marks]

10. (a) $6+6\cos x=0$ (or setting their $f'(x)=0$) ***(M1)***

$\cos x=-1$ (or $\sin x=0$)

$x=\pi,\ x=3\pi$ ***A1A1***

[3 marks]

(b) attempt to integrate $\int_{\pi}^{3\pi}(6+6\cos x)\mathrm{d}x$ ***(M1)***

$=\left[6x+6\sin x\right]_{\pi}^{3\pi}$ ***A1A1***

substitute their limits into their integrated expression and subtract ***(M1)***

$=(18\pi+6\sin 3\pi)-(6\pi+6\sin\pi)$

$=(6(3\pi)+0)-(6\pi+0)\ \ (=18\pi-6\pi)$ ***A1***

area $=12\pi$ ***AG***

[5 marks]

(c) attempt to substitute into the formula for surface area (including base) ***(M1)***

$\pi(2^2)+\pi(2)(l)=12\pi$ ***(A1)***

$4\pi+2\pi l=12\pi$

$2\pi l=8\pi$

$l=4$ ***A1***

[3 marks]

continued…

Question 10 continued

(d) valid attempt to find the height of the cone ***(M1)***

e.g. $2^2 + h^2 = (\text{their } l)^2$

$h = \sqrt{12}\ \left(= 2\sqrt{3}\right)$ ***(A1)***

attempt to use $V = \frac{1}{3}\pi r^2 h$ with their values substituted ***M1***

$\left(\frac{1}{3}\pi\left(2^2\right)\left(\sqrt{12}\right)\right)$

$\text{volume} = \frac{4\pi\sqrt{12}}{3}\left(= \frac{8\pi\sqrt{3}}{3} = \frac{8\pi}{\sqrt{3}}\right)$ ***A1***

[4 marks]

Total [15 marks]

12. (a) **EITHER**

horizontal stretch/scaling with scale factor $\frac{1}{2}$

Note: Do not allow 'shrink' or 'compression'

followed by a horizontal translation/shift $\frac{1}{2}$ units to the left ***A2***

Note: Do not allow 'move'

OR

horizontal translation/shift 1 unit to the left

followed by horizontal stretch/scaling with scale factor $\frac{1}{2}$ ***A2***

THEN

vertical translation/shift up by $\frac{\pi}{4}$ (or translation through $\begin{pmatrix} 0 \\ \frac{\pi}{4} \end{pmatrix}$) ***A1***

(may be seen anywhere)

[3 marks]

(b) let $\alpha = \arctan p$ and $\beta = \arctan q$ ***M1***

$p = \tan\alpha$ and $q = \tan\beta$ ***(A1)***

$\tan(\alpha+\beta) = \frac{p+q}{1-pq}$ ***A1***

$\alpha+\beta = \arctan\left(\frac{p+q}{1-pq}\right)$ ***A1***

so $\arctan p + \arctan q \equiv \arctan\left(\frac{p+q}{1-pq}\right)$ where $p, q > 0$ and $pq < 1$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) **METHOD 1**

$\frac{\pi}{4} = \arctan 1$ (or equivalent) ***A1***

$$\arctan\left(\frac{x}{x+1}\right) + \arctan 1 = \arctan\left(\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}(1)}\right)$$ ***A1***

$$= \arctan\left(\frac{\frac{x+x+1}{x+1}}{\frac{x+1-x}{x+1}}\right)$$ ***A1***

$$= \arctan\left(2x+1\right)$$ ***AG***

[3 marks]

METHOD 2

$\tan\frac{\pi}{4} = 1$ (or equivalent) ***A1***

Consider $\arctan\left(2x+1\right) - \arctan\left(\frac{x}{x+1}\right) = \frac{\pi}{4}$

$$\tan\left(\arctan\left(2x+1\right) - \arctan\left(\frac{x}{x+1}\right)\right)$$

$$= \arctan\left(\frac{2x+1-\frac{x}{x+1}}{1+\frac{x(2x+1)}{x+1}}\right)$$ ***A1***

$$= \arctan\left(\frac{(2x+1)(x+1)-x}{x+1+x(2x+1)}\right)$$ ***A1***

$$= \arctan 1$$ ***AG***

[3 marks]

continued…

Question 12 continued

METHOD 3

$\tan\left(\arctan\left(2x+1\right)\right) = \tan\left(\arctan\left(\frac{x}{x+1}\right)+\frac{\pi}{4}\right)$

$\tan\frac{\pi}{4}=1$ (or equivalent) ***A1***

LHS = $2x+1$ ***A1***

RHS = $\frac{\frac{x}{x+1}+1}{1-\frac{x}{x+1}}\left(=2x+1\right)$ ***A1***

[3 marks]

Question 12 continued

(d) let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{n}{n+1}\right)$ for $n\in\mathbb{Z}^+$

consider $\mathrm{P}(1)$:

when $n=1$, $\sum_{r=1}^{1}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{1}{2}\right)=\text{RHS}$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true, ie. $\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)=\arctan\left(\frac{k}{k+1}\right)\ \left(k\in\mathbb{Z}^+\right)$ ***M1***

Note: Award ***M0*** for statements such as "let $n=k$".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\arctan\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{k}\arctan\left(\frac{1}{2r^2}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ (***M1)***

$$=\arctan\left(\frac{k}{k+1}\right)+\arctan\left(\frac{1}{2(k+1)^2}\right)$$ ***A1***

$$=\arctan\left(\frac{\frac{k}{k+1}+\frac{1}{2(k+1)^2}}{1-\left(\frac{k}{k+1}\right)\left(\frac{1}{2(k+1)^2}\right)}\right)$$ ***M1***

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2(k+1)^3-k}\right)$$ ***A1***

Note: Award ***A1*** for correct numerator, with $(k + 1)$ factored. Denominator does not need to be simplified

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{2k^3+6k^2+5k+2}\right)$$ ***A1***

Note: Award ***A1*** for denominator correctly expanded. Numerator does not need to be simplified. These two ***A*** marks may be awarded in any order

$$=\arctan\left(\frac{(k+1)(2k^2+2k+1)}{(k+2)(2k^2+2k+1)}\right)=\arctan\left(\frac{k+1}{k+2}\right)$$ ***A1***

Note: The word 'arctan' must be present to be able to award the last three A marks

continued…

Question 12 continued

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so

$\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks have been awarded.

Note: To award the final ***R1***, the truth of $\mathrm{P}(k)$ must be mentioned. '$\mathrm{P}(k)$ implies $\mathrm{P}(k+1)$' is insufficient to award the mark.

[9 marks]
Total [19 marks]

3. Amplitude is $\frac{110}{2} = 55$ ***(A1)***

$a = -55$ ***A1***

$c = 65$ ***A1***

$\frac{2\pi}{b} = 20$ OR $-55\cos(20b) + 65 = 10$ ***(M1)***

$b = \frac{\pi}{10}\left(= 0.314\right)$ ***A1***

Total [5 marks]

6 (a) attempt to find a vector perpendicular to Π_1 and Π_2 using a cross product ***(M1)***

$$\begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}=\left(2-(-2)\right)\boldsymbol{i}+(1-3)\,\boldsymbol{j}+(-6-2)\,\boldsymbol{k}$$

$$=\begin{pmatrix}4\\-2\\-8\end{pmatrix}\left(=2\begin{pmatrix}2\\-1\\-4\end{pmatrix}\right) \quad \textit{(A1)}$$

equation is $4x-2y-8z=0\left(\Rightarrow 2x-y-4z=0\right)$ ***A1***

[3 marks]

(b) attempt to solve 3 simultaneous equations in 3 variables ***(M1)***

$$\left(\frac{41}{21},-\frac{10}{21},\frac{23}{21}\right)\left(=(1.95,-0.476,1.10)\right) \quad \textit{A1}$$

[2 marks]
Total [5 marks]

2. (a) $A = s^2$ and $P = 4s$ ***(A1)***

$A = P \Rightarrow s^2 = 4s$ ***(M1)***

$s(s-4) = 0$

$\Rightarrow s = 4 (s > 0)$ ***A1***

Note: Award ***A1M1A0*** if both $s = 4$ and $s = 0$ are stated as final answers.

[3 marks]

(b) $A_T = \frac{1}{2}x^2 \sin\frac{2\pi}{n}$ ***A1***

Note: Award ***A1*** for a correct alternative form expressed in terms of x and n only.

For example, using Pythagoras' theorem, $A_T = x\sin\frac{\pi}{n}\sqrt{x^2 - x^2\sin^2\frac{\pi}{n}}$ or $A_T = 2\left(\frac{1}{2}\left(x\sin\frac{\pi}{n}\right)\left(x\cos\frac{\pi}{n}\right)\right)$ or $A_T = x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$.

[1 mark]

continued…

Question 2 continued

(c) **METHOD 1**

uses $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ ***(M1)***

$\frac{\frac{y}{2}}{x} = \sin\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

METHOD 2

uses Pythagoras' theorem $\left(\frac{y}{2}\right)^2 + h^2 = x^2$ and $h = x\cos\frac{\pi}{n}$ ***(M1)***

$\left(\frac{y}{2}\right)^2 + \left(x\cos\frac{\pi}{n}\right)^2 = x^2 \quad \left(y^2 = 4x^2\left(1 - \cos^2\frac{\pi}{n}\right)\right)$

$= 4x^2\sin^2\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

continued…

Question 2 continued

METHOD 3

uses the cosine rule ***(M1)***

$y^2 = 2x^2 - 2x^2\cos\frac{2\pi}{n}\ \left(= 2x^2\left(1-\cos\frac{2\pi}{n}\right)\right)$

$= 4x^2\sin^2\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

METHOD 4

uses the sine rule ***(M1)***

$\frac{y}{\sin\frac{2\pi}{n}} = \frac{x}{\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)}$

$y\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

continued…

Question 2 continued

(d) $A = P \Rightarrow nA_T = ny$ ***(M1)***

Note: Award ***M1*** for equating correct expressions for A and P.

$\frac{1}{2}nx^2 \sin\frac{2\pi}{n} = 2nx\sin\frac{\pi}{n} \left(nx^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2nx\sin\frac{\pi}{n}\right)$

$\frac{1}{2}x^2 \sin\frac{2\pi}{n} = 2x\sin\frac{\pi}{n} \left(x^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\right)$ ***A1***

uses $\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ (seen anywhere in part (d) or in part (b)) ***(M1)***

$x^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}$

attempts to either factorise or divide their expression ***(M1)***

$x\sin\frac{\pi}{n}\left(x\cos\frac{\pi}{n} - 2\right) = 0$

$x = \frac{2}{\cos\frac{\pi}{n}}, \left(x\sin\frac{\pi}{n} \neq 0\right)$ (or equivalent) ***A1***

continued…

Question 2 continued

EITHER

substitutes $x = \dfrac{2}{\cos\dfrac{\pi}{n}}$ (or equivalent) into $P = ny$ ***(M1)***

$P = 2n\left(\dfrac{2}{\cos\dfrac{\pi}{n}}\right)\left(\sin\dfrac{\pi}{n}\right)$ ***A1***

Note: Other approaches are possible. For example, award ***A1*** for $P = 2nx\cos\dfrac{\pi}{n}\tan\dfrac{\pi}{n}$ and ***M1*** for substituting $x = \dfrac{2}{\cos\dfrac{\pi}{n}}$ into P.

OR

substitutes $x = \dfrac{2}{\cos\dfrac{\pi}{n}}$ (or equivalent) into $A = nA_T$ ***(M1)***

$A = \dfrac{1}{2}n\left(\dfrac{2}{\cos\dfrac{\pi}{n}}\right)^2\left(\sin\dfrac{2\pi}{n}\right)$

$A = \dfrac{1}{2}n\left(\dfrac{2}{\cos\dfrac{\pi}{n}}\right)^2\left(2\sin\dfrac{\pi}{n}\cos\dfrac{\pi}{n}\right)$ ***A1***

THEN

$A = P = 4n\tan\dfrac{\pi}{n}$ ***AG***

[7 marks]

continued…

Question 2 continued

(e) (i) attempts to use the Maclaurin series for $\tan x$ with $x = \frac{\pi}{n}$ ***(M1)***

$$\tan\frac{\pi}{n} = \frac{\pi}{n} + \frac{\left(\frac{\pi}{n}\right)^3}{3} + \frac{2\left(\frac{\pi}{n}\right)^5}{15}(+...)$$

$$4n\tan\frac{\pi}{n} = 4n\left(\frac{\pi}{n} + \frac{\pi^3}{3n^3} + \frac{2\pi^5}{15n^5}(+...)\right) \text{ (or equivalent)} \quad \textbf{A1}$$

$$= 4\left(\pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + ...\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right) = 4\pi \quad \textbf{A1}$$

Note: Award a maximum of ***M1A1A0*** if $\lim_{n\to\infty}$ is not stated anywhere.

[3 marks]

(ii) (as $n \to \infty$, $P \to 4\pi$ and $A \to 4\pi$)

the polygon becomes a circle of radius 2 ***R1***

Note: Award ***R1*** for alternative responses such as:
the polygon becomes a circle of area 4π OR
the polygon becomes a circle of perimeter 4π OR
the polygon becomes a circle with $A = P = 4\pi$.
Award ***R0*** for polygon becomes a circle.

[1 mark]

(f) $A = \frac{1}{2}ab$ and $P = a + b + \sqrt{a^2 + b^2}$ ***(A1)(A1)***

equates their expressions for A and P ***M1***

$A = P \Rightarrow a + b + \sqrt{a^2 + b^2} = \frac{1}{2}ab$

$\sqrt{a^2 + b^2} = \frac{1}{2}ab - (a + b)$ ***M1***

Note: Award ***M1*** for isolating $\sqrt{a^2 + b^2}$ or $\pm 2\sqrt{a^2 + b^2}$. This step may be seen later.

$a^2 + b^2 = \left(\frac{1}{2}ab - (a + b)\right)^2$

$a^2 + b^2 = \frac{1}{4}a^2b^2 - 2\left(\frac{1}{2}ab\right)(a + b) + (a + b)^2$ ***M1***

$\left(= \frac{1}{4}a^2b^2 - a^2b - ab^2 + a^2 + 2ab + b^2\right)$

Note: Award ***M1*** for attempting to expand their RHS of either $a^2 + b^2 = \ldots$ or $4(a^2 + b^2) = \ldots$.

EITHER

$ab\left(\frac{1}{4}ab - a - b + 2\right) = 0 \ (ab \neq 0)$ ***A1***

$\frac{1}{4}ab - a - b + 2 = 0$

$ab - 4a = 4b - 8$

OR

$\frac{1}{4}a^2b^2 - a^2b - ab^2 + 2ab = 0$

$a\left(\frac{1}{4}b^2 - b\right) + (2b - b^2) = 0 \ \left(a(b^2 - 4b) + (8b - 4b^2) = 0\right)$ ***A1***

$a = \frac{4b^2 - 8b}{b^2 - 4b}$

continued…

Question 2 continued

THEN

$\Rightarrow a = \frac{4b-8}{b-4}$ ***A1***

$a = \frac{4b-16+8}{b-4}$

$a = \frac{8}{b-4} + 4$ ***AG***

Note: Award a maximum of ***A1A1M1M1M0A0A0*** for attempting to verify. For example, verifying that $A = P = \frac{16}{b-4} + 2b + 4$ gains 4 of the 7 marks.

[7 marks]

continued…

Question 2 continued

(g) (i) using an appropriate method ***(M1)***

eg substituting values for b or using divisibility properties

$(5,12,13)$ and $(6,8,10)$ ***A1A1***

Note: Award ***A1A0*** for either one set of three correct side lengths or two sets of two correct side lengths.

[3 marks]

(ii) $A = P = 30$ and $A = P = 24$ ***A1***

Note: Do not award ***A1FT***.

[1 mark]

Total [28 marks]

6. (a) **METHOD 1**

attempt to write all LHS terms over a common denominator of $x-1$ ***(M1)***

$2x-3-\frac{6}{x-1}=\frac{2x(x-1)-3(x-1)-6}{x-1}$ OR $\frac{(2x-3)(x-1)}{x-1}-\frac{6}{x-1}$

$=\frac{2x^2-2x-3x+3-6}{x-1}$ OR $\frac{2x^2-5x+3}{x-1}-\frac{6}{x-1}$ ***A1***

$=\frac{2x^2-5x-3}{x-1}$ ***AG***

[2 marks]

METHOD 2

attempt to use algebraic division on RHS ***(M1)***

correctly obtains quotient of $2x-3$ and remainder -6 ***A1***

$=2x-3-\frac{6}{x-1}$ as required. ***AG***

[2 marks]

continue…

Question 6 continued.

(b) consider the equation $\frac{2\sin^2 2\theta - 5\sin 2\theta - 3}{\sin 2\theta - 1} = 0$ ***(M1)***

$\Rightarrow 2\sin^2 2\theta - 5\sin 2\theta - 3 = 0$

EITHER

attempt to factorise in the form $(2\sin 2\theta + a)(\sin 2\theta + b)$ ***(M1)***

Note: Accept any variable in place of $\sin 2\theta$.

$(2\sin 2\theta + 1)(\sin 2\theta - 3) = 0$

OR

attempt to substitute into quadratic formula ***(M1)***

$\sin 2\theta = \frac{5 \pm \sqrt{49}}{4}$

THEN

$\sin 2\theta = -\frac{1}{2}$ or $\sin 2\theta = 3$ ***(A1)***

Note: Award ***A1*** for $\sin 2\theta = -\frac{1}{2}$ only.

one of $\frac{7\pi}{6}$ OR $\frac{11\pi}{6}$ (accept 210 or 330) ***(A1)***

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$ (must be in radians) ***A1***

Note: Award ***A0*** if additional answers given.

[5 marks]

Total [7 marks]

2. EITHER

attempt to use cosine rule ***(M1)***

$12^2 + \text{AB}^2 - 2\times 12\times\cos 25^\text{o}\times\text{AB} = 7^2$ OR $\text{AB}^2 - 21.7513...\text{AB} + 95 = 0$ ***(A1)***

at least one correct value for AB ***(A1)***

$\text{AB} = 6.05068...$ OR $\text{AB} = 15.7007...$

using their smaller value for AB to find minimum perimeter ***(M1)***

$12 + 7 + 6.05068...$

OR

attempt to use sine rule ***(M1)***

$\frac{\sin \text{B}}{12} = \frac{\sin 25^\text{o}}{7}$ OR $\sin \text{B} = 0.724488...$ OR $\hat{\text{B}} = 133.573...^\text{o}$ OR $\hat{\text{B}} = 46.4263...^\text{o}$ ***(A1)***

at least one correct value for $\hat{C}$

$\hat{C} = 21.4263...^\text{o}$ OR $\hat{C} = 108.573...^\text{o}$ ***(A1)***

using their acute value for $\hat{C}$ to find minimum perimeter ***(M1)***

$12 + 7 + \sqrt{12^2 + 7^2 - 2\times 12\times 7\cos 21.4263...^\circ}$ OR $12 + 7 + \frac{7\sin 21.4263...^\text{o}}{\sin 25^\text{o}}$

THEN

$25.0506\ldots$

minimum perimeter = 25.1. ***A1***

Total [5 marks]

4. (a) attempt to find the area of either shaded region in terms of r and θ ***(M1)***

> **Note*:*** Do not award ***M1*** if they have only copied from the booklet and not applied to the shaded area.

Area of segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$ ***A1***

Area of triangle $= \frac{1}{2}r^2\sin(\pi - \theta)$ ***A1***

correct equation in terms of θ only ***(A1)***

$\theta - \sin\theta = \sin(\pi - \theta)$

$\theta - \sin\theta = \sin\theta$ ***A1***

$\theta = 2\sin\theta$ ***AG***

> **Note*:*** Award a maximum of ***M1A1A0A0A0*** if a candidate uses degrees (i.e., $\frac{1}{2}r^2\sin(180° - \theta)$), even if later work is correct.

> **Note*:*** If a candidate directly states that the area of the triangle is $\frac{1}{2}r^2\sin\theta$, award a maximum of ***M1A1A0A1A1***.

[5 marks]

(b) $\theta = 1.89549...$

$\theta = 1.90$ ***A1***

> **Note:** Award ***A0*** if there is more than one solution. Award ***A0*** for an answer in degrees.

[1 mark]

Total [6 marks]

9. (a) $12 = \frac{2\pi}{b}$ OR $b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG***

[1 mark]

(b) $a = \frac{6.8 - 2.2}{2}$ OR $a = \frac{\max - \min}{2}$ ***(M1)***

$= 2.3$ (m) ***A1***

[2 marks]

(c) $d = \frac{6.8 + 2.2}{2}$ OR $d = \frac{\max + \min}{2}$ ***(M1)***

$= 4.5$ (m) ***A1***

[2 marks]

continued…

Question 9 continued.

(d) **METHOD 1**

substituting $t = 4.5$ and $H = 6.8$ for example into their equation for H ***(A1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}(4.5 - c)\right) + 4.5$$

attempt to solve their equation ***(M1)***

$c = 1.5$ ***A1***

METHOD 2

using horizontal translation of $\frac{12}{4}$ ***(M1)***

$4.5 - c = 3$ ***(A1)***

$c = 1.5$ ***A1***

METHOD 3

$$H'(t) = (2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(t - c)\right)$$ ***(A1)***

attempts to solve their $H'(4.5) = 0$ for c ***(M1)***

$$(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(4.5 - c)\right) = 0$$

$c = 1.5$ ***A1***

[3 marks]

(e) attempt to find H when $t = 12$ or $t = 0$, graphically or algebraically ***(M1)***

$H = 2.87365...$

$H = 2.87\,(\text{m})$ ***A1***

[2 marks]

continue…

Question 9 continued.

(f) attempt to solve $5 = 2.3\sin\left(\frac{\pi}{6}(t-1.5)\right)+4.5$ ***(M1)***

times are $t = 1.91852...$ and $t = 7.08147...$, $(t = 13.9185..., t = 19.0814...)$ ***(A1)***

total time is $2\times(7.081...-1.919...)$

$10.3258...$

$= 10.3$ (hours) ***A1***

Note: Accept 10.

[3 marks]

(g) **METHOD 1**

substitutes $t = \frac{11}{3}$ and $H = 6.8$ into their equation for H and attempts to solve for c ***(M1)***

$6.8 = 2.3\sin\left(\frac{\pi}{6}\left(\frac{11}{3}-c\right)\right)+4.5 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

METHOD 2

uses their horizontal translation $\left(\frac{12}{4} = 3\right)$ ***(M1)***

$\frac{11}{3}-c = 3 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

[2 marks]

Total [15 marks]

11. (a) (i) attempts to find either $\vec{AB}$ or $\vec{AC}$ ***(M1)***

$\vec{AB} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

(ii) **METHOD 1**

attempts to find $\vec{AB} \times \vec{AC}$ ***(M1)***

$\vec{AB} \times \vec{AC} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ ***A1***

EITHER

equation of plane is of the form $14x - 21y - 7z = d$ $\left(2x - 3y - z = d\right)$ (***A1)***

substitutes a valid point e.g $(3, 0, 0)$ to obtain a value of d ***M1***

$d = 42$ $\left(d = 6\right)$

OR

attempts to use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ ***(M1)***

$\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} \left(\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = 42 \right)$ ***A1***

$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \left(\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6 \right)$

THEN

$14x - 21y - 7z = 42$ $\left(2x - 3y - z = 6\right)$ ***A1***

METHOD 2

equation of plane is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

attempts to form equations for x, y, z in terms of their parameters ***(M1)***

$x = 3 - 3s - 2t$, $y = -2s + t$, $z = -7t$

A1

eliminates at least one of their parameters ***(M1)***

for example, $2x - 3y = 6 - 7t \left(\Rightarrow 2x - 3y = 6 + z\right)$

$2x - 3y - z = 6$ ***A1***

[7 marks]

(b) **METHOD 1**

substitutes $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ into their Π_1 and Π_2 (given) ***(M1)***

$\Pi_1 \colon 2\lambda - 3(-2+\lambda) - (-\lambda) = 6$ and $\Pi_2 \colon 3\lambda - (-2+\lambda) + 2(-\lambda) = 2$ ***A1***

Note: Award ***(M1)A0*** for correct verification using a specific value of λ.

so the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

METHOD 2

EITHER

attempts to find $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} -7 \\ -7 \\ 7 \end{pmatrix}$

OR

$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (2-3+1) = 0$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (3-1-2) = 0$ ***M1***

THEN

substitutes $(0, -2, 0)$ into Π_1 and Π_2

$\Pi_1 \colon 2(0) - 3(-2) - (0) = 6$ and $\Pi_2 \colon 3(0) - (-2) + 2(0) = 2$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

METHOD 3

attempts to solve $2x - 3y - z = 6$ and $3x - y + 2z = 2$ ***(M1)***

for example, $x = -\lambda, y = -2 - \lambda, z = \lambda$ ***A1***

Note: Award ***A1*** for substituting $x=0$ (or $y=-2$ or $z=0$) into Π_1 and Π_2 and solving simultaneously. For example, solving $-3y-z=6$ and $-y+2z=2$ to obtain $y=-2$ and $z=0$.

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

[2 marks]

(c) (i) substitutes the equation of L into the equation of Π_3 ***(M1)***

$2\lambda+2\lambda=3\Rightarrow 4\lambda=3$ ***A1***

$\lambda=\frac{3}{4}$ ***AG***

(ii) P has coordinates $\left(\frac{3}{4},-\frac{5}{4},-\frac{3}{4}\right)$ ***A1***

[3 marks]

(d) (i) normal to Π_3 is $\boldsymbol{n}=\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***(A1)***

Note: May be seen or used anywhere.

considers the line normal to Π_3 passing through $\mathrm{B}(0,-2,0)$ ***(M1)***

$\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***A1***

EITHER

finding the point on the normal line that intersects Π_3

attempts to solve simultaneously with plane $2x-2z=3$ ***(M1)***

$4\mu+4\mu=3$

$\mu=\frac{3}{8}$ ***A1***

point is $\left(\frac{3}{4},-2,-\frac{3}{4}\right)$

OR

$$\left(\begin{pmatrix}2\mu\\-2\\-2\mu\end{pmatrix}-\begin{pmatrix}\frac{3}{4}\\-\frac{5}{4}\\-\frac{3}{4}\end{pmatrix}\right)\cdot\begin{pmatrix}2\\0\\-2\end{pmatrix}=0 \quad \textbf{(M1)}$$

$$4\mu-\frac{3}{2}+4\mu-\frac{3}{2}=0$$

$$\mu=\frac{3}{8} \quad \textbf{A1}$$

OR

attempts to find the equation of the plane parallel to Π_3 containing B' $(x-z=3)$ and solve simultaneously with L **(M1)**

$$2\mu'+2\mu'=3$$

$$\mu'=\frac{3}{4} \quad \textbf{A1}$$

THEN

so, another point on the reflected line is given by

$$\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\frac{3}{4}\begin{pmatrix}2\\0\\-2\end{pmatrix} \quad \textbf{(A1)}$$

$$\Rightarrow \mathrm{B}'\left(\frac{3}{2},-2,-\frac{3}{2}\right) \quad \textbf{A1}$$

(ii) **EITHER**

attempts to find the direction vector of the reflected line using their P and B' **(M1)**

$$\overrightarrow{\mathrm{PB}'}=\begin{pmatrix}\frac{3}{4}\\-\frac{3}{4}\\-\frac{3}{4}\end{pmatrix}$$

OR

attempts to find their direction vector of the reflected line using a vector approach **(M1)**

$$\overrightarrow{\mathrm{PB}'}=\overrightarrow{\mathrm{PB}}+\overrightarrow{\mathrm{BB}'}=-\frac{3}{4}\begin{pmatrix}1\\1\\-1\end{pmatrix}+\frac{3}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

THEN

$$\boldsymbol{r}=\begin{pmatrix}\frac{3}{2}\\-2\\-\frac{3}{2}\end{pmatrix}+\lambda\begin{pmatrix}\frac{3}{4}\\-\frac{3}{4}\\-\frac{3}{4}\end{pmatrix} \text{ (or equivalent)} \quad \textbf{A1}$$

Note: Award ***A0*** for either '$\boldsymbol{r} =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'.

[9 marks]
Total [21 marks]

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6. (a) attempting to use the change of base rule ***M1***

$\log_9(\cos 2x+2)=\dfrac{\log_3(\cos 2x+2)}{\log_3 9}$ ***A1***

$=\dfrac{1}{2}\log_3(\cos 2x+2)$ ***A1***

$=\log_3\sqrt{\cos 2x+2}$ ***AG***

[3 marks]

(b) $\log_3(2\sin x)=\log_3\sqrt{\cos 2x+2}$

$2\sin x=\sqrt{\cos 2x+2}$ ***M1***

$4\sin^2 x=\cos 2x+2$ (or equivalent) ***A1***

use of $\cos 2x=1-2\sin^2 x$ ***(M1)***

$6\sin^2 x=3$

$\sin x=(\pm)\dfrac{1}{\sqrt{2}}$ ***A1***

$x=\dfrac{\pi}{4}$ ***A1***

Note: Award ***A0*** if solutions other than $x=\dfrac{\pi}{4}$ are included.

[5 marks]

Total [8 marks]

8. recognition that the angle between the normal and the line is $60°$ (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9}\times\sqrt{1+4+p^2}}$$ ***A1***

$$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$$ ***A1***

$$3\sqrt{5+p^2} = 4|p|$$

attempt to square both sides ***M1***

$$9\left(5+p^2\right) = 16p^2 \Rightarrow 7p^2 = 45$$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

Total [7 marks]

1. (a) ***METHOD 1***

attempt to use the cosine rule ***(M1)***

$\cos\theta = \frac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$ (or equivalent) ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

METHOD 2

attempt to split triangle AOB into two congruent right triangles ***(M1)***

$\sin\left(\frac{\theta}{2}\right) = \frac{2.5}{4}$ ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

(b) attempt to find the area of the shaded region ***(M1)***

$\frac{1}{2} \times 4 \times 4 \times (2\pi - 1.35...)$ ***A1***

$= 39.5\ (\text{cm}^2)$ ***A1***

[3 marks]

Total [6 marks]

7. attempting to find $\boldsymbol{r}_{\mathrm{B}}-\boldsymbol{r}_{\mathrm{A}}$ for example ***(M1)***

$\boldsymbol{r}_{\mathrm{B}}-\boldsymbol{r}_{\mathrm{A}}=\begin{pmatrix}3\\-6\end{pmatrix}+t\begin{pmatrix}-5\\4\end{pmatrix}$

attempting to find $\left|\boldsymbol{r}_{\mathrm{B}}-\boldsymbol{r}_{\mathrm{A}}\right|$ ***M1***

distance $d(t)=\sqrt{(3-5t)^2+(4t-6)^2}\left(=\sqrt{41t^2-78t+45}\right)$ ***A1***

using a graph to find the d – coordinate of the local minimum ***M1***

the minimum distance between the ships is $2.81\ (\mathrm{km})\left(=\frac{11\sqrt{41}}{41}\ (\mathrm{km})\right)$ ***A1***

Total [5 marks]

12. (a) stating the relationship between $\cot$ and $\tan$ and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \dfrac{1}{\tan 2\theta}$ and $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b) ***METHOD 1***

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \tan^2\theta + 2\tan\theta\left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0 (= \text{RHS})$ ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \cot^2\theta - 2\cot\theta\left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \dfrac{1}{\tan^2\theta} - \left(\dfrac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\dfrac{1}{\tan^2\theta} - \dfrac{1}{\tan^2\theta} + 1 - 1 = 0 (= \text{RHS})$ ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \dfrac{1}{\tan\theta}$

$= \dfrac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

$= -1$ ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and -1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) ***METHOD 1***

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}}$$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2 - \sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

Total [19 marks]

1. (a) ***METHOD 1***

consider right-angled triangle OCX where $\mathrm{CX} = \frac{x}{2}$

$\sin\frac{\pi}{3} = \frac{\frac{x}{2}}{1}$ ***M1A1***

$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2(1)(1)\cos\frac{2\pi}{3}$ ***M1A1***

$x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

Note: Accept use of sine rule.

[3 marks]

(b) $\sin\frac{\pi}{4} = \frac{1}{x}$ where $x =$ side of square ***M1***

$x = \sqrt{2}$ ***A1***

$P_i = 4\sqrt{2}$ ***A1***

[3 marks]

(c) 6 equilateral triangles $\Rightarrow x = 1$ ***A1***

$P_i = 6$ ***A1***

[2 marks]

(d) in right-angled triangle $\sin\left(\frac{\pi}{n}\right) = \frac{\frac{x}{2}}{1}$ ***M1***

$\Rightarrow x = 2\sin\left(\frac{\pi}{n}\right)$ ***A1***

$P_i = n \times x$

$P_i = n \times 2\sin\left(\frac{\pi}{n}\right)$ ***M1***

$P_i = 2n\sin\left(\frac{\pi}{n}\right)$ ***AG***

[3 marks]

continued…

Question 1 continued

(e) consider $\lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ ***M1***

$2n\sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \ldots\right)$ ***(A1)***

$= 2\left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \ldots\right)$ ***A1***

$\Rightarrow \lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\pi$ ***A1***

as $n \to \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$ ***R1***

[5 marks]

(f) consider an n-sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre ***M1A1***

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2\tan\left(\frac{\pi}{n}\right)$ ***M1A1***

Perimeter $P_c = 2n\tan\left(\frac{\pi}{n}\right)$ ***AG***

[4 marks]

(g) consider $\lim_{n\to\infty} 2n\tan\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right) = \frac{0}{0}$ ***R1***

attempt to use L'Hopital's rule ***M1***

$= \lim_{n\to\infty}\left(\frac{-\frac{2\pi}{n^2}\sec^2\left(\frac{\pi}{n}\right)}{-\frac{1}{n^2}}\right)$ ***A1A1***

$= 2\pi$ ***A1***

[5 marks]

continued…

Question 1 continued

(h) $P_i < 2\pi < P_c$

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$ ***M1***

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$ ***A1***

[2 marks]

(i) attempt to find the lower bound and upper bound approximations within 0.005 of π ***(M1)***

$n = 46$ ***A2***

[3 marks]

Total [30 marks]

3. (a) $(f \circ g)(x) = f(2x)$ ***(A1)***

$f(2x) = \sqrt{3}\sin 2x + \cos 2x$ ***A1***

[2 marks]

(b) $\sqrt{3}\sin 2x + \cos 2x = 2\cos 2x$

$\sqrt{3}\sin 2x = \cos 2x$

recognizing to use $\tan$ or $\cot$ ***M1***

$\tan 2x = \frac{1}{\sqrt{3}}$ OR $\cot 2x = \sqrt{3}$ (values may be seen in right triangle) ***(A1)***

$\left(\arctan\left(\frac{1}{\sqrt{3}}\right) = \right) \frac{\pi}{6}$ (seen anywhere) (accept degrees) ***(A1)***

$2x = \frac{\pi}{6}, \frac{7\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}$ ***A1A1***

Note:	Do not award the final ***A1*** if any additional solutions are seen. Award ***A1A0*** for correct answers in degrees. Award ***A0A0*** for correct answers in degrees with additional values.

[5 marks]

Total [7 marks]

11. (a) **METHOD 1**

attempt to eliminate a variable ***M1***

obtain a pair of equations in two variables

EITHER

$-3x+z=-3$ and ***A1***

$-3x+z=44$ ***A1***

OR

$-5x+y=-7$ and ***A1***

$-5x+y=40$ ***A1***

OR

$3x-z=3$ and ***A1***

$3x-z=-\frac{79}{5}$ ***A1***

THEN

the two lines are parallel ($-3\neq 44$ or $-7\neq 40$ or $3\neq -\frac{79}{5}$) ***R1***

Note: There are other possible pairs of equations in two variables. To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

METHOD 2

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$" is missing. Subsequent marks may still be awarded.

Attempt to substitute $(1+\lambda, -2+5\lambda, 3\lambda)$ in Π_3 ***M1***

$-9(1+\lambda)+3(-2+5\lambda)-2(3\lambda)=32$

$-15=32,$ a contradiction ***R1***

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

METHOD 3

attempt to eliminate a variable ***M1***

$-3y+5z=6$ ***A1***

$-3y+5z=100$ ***A1***

$0=94,$ a contradiction ***R1***

Note: Accept other equivalent alternatives. Accept other valid methods. To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

continued…

Question 11 continued

(b) (i) $\Pi_1 : 2+2+0=4$ and $\Pi_2 : 1+4+0=5$ ***A1***

(ii) **METHOD 1**

attempt to find the vector product of the two normals ***M1***

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$$ ***A1***

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r =$" is missing.
Accept any multiple of the direction vector.

Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "$r =$" only once.

continued…

Question 11 continued

METHOD 2

attempt to eliminate a variable from Π_1 and Π_2 ***M1***

$3x-z=3$ OR $3y-5z=-6$ OR $5x-y=7$

Let $x=t$

substituting $x=t$ in $3x-z=3$ to obtain

$z=-3+3t$ and $y=5t-7$ (for all three variables in parametric form) ***A1***

$$\boldsymbol{r}=\begin{pmatrix}0\\-7\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\5\\3\end{pmatrix}$$ ***A1A1***

Note: Award ***A1A0*** if "$r=$" is missing.
Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes Π_1 and Π_2.

[5 marks]

continued…

Question 11 continued

(c) **METHOD 1**

the line connecting L and Π_3 is given by L_1

attempt to substitute position and direction vector to form L_1 ***(M1)***

$\boldsymbol{s} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$ ***A1***

substitute $(1-9t, -2+3t, -2t)$ in Π_3 ***M1***

$-9(1-9t) + 3(-2+3t) - 2(-2t) = 32$

$94t = 47 \Rightarrow t = \frac{1}{2}$ ***A1***

attempt to find distance between $(1, -2, 0)$ and their point $\left(-\frac{7}{2}, -\frac{1}{2}, -1\right)$ ***(M1)***

$= \left| \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right| = \frac{1}{2}\sqrt{(-9)^2 + 3^2 + (-2)^2}$

$= \frac{\sqrt{94}}{2}$ ***A1***

[6 marks]

continued…

Question 11 continued

METHOD 2

unit normal vector equation of Π_3 is given by $\frac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}$ ***(M1)***

$=\frac{32}{\sqrt{94}}$ ***A1***

let Π_4 be the plane parallel to Π_3 and passing through P,

then the normal vector equation of Π_4 is given by

$\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=-15$ ***M1***

unit normal vector equation of Π_4 is given by

$\frac{\begin{pmatrix}-9\\3\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}}{\sqrt{81+9+4}}=\frac{-15}{\sqrt{94}}$ ***A1***

distance between the planes is $\frac{32}{\sqrt{94}}-\frac{-15}{\sqrt{94}}$ ***(M1)***

$=\frac{47}{\sqrt{94}}\left(=\frac{\sqrt{94}}{2}\right)$ ***A1***

[6 marks]

Total [15 marks]

4. determines $\frac{\pi}{4}$ (or $45°$) as the first quadrant (reference) angle ***(A1)***

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ ***(M1)***

Note: Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}, \frac{7\pi}{4}(,...)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4} \Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected ***(R1)***

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\left(=\frac{7\pi}{4}\right)$ ***A1***

$x=\frac{17\pi}{6}$ (must be in radians) ***A1***

[5 marks]

2. (a) valid approach to find area of segment by finding area of sector – area of triangle ***(M1)***

$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$\frac{1}{2}(2)^2\theta - \frac{1}{2}(2)^2\sin\theta$ ***(A1)***

area = $2\theta - 2\sin\theta$ ***A1***

[3 marks]

(b) **EITHER**

area of logo = area of rectangle – area of segments ***(M1)***

$5\times 4 - 2\times(2\theta - 2\sin\theta) = 13.4$ ***(A1)***

OR

area of one segment = $\frac{20-13.4}{2}$ $(=3.3)$ ***(M1)***

$2\theta - 2\sin\theta = 3.3$ ***(A1)***

THEN

$\theta = 2.35672...$

$\theta = 2.36$ (do not accept an answer in degrees) ***A1***

Note: Award ***(M1)(A1)A0*** if there is more than one solution.
Award ***(M1)(A1FT)A0*** if the candidate works in degrees and obtains a final answer of $135.030\ldots$

[3 marks]

Total [6 marks]

7. (a) $|\boldsymbol{a}| = \sqrt{12^2 + (-5)^2}\ (=13)$ ***(A1)***

$2 \leq |\boldsymbol{a} + \boldsymbol{b}| \leq 28$ (accept min 2 and max 28) ***A1***

Note: Award ***(A1)A0*** for 2 and 28 seen with no indication that they are the endpoints of an interval.

[2 marks]

(b) recognition that $\boldsymbol{p}$ or $\boldsymbol{b}$ is a negative multiple of $\boldsymbol{a}$ ***(M1)***

$\boldsymbol{p} = -2\hat{\boldsymbol{a}}$ OR $\boldsymbol{b} = -\frac{15}{13}\boldsymbol{a}\left(= -\frac{15}{13}\begin{pmatrix}12\\-5\end{pmatrix}\right)$

$\boldsymbol{p} = -\frac{2}{13}\begin{pmatrix}12\\-5\end{pmatrix}\left(= \begin{pmatrix}-1.85\\0.769\end{pmatrix}\right)$ ***A1***

[2 marks]

(c) **METHOD 1**

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix}12\\-5\end{pmatrix}$

$\Rightarrow \boldsymbol{q}$ is in the direction $\begin{pmatrix}5\\12\end{pmatrix}$ ***(M1)***

$\boldsymbol{q} = k\begin{pmatrix}5\\12\end{pmatrix}$ ***(A1)***

$\left(|\boldsymbol{q}| =\right)\sqrt{(5k)^2 + (12k)^2} = 15$ ***(M1)***

$k = \frac{15}{13}$ ***(A1)***

$\boldsymbol{q} = \frac{15}{13}\begin{pmatrix}5\\12\end{pmatrix}\left(= \begin{pmatrix}\frac{75}{13}\\\frac{180}{13}\end{pmatrix} = \begin{pmatrix}5.77\\13.8\end{pmatrix}\right)$ ***A1***

[5 marks]

continued…

Question 7 continued

METHOD 2

$\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

attempt to set scalar product $\boldsymbol{q}.\boldsymbol{a} = 0$ OR product of gradients $= -1$ ***(M1)***

$12x - 5y = 0$ ***(A1)***

$(|\boldsymbol{q}| =)\sqrt{x^2 + y^2} = 15$

attempt to solve simultaneously to find a quadratic in x or in y ***(M1)***

$x^2 + \left(\frac{12x}{5}\right)^2 = 15^2$ OR $\left(\frac{5y}{12}\right)^2 + y^2 = 15^2$

$\boldsymbol{q} = \begin{pmatrix} \frac{75}{13} \\ \frac{180}{13} \end{pmatrix} \left(= \begin{pmatrix} 5.77 \\ 13.8 \end{pmatrix}\right)$ ***A1A1***

Note: Award ***A1*** independently for each value. Accept values given as $x = \frac{75}{13}$ and $y = \frac{180}{13}$ or equivalent.

[5 marks]

Total [9 marks]

1. (a) **EITHER**

uses the cosine rule ***(M1)***

$\mathrm{AB}^2 = 5^2 + 5^2 - 2\times 5\times 5\times \cos 1.9$ ***(A1)***

OR

uses right-angled trigonometry ***(M1)***

$\frac{\frac{\mathrm{AB}}{2}}{5} = \sin 0.95$ ***(A1)***

OR

uses the sine rule ***(M1)***

$\alpha = \frac{1}{2}(\pi - 1.9)(= 0.6207...)$

$\frac{\mathrm{AB}}{\sin 1.9} = \frac{5}{\sin 0.6207...}$ ***(A1)***

THEN

$\mathrm{AB} = 8.1341...$

$\mathrm{AB} = 8.13$ (m) ***A1***

[3 marks]

continued…

Question 1 continued

(b) let the shaded area be A

METHOD 1

Attempt at finding reflex angle ***(M1)***

$\text{A}\hat{\text{O}}\text{B} = 2\pi - 1.9 \ (= 4.3831...)$

substitution into area formula ***(A1)***

$A = \frac{1}{2} \times 5^2 \times 4.3831...$ OR $\left(\frac{2\pi - 1.9}{2\pi}\right) \times \pi\left(5^2\right)$

$= 54.7898...$

$= 54.8$ (m^2) ***A1***

METHOD 2

let the area of the circle be A_C and the area of the unshaded sector be A_U

$A = A_C - A_U$ ***(M1)***

$A = \pi \times 5^2 - \frac{1}{2} \times 5^2 \times 1.9 \ \left(= 78.5398... - 23.75\right)$ ***(A1)***

$= 54.7898...$

$= 54.8$ (m^2) ***A1***

[3 marks]

Total [6 marks]

11. (a) let ϕ be the required angle (bearing)

EITHER

$\phi = 90° - \arctan\frac{1}{2}\ \left(= \arctan 2\right)$ ***(M1)***

Note: Award ***M1*** for a labelled sketch.

OR

$\cos\phi = \frac{\begin{pmatrix}0\\1\end{pmatrix}.\begin{pmatrix}4\\2\end{pmatrix}}{\sqrt{1}\times\sqrt{20}}\ \left(= 0.4472..., = \frac{1}{\sqrt{5}}\right)$ ***(M1)***

$\phi = \arccos\left(0.4472...\right)$

THEN

$063°$ ***A1***

Note: Do not accept $063.4°$ or $63.4°$ or 1.10^{c}.

[2 marks]

(b) **Method 1**

let $|\boldsymbol{b}_A|$ be the speed of A and let $|\boldsymbol{b}_B|$ be the speed of B

attempts to find the speed of one of A or B ***(M1)***

$|\boldsymbol{b}_A| = \sqrt{(-6)^2 + 2^2 + 4^2}$ or $|\boldsymbol{b}_B| = \sqrt{4^2 + 2^2 + (-2)^2}$

Note: Award ***M0*** for $|\boldsymbol{b}_A| = \sqrt{19^2 + (-1)^2 + 1^2}$ and $|\boldsymbol{b}_B| = \sqrt{1^2 + 0^2 + 12^2}$.

$|\boldsymbol{b}_A| = 7.48...\ \left(= \sqrt{56}\right)$ (km min^{-1}) and $|\boldsymbol{b}_B| = 4.89...\ \left(= \sqrt{24}\right)$ (km min^{-1}) ***A1***

$|\boldsymbol{b}_A| > |\boldsymbol{b}_B|$ so A travels at a greater speed than B ***AG***

[2 marks]

continued…

Question 11 continued

Method 2

attempts to use $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$\text{speed}_A = \dfrac{|r_A(t_2) - r_A(t_1)|}{t_2 - t_1}$ and $\text{speed}_B = \dfrac{|r_B(t_2) - r_B(t_1)|}{t_2 - t_1}$ ***(M1)***

for example:

$\text{speed}_A = \dfrac{|r_A(1) - r_A(0)|}{1}$ and $\text{speed}_B = \dfrac{|r_B(1) - r_B(0)|}{1}$

$\text{speed}_A = \dfrac{\sqrt{(-6)^2 + 2^2 + 4^2}}{1}$ and $\text{speed}_B = \dfrac{\sqrt{4^2 + 2^2 + 2^2}}{1}$

$\text{speed}_A = 7.48... \left(2\sqrt{14}\right)$ and $\text{speed}_B = 4.89... \left(\sqrt{24}\right)$ ***A1***

$\text{speed}_A > \text{speed}_B$ so A travels at a greater speed than B ***AG***

[2 marks]

continued…

Question 11 continued

(c) attempts to use the angle between two direction vectors formula ***(M1)***

$$\cos\theta = \frac{(-6)(4)+(2)(2)+(4)(-2)}{\sqrt{(-6)^2+2^2+4^2}\sqrt{4^2+2^2+(-2)^2}}$$ ***(A1)***

$\cos\theta = -0.7637... \left(= -\frac{7}{\sqrt{84}}\right)$ or $\theta = \arccos(-0.7637...)$ $(= 2.4399...)$

attempts to find the acute angle $180° - \theta$ using their value of θ ***(M1)***

$= 40.2°$ ***A1***

[4 marks]

continued…

Question 11 continued

(d) (i) for example, sets $\boldsymbol{r_A}(t_1) = \boldsymbol{r_B}(t_2)$ and forms at least two equations ***(M1)***

$19 - 6t_1 = 1 + 4t_2$

$-1 + 2t_1 = 2t_2$

$1 + 4t_1 = 12 - 2t_2$

Note: Award ***M0*** for equations involving t only.

EITHER

attempts to solve the system of equations for one of t_1 or t_2 ***(M1)***

$t_1 = 2$ or $t_2 = \frac{3}{2}$ ***A1***

OR

attempts to solve the system of equations for t_1 and t_2 ***(M1)***

$t_1 = 2$ and $t_2 = \frac{3}{2}$ ***A1***

THEN

substitutes their t_1 or t_2 value into the corresponding $\boldsymbol{r_A}$ or $\boldsymbol{r_B}$ ***(M1)***

$\mathrm{P}(7, 3, 9)$ ***A1***

Note: Accept $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}$. Accept 7 km east of 0, 3 km north of 0 and 9 km above sea level.

continued…

Question 11 continued

(ii) attempts to find the value of $t_1 - t_2$ (***M1)***

$t_1 - t_2 = 2 - \frac{3}{2}$

0.5 minutes (30 seconds) ***A1***

[7 marks]

continued…

Question 11 continued

(e) **EITHER**

attempts to find $\boldsymbol{r_B} - \boldsymbol{r_A}$ ***(M1)***

$$\boldsymbol{r_B} - \boldsymbol{r_A} = \begin{pmatrix} -18 \\ 1 \\ 11 \end{pmatrix} + t\begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$D(t) = \sqrt{(10t-18)^2 + 1 + (11-6t)^2}$ ***A1***

OR

attempts to find $\boldsymbol{r_A} - \boldsymbol{r_B}$ ***(M1)***

$$\boldsymbol{r_A} - \boldsymbol{r_B} = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t\begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$D(t) = \sqrt{(18-10t)^2 + (-1)^2 + (-11+6t)^2}$ ***A1***

Note: Award ***M0M0A0*** for expressions using two different time parameters.

THEN

either attempts to find the local minimum point of $D(t)$ or attempts to find the value of t such that $D'(t) = 0$ (or equivalent) ***(M1)***

$t = 1.8088... \left(= \frac{123}{68} \right)$

$D(t) = 1.01459...$

minimum value of $D(t)$ is $1.01 \left(= \frac{\sqrt{1190}}{34} \right)$ (km) ***A1***

[5 marks]

Note: Award ***M0*** for attempts at the shortest distance between two lines.

Total [20 marks]

2.

(a) (i) attempt to find midpoint of A and B ***(M1)***

centre $(-1,3,-2)$ (accept vector notation and/or missing brackets) ***A1***

(ii) attempt to find AB or half of AB or distance between the centre and A (or B) ***(M1)***

$\frac{\sqrt{4^2+2^2+4^2}}{2}$ or $\sqrt{2^2+1^2+2^2}$

$=3$ ***A1***

[4 marks]

(b) attempt to find the distance between their centre and V
(the perpendicular height of the cone) ***(M1)***

$\sqrt{0^2+4^2+2^2}$ OR $\sqrt{(\text{their slant height})^2-(\text{their radius})^2}$

$=\sqrt{20}\left(=2\sqrt{5}\right)$ ***(A1)***

Volume $=\frac{1}{3}\pi 3^2\sqrt{20}$

$=3\pi\sqrt{20}\left(=6\pi\sqrt{5}\right)$ ***A1***

[3 marks]

Total [7 marks]

10. (a) $\cos^2 x - 3\sin^2 x = 0$

valid attempt to reduce equation to one involving one trigonometric function ***(M1)***

$\frac{\sin^2 x}{\cos^2 x} = \frac{1}{3}$ OR $1 - \sin^2 x - 3\sin^2 x = 0$ OR $\cos^2 x - 3\left(1 - \cos^2 x\right) = 0$

OR $\cos 2x - 1 + \cos 2x = 0$

correct equation ***(A1)***

$\tan^2 x = \frac{1}{3}$ OR $\cos^2 x = \frac{3}{4}$ OR $\sin^2 x = \frac{1}{4}$ OR $\cos 2x = \frac{1}{2}$

$\tan x = \pm\frac{1}{\sqrt{3}}$ OR $\cos x = \pm\frac{\sqrt{3}}{2}$ OR $\sin x = (\pm)\frac{1}{2}$ OR $2x = \frac{\pi}{3}\left(, \frac{5\pi}{3}\right)$ ***(A1)***

$x = \frac{\pi}{6},\ x = \frac{5\pi}{6}$ ***A1A1***

Note: Award ***M1A1A0A1A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = \frac{\pi}{6}$.

Award ***M1A1A0A0A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = 30°$.

Award ***M1A1A1A1A0*** for candidates who give both answers in degrees.

Award ***M1A1A1A1A0*** for candidates who give both correct answers in radians, but who include additional solutions outside the domain.

Award a maximum of ***M1A0A0A1A1*** for correct answers with no working.

[5 marks]

continued…

Question 10 continued

(b) (i) attempt to use the chain rule (may be evidenced by at least one $\cos x \sin x$ term) ***(M1)***

$f'(x) = -2\cos x \sin x - 6\sin x \cos x \left(= -8\sin x \cos x = -4\sin 2x\right)$ ***A1***

(ii) valid attempt to solve their $f'(x) = 0$ ***(M1)***

at least 2 correct x-coordinates (may be seen in coordinates) ***(A1)***

$x = 0,\ x = \frac{\pi}{2},\ x = \pi$

Note: Accept additional correct solutions outside the domain.
Award ***A0*** if any additional incorrect solutions are given.

correct coordinates (may be seen in graph for part (c)) ***A1A1A1***

$(0,1),\ (\pi,1),\ \left(\frac{\pi}{2},-3\right)$

Note: Award a maximum of ***M1A1A1A1A0*** if any additional solutions are given.

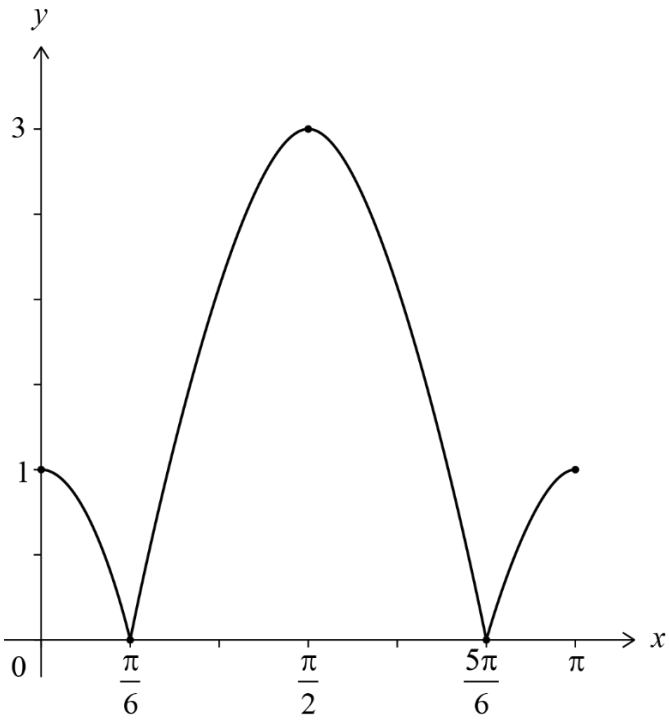
Note: If candidates do not find at least two correct x-coordinates, it is possible to award the appropriate final marks for their correct coordinates, such as ***M1A0A0A1A0.***

[7 marks]

continued…

Question 10 continued

(c)



attempt to reflect the negative part of the graph of f in the x-axis ***M1***

endpoints have coordinates $(0,1)$, $(\pi,1)$ ***A1***

smooth maximum at $\left(\frac{\pi}{2},3\right)$ ***A1***

sharp points (cusps) at x-intercepts $\frac{\pi}{6}$, $\frac{5\pi}{6}$ ***A1***

[4 marks]

continued…

Question 10 continued

(d) considers points of intersection of $y = \left|f\left(x\right)\right|$ and $y = 1$ on graph or algebraically ***(M1)***

$-\left(\cos^2 x - 3\sin^2 x\right) = 1$ or $-\left(1 - 4\sin^2 x\right) = 1$ or $-\left(4\cos^2 x - 3\right) = 1$ or $-\left(2\cos 2x - 1\right) = 1$

$\tan^2 x = 1$ or $\sin^2 x = \frac{1}{2}$ or $\cos^2 x = \frac{1}{2}$ or $\cos 2x = 0$ ***(A1)***

$x = \frac{\pi}{4}, \frac{3\pi}{4}$ ***(A1)***

For $\left|f\left(x\right)\right| > 1$

$\frac{\pi}{4} < x < \frac{3\pi}{4}$ ***A1***

[4 marks]

Total [20 marks]

7. (a) **METHOD 1**

attempt to use scalar product or formula for angle between two vectors ***(M1)***

$\boldsymbol{u}.\boldsymbol{v} = \cos\frac{1}{n} + \sin\frac{1}{n}$ (seen anywhere) ***(A1)***

$$\cos\theta = \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}\sqrt{\left(\cos^2\frac{1}{n} + \sin^2\frac{1}{n}\right)}}\left(= \frac{\cos\frac{1}{n} + \sin\frac{1}{n}}{\sqrt{2}}\right)$$ ***A1***

METHOD 2

attempt to use an Argand diagram showing two complex numbers in the first quadrant with the angle between them marked as θ ***(M1)***

$\arg(\boldsymbol{u}) = \frac{\pi}{4}$ (accept 45° or $\arctan(1)$) **and** $\arg(\boldsymbol{v}) = \frac{1}{n}$ ***(A1)***

$\cos\theta = \cos\left|\frac{\pi}{4} - \frac{1}{n}\right|$ ***A1***

continued…

Question 7 continued

(b) use of $\frac{1}{n} \to 0$ as $n \to \infty$ ***(M1)***

EITHER

$\left(\cos\theta \to\right)\frac{1}{\sqrt{2}}$ ***(A1)***

OR

$\left(v \to\right)i$ ***(A1)***

THEN

the limit is $\frac{\pi}{4}$ ***A1***

Note: Accept $45°$. Do not accept rounded values such as 0.785.

[3 marks]

Total [6 marks]

9. (a) (i) attempt to use the cosine rule ***(M1)***

$\text{AC} = \sqrt{2^2 + 4^2 - 2(2)(4)\cos\alpha}\left(= \sqrt{20 - 16\cos\alpha} = 2\sqrt{5 - 4\cos\alpha}\right)$ ***A1***

(ii) $\text{AC} = \sqrt{6^2 + 8^2 - 2(6)(8)\cos\beta}\left(= \sqrt{100 - 96\cos\beta} = 2\sqrt{25 - 24\cos\beta}\right)$ ***A1***

(iii) $5 - 4\cos\alpha = 25 - 24\cos\beta$

$\alpha = \arccos\left(6\cos\beta - 5\right)$ ***A1***

[4 marks]

(b) attempt to find the sum of two triangle areas using $A = \frac{1}{2}ab\sin C$ ***(M1)***

Note: Do not award this ***M1*** if the triangle is assumed to be right angled.

$\text{Area} = \frac{1}{2}(8)\sin\alpha + \frac{1}{2}(48)\sin\beta$ ***(A1)***

attempt to express the area in terms of one variable only ***(M1)***

$= 4\sqrt{1 - (6\cos\beta - 5)^2} + 24\sin\beta$ or $4\sin\left(\arccos\left(6\cos\beta - 5\right)\right) + 24\sin\beta$ OR

$4\sin\alpha + 24\sqrt{1 - \left(\frac{5 + \cos\alpha}{6}\right)^2}$ or $4\sin\alpha + 24\sin\left(\arccos\left(\frac{5 + \cos\alpha}{6}\right)\right)$

Max area $= 19.5959\ldots$

$= 19.6$ ***A1***

[4 marks]

Total [8 marks]

Section B

12.

(a) $\vec{AB} = \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix}, \vec{AC} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ ***A1A1***

[2 marks]

(b) **METHOD 1**

$k - 1 = 2 \times 4$ ***M1***

$k = 9$ ***AG***

METHOD 2

in order by y or z-ordinate, the points are $(k, -2, 1), (5, 0, 2)\ (1, 2, 3)$

$k - 5 = 5 - 1$ ***M1***

$k = 9$ ***AG***

[1 mark]

continued…

Question 12 continued

(c) (i) attempt to set up a vector equation using a point, a parameter and a direction vector ***(M1)***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$ (or equivalent) ***A1***

Note: "$\boldsymbol{r} =$" or equivalent must be seen for ***A1***.

continued…

Question 12 continued

(ii) **METHOD 1**

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use a different parameter for L_2 ***(M1)***

$\frac{x-1}{2} = \frac{y}{3} = 1-z = \mu$ or $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

point on line L_2 has coordinates $(1+2\mu, 3\mu, 1-\mu)$ ***(A1)***

Note: This ***A1*** may be implied by $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

$1+4\lambda = 1+2\mu$

$2-2\lambda = 3\mu$

$3-\lambda = 1-\mu$

any two of the above equations ***A1***

attempt to solve two simultaneous equations with two parameters ***(M1)***

eg $\lambda = 0.25$, $\mu = 0.5$ or $\lambda = 1.6$, $\mu = -0.4$ or $\lambda = -2$, $\mu = -4$ ***A1***

substitute into third equation or solve a different pair of simultaneous equations ***M1***

obtain contradiction *eg* $3-0.25 \neq 1-0.5$ or $1+4(1.6) \neq 1+2(-0.4)$ or $2-2(-2) \neq 3(-4)$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

continued…

Question 12 continued

METHOD 2

point on line L_1 has coordinates $(1+4\lambda, 2-2\lambda, 3-\lambda)$

attempt to use the equation of L_2 to generate at least two equations in λ ***(M1)***

if the two lines intersect,

$$\frac{(1+4\lambda)-1}{2} = \frac{2-2\lambda}{3} \left(\Rightarrow 2\lambda = \frac{2-2\lambda}{3} \right)$$

$$\frac{(1+4\lambda)-1}{2} = 1-(3-\lambda) (\Rightarrow 2\lambda = \lambda - 2)$$

$$\frac{2-2\lambda}{3} = 1-(3-\lambda) \Rightarrow \left(\frac{2-2\lambda}{3} = \lambda - 2 \right)$$

any two of the above equations ***A1A1***

attempt to solve at least one equation in λ ***(M1)***

one of $\lambda = \frac{1}{4}$, $\lambda = -2$, $\lambda = \frac{8}{5}$ seen ***A1***

substitute into second equation or solve second equation ***M1***

obtain contradiction *eg* $\lambda = \frac{1}{4} \neq -2$ or $2\left(\frac{1}{4}\right) \neq \frac{1}{4} - 2$ (so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

continued…

Question 12 continued

METHOD 3

attempt to use a find Cartesian equation for L_1 ***(M1)***

$\frac{x-1}{4}=\frac{y-2}{-2}=\frac{z-3}{-1}$ ***A1***

attempt to isolate one variable in both equations ***(M1)***

$L_1: z=\frac{1-x}{4}+3=\frac{y-2}{2}+3 \quad L_2: z=\frac{1-x}{2}+1=\frac{-y}{3}+1$ OR

$L_1: y=\frac{1-x}{2}+2=2(z-3)+2 \quad L_2: y=\frac{3(x-1)}{2}=3(1-z)$ OR

$L_1: x=1-2(y-2)=1-4(z-3) \quad L_2: x=\frac{2y}{3}+1=1-2(z-1)$ ***A1***

attempt to solve for each of the other two variables ***(M1)***

e.g. $\frac{1-x}{2}+1=\frac{1-x}{4}+3$ and $\frac{-y}{3}+1=\frac{y-2}{2}+3$

$x=-7,\ y=-1.2$ OR $x=2,\ z=1.4$ OR $y=1.5,\ z=5$ ***A1***

obtain contradiction *eg* $z=5\neq 1.4$ OR $y=1.5\neq -1.2$ OR $x=2\neq -7$

(so the lines do not intersect) ***R1***

Note: Do not award this ***R1*** if it is based on incorrect values.

lines are not parallel ***R1***

so lines are skew ***AG***

[10 marks]

continued…

Question 12 continued

(d) (i) **METHOD 1**

attempt to find cross product of two of $\vec{AB}, \vec{AC}$ and $\vec{BC}$ or their opposites ***M1***

eg $\vec{AB} \times \vec{AC} = \begin{pmatrix} 0 \\ k-9 \\ 18-2k \end{pmatrix} \left(= (k-9) \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$ ***A1***

attempt to substitute their cross product and a point into the equation of a plane ***(M1)***

$(k-9)y + 2(9-k)z = 2(k-9) + 6(9-k)$

$(k-9)y + 2(9-k)z = 36 - 4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

METHOD 2

attempt to find vector equation of Π and write x, y and z in parametric form ***M1***

$$\left(\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} k-1 \\ -4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \Rightarrow \right) \ x = 1 + \lambda(k-1) + 4\mu, \ y = 2 - 4\lambda - 2\mu,$$

$z = 3 - 2\lambda - \mu$ or equivalent ***A1***

attempt to eliminate both parameters to work towards Cartesian form ***M1***

$(k-9)y + 2(9-k)z = 36 - 4k \ (\Rightarrow y - 2z = -4 \text{ since } k \neq 9)$ ***A1***

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to find the equation of the line through (0, 0, 0) perpendicular to the plane ***(M1)***

EITHER

$\left(\boldsymbol{r} =\right) t\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$t + 4t + 4 = 0$

$t = -\frac{4}{5}$ ***(A1)***

OR

$\left(\boldsymbol{r} =\right) t(k-9)\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ ***(A1)***

attempt to find the point where the line and plane intersect ***(M1)***

$t(k-9)^2 + 4t(k-9)^2 + 4(k-9) = 0$

$t = -\frac{4}{5(k-9)}$ ***(A1)***

THEN

so the point on the plane closest to the origin is $(0,\ -0.8,\ 1.6)$ ***A1***

continued…

Question 12 continued

METHOD 2

choose a point on the plane (p , q , r)

$q-2r+4=0$ OR $q(k-9)-2r(k-9)+4(k-9)=0 \Rightarrow q=2r-4$

distance to the origin is $\sqrt{p^2+(2r-4)^2+r^2}$ ***(A1)***

since ***p*** is independent of *r*, distance is minimised when $p=0$ ***(R1)***

attempt to find the value of *r* for which their $\sqrt{(2r-4)^2+r^2}$ is minimised ***(M1)***

$r=1.6$ ***(A1)***

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

continued…

Question 12 continued

METHOD 3

attempt to find a vector from the origin to the closest point on the plane ***(M1)***

EITHER

$\left(\boldsymbol{r} =\right)t\begin{pmatrix}0\\1\\-2\end{pmatrix}$ ***(A1)***

distance to the origin $=\left(\dfrac{4}{\sqrt{1^2+(-2)^2}}=\dfrac{4}{\sqrt{5}}\right)=\dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t=\pm\dfrac{4}{5}$

check in equation of plane $y-2z=-4$ to get $t=-\dfrac{4}{5}$ ***(R1)***

OR

$\left(\boldsymbol{r} =\right)t(k-9)\begin{pmatrix}0\\1\\-2\end{pmatrix}$ ***(A1)***

distance to the origin $=\left(\dfrac{4}{\sqrt{1^2+(-2)^2}}=\dfrac{4}{\sqrt{5}}\right)=\dfrac{4\sqrt{5}}{5}$ ***(A1)***

$t=\pm\dfrac{4}{5(k-9)}$

check in equation of plane $y-2z=-4$ to get $t=-\dfrac{4}{5(k-9)}$ ***(R1)***

THEN

so the point on the plane closest to the origin is $(0, -0.8, 1.6)$ ***A1***

[9 marks]

Total [22 marks]

3. $1-2\sin^2 x = \sin x$ ***A1***

$2\sin^2 x + \sin x - 1 = 0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x - 1)(\sin x + 1)$ OR $\frac{-1 \pm \sqrt{1-4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x = \frac{1}{2}$ OR $\sin x = -1$

any correct solution from $\sin x = -1$ ***A1***

any correct solution from $\sin x = \frac{1}{2}$ ***A1***

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1***

Note: If no working shown, award no marks for a final value(s).
Award ***A0*** for $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ if additional values also given.

Total [6 marks]

5. (a) recognition that period is $4m$ OR substitution of a point on f (except the origin) ***(M1)***

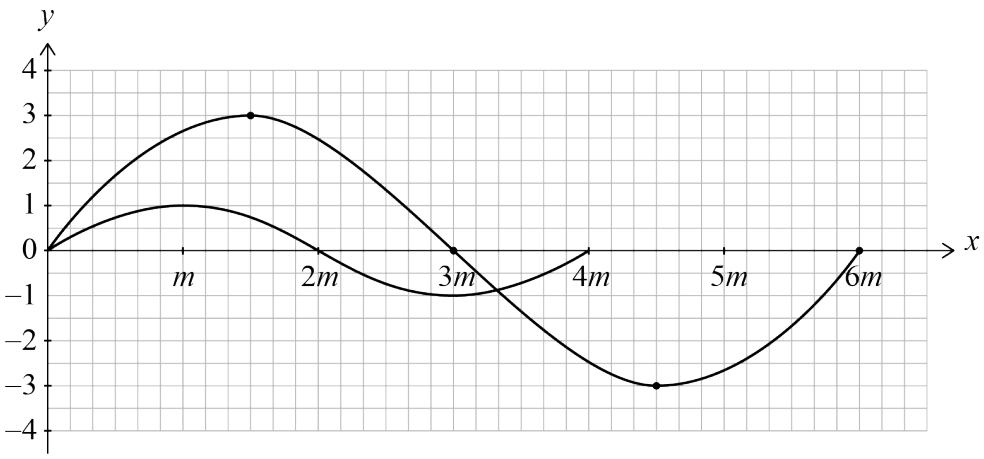
$4m = \frac{2\pi}{q}$ OR $1 = \sin qm$

$m = \frac{\pi}{2q}$ ***A1***

[2 marks]

(b) horizontal scale factor is $\frac{3}{2}$ (seen anywhere) ***(A1)***

Note: This ***(A1)*** may be earned by seeing a period of $6m$, half period of $3m$ or the correct x-coordinate of the maximum/minimum point.



A1A1A1

Note: Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
A1 for correct amplitude.
A1 for correct domain.
A1 for correct max and min points **and** correct x-intercepts.

[4 marks]

Total [6 marks]

12. (a) $2t+1\times 0+0\times(3+t)\quad(=2t)$ (seen anywhere) ***(A1)***

one correct magnitude $\sqrt{1^2+1^2+0^2}, \sqrt{(2t)^2+(3+t)^2}$ ***(A1)***

correct substitution of their magnitudes and scalar product ***M1***

$2t=\sqrt{2}\times\sqrt{(2t)^2+(3+t)^2}\times\cos\frac{\pi}{3}$ OR $\cos\frac{\pi}{3}=\frac{2t}{\sqrt{2}\times\sqrt{5t^2+6t+9}}$

$4t=\sqrt{2\left(4t^2+9+6t+t^2\right)}$ OR $\frac{1}{2}=\frac{2t}{\sqrt{2\left(5t^2+6t+9\right)}}$ (or equivalent) ***A1***

$4t=\sqrt{10t^2+12t+18}$ ***AG***

[4 marks]

(b) correct quadratic equation ***A1***

$16t^2=10t^2+12t+18,\ 6t^2-12t-18=0\ ,\ t^2-2t-3=0$

valid attempt to solve their quadratic set =0 ***(M1)***

$(t+1)(t-3)$ OR $\frac{12\pm\sqrt{(-12)^2-4\times6\times(-18)}}{12}$ OR $(t-1)^2-4$ ***(A1)***

$t=3$ ***A1***

Note: Award ***A0*** if additional answer(s) given.

[4 marks]

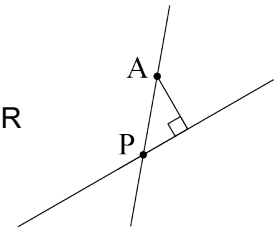
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Question 12 continued

(c) **METHOD 1**

recognizing shortest distance from A is perpendicular to L_1 ***(M1)***

OR



$\sin\frac{\pi}{3} = \frac{x}{\left|\vec{\mathrm{PA}}\right|}$

$\left|\vec{\mathrm{PA}}\right| = \sqrt{6^2+6^2}$ $\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

$\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{72}}$ ***(A1)***

$x = \frac{\sqrt{216}}{2}$ $\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$

shortest distance is $\frac{\sqrt{216}}{2}\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

METHOD 2

recognition that the distance required is $\frac{\left|\boldsymbol{v}\times\vec{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$$=\frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}\right| \quad \textbf{(A1)}$$

$$=\frac{1}{\sqrt{2}}\left|\begin{pmatrix}6\\-6\\-6\end{pmatrix}\right| \quad \textbf{(A1)}$$

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

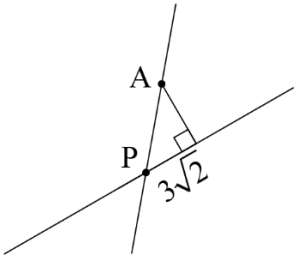
continued…

Question 12 continued

METHOD 3

recognition that the base of the triangle is $\dfrac{\left|\boldsymbol{v}\cdot\vec{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$\dfrac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$

$=\dfrac{6}{\sqrt{2}}\left(=3\sqrt{2}\right)$ OR 

(A1)

$\left|\vec{\mathrm{PA}}\right|=\sqrt{6^2+6^2}\quad\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

Note: The value of $\left|\vec{\mathrm{PA}}\right|=\sqrt{6^2+6^2}$ may be seen as part of the working of their shortest distance, $d=\sqrt{\left|\vec{\mathrm{PA}}\right|^2-b^2}=\sqrt{\left(\sqrt{72}\right)^2-\left(3\sqrt{2}\right)^2}$

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

METHOD 4

Let B be a general point on L_1 $(\lambda, 8+\lambda, -3)$ such that AB is perpendicular to L_1

attempt to find vector $\vec{AB}$ OR $\left|\vec{AB}\right|$ (the shortest distance from A to L_1) ***(M1)***

$$\vec{AB} = \vec{OB} - \vec{OA} = \vec{OP} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \vec{OA}\left(= \begin{pmatrix}0\\8\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}6\\8\\3\end{pmatrix} = \vec{AP} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right) \quad (\lambda \in \mathbb{R})$$

$$\vec{AB} = \begin{pmatrix}-6\\0\\-6\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ OR } \left|\vec{AB}\right| = \sqrt{(\lambda-6)^2 + (8+\lambda-8)^2 + (-3-3)^2} \quad \textbf{A1}$$

$$\left|\vec{AB}\right| = \sqrt{(\lambda-6)^2 + \lambda^2 + (-6)^2} \left(= \sqrt{2\lambda^2 - 12\lambda + 72}\right)$$

EITHER

$$\frac{d}{d\lambda}\left(\left|\vec{AB}\right|^2\right) = 0 \Rightarrow 4\lambda - 12 = 0 \Rightarrow \lambda = 3 \quad \textbf{A1}$$

OR

$\left|\vec{AB}\right| = \sqrt{2(\lambda-3)^2 + 54}$ to obtain $\lambda = 3$ ***A1***

OR

$$\begin{pmatrix}-6+\lambda\\ \lambda\\ -6\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = 0 \Rightarrow -6 + \lambda + \lambda = 0 \Rightarrow \lambda = 3 \quad \textbf{A1}$$

THEN

shortest distance is $\sqrt{54}$ $\left(= 3\sqrt{6}\right)$ ***A1***

[4 marks]

continued…

Question 12 continued

(d) attempt to find the vector product of two direction vectors ***(M1)***

$$\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}$$

$\boldsymbol{n}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ (or any scalar multiple of this) (accept $\boldsymbol{n}=<1,-1,-1>$ or equivalent) ***A1***

Note: Award ***A0*** for a final answer given in coordinate form.

[2 marks]

continued…

Question 12 continued

(e) substituting their x into volume formula and equating ***(M1)***

$\frac{1}{3}\pi\left(3\sqrt{6}\right)^2 h = 90\sqrt{3}\pi$

$h = 5\sqrt{3}$ (seen anywhere) ***A1***

recognition that the position vector of vertex is given by $\vec{\mathrm{OA}} + \mu\boldsymbol{n}$ OR $\vec{\mathrm{OA}} + h \times \hat{\boldsymbol{n}}$ ***(M1)***

$\begin{pmatrix}6\\8\\3\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ OR $(6+\mu, 8-\mu, 3-\mu)$

EITHER

recognition that $\mu|\boldsymbol{n}| = h$ (where μ is a parameter) ***(M1)***

$\mu|\boldsymbol{n}| = 5\sqrt{3}$ OR $\sqrt{\mu^2 + (-\mu)^2 + (-\mu)^2} = 5\sqrt{3}$ OR $3\mu^2 = 75$ $\left(\Rightarrow \sqrt{3}\mu = 5\sqrt{3}\right)$

$\mu = \pm 5$ (accept $\mu = 5$) ***(A1)***

OR

attempt to find cone's height vector $h \times \hat{\boldsymbol{n}}$ ***(M1)***

$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ ***(A1)***

$5\sqrt{3} \times \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

THEN

$= \begin{pmatrix}6\\8\\3\end{pmatrix} \pm 5\begin{pmatrix}1\\-1\\-1\end{pmatrix} \left(= \begin{pmatrix}6\\8\\3\end{pmatrix} \pm \begin{pmatrix}5\\-5\\-5\end{pmatrix}\right)$

vertex = $(11, 3, -2)$ and $(1, 13, 8)$ (accept position vectors) ***A1A1***

Note: Award a maximum of ***(M0)A0(M1)(M1)(A1)A1A1FT*** for $\left(\frac{39}{4}, \frac{17}{4}, -\frac{3}{4}\right)$ and $\left(\frac{9}{4}, \frac{47}{4}, \frac{27}{4}\right)$ obtained using $x = \left|\vec{\mathrm{PA}}\right|$ from part (c).

[7 marks]

Total [21 marks]

1. (a) attempts to find perimeter ***(M1)***

$\text{arc} + 2 \times \text{radius}$ OR $10 + 4 + 4$

$= 18 \text{ (cm)}$ ***A1***

[2 marks]

(b) $10 = 4\theta$ ***(A1)***

$\theta = \frac{10}{4} \left(= \frac{5}{2}, 2.5 \right)$ ***A1***

[2 marks]

(c) area $= \frac{1}{2}\left(\frac{10}{4}\right)\left(4^2\right) \left(= 1.25 \times 16\right)$ ***(A1)***

$= 20 \text{ (cm}^2\text{)}$ ***A1***

[2 marks]

Total [6 marks]

8. (a) $\cos k = \frac{\sin k}{\cos k}$ ***A1***

$\cos^2 k = \sin k$ ***AG***

[1 mark]

(b) $f'(k) = -\sin k$ and $g'(k) = \sec^2 k$ ***A1***

Note: Award ***A1*** for $f'(x) = -\sin x$ and $g'(x) = \sec^2 x$.

EITHER

$f'(k)g'(k) = -\frac{\sin k}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow f'(k)g'(k)\left(= -\frac{\sin k}{\sin k}\right) = -1$ ***R1***

OR

$g'(k) = \frac{1}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow g'(k) = \frac{1}{\sin k} = -\frac{1}{f'(k)}$ ***R1***

Note: Accept showing that $f'(k) = -\frac{1}{g'(k)}$.

Note: Allow'backwards methods' such as starting with $f'(k) = -\frac{1}{g'(k)}$ leading to $\cos^2 k = \sin k$

THEN

$\Rightarrow$ the two tangents intersect at right angles at P ***AG***

Note: To obtain the final ***R1***, all of the previous marks must have been awarded.

[3 marks]

continued…

Question 8 continued

(c) $1-\sin^2 k = \sin k$ (from part (a)) ***A1***

$\sin^2 k + \sin k - 1 = 0$

attempts to solve for $\sin k$ ***(M1)***

$$\sin k = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

(for $0 < k < \frac{\pi}{2}$, $\sin k > 0$) $\Rightarrow \sin k = \frac{-1+\sqrt{5}}{2}$ ***A1***

$(a = -1, b = 5, c = 2)$

Note: Award ***A0*** if more than one solution is given

[3 marks]

Total [7 marks]

9. (a) $\vec{\mathrm{OM}} = \boldsymbol{a} + k\boldsymbol{c}$ ***A1***

$\vec{\mathrm{MC}} = (1-k)\boldsymbol{c} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) attempts to expand their dot product $\vec{\mathrm{OM}} \bullet \vec{\mathrm{MC}} = (\boldsymbol{a} + k\boldsymbol{c}) \bullet ((1-k)\boldsymbol{c} - \boldsymbol{a})$ ***M1***

$= (1-2k)(\boldsymbol{a} \bullet \boldsymbol{c}) - |\boldsymbol{a}|^2 + k(1-k)|\boldsymbol{c}|^2$ (or equivalent)

uses $|\boldsymbol{c}| = 2|\boldsymbol{a}|$ and $\boldsymbol{a} \bullet \boldsymbol{c} = 2|\boldsymbol{a}|^2 \cos\theta$ ***M1***

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - |\boldsymbol{a}|^2 + 4k(1-k)|\boldsymbol{a}|^2$

$= 2(1-2k)|\boldsymbol{a}|^2 \cos\theta - (1-2k)^2 |\boldsymbol{a}|^2$ ***A1***

$|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ ***AG***

[3 marks]

(c) attempts to solve $|\boldsymbol{a}|^2 (1-2k)(2\cos\theta - (1-2k)) = 0$ for k ***(M1)***

$k = \frac{1}{2}$ or $k = \frac{1}{2} - \cos\theta$ $\left(|\boldsymbol{a}|^2 > 0\right)$

Note: Award ***(M1)*** for their '$k =$' or their '$\cos\theta =$'. For example, $\cos\theta = \frac{1-2k}{2}$ or equivalent.

as $0 \le k \le 1$, $0 \le \frac{1}{2} - \cos\theta \le 1$

$-\frac{1}{2} \le \cos\theta \le \frac{1}{2}$ ***A1***

$\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $\theta \ne \frac{\pi}{2}$ ***A1A1***

($\theta = \frac{\pi}{2}$ corresponds to only one possible position for M when $k = \frac{1}{2}$)

[4 marks]

Total [9 marks]

Section B

1. (a) Let N be North

$\mathrm{N\hat{J}D} = 34°$ OR $\mathrm{D\hat{J}L} = 56°$ (must be labelled or indicated in diagram): ***(A1)***

$\mathrm{J\hat{D}L} = 99(°)$ ***A1***

Note: Accept $\frac{11\pi}{20}, 1.73$ (radians).

[2 marks]

(b) attempt to apply the sine rule ***(M1)***

$\frac{\mathrm{DL}}{\sin 56°} = \frac{500}{\sin 99°}$ OR $\frac{\mathrm{DL}}{\sin 0.977384...} = \frac{500}{\sin 1.72787...}$ ***(A1)***

$419.685\ldots$

$\mathrm{DL} = 420\ (\mathrm{km})$ ***A1***

Note: Award ***M1A1A0*** for $261\ (\mathrm{km})$ from use of degrees with GDC set in radians (with or without working).

[3 marks]

Total [5 marks]

8. direction vector of the line is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (seen anywhere) ***(A1)***

normal vector of the plane is $\begin{pmatrix} 4 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$ (seen anywhere) ***(A1)***

EITHER

correct scalar product $12+2\cos\alpha-\sin\alpha$ (seen anywhere) ***(A1)***

one correct magnitude (seen anywhere) ***(A1)***

$\sqrt{16+\cos^2\alpha+\sin^2\alpha}\left(=\sqrt{17}\right)$, $\sqrt{9+4+1}\left(=\sqrt{14}\right)$

recognizing angle between normal and direction vector is $\frac{\pi}{2}-\alpha$ (seen anywhere) ***(M1)***

Note: angle $\frac{\pi}{2}-\alpha$ may be implied by use of $\sin\alpha$ on the RHS of the step below

attempt to substitute into the formula for the angle between two vectors to form an equation in α ***(M1)***

$12+2\cos\alpha-\sin\alpha=\sqrt{17}\sqrt{14}\cos\left(\frac{\pi}{2}-\alpha\right)$ OR $12+2\cos\alpha-\sin\alpha=\sqrt{17}\sqrt{14}\sin\alpha$

continued…

Question 8 continued

OR

correct expression for the magnitude of the vector product

$\left\| \begin{pmatrix} 2\sin\alpha+\cos\alpha \\ -4-3\sin\alpha \\ 3\cos\alpha-8 \end{pmatrix} \right\| \left(= \sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2} \right)$ (seen anywhere) ***(A1)***

one correct magnitude (seen anywhere) ***(A1)***

$\sqrt{16+\cos^2\alpha+\sin^2\alpha}\left(=\sqrt{17}\right)$, $\sqrt{9+4+1}\left(=\sqrt{14}\right)$

recognizing angle between normal and direction vector is $\frac{\pi}{2}-\alpha$ (seen anywhere) ***(M1)***

Note: angle $\frac{\pi}{2}-\alpha$ may be implied by use of $\cos\alpha$ on the RHS of the step below

attempt to substitute into the formula for the angle between two vectors to form an equation in α ***(M1)***

$\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}=\sqrt{17}\sqrt{14}\sin\left(\frac{\pi}{2}-\alpha\right)$ OR

$\sqrt{(2\sin\alpha+\cos\alpha)^2+(-4-3\sin\alpha)^2+(3\cos\alpha-8)^2}=\sqrt{17}\sqrt{14}\cos\alpha$

THEN

$\alpha = 0.932389...$

$\alpha = 0.932$ ***A1***

Note: Award maximum ***(A1)(A1)(A1)(A1)(M1)(M1)A0*** for a correct answer given in degrees $\alpha = 54.4219...^\circ$.

[7 marks]

10. (a) recognition that $45 = 10 + 10 + \text{arc length}$ ***(M1)***

arc length $= 25$ (cm) ***(A1)***

$25 = 12\theta$ ***A1***

$\theta = 2.08$ correct to 3 significant figures ***AG***

[3 marks]

(b)

Note: There are many different ways to dissect the cross-section to determine its area. In all approaches, candidates will need to find w or $\frac{w}{2}$. Award the first three marks for work seen anywhere.

EITHER

evidence of using the cosine rule OR sine rule ***(M1)***

$w^2 = 12^2 + 12^2 - 2 \cdot 12 \cdot 12\cos(2.08)$ OR $\frac{w}{\sin(2.08)} = \frac{12}{\sin(0.530796...)}$ ***(A1)***

$w = 20.6977\ldots$ or $\frac{w}{2} = 10.3488...$ ***(A1)***

OR

using trig ratios in a right triangle with angle $\frac{2.08}{2}$ and side length $\frac{w}{2}$ ***(M1)***

$\sin\left(\frac{2.08}{2}\right) = \frac{\frac{w}{2}}{12}$ ***(A1)***

$w = 20.6977\ldots$ or $\frac{w}{2} = 10.3488...$ ***(A1)***

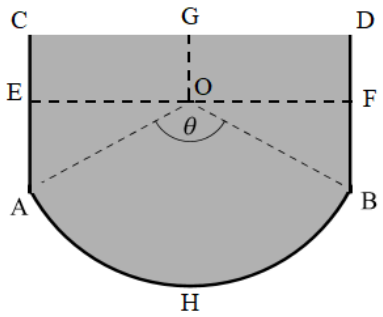
Note: Accept $w = 20.7179\ldots$ from use of $\frac{\theta}{2} = \frac{25}{24}$.

continued…

Question 10 continued

THEN

Let the points A, B, C, D, E, F, G, H lie on the figure as follows:



EITHER

(segment AHB =) sector OAB – triangle OAB ***(M1)***

$= \frac{1}{2}\times 12^2 \times 2.08 - \frac{1}{2}\times 12^2 \times \sin 2.08 \left(= 149.76 - 62.8655... = 86.8944...\right)$ ***(A1)***

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

sector OAB – triangle OAB + rectangle CDBA

$= 86.8944... + 10w \left(= 86.8944... + 206.977...\right)$

Note: Use of $\theta = \frac{25}{12}$ throughout leads to segment $\text{OAB} = 87.2517...$ and cross-sectional area $= 87.2517... + 207.179...$.

continued…

Question 10 continued

OR

trapezium CGOA (= rectangle CGOE + triangle EOA) ***(M1)***

$= \frac{1}{2} \times \left(10 + \left(10 - 12\cos(1.04)\right)\right) \times \frac{20.6977...}{2}$ $(= 72.0557)$ ***(A1)***

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

2 × trapezium CGOA + sector OAB

$= 2(72.0557...) + \frac{1}{2} \times 12^2 \times 2.08 \left(= 144.111... + 149.76\right)$

Note: Use of $\theta = \frac{25}{12}$ leads to area of trapezium CGOA $= 72.2154...$ and cross-sectional area $= 144.430... + 150$.

OR

2 x area of trapezium CGOA (= area of rectangle CDFE + 2 x triangle EOA) ***(M1)***

$20.6977... \times \left(10 - 12\cos(1.04)\right) + 2 \times \frac{1}{2} \times 12\cos(1.04) \times 12\sin(1.04)$ ***(A1)***

$\left(= 81.2458... + 62.8655...\right)$

valid approach to find total cross-sectional area (seen anywhere) ***(M1)***

2 x trapezium CGOA + sector OAB

$= 144.111... + \frac{1}{2} \times 12^2 \times 2.08 \ \left(= 144.111... + 149.76\right)$

Note: Use of $\theta = \frac{25}{12}$ leads to 2 x area of trapezium CGOA $= 144.430...$ and cross-sectional area $= 144.430... + 150$.

THEN

area of cross-section $= 293.871...$ ($294.430...$ from exact answer)

$= 294$ (cm^2) ***A1***

[7 marks]

continued…

Question 10 continued

(c) **METHOD 1**

$-4.71976...$ volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognising rainfall can be represented by an integral ***(M1)***

$\int_0^{60} R'(t)\,\mathrm{d}t \left(=\frac{250}{2\pi}\sin\left(\frac{2\pi\times 60}{5}\right)+3000\times 60\right)$ ***(A1)***

Note: Accept any 60 second interval or any interval which is a multiple of 5 seconds (one period) scaled up to 60 seconds e.g. $12\int_0^5 R'(t)\,\mathrm{d}t$.

rainfall over 60 seconds $=180000$ (cm^3) ***A1***

the gutter will overflow because the rainfall > gutter volume ***A1***

METHOD 2

volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognition that cosine has a minimum value of -1 ***(M1)***

$R'(t)\geq -1\times 50+3000\left(\mathrm{cm}^3\mathrm{s}^{-1}\right)$ ***(A1)***

rainfall over 60 seconds ≥ 177000 ***(A1)***

the gutter will overflow because the rainfall > gutter volume ***A1***

METHOD 3

volume of gutter $=176323$ OR 176658 (OR $600\times$ their area) (seen anywhere) ***A1***

recognising rainfall can be represented by an integral ***(M1)***

attempt to solve $60>58.8$ OR $\int_0^T R'(t)\,\mathrm{d}t=176658$ ***(M1)***

time to reach overflow point = $58.7875...$ OR $58.8990...$ ***A1***

the gutter will overflow because $60>58.8$ OR $60>58.9$ ***A1***

[5 marks]

Total [15 marks]

2. METHOD 1

attempt to substitute into cosine rule ***(M1)***

$154^2 = 150^2 + 90^2 - 2(150)(90)\cos \hat{APB}$ OR $\cos \hat{APB} = \frac{150^2 + 90^2 - 154^2}{2(150)(90)}$ ***(A1)***

$\hat{APB} = 75.2286...°$ OR $1.31298...$ radians

$\hat{APB} = 75.2°$ OR 1.31 radians ***(A1)***

valid approach to find θ ***(M1)***

$\theta = \frac{180° - \hat{APB}}{2}$ OR $\theta = \frac{180° - 75.2286...°}{2}$ $(= 52.3856...)$ OR

$\theta = \frac{\pi - 1.31298...}{2} (= 0.914302...)$

valid approach to express h in terms of θ ***(M1)***

$\sin\theta = \frac{h}{150}$ OR $h = 150\sin 52.3856...°$

$h = 118.820...$

$h = 119$ (m) ***A1***

[6 marks]

METHOD 2

attempts to find either the distance between the buildings or the difference in height between the buildings in terms of θ ***(M1)***

distance between the buildings is $(150 + 90)\cos\theta$ and the difference in height between the buildings is $(150 - 90)\sin\theta$ ***(A1)***

uses Pythagoras and attempts to solve for θ ***(M1)***

$(60\sin\theta)^2 + (240\cos\theta)^2 = 154^2$

$\theta = 0.914302... (= 52.3856...°)$ ***(A1)***

$\frac{h}{150} = \sin(0.914302...)$ ***(M1)***

$h = 118.820...$

$h = 119$ (m) ***A1***

[6 marks]

6. (a) **METHOD 1**

the general point on L has coordinates $(\lambda, 2-2\lambda, 4-2\lambda)$

substitutes this general point into both Π_1 and Π_2 ***(M1)***

$2\lambda-(2-2\lambda)+2(4-2\lambda)(=2\lambda-2+2\lambda+8-4\lambda)$ ***A1***

$=6$ ***AG***

$4\lambda+3(2-2\lambda)-(4-2\lambda)(=4\lambda+6-6\lambda-4+2\lambda)$ ***A1***

$=2$ ***AG***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Award ***(M1)A0A0*** for correct verification using a specific value of λ.

continued…

Question 6 continued

METHOD 2

substitutes $(0,2,4)$ into both Π_1 and Π_2 and shows that

$0-2+8=6$ and $0+6-4=2$ ***A1***

hence $(0,2,4)$ lies in both Π_1 and Π_2 ***AG***

EITHER

attempts to find $\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}4\\3\\-1\end{pmatrix}$ ***M1***

$=\begin{pmatrix}-5\\10\\10\end{pmatrix}$ ***A1***

OR

attempts to find $\begin{pmatrix}1\\-2\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\-2\end{pmatrix}\bullet\begin{pmatrix}4\\3\\-1\end{pmatrix}$ ***M1***

$(2+2-4)=0$ and $(4-6+2)=0$ ***A1***

THEN

(so $\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ is perpendicular to both normal vectors)

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Award ***M1*** for substituting $x=0$ (or $y=2$ or $z=4$) into Π_1 and Π_2 and solving simultaneously, for example, solving $-y+2z=6$ and $3y-z=2$. Award ***A1*** for $y=2$ and $z=4$, for example.

continued…

Question 6 continued

METHOD 3

attempts row reduction to obtain *eg*,

$x+\frac{z}{2}=2$ and $y-z=-2$ ***(M1)***

substitutes $x=\lambda$ into $x+\frac{z}{2}=2$, solves for z and obtains $z=4-2\lambda$ ***A1***

substitutes $z=4-2\lambda$ into $y-z=-2$, solves for y and obtains $y=2-2\lambda$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

METHOD 4

attempts to solve $2x-y+2z=6$ and $4x+3y-z=2$ ***(M1)***

for example, $x=\lambda, y=2-2\lambda, z=4-2\lambda$ ***A2***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Only award marks for convincing use of a GDC.

[3 marks]

continued…

Question 6 continued

(b) **EITHER**

the position vector for point P nearest to the origin is perpendicular to the direction of L

$\begin{pmatrix} \lambda \\ 2-2\lambda \\ 4-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 0$ ***(M1)***

$\lambda - 2(2-2\lambda) - 2(4-2\lambda) = 0$ ***(A1)***

$9\lambda - 12 = 0$ ***(A1)***

OR

let s be the distance from the origin to a point P on L, then

$s^2 = \lambda^2 + (2-2\lambda)^2 + (4-2\lambda)^2$ ***(A1)***

attempts to find λ such that $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 0$ ***(M1)***

either $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 18\lambda - 24 (= 0)$ or a graph of s^2 versus λ ***(A1)***

Note: Award as above for use of $s = \sqrt{\lambda^2 + (2-2\lambda)^2 + (4-2\lambda)^2}$.

THEN

$\lambda = \frac{4}{3}$ ***A1***

$\mathrm{P}\left(\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) (\mathrm{P}(1.33, -0.667, 1.33))$ ***A1***

[5 marks]

Total [8 marks]

10. (a) $7.8 = \frac{2\pi}{\text{period}}$ ***(M1)***

$\frac{2\pi}{7.8} = 0.805536...$

$\text{period} = 0.806 \left(= \frac{20\pi}{78} \right)$ ***A1***

[2 marks]

(b) **METHOD 1**

(i) $\text{amplitude} = \frac{\max - \min}{2}$ ***(M1)***

$\frac{1.8 - 1}{2}$

$a = -0.4$ ***A1***

(ii) $b = 1.4$ ***A1***

METHOD 2

attempt to form two simultaneous equations in a and b ***(M1)***

$H(0) = 1 \Rightarrow a + b = 1$, $H\left(\frac{\pi}{7.8} \right) = 1.8 \Rightarrow -a + b = 1.8$

$a = -0.4, b = 1.4$ ***A1A1***

[3 marks]

continued…

Question 10 continued

(c) **EITHER**

$\frac{5}{\text{period}} = 6.207... < 6\frac{1}{2}$ ***(A1)***

OR

consideration of number of maximums on graph in first 5 seconds ***(A1)***

OR

maximums when $t = 0.403, 1.21, 2.01, 2.82, 3.62, 4.43$ ***(A1)***

THEN

6 times ***A1***

[2 marks]

(d) recognizing that $H(t) = 1.5$ ***(M1)***

$-0.4\cos(7.8t) + 1.4 = 1.5$

$0.233779...$

$t = 0.234$ (seconds) ***A1***

[2 marks]

continued…

Question 10 continued

(e) finding second time height is 1.5 metres ***(M1)***

$t = 0.571757...$

in each period, height is greater than 1.5 metres for 0.337978… seconds ***(A1)***

Note: Award ***(M1)(A1)*** for total time $2.02787...$ seen.

multiplying their value by 6 and divide by 5 ***(M1)***

$\frac{0.337978... \times 6}{5}$ OR $\frac{2.02787...}{5}$

$= 0.405574...$

$\text{P(height is greater than 1.5 m)} = 0.406$ ***A1***

[4 marks]

Total [13 marks]

4. METHOD 1

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2-1^2}=\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha+\sin^2\alpha=1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C}=1-\left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C}=\frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $=\frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$=\frac{1}{2}\times 10\times\sqrt{6}\times\frac{\sqrt{24}}{5}$ ***(A1)***

$=12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \hat{\mathrm{BAC}}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use '$\text{Area} = \frac{1}{2} \text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm²) ***A1***

[6 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta+{}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta+{}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta \quad \textbf{A1}$$

$$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)+\mathrm{i}\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right) \quad \textbf{A1A1}$$

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos5\theta+\mathrm{i}\sin5\theta$ (***A1)***

equate imaginary parts: ***(M1)***

$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$ ***A1***

substitute $\cos^2\theta=1-\sin^2\theta$ ***(M1)***

$\sin5\theta=5\left(1-\sin^2\theta\right)^2\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$\sin5\theta=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ ***AG***

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \frac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \frac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\frac{20}{64}}$ ***A1***

$= \frac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5}, \sin\frac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\frac{5}{16}$ ***M1***

$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$ ***A1***

recognition that $\sin\frac{\pi}{5} = \sin\frac{4\pi}{5}$ and $\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$

$\sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]

4. METHOD 1

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2-1^2}=\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha+\sin^2\alpha=1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C}=1-\left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C}=\frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $=\frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$=\frac{1}{2}\times 10\times\sqrt{6}\times\frac{\sqrt{24}}{5}$ ***(A1)***

$=12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \hat{\mathrm{BAC}}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use '$\text{Area} = \frac{1}{2} \text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm²) ***A1***

[6 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta+{}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta+{}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta \quad \textbf{A1}$$

$$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)+\mathrm{i}\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right) \quad \textbf{A1A1}$$

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos5\theta+\mathrm{i}\sin5\theta$ (***A1)***

equate imaginary parts: ***(M1)***

$$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta \quad \textbf{A1}$$

substitute $\cos^2\theta=1-\sin^2\theta$ ***(M1)***

$$\sin5\theta=5\left(1-\sin^2\theta\right)^2\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta \quad \textbf{A1}$$

$$\sin5\theta=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta \quad \textbf{A1}$$

$$=16\sin^5\theta-20\sin^3\theta+5\sin\theta \quad \textbf{AG}$$

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \frac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \frac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\frac{20}{64}}$ ***A1***

$= \frac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5}, \sin\frac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\frac{5}{16}$ ***M1***

$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$ ***A1***

recognition that $\sin\frac{\pi}{5} = \sin\frac{4\pi}{5}$ and $\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$

$\sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]

1. (a) $\mathrm{BV} = \sqrt{(6-3)^2 + (8-4)^2 + (0-9)^2}$ ***(A1)***

$= 10.2956...$

$= 10.3\ (=\sqrt{106})$ ***A1***

Note: Award ***SC(A0)A1*** for $\mathrm{BV} = \begin{pmatrix} -3 \\ -4 \\ 9 \end{pmatrix}$ where a candidate has misinterpreted notation.

[2 marks]

(b) **METHOD 1**

$\mathrm{BV} = \mathrm{VC}$ AND $\mathrm{BC} = 8$ (seen anywhere) ***(A1)***

attempt to use the cosine rule on triangle BVC for any angle ***(M1)***

Note: Recognition must be shown in context either in terms of labelled sides or in side lengths.

$\cos \hat{\mathrm{BVC}} = \dfrac{10.2...^2 + 10.2...^2 - 8^2}{2 \times 10.2... \times 10.2...}$ OR

$8^2 = 10.2....^2 + 10.2...^2 - 2 \times 10.2... \times 10.2... \cos \hat{\mathrm{BVC}}$ ***(A1)***

$\hat{\mathrm{BVC}} = 0.798037...$

$\hat{\mathrm{BVC}} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\hat{\mathrm{BVC}} = 0.80$ (or 46°) (2sf).

METHOD 2

let M be the midpoint of BC

$\mathrm{BM} = 4$ (seen anywhere) ***(A1)***

attempt to use sine or cosine in triangle BMV or CMV ***(M1)***

$\arcsin \dfrac{4}{\sqrt{106}}$ OR $\dfrac{\pi}{2} - \arccos \dfrac{4}{\sqrt{106}}$ OR 0.399018 ***(A1)***

$\hat{\mathrm{BVC}} = 0.798037...$

$\hat{\mathrm{BVC}} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\hat{\mathrm{BVC}} = 0.80$ (or 46°) (2sf).

METHOD 3

$\overrightarrow{VC} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix}$ and $\overrightarrow{VB} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$ ***(A1)***

attempt to use the cosine formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$\cos B\hat{V}C = \dfrac{3(3)-4(4)-9(-9)}{\sqrt{3^2+(-4)^2+(-9)^2}\sqrt{3^2+4^2+(-9)^2}} \left(= \dfrac{74}{106}\right)$ ***(A1)***

$B\hat{V}C = 0.798037...$

$B\hat{V}C = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $B\hat{V}C = 0.80$ (or 46°) (2sf).

METHOD 4

$\overrightarrow{VC} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix}$ and $\overrightarrow{VB} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$ ***(A1)***

attempt to use the cross product formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$\sin B\hat{V}C = \dfrac{\left\| \begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix} \right\|}{\sqrt{3^2+(-4)^2+(-9)^2}\sqrt{3^2+4^2+(-9)^2}} \left(= \dfrac{\sqrt{5760}}{106}\right)$ ***(A1)***

$B\hat{V}C = 0.798037...$

$B\hat{V}C = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $B\hat{V}C = 0.80$ (or 46°) (2sf).

Award ***SC(A1)(M1)(A0)A0*** for $\text{area} = \dfrac{1}{2}\left\| \begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix} \right\| = \dfrac{\sqrt{5760}}{2}\left(= 37.9\right)$ where a candidate has misinterpreted notation.

[4 marks]

Total [6 marks]

3. (a) use of sector area formula to find area of at least one sector ***(M1)***

$\frac{1}{2}\times 5.2\times 100-\frac{1}{2}\times 5.2\times r^2$ OR $10^2\pi-\frac{1}{2}10^2\times(2\pi-5.2)-\left(\pi r^2-\frac{1}{2}\times(2\pi-5.2)\times r^2\right)$ ***A1***

$(\text{area})=260-2.6r^2$ ***AG***

Note: There are many different ways to find the area of the "C". In all methods, the ***A*** mark is awarded for working which leads directly to the ***AG***.
Many candidates are working with rounded intermediate values. Award the ***A*** mark to correct work with values that round to the 3sf value of 260 and the 2sf value of 2.6 eg $259.99-2.6015r^2$.

[2 marks]

(b) (i) $260-2.6r^2=64$ ***(A1)***

$r=8.68243...$

$=8.68\,(\text{cm})\left(\frac{14\sqrt{65}}{13}\text{ exact}\right)$ ***A1***

(ii) 10×5.2 OR $8.68...\times 5.2$ ***(A1)***

substituting their value of r into $10\times 5.2+r\times 5.2+2(10-r)$ (or equivalent) ***(M1)***

Perimeter = $10\times 5.2+8.68...\times 5.2+2(10-8.68...)$ $(=52+45.1486...+2.63513...)$

= $99.7837\ldots$

= 99.8 (cm) ***A1***

[5 marks]

Total [7 marks]

8. (a) $\overrightarrow{AB} = \begin{pmatrix} 1-p \\ -1 \end{pmatrix}$ ***A1***

$\overrightarrow{AC} = \begin{pmatrix} p \\ -p \\ 2 \end{pmatrix}$ ***A1***

attempt to evaluate their $\overrightarrow{AB} \times \overrightarrow{AC}$ by use of formula or determinant ***M1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2(1-p)-p \\ -(2+p) \\ -p-p(1-p) \end{pmatrix}$ OR $(2(1-p)-p)\boldsymbol{i} - (2+p)\boldsymbol{j} + (-p-p(1-p))\boldsymbol{k}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$ ***AG***

[4 marks]

(b) $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$

$= (2-3p)^2 + (-2-p)^2 + (p^2-2p)^2 \left(= p^4 - 4p^3 + 14p^2 - 8p + 8\right)$ ***(A1)***

attempt to find minimum of their $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$ ***(M1)***

$6.75257...$ OR $p = 0.3264...$

min value is 6.75 ***A1***

[3 marks]

continued…

Question 8 continued

(c) **METHOD 1**

valid attempt to find $\text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ using their answer to part b) ***(M1)***

$\text{area} = \frac{1}{2}\sqrt{6.75257...}$

$= 1.299285...$

$= 1.30$ (units^2) ***A1***

[2 marks]

Total [9 marks]

12. (a) attempt to set at least two components of L and M equal ***M1***

$1+2s=9+4t$

$2+3s=9+t$

$-3+6s=11+2t$

attempt to solve two of their equations simultaneously ***(M1)***

$s=2$ OR $t=-1$ ***A1***

EITHER

substitute $s=2$ and $t=-1$ into remaining component e.g. $-3+6(2)=11+2(-1)$ ***R1***

OR

recognition that 2nd and 3rd equations are equivalent ***R1***

THEN

position vector of A is $\begin{pmatrix}5\\8\\9\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.
The final ***A1*** is independent of ***R1***.

[5 marks]

(b) **METHOD 1**

attempt to substitute at least one line into the equation of the plane ***(M1)***

$\begin{pmatrix}1+2s\\2+3s\\-3+6s\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(2+3s)-1(-3+6s)=7$ ***A1***

$\begin{pmatrix}9+4t\\9+t\\11+2t\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(9+t)-1(11+2t)=7$ ***A1***

continued…

Question 12 continued

METHOD 2

consideration the direction of one line and a point on that line ***(M1)***

direction $\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

direction $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

METHOD 3

consideration of direction of both lines ***(M1)***

EITHER

$\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ and $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ (hence L and M are parallel to the plane) ***A1***

OR

$\begin{pmatrix}2\\3\\6\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}0\\20\\-10\end{pmatrix}=k\begin{pmatrix}0\\2\\-1\end{pmatrix}$ (hence L and M are parallel to the plane) ***A1***

THEN

$\begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ ***A1***

[3 marks]

continued…

Question 12 continued

(c) (i) position vector of point on the line is $\left(\boldsymbol{r} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \left(= \begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} \right)\right)$ ***(A1)***

attempt to substitute position vector into equation of plane Π ***(M1)***

meets Π when $\begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$

$2(12+2\lambda) - (2-\lambda) = 7$

$22 + 5\lambda = 7$

$\lambda = -3$ ***(A1)***

position vector of $\left(\boldsymbol{r} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \right) \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.

(ii) **METHOD 1**

attempt to find $\overrightarrow{BC}$ using $\overrightarrow{OC} - \overrightarrow{OB}$ ***(M1)***

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$

attempt to use distance formula to find $\left|\overrightarrow{BC}\right|$ ***(M1)***

$\left|\overrightarrow{BC}\right| = \sqrt{(-6)^2 + 3^2}$

$= 6.71 \left(= \sqrt{45} = 3\sqrt{5} \right)$ ***A1***

METHOD 2

recognition that $\left|\overrightarrow{\mathrm{BC}}\right| = 3\times\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|$ ***(M1)***

attempt to use distance formula to find $\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|$ ***(M1)***

$\left|\overrightarrow{\mathrm{BC}}\right| = 3\sqrt{2^2+(-1)^2}$

$= 6.71\left(=\sqrt{45}=3\sqrt{5}\right)$ ***A1***

[7 marks]

continued…

Question 12 continued

(d) let B' be the image of B

METHOD 1

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ ***(A1)***

recognition that $\mu = 2\lambda \left(= -6\right)$ OR $\left|\mathrm{BC}\right| = \left|\mathrm{CB}'\right|$ (may be seen in a diagram) ***(M1)***

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$

so coordinates are $\mathrm{B}'\left(-3, 0, 8\right)$ ***A1***

METHOD 2

$\overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{CB}'} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ ***(A1)***

Note: This may come from $\overrightarrow{\mathrm{BC}} = -3\sqrt{5}\boldsymbol{n}$ using the unit normal vector $\boldsymbol{n} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{CB}'}$ OR $\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OB}} + 2\overrightarrow{\mathrm{BC}}$ OR $\overrightarrow{\mathrm{OB}'} = 2\overrightarrow{\mathrm{OC}} - \overrightarrow{\mathrm{OB}}$ ***(M1)***

$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -6 \\ 12 \\ 10 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$ (so coordinates are $\mathrm{B}'\left(-3, 0, 8\right)$) ***A1***

[3 marks]

Total [18 marks]

1. (a) $\mathrm{BV} = \sqrt{(6-3)^2 + (8-4)^2 + (0-9)^2}$ ***(A1)***

$= 10.2956...$

$= 10.3\ (=\sqrt{106})$ ***A1***

Note: Award ***SC(A0)A1*** for $\mathrm{BV} = \begin{pmatrix} -3 \\ -4 \\ 9 \end{pmatrix}$ where a candidate has misinterpreted notation.

[2 marks]

(b) **METHOD 1**

$\mathrm{BV} = \mathrm{VC}$ AND $\mathrm{BC} = 8$ (seen anywhere) ***(A1)***

attempt to use the cosine rule on triangle BVC for any angle ***(M1)***

Note: Recognition must be shown in context either in terms of labelled sides or in side lengths.

$\cos \hat{\mathrm{BVC}} = \dfrac{10.2...^2 + 10.2...^2 - 8^2}{2 \times 10.2... \times 10.2...}$ OR

$8^2 = 10.2....^2 + 10.2...^2 - 2 \times 10.2... \times 10.2... \cos \hat{\mathrm{BVC}}$ ***(A1)***

$\hat{\mathrm{BVC}} = 0.798037...$

$\hat{\mathrm{BVC}} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\hat{\mathrm{BVC}} = 0.80$ (or 46°) (2sf).

METHOD 2

let M be the midpoint of BC

$\mathrm{BM} = 4$ (seen anywhere) ***(A1)***

attempt to use sine or cosine in triangle BMV or CMV ***(M1)***

$\arcsin \dfrac{4}{\sqrt{106}}$ OR $\dfrac{\pi}{2} - \arccos \dfrac{4}{\sqrt{106}}$ OR 0.399018 ***(A1)***

$\hat{\mathrm{BVC}} = 0.798037...$

$\hat{\mathrm{BVC}} = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\hat{\mathrm{BVC}} = 0.80$ (or 46°) (2sf).

METHOD 3

$\overrightarrow{VC} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix}$ and $\overrightarrow{VB} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$ ***(A1)***

attempt to use the cosine formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$\cos B\hat{V}C = \dfrac{3(3)-4(4)-9(-9)}{\sqrt{3^2+(-4)^2+(-9)^2}\sqrt{3^2+4^2+(-9)^2}}\left(=\dfrac{74}{106}\right)$ ***(A1)***

$B\hat{V}C = 0.798037...$

$B\hat{V}C = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $B\hat{V}C = 0.80$ (or 46°) (2sf).

METHOD 4

$\overrightarrow{VC} = \begin{pmatrix} 3 \\ -4 \\ -9 \end{pmatrix}$ and $\overrightarrow{VB} = \begin{pmatrix} 3 \\ 4 \\ -9 \end{pmatrix}$ ***(A1)***

attempt to use the cross product formula for an angle using two vectors representing the edges of triangle BVC ***(M1)***

$\sin B\hat{V}C = \dfrac{\left\|\begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix}\right\|}{\sqrt{3^2+(-4)^2+(-9)^2}\sqrt{3^2+4^2+(-9)^2}}\left(=\dfrac{\sqrt{5760}}{106}\right)$ ***(A1)***

$B\hat{V}C = 0.798037...$

$B\hat{V}C = 0.798$ (accept 45.7°) ***A1***

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $B\hat{V}C = 0.80$ (or 46°) (2sf).

Award ***SC(A1)(M1)(A0)A0*** for $\text{area} = \dfrac{1}{2}\left\|\begin{pmatrix} 72 \\ 0 \\ 24 \end{pmatrix}\right\| = \dfrac{\sqrt{5760}}{2}\left(=37.9\right)$ where a candidate has misinterpreted notation.

[4 marks]

Total [6 marks]

3. (a) use of sector area formula to find area of at least one sector ***(M1)***

$\frac{1}{2}\times 5.2\times 100-\frac{1}{2}\times 5.2\times r^2$ OR $10^2\pi-\frac{1}{2}10^2\times(2\pi-5.2)-\left(\pi r^2-\frac{1}{2}\times(2\pi-5.2)\times r^2\right)$ ***A1***

$(\text{area})=260-2.6r^2$ ***AG***

Note: There are many different ways to find the area of the "C". In all methods, the ***A*** mark is awarded for working which leads directly to the ***AG***.
Many candidates are working with rounded intermediate values. Award the ***A*** mark to correct work with values that round to the 3sf value of 260 and the 2sf value of 2.6 eg $259.99-2.6015r^2$.

[2 marks]

(b) (i) $260-2.6r^2=64$ ***(A1)***

$r=8.68243...$

$=8.68\,(\text{cm})\left(\frac{14\sqrt{65}}{13}\text{ exact}\right)$ ***A1***

(ii) 10×5.2 OR $8.68...\times 5.2$ ***(A1)***

substituting their value of r into $10\times 5.2+r\times 5.2+2(10-r)$ (or equivalent) ***(M1)***

Perimeter = $10\times 5.2+8.68...\times 5.2+2(10-8.68...)$ $(=52+45.1486...+2.63513...)$

= $99.7837\ldots$

= 99.8 (cm) ***A1***

[5 marks]

Total [7 marks]

8. (a) $\overrightarrow{AB} = \begin{pmatrix} 1-p \\ -1 \end{pmatrix}$ ***A1***

$\overrightarrow{AC} = \begin{pmatrix} p \\ -p \\ 2 \end{pmatrix}$ ***A1***

attempt to evaluate their $\overrightarrow{AB} \times \overrightarrow{AC}$ by use of formula or determinant ***M1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2(1-p)-p \\ -(2+p) \\ -p-p(1-p) \end{pmatrix}$ OR $(2(1-p)-p)\boldsymbol{i} - (2+p)\boldsymbol{j} + (-p-p(1-p))\boldsymbol{k}$ ***A1***

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2-3p \\ -2-p \\ p^2-2p \end{pmatrix}$ ***AG***

[4 marks]

(b) $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$

$= (2-3p)^2 + (-2-p)^2 + (p^2-2p)^2 \left(= p^4 - 4p^3 + 14p^2 - 8p + 8\right)$ ***(A1)***

attempt to find minimum of their $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|^2$ ***(M1)***

$6.75257...$ OR $p = 0.3264...$

min value is 6.75 ***A1***

[3 marks]

continued…

Question 8 continued

(c) **METHOD 1**

valid attempt to find $\text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ using their answer to part b) ***(M1)***

$\text{area} = \frac{1}{2}\sqrt{6.75257...}$

$= 1.299285...$

$= 1.30$ (units^2) ***A1***

[2 marks]

Total [9 marks]

12. (a) attempt to set at least two components of L and M equal ***M1***

$1+2s=9+4t$

$2+3s=9+t$

$-3+6s=11+2t$

attempt to solve two of their equations simultaneously ***(M1)***

$s=2$ OR $t=-1$ ***A1***

EITHER

substitute $s=2$ and $t=-1$ into remaining component e.g. $-3+6(2)=11+2(-1)$ ***R1***

OR

recognition that 2nd and 3rd equations are equivalent ***R1***

THEN

position vector of A is $\begin{pmatrix}5\\8\\9\end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.
The final ***A1*** is independent of ***R1***.

[5 marks]

(b) **METHOD 1**

attempt to substitute at least one line into the equation of the plane ***(M1)***

$\begin{pmatrix}1+2s\\2+3s\\-3+6s\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(2+3s)-1(-3+6s)=7$ ***A1***

$\begin{pmatrix}9+4t\\9+t\\11+2t\end{pmatrix}\bullet\begin{pmatrix}0\\2\\-1\end{pmatrix}=2(9+t)-1(11+2t)=7$ ***A1***

continued…

Question 12 continued

METHOD 2

consideration the direction of one line and a point on that line ***(M1)***

direction $\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

direction $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ AND point $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ $\left(\text{or } \begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7\right)$ ***A1***

METHOD 3

consideration of direction of both lines ***(M1)***

EITHER

$\begin{pmatrix}2\\3\\6\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ and $\begin{pmatrix}4\\1\\2\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=0$ (hence L and M are parallel to the plane) ***A1***

OR

$\begin{pmatrix}2\\3\\6\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}0\\20\\-10\end{pmatrix}=k\begin{pmatrix}0\\2\\-1\end{pmatrix}$ (hence L and M are parallel to the plane) ***A1***

THEN

$\begin{pmatrix}5\\8\\9\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}1\\2\\-3\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ OR $\begin{pmatrix}9\\9\\11\end{pmatrix}.\begin{pmatrix}0\\2\\-1\end{pmatrix}=7$ ***A1***

[3 marks]

continued…

Question 12 continued

(c) (i) position vector of point on the line is $\left(\boldsymbol{r} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \left(= \begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} \right)\right)$ ***(A1)***

attempt to substitute position vector into equation of plane Π ***(M1)***

meets Π when $\begin{pmatrix} -3 \\ 12+2\lambda \\ 2-\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 7$

$2(12+2\lambda) - (2-\lambda) = 7$

$22 + 5\lambda = 7$

$\lambda = -3$ ***(A1)***

position vector of $\left(\boldsymbol{r} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \right) \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$ ***A1***

Note: Accept a row vector and/or coordinates.

(ii) **METHOD 1**

attempt to find $\overrightarrow{BC}$ using $\overrightarrow{OC} - \overrightarrow{OB}$ ***(M1)***

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$

attempt to use distance formula to find $\left|\overrightarrow{BC}\right|$ ***(M1)***

$\left|\overrightarrow{BC}\right| = \sqrt{(-6)^2 + 3^2}$

$= 6.71 \left(= \sqrt{45} = 3\sqrt{5} \right)$ ***A1***

METHOD 2

recognition that $\left|\overrightarrow{\mathrm{BC}}\right| = 3\times\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|$ ***(M1)***

attempt to use distance formula to find $\left|\begin{pmatrix}0\\2\\-1\end{pmatrix}\right|$ ***(M1)***

$\left|\overrightarrow{\mathrm{BC}}\right| = 3\sqrt{2^2+(-1)^2}$

$= 6.71\left(=\sqrt{45}=3\sqrt{5}\right)$ ***A1***

[7 marks]

continued…

Question 12 continued

(d) let B' be the image of B

METHOD 1

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ ***(A1)***

recognition that $\mu = 2\lambda \left(= -6\right)$ OR $\left|\mathrm{BC}\right| = \left|\mathrm{CB}'\right|$ (may be seen in a diagram) ***(M1)***

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$

so coordinates are $\mathrm{B}'\left(-3, 0, 8\right)$ ***A1***

METHOD 2

$\overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{CB}'} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ ***(A1)***

Note: This may come from $\overrightarrow{\mathrm{BC}} = -3\sqrt{5}\boldsymbol{n}$ using the unit normal vector $\boldsymbol{n} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OC}} + \overrightarrow{\mathrm{CB}'}$ OR $\overrightarrow{\mathrm{OB}'} = \overrightarrow{\mathrm{OB}} + 2\overrightarrow{\mathrm{BC}}$ OR $\overrightarrow{\mathrm{OB}'} = 2\overrightarrow{\mathrm{OC}} - \overrightarrow{\mathrm{OB}}$ ***(M1)***

$= \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$ OR $\begin{pmatrix} -6 \\ 12 \\ 10 \end{pmatrix} - \begin{pmatrix} -3 \\ 12 \\ 2 \end{pmatrix}$

$\overrightarrow{\mathrm{OB}'} = \begin{pmatrix} -3 \\ 0 \\ 8 \end{pmatrix}$ (so coordinates are $\mathrm{B}'\left(-3, 0, 8\right)$) ***A1***

[3 marks]

Total [18 marks]

3. (a) $2r + r\theta = 10$ ***A1***

$\frac{1}{2}r^2\theta = 6.25$ ***A1***

attempt to eliminate θ to obtain an equation in r ***M1***

correct intermediate equation in r ***A1***

$10 - 2r = \frac{25}{2r}$ OR $\frac{10}{r} - 2 = \frac{25}{2r^2}$ OR $\frac{1}{2}r^2\left(\frac{10}{r} - 2\right) = 6.25$ OR $12.5 + 2r^2 = 10r$

$4r^2 - 20r + 25 = 0$ ***AG***

[4 marks]

(b) attempt to solve quadratic by factorizing or use of formula or completing the square ***(M1)***

$(2r-5)^2 = 0$ OR $r = \frac{20 \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}\left(= \frac{20 \pm \sqrt{400-400}}{8}\right)$

$r = \frac{5}{2}$ ***A1***

attempt to substitute their value of r into their perimeter or area equation ***(M1)***

$\theta = \frac{10 - 2\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)}$ or $\theta = \frac{25}{2\left(\frac{5}{2}\right)^2}$

$\theta = 2$ ***A1***

[4 marks]

Total [8 marks]

1. $\tan^{-1}1 = 45^\circ$ or equivalent ***(A1)***

attempt to equate $2x - 5^\circ$ to their reference angle ***(M1)***

Note: Do not accept $2x - 5° = 1$.

$2x - 5^\circ = 45^\circ, \left(225^\circ\right)$

$x = 25^\circ, 115^\circ$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.

[4 marks]

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11.

(a) $2\times 2+6\times\frac{1}{2}-2\times 1=5$ ***A1***

[1 mark]

(b) $2\left(k^2-6\right)+6\left(2k+3\right)+12\left(=0\right)$ ***(A1)***

equating their scalar product of the direction normals to zero ***(M1)***

$2\left(k^2-6\right)+6\left(2k+3\right)+12=0$

$k^2-6+6k+9+6=0$ OR $\left(k+3\right)^2=0$

$k=-3$ ***A1***

attempt to substitute k, p and coordinates of A into Π_2 ***(M1)***

$q=3\times 2-3\times\frac{1}{2}-6\times 1$

$q=-\frac{3}{2}$ ***A1***

[5 marks]

continued…

Question 11 continued.

(c) attempt to equate a pair of ratios or equate vector product to zero vector ***(M1)***

$$\begin{pmatrix}2\\6\\-2\end{pmatrix}=\mu\begin{pmatrix}3\\9\\p\end{pmatrix}\Rightarrow\mu=\frac{2}{3}\text{ OR }\frac{2}{-2}=\frac{3}{p}\text{ OR }\frac{6}{-2}=\frac{9}{p}\text{ OR }6p+18=0\text{ OR }-6-2p=0$$

$p=-3$ ***A1***

[2 marks]

(d) (i) attempt to find the vector equation of the line through A perpendicular to Π_1 ***(M1)***

$$(\boldsymbol{r}=)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\\frac{1}{2}\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\6\\-2\end{pmatrix}$$

attempt to substitute their vector equation into Π_2 ***(M1)***

$$3\left(2+2\lambda\right)+9\left(\frac{1}{2}+6\lambda\right)-3\left(1-2\lambda\right)\left(=-\frac{51}{2}\right)$$

$6+6\lambda+\frac{9}{2}+54\lambda-3+6\lambda=-\frac{51}{2}$ (or equivalent) ***(A1)***

$66\lambda=-\frac{51}{2}-6-\frac{9}{2}+3$ (or equivalent)

$\lambda=-\frac{1}{2}$ ***A1***

$\left(1,-\frac{5}{2},2\right)$ ***A1***

continued…

Question 11 continued.

(ii) distance AB or $\left|\lambda\begin{pmatrix}2\\6\\-2\end{pmatrix}\right|$ ***(M1)***

$\sqrt{(2-1)^2+\left(\frac{1}{2}+\frac{5}{2}\right)^2+(1-2)^2}$ OR $\sqrt{1+9+1}$ ***A1***

$=\sqrt{11}$ ***AG***

[7 marks]

(e) Valid method to find a point C on Π_3 using $\overrightarrow{\mathrm{AC}}=\overrightarrow{\mathrm{BA}}$ or $\overrightarrow{\mathrm{BC}}=2\overrightarrow{\mathrm{BA}}$ or A as the midpoint of BC. ***(M1)***

$\lambda=\frac{1}{2}$, $\overrightarrow{\mathrm{OC}}=\begin{pmatrix}2\\\frac{1}{2}\\1\end{pmatrix}+\begin{pmatrix}1\\3\\-1\end{pmatrix}$, $\overrightarrow{\mathrm{OC}}=\begin{pmatrix}1\\-\frac{5}{2}\\2\end{pmatrix}+\begin{pmatrix}2\\6\\-2\end{pmatrix}$, $\overrightarrow{\mathrm{OA}}=\frac{1}{2}\left(\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OC}}\right)$

point on Π_3 is $\left(3,\frac{7}{2},0\right)$ ***A1***

attempt to substitute their $\left(3,\frac{7}{2},0\right)$ into $\Pi_3: x+3y-z=d$ (or equivalent) ***(M1)***

$1\times3+3\times\frac{7}{2}-1\times0\left(=\frac{27}{2}\right)$

$\Pi_3: x+3y-z=\frac{27}{2}$ $\left(2x+6y-2z=27\right)$ (or equivalent) ***A1***

[4 marks]

Total [19 marks]

2. (a) attempt to use trigonometry to find the radius of the cone OR Oliver's distance from centre $(r+5)$ ***(M1)***

$\tan 58^\circ = \frac{18.2}{r+5}$ OR $\frac{r+5}{\sin 32^\circ} = \frac{18.2}{\sin 58^\circ}$ OR $(r+5=)\,11.3726...$ ***(A1)***

$r = 6.37262...$ (m)

$(r=)\,6.37$ (m) ***A1***

[3 marks]

(b) attempt to substitute $h = 20$ and their radius into the correct volume of cone formula ***(M1)***

$$V = \frac{\pi(6.37262...)^2(20)}{3}$$

$$= 850.540\ldots$$

$$= 851\ (\text{m}^3)$$ ***A1***

Note: Accept $849.840\ldots\ (850)$ obtained from using $r = 6.37$.

[2 marks]

Total [5 marks]

10. (a) **EITHER**

attempt to find value of t for the first low tide OR the first high tide ***(M1)***

$11.2619\ldots - 5.13801\ldots$

$= 6.12396\ldots$ ***(A1)***

OR

attempt to find half of the period ***(M1)***

$\frac{1}{2} \times \frac{2\pi}{0.513}$

$= 6.12396\ldots$ ***(A1)***

THEN

$m = (6.12396\ldots - 6) \times 60 = 7.43773...$

$m = 7$ ***A1***

[3 marks]

(b) attempt to solve $H(t) = 1$ ***(M1)***

$3.56919\ldots$ OR $6.70684\ldots$ OR $15.8171\ldots$ OR $18.9547\ldots$

$(6.70684\ldots - 3.56919\ldots =)\ 3.13764...$

$= 3.14$ (hours) ***A1***

[2 marks]

(c) recognition that $H'(13)$ is required ***(M1)***

$= -0.650622\ldots$

$= -0.651$ (m/h) ***A1***

[2 marks]

continued…

Question 10 continued

(d)

Note: In part (d), award the marks for a, b, c and d independent of each other.

METHOD 1

$a = 1.17$ ***A1***

$d = 1.57$ ***A1***

attempt to find time between low and high tide in hours ***(M1)***

6 hours and 21 minutes = 6.35 hours

(period =) 12.7 ***(A1)***

$b = \frac{2\pi}{12.7} = 0.494739\ldots$

$b = 0.495 \left(= \frac{60\pi}{381} \right)$ ***A1***

attempt to find mean of low and high tide times OR substitute values of a known point ***(M1)***

$c = \frac{1}{2}\left(2\frac{41}{60} + 9\frac{2}{60} \right)$ OR eg $0.40 = 1.17\sin\left(0.495\left(2.68333... - c\right)\right) + 1.57$

$c = 5.85833...$

$c = 5.86$ ***A1***

Note: Award ***(M1)A1*** for $c = 18.6$.
Award ***(M1)A0*** for $c = -6.84$.

[7 marks]

continued…

Question 10 continued

METHOD 2

$a = 1.17$ ***A1***

$d = 1.57$ ***A1***

substituting at least one point into $h(t)$ ***(M1)***

$1.17\sin\left(b\left(2\frac{41}{60} - c\right)\right) + 1.57 = 0.4$ OR $1.17\sin\left(b\left(9\frac{2}{60} - c\right)\right) + 1.57 = 2.74$

$b\left(2\frac{41}{60} - c\right) = -\frac{\pi}{2}\left(= -1.57\right)$ **AND** $b\left(9\frac{2}{60} - c\right) = \frac{\pi}{2}\left(= 1.57\right)$ ***(A1)***

Note: accept any angles of the form $-\frac{\pi}{2} + c\pi k$ and $\frac{\pi}{2} + c\pi k$.

EITHER

use of graph or table to find their intersection ***(M1)***

OR

attempt to solve their equations simultaneously ***(M1)***

$$\frac{2\frac{41}{60} - c}{9\frac{2}{60} - c} = -1$$

THEN

$c = 5.85833...$

$c = 5.86$ ***A1***

$b = 0.494739...$

$b = 0.495$ ***A1***

[7 marks]

(e) attempt to find point of intersection of two graphs ***(M1)***

$T = 4.16292\ldots$ OR $T = 4.16417\ldots$ (using 3 sf)

$T = 4.16$ ***A1***

[2 marks]

Total [16 marks]

12. (a) Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$\left(\boldsymbol{u}\cdot\boldsymbol{v}\right)^2+\left|\boldsymbol{u}\times\boldsymbol{v}\right|^2$

$=\left(\left|\boldsymbol{u}\right|\left|\boldsymbol{v}\right|\cos\theta\right)^2+\left(\left|\boldsymbol{u}\right|\left|\boldsymbol{v}\right|\sin\theta\right)^2$ OR $\left|\boldsymbol{u}\right|^2\left|\boldsymbol{v}\right|^2\cos^2\theta+\left|\boldsymbol{u}\right|^2\left|\boldsymbol{v}\right|^2\sin^2\theta$ ***(A1)***

$=\left|\boldsymbol{u}\right|^2\left|\boldsymbol{v}\right|^2\left(\cos^2\theta+\sin^2\theta\right)$ ***A1***

$=\left|\boldsymbol{u}\right|^2\left|\boldsymbol{v}\right|^2$ ***AG***

[2 marks]

(b) (i) $\left|\boldsymbol{u}\times\boldsymbol{v}\right|=2\sqrt{6}\left(=4.89897...=4.90\right)$ ***A1***

(ii) $\boldsymbol{v}=\begin{pmatrix}3\\1\\-1\end{pmatrix}$ ***(A1)***

$\left|\boldsymbol{v}\right|=\sqrt{3^2+1^2+\left(-1\right)^2}\ \left(=\sqrt{11}=3.31662\ldots\right)$ ***(A1)***

substitution of values $\boldsymbol{u}\cdot\boldsymbol{v},\left|\boldsymbol{u}\times\boldsymbol{v}\right|$ and $\left|\boldsymbol{v}\right|$ into $\left(\boldsymbol{u}\cdot\boldsymbol{v}\right)^2+\left|\boldsymbol{u}\times\boldsymbol{v}\right|^2=\left|\boldsymbol{u}\right|^2\left|\boldsymbol{v}\right|^2$ ***M1***

$3^2+\left(2\sqrt{6}\right)^2=\left|\boldsymbol{u}\right|^2\ \left|\sqrt{11}\right|^2$

$\left|\boldsymbol{u}\right|=\sqrt{3}\left(=1.73205...=1.73\right)$ ***A1***

continued…

Question 12 continued

(iii) $\boldsymbol{u} = \begin{pmatrix} p \\ q-1 \\ 1 \end{pmatrix}$ ***(A1)***

attempt to use $\boldsymbol{u} \cdot \boldsymbol{v} = 3$ ***(M1)***

$$\begin{pmatrix} p \\ q-1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 3$$

$3p + q = 5 \left(\Rightarrow q = 5 - 3p \right)$ ***A1***

attempt to use $|\boldsymbol{u}| = \sqrt{3}$ ***(M1)***

$p^2 + (q-1)^2 + 1^2 = \sqrt{3}^2 \left(\Rightarrow p^2 + q^2 - 2q + 1 + 1 = 3 \right)$

Note: Award ***M1*** for use of $|\boldsymbol{u} \times \boldsymbol{v}|^2 = q^2 + (p+3)^2 + (p - 3q + 3)^2 \left(= \left(2\sqrt{6} \right)^2 \right)$.

attempt to form quadratic in one variable, p or q ***(M1)***

$p^2 + (4 - 3p)^2 = 2$ OR $p^2 + (5 - 3p)^2 - 2p = 1$ OR $10p^2 - 24p + 14 = 0$ OR

$\left(\frac{5-q}{3} \right)^2 + (q-1)^2 = 2$ OR $\left(\frac{5-q}{3} \right)^2 + q^2 - 2q = 1$ OR $10q^2 - 28q + 16 = 0$ ***(A1)***

$p = 1$ or $p = 1.4 \left(= \frac{7}{5} \right)$ ***A1***

$q = 2$ or $q = 0.8 \left(= \frac{4}{5} \right)$ ***A1***

Note: Award final ***A1*** marks for correct values, even if the p values and q values are not explicitly paired.

[13 marks]

continued…

Question 12 continued

(c) **METHOD 1**

attempt to express $\boldsymbol{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of one variable ***(M1)***

$\boldsymbol{v} \cdot \boldsymbol{w} = 3x + y - z = 0,\ \boldsymbol{u} \cdot \boldsymbol{w} = x + y + z = 0$

$y = -2x$ and $z = x$ OR $x = z = -\frac{1}{2}y$ OR $x = z$ and $y = -2z$ ***(A1)***

attempt to use area of a triangle = $\frac{\text{base} \times \text{height}}{2} = \frac{|\boldsymbol{w}||\boldsymbol{v}|}{2}$ ***(M1)***

$\frac{\sqrt{x^2 + y^2 + z^2}\sqrt{11}}{2} = 5$

$6x^2 = \frac{100}{11}$ or $\frac{3y^2}{2} = \frac{100}{11}$ or $6z^2 = \frac{100}{11}$

$x = \pm 1.2309\ldots \left(= \pm \frac{5\sqrt{66}}{33} \right)$ OR $y = \mp 2.4618\ldots \left(\mp \frac{10\sqrt{66}}{33} \right)$ OR

$z = \pm 1.2309\ldots \left(= \pm \frac{5\sqrt{66}}{33} \right)$ ***A1***

$\boldsymbol{w} = \pm \begin{pmatrix} 1.23091\ldots \\ -2.46182\ldots \\ 1.23091\ldots \end{pmatrix}$

$\boldsymbol{w} = \pm \begin{pmatrix} 1.23 \\ -2.46 \\ 1.23 \end{pmatrix}$ or $\boldsymbol{w} = \pm \frac{5}{\sqrt{66}} \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$ ***A1***

Note: If no working shown, award ***M1A1(M1)A1A0*** for $\boldsymbol{w} = \pm \begin{pmatrix} 1.2 \\ -2.5 \\ 1.2 \end{pmatrix}$ (2 sf)

continued…

Question 12 continued

METHOD 2

attempt to write $\boldsymbol{u}\times\boldsymbol{v}$ as a multiple of $\boldsymbol{w}$ or recognizing that $\boldsymbol{w}$ is normal to $\boldsymbol{u}$ and $\boldsymbol{v}$ ***(M1)***

$\boldsymbol{w}=\boldsymbol{u}\times\boldsymbol{v}=\lambda\begin{pmatrix}-2\\4\\-2\end{pmatrix}\left(=\begin{pmatrix}-2\lambda\\4\lambda\\-2\lambda\end{pmatrix}\right)$ ***(A1)***

attempt to use area of a triangle = $\frac{\text{base}\times\text{height}}{2}$ OR area = $\frac{1}{2}|\boldsymbol{v}\times\boldsymbol{w}|$ ***(M1)***

$\frac{\sqrt{(-2\lambda)^2+(4\lambda)^2+(-2\lambda)^2}\sqrt{11}}{2}=5$ OR $\frac{1}{2}|\boldsymbol{v}\times\boldsymbol{w}|=\lambda\begin{pmatrix}1\\4\\7\end{pmatrix}=5$

$(-2\lambda)^2+(4\lambda)^2+(-2\lambda)^2=\frac{100}{11}$ OR $\lambda^2+(4\lambda)^2+(7\lambda)^2=25$

$24\lambda^2=\frac{100}{11}$ OR $66\lambda^2=25$

$\lambda=\pm0.61545\ldots\left(=\pm\frac{5\sqrt{66}}{66}\right)$ ***A1***

$\boldsymbol{w}=\pm\begin{pmatrix}1.23091\ldots\\-2.46182\ldots\\1.23091\ldots\end{pmatrix}$

$\boldsymbol{w}=\pm\begin{pmatrix}1.23\\-2.46\\1.23\end{pmatrix}$ or $\boldsymbol{w}=\pm\frac{5}{\sqrt{66}}\begin{pmatrix}-2\\4\\-2\end{pmatrix}$ ***A1***

Note: If no working shown, award ***M1A1(M1)A1A0*** for $\boldsymbol{w}=\pm\begin{pmatrix}1.2\\-2.5\\1.2\end{pmatrix}$ (2 sf)

continued…

Question 12 continued

METHOD 3

recognising $\frac{1}{2}|\boldsymbol{u}\times\boldsymbol{w}| = 5\times\frac{|\boldsymbol{u}|}{|\boldsymbol{v}|}\ \left(=\frac{5\sqrt{3}}{\sqrt{11}}\right)$ ***(M1)***

since $\boldsymbol{w}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ is a mulitple of their normal

$\Rightarrow \boldsymbol{w} = \lambda\begin{pmatrix}-2\\4\\-2\end{pmatrix}$ ***(A1)***

attempt to find $\boldsymbol{u}\times\boldsymbol{w}$ ***(M1)***

$$\boldsymbol{u}\times\boldsymbol{w} = \begin{pmatrix}1\\1\\1\end{pmatrix}\times\lambda\begin{pmatrix}-2\\4\\-2\end{pmatrix} = \lambda\begin{pmatrix}-6\\0\\6\end{pmatrix}$$

$$|\boldsymbol{u}\times\boldsymbol{w}| = \lambda\sqrt{72}$$

$\frac{1}{2}\lambda\sqrt{72} = 5\frac{\sqrt{3}}{\sqrt{11}} \Rightarrow \lambda = \frac{10}{\sqrt{264}}\ \left(=0.615457...\right)$ ***A1***

$$\boldsymbol{w} = \pm\begin{pmatrix}1.23091\ldots\\-2.46182\ldots\\1.23091\ldots\end{pmatrix}$$

$\boldsymbol{w} = \pm\begin{pmatrix}1.23\\-2.46\\1.23\end{pmatrix}$ or $\boldsymbol{w} = \pm\frac{5}{\sqrt{66}}\begin{pmatrix}-2\\4\\-2\end{pmatrix}$ ***A1***

Note: If no working shown, award ***M1A1(M1)A1A0*** for $\boldsymbol{w} = \pm\begin{pmatrix}1.2\\-2.5\\1.2\end{pmatrix}$ (2 sf)

[5 marks]

Total [20 marks]

11. (a) **METHOD 1**

let M be the midpoint of $[\mathrm{AB}]$ and so $\mathrm{AB} = 2\mathrm{AM}$

attempts to use Pythagoras' theorem to find AM^2 OR AM ***(M1)***

$\mathrm{AM}^2 = 20^2 - 14^2 \ (= 204)$ OR $\mathrm{AM} = \sqrt{20^2 - 14^2} \ \left(= 14.2828... = \sqrt{204} = 2\sqrt{51}\right)$

recognizes that $\mathrm{AB} = 2\mathrm{AM}$ ***(A1)***

$\mathrm{AB} = 2 \times 14.2828... \ (= 28.5657...) \ \left(= 2\sqrt{204} = 4\sqrt{51}\right)$ ***A1***

$\mathrm{AB} = 28.5657...$

$\mathrm{AB} = 28.57$ (m) ***AG***

METHOD 2

let M be the midpoint of $[\mathrm{AB}]$ and so $\mathrm{AB} = 2\mathrm{AM}$

let $\theta = \mathrm{A\hat{S}M}$

$\theta = 0.795398... \left(= \cos^{-1}\frac{14}{20}\right)$ ***(A1)***

attempts to use a valid trigonometric ratio ***M1***

EITHER

$\mathrm{AM} = 14\tan(0.795398...) \left(= 14.2828... = 14\tan\left(\cos^{-1}\frac{14}{20}\right)\right)$ ***A1***

OR

$\mathrm{AM} = 20\sin(0.795398...) \left(= 14.2828... = 20\sin\left(\cos^{-1}\frac{14}{20}\right)\right)$ ***A1***

THEN

$\mathrm{AB} = 28.5657...$

$\mathrm{AB} = 28.57$ (m) ***AG***

[3 marks]

continued…

Question 11 continued.

(b) **EITHER**

the sprinkler rotates through (an angle of) 2π (radians) every 16 seconds and

hence rotates through $\frac{2\pi}{16}$ (radians) in 1 second ***A1***

OR

$\left(\frac{2\pi}{n}=16 \Rightarrow n=\right)\frac{2\pi}{16}\ \left(=\frac{\pi}{8}\right)$ ***A1***

THEN

sprinkler rotates through an angle of $\frac{\pi}{8}$ radians in one second ***AG***

[1 mark]

continued…

Question 11 continued.

(c)

Note: For candidates that used Method 2 in part (a) apply full FT from their value of θ.

attempts to find 2θ where $\theta = \text{A}\hat{\text{S}}\text{M}$ ***(M1)***

$= 2(0.795398...)$ $\left(1.59079... = 2\cos^{-1}\frac{14}{20}\right)$

uses $\frac{\theta}{t}$ (rad/s) or similar to form an equation involving T ***(M1)***

$\frac{2\pi}{16} = \frac{1.59079...}{T}$ $\left(\frac{2\pi}{16} = \frac{2\cos^{-1}\frac{14}{20}}{T}\right)$ ***(A1)***

$T = 4.05093...$ $\left(= \frac{1.59079...}{\frac{2\pi}{16}}\right)$ $\left(= \frac{2\cos^{-1}\frac{14}{20}}{\frac{2\pi}{16}}\right)$

$T = 4.05$ (s) ***A1***

[4 marks]

continued…

Question 11 continued.

(d) $\alpha = \frac{\pi t}{8}$ ***A1***

[1 mark]

(e) applies sine rule in ΔASD ***A1***

$\frac{d}{\sin\alpha} = \frac{20}{\sin A\hat{D}S}$

attempts to find $A\hat{D}S$ in terms of α ***M1***

$A\hat{D}S = \pi - \beta - \alpha \ \left(= \pi - 0.7754 - \alpha\right)\left(= 2.366... - \alpha\right) \ \left(= 2.37 - \alpha\right)$

$d = \frac{20\sin\alpha}{\sin\left(2.366... - \alpha\right)} \left(= \frac{20\sin\alpha}{\sin\left(2.37 - \alpha\right)}\right)$ (accept $d = \frac{20\sin\alpha}{\sin\left(\pi - \beta - \alpha\right)}$) ***A1***

$d = \frac{20\sin\left(\frac{\pi t}{8}\right)}{\sin\left(2.37 - \frac{\pi t}{8}\right)}$ ***AG***

[3 marks]

(f) 18 (m) ***A1***

[1 mark]

continued…

Question 11 continued.

(g) (i) $w = \left| 0.05t^2 + 1.1t + 18 - \dfrac{20\sin\left(\dfrac{\pi t}{8}\right)}{\sin\left(2.37 - \dfrac{\pi t}{8}\right)} \right|$ ***A1***

(ii) attempts to solve $w = 0$ for t ***(M1)***

$t = 3.34880...\left(12.7765...\right)$

$t = 3.35$ (s) ***A1***

$22.2444...$

22.2 (m) (south of A) ***A1***

[4 marks]

Total [17 marks]

1. (a) $\frac{1}{2}r^2\theta = 48$ OR $\frac{1}{2}r^2(1.5) = 48$ ***(A1)***

attempt to solve their equation to find r or r^2 ***(M1)***

Note: To award the ***M1***, candidate's equation must include r^2 and $\theta = 1.5$, and they must attempt to isolate r^2 or r.

$r^2 = 64$

$r = 8$ (cm) ***A1***

[3 marks]

(b) evidence of summing the two radii and the arc length ***(M1)***

perimeter $= 2r + r\theta$

$= 16 + 8(1.5)$

$= 28$ (cm) ***A1***

[2 marks]

Total [5 marks]

4. attempt to substitute into cosine rule ***(M1)***

$\left(\cos 2\theta =\right)\dfrac{4^2+6^2-5^2}{2\times 4\times 6}$ OR $5^2=4^2+6^2-2\times 4\times 6\times\cos 2\theta$

$\left(\cos 2\theta =\right)\dfrac{27}{48}\left(=\dfrac{9}{16}\right)$ ***(A1)***

attempt to use $\cos 2\theta = 2\cos^2\theta - 1$ ***(M1)***

$\cos^2\theta = \dfrac{1+\dfrac{27}{48}}{2}\quad\left(=\dfrac{1+\dfrac{9}{16}}{2}\right)$

$\cos^2\theta = \dfrac{75}{96}\quad\left(=\dfrac{25}{32}\right)$ ***A1***

$\cos\theta = (\pm)\sqrt{\dfrac{75}{96}}\quad\left(=\sqrt{\dfrac{25}{32}}=\dfrac{5}{\sqrt{32}}\right)$ ***(A1)***

$=\dfrac{5}{4\sqrt{2}}$

$=\dfrac{5\sqrt{2}}{8}\quad(p=5, q=8)$ ***A1***

Note: The final answer must be positive.

[6 marks]

8. (a) $\vec{OP} = \boldsymbol{a} + \lambda(\boldsymbol{b} - \boldsymbol{a})$ ***A1***

$= (1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}$ ***AG***

[1 mark]

(b) **METHOD 1**

recognition that $\vec{OP} \cdot \vec{AB} = 0$ (may be seen anywhere) ***(M1)***

$[(1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{b}].[\boldsymbol{b} - \boldsymbol{a}](=0)$ ***A1***

attempt to multiply out scalar product ***M1***

$(1-\lambda)\boldsymbol{a}.\boldsymbol{b} + \lambda\boldsymbol{b}.\boldsymbol{b} - (1-\lambda)\boldsymbol{a}.\boldsymbol{a} - \lambda\boldsymbol{b}.\boldsymbol{a}(=0)$ ***(A1)***

attempt to substitute for $\boldsymbol{a}.\boldsymbol{b}$ and $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ ***(M1)***

$\frac{1}{4}(1-\lambda) + 4\lambda - (1-\lambda) - \frac{\lambda}{4}(=0)$ ***(A1)***

$1 - \lambda + 16\lambda - 4 + 4\lambda - \lambda = 0$

$18\lambda - 3 = 0$

$\lambda = \frac{1}{6}$ ***A1***

continued…

Question 8 continued

METHOD 2

$\cos AOB = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{1}{8}$ ***A1***

attempt to use cosine rule to find AB ***M1***

$|AB|^2 = 1^2 + 2^2 - 2(1)(2)\left(\dfrac{1}{8}\right)$

$AB = \dfrac{3\sqrt{2}}{2}$ ***A1***

attempt to apply Pythagoras' Theorem twice: ***M1***

$|OP|^2 + \left(\dfrac{3\sqrt{2}}{2}\lambda\right)^2 = 1$ and

$|OP|^2 + \left(\dfrac{3\sqrt{2}}{2}(1-\lambda)\right)^2 = 4$ ***A1***

attempt to solve simultaneously: ***M1***

$\dfrac{9}{2}(1-\lambda)^2 - \dfrac{9}{2}\lambda^2 = 3$

$\lambda = \dfrac{1}{6}$ ***A1***

[7 marks]

Total [8 marks]

9. (a) **METHOD 1**

attempt to use identity $\tan(A-B) \equiv \frac{\tan A - \tan B}{1+\tan A \tan B}$ ***M1***

$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - 1}{1+\tan\theta}$ ***A1***

attempt to write their RHS in terms of $\sin\theta$ and $\cos\theta$ ***M1***

$\frac{\frac{\sin\theta}{\cos\theta} - 1}{1+\frac{\sin\theta}{\cos\theta}}$ OR $\frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$

multiply through by the conjugate of the denominator $\frac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}$ ***M1***

$= \frac{-\sin^2\theta - \cos^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \frac{-\left(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta\right)}{\cos^2\theta - \sin^2\theta}$

$= \frac{-(1-\sin 2\theta)}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \frac{\sin 2\theta - 1}{\cos 2\theta}$ ***AG***

continued…

Question 9 continued.

METHOD 2

attempt to write $\tan\left(\theta-\frac{\pi}{4}\right)$ in terms of sin and cos: ***M1***

$$\left(\tan\left(\theta-\frac{\pi}{4}\right)\right)=\frac{\sin\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta-\frac{\pi}{4}\right)}$$

attempt to use both sin and cos addition formulae: ***M1***

$$=\frac{\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}}{\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}}$$

$$=\frac{\sin\theta-\cos\theta}{\cos\theta+\sin\theta}$$ ***A1***

multiply through by the conjugate of the denominator $\frac{\cos\theta-\sin\theta}{\cos\theta-\sin\theta}$ ***M1***

$$=\frac{-\sin^2\theta-\cos^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$ ***A1***

$$=\frac{-\left(\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta\right)}{\cos^2\theta-\sin^2\theta}$$

$$=\frac{-\left(1-\sin 2\theta\right)}{\cos^2\theta-\sin^2\theta}$$ ***A1***

$$=\frac{\sin 2\theta-1}{\cos 2\theta}$$ ***AG***

continued…

Question 9 continued.

METHOD 3

attempt to use given identity $\tan(A-B) \equiv \frac{\tan A - \tan B}{1+\tan A \tan B}$ ***M1***

$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - 1}{1+\tan\theta}$ ***A1***

multiply through by the conjugate of the denominator $\frac{1-\tan\theta}{1-\tan\theta}$ ***M1***

$\frac{(\tan\theta - 1)}{(1+\tan\theta)} \cdot \frac{(1-\tan\theta)}{(1-\tan\theta)}$ ***A1***

$\frac{2\tan\theta - \tan^2\theta - 1}{1-\tan^2\theta} \left(= \frac{2\tan\theta - \sec^2\theta}{1-\tan^2\theta}\right)$

attempt to write their RHS in terms of $\sin\theta$ and $\cos\theta$ ***M1***

$= \frac{2\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos^2\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}$

$= \frac{2\sin\theta\cos\theta - 1}{\cos^2\theta - \sin^2\theta}$ ***A1***

$= \frac{\sin 2\theta - 1}{\cos 2\theta}$ ***AG***

[6 marks]

continued…

Question 9 continued.

(b) recognition that $x = 2\theta \Rightarrow \theta = \frac{x}{2}$ ***M1***

$$\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) = \sqrt{3}$$

$$\frac{x}{2} - \frac{\pi}{4} = \frac{\pi}{3}, \left(\frac{4\pi}{3}\right)$$ ***A1***

$$x = \frac{7\pi}{6}$$ ***A1***

Note: Award ***A0*** if extra solutions outside the domain are seen.

[3 marks]

Total [9 marks]

SECTION B

4. (a) **METHOD 1**

attempt to use right triangle trigonometry ***(M1)***

$\tan \hat{BAE} = \frac{12}{7}$ OR $\tan\left(90° - \hat{BAE}\right) = \frac{7}{12}$ ***(A1)***

$59.7435...$

$\hat{BAE} = 59.7°$ ***A1***

Note: Award ***(M1)(A1)A0*** for the equivalent radian value of 1.04.

METHOD 2

attempt to find $\hat{BAE}$ using sine rule OR cosine rule (***M1)***

$\frac{\sin \hat{BAE}}{12} = \frac{\sin 90}{\sqrt{12^2 + 7^2}}$ OR $12^2 = 7^2 + 193 - 2 \times 7 \times \sqrt{12^2 + 7^2} \times \cos \hat{BAE}$ ***(A1)***

$\hat{BAE} = 59.7435...$

$\hat{BAE}=59.7°$ ***A1***

Note: Award ***(M1)(A1)A0*** for the equivalent radian value of 1.04.

[3 marks]

(b) (i) **METHOD 1**

attempt to find DE using right angle trigonometry ***(M1)***

$\sin 59.7435...° = \frac{350}{DE}$ OR equivalent ***(A1)***

$DE = 405.196...$

$CE = 405.196... + 50$

$= 455.196...$

$= 455$ (cm) ***A1***

METHOD 2

Let $DE = EF = x$

attempt to find DE using their $D\hat{E}F$ and the sine rule OR cosine rule (***M1)***

$\frac{700}{\sin(119.487...)} = \frac{DE}{\sin(30.2564...)}$ OR $x^2 = 700^2 + x^2 - 2 \times 700 \times x \times \cos 30.2564...$ ***(A1)***

$DE = 405.196...$

$CE = 405.196... + 50$

$= 455.196...$

$= 455$ (cm) ***A1***

continued…

Question 4 continued.

METHOD 3

Let G be the midpoint of DF

$\mathrm{EG} = \frac{7}{12} \times 350 \left(= \frac{1225}{6} = 204.166... \right)$ ***(A1)***

use of Pythagoras' with their EG to find DE ***(M1)***

$\mathrm{DE} = \sqrt{204.166...^2 + 350^2} \ \left(= 405.196... \right)$

$\mathrm{CE} = 405.196... + 50$

$= 455.196...$

$= 455$ (cm) ***A1***

(ii) $\tan(59.7435...°) = \frac{30}{x}$ OR $\frac{12}{7} = \frac{30}{x}$ ***(A1)***

$x = 17.5$

$\mathrm{BA} = 455.196... + 17.5$

$= 472.696...$

$= 473$ (cm) ***A1***

[5 marks]

Total [8 marks]

11. (a) attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$-\frac{x}{2}+1=\lambda \Rightarrow x=2-2\lambda$, $y+4=\lambda \Rightarrow y=-4+\lambda$, $\frac{z}{3}=\lambda \Rightarrow z=3\lambda$ OR

$\frac{x-2}{-2}=\frac{y+4}{1}=\frac{z-0}{3}$

correct direction vector x, y, z or equivalent seen in vector form ***(A1)***

$\boldsymbol{r}=\begin{pmatrix}2\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\3\end{pmatrix}$ ***A1***

Note: Award ***(M1)(A1)A0*** if "$\boldsymbol{r}=$" is omitted.

[3 marks]

(b) **METHOD 1**

L passes through $\mathrm{P}(2-2\lambda,-4+\lambda,3\lambda)$ ***(A1)***

attempt to apply distance formula to find their $\left|\overrightarrow{\mathrm{OP}}\right|$ or their $\left|\overrightarrow{\mathrm{OP}}\right|^2$ ***(M1)***

$\left|\overrightarrow{\mathrm{OP}}\right|^2=(2-2\lambda)^2+(-4+\lambda)^2+(3\lambda)^2$ ***(A1)***

attempt to find their minimum value of $\left|\overrightarrow{\mathrm{OP}}\right|$ or $\left|\overrightarrow{\mathrm{OP}}\right|^2$ using GDC ***(M1)***

$\lambda=0.571428...\left(\Rightarrow\left|\overrightarrow{\mathrm{OP}}\right|=\sqrt{15.4285...}\ \text{OR}\ \left|\overrightarrow{\mathrm{OP}}\right|^2=15.4285...\right)$

$3.92792...$

$3.93\left(=\frac{6\sqrt{21}}{7}\right)$ ***A1***

continued…

Question 11 continued.

METHOD 2

setting the scalar product of their line and their direction vector to zero ***(M1)***

$$\begin{pmatrix} 2-2\lambda \\ -4+\lambda \\ 3\lambda \end{pmatrix}.\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}=0$$

$$-4+4\lambda-4+\lambda+9\lambda=0$$

$$-8+14\lambda=0$$

$$\lambda=\frac{4}{7}\left(=0.571428...\right)$$ ***(A1)***

attempt to substitute their value for λ into their $\boldsymbol{r}$ to find the position vector of the closest point and find $|\boldsymbol{r}|$ ***(M1)***

$$\boldsymbol{r}=\begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}+\frac{4}{7}\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}=\begin{pmatrix} \frac{6}{7} \\ -\frac{24}{7} \\ \frac{12}{7} \end{pmatrix}$$

$$|\boldsymbol{r}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(-\frac{24}{7}\right)^2+\left(\frac{12}{7}\right)^2}$$ ***(A1)***

$$=3.92792\ldots=\frac{6\sqrt{21}}{7}$$

$$=3.93\left(=\frac{6\sqrt{21}}{7}\right)$$ ***A1***

continued…

Question 11 continued.

METHOD 3

Let P be a point on L

attempt to find the cross product between $\overrightarrow{OP}$ and the direction, $\boldsymbol{b}$, of L ***(M1)***`

$$\begin{pmatrix}2\\-4\\-0\end{pmatrix}\times\begin{pmatrix}-2\\1\\3\end{pmatrix}\left(=(-12-0)\boldsymbol{i}+(0-6)\boldsymbol{j}+(2-8)\boldsymbol{k}\right)$$

$$=\begin{pmatrix}-12\\-6\\-6\end{pmatrix}\left(=-6\begin{pmatrix}2\\1\\1\end{pmatrix}\right)$$ ***(A1)***

attempt to find shortest distance using $\dfrac{\left|\overrightarrow{OP}\times\boldsymbol{b}\right|}{|\boldsymbol{b}|}$ ***(M1)***

$\dfrac{\sqrt{216}}{\sqrt{14}}$ ***(A1)***

3.92792...

$3.93\left(=\dfrac{6\sqrt{21}}{7}\right)$ ***A1***

[5 marks]

continued…

Question 11 continued.

(c) **METHOD 1**

substitute their x, y, z into equation of plane $6x-3y+5z(=24)$ ***M1***

$6(2-2\lambda)-3(-4+\lambda)+5(3\lambda)$ ***(A1)***

$=12-12\lambda+12-3\lambda+15\lambda$ ***A1***

$=24$

so the line is contained in the plane ***AG***

Note: For FT from an incorrect part a), award ***M1A0A0.***

METHOD 2

consider the direction of the line **and** a point on the line ***M1***

$\begin{pmatrix}-2\\1\\3\end{pmatrix}.\begin{pmatrix}6\\-3\\5\end{pmatrix}=-12-3+15=0$ (so line is parallel to plane) ***A1***

$6(2)-3(-4)+5(0)=12+12=24$ (so line lies in the plane) ***A1***

so the line is contained in the plane ***AG***

Note: Both ***A*** marks are dependent on the ***M*** mark.

Note: For FT from an incorrect part a), award ***M1A0A0.***

[3 marks]

continued…

Question 11 continued.

(d) **METHOD 1**

recognition that the direction M is from the point $\begin{pmatrix}4\\1\\2\end{pmatrix}$ and a point on the z-axis $\begin{pmatrix}0\\0\\z\end{pmatrix}$ ***(M1)***

direction of M is $(\pm)\begin{pmatrix}4\\1\\2-z\end{pmatrix}$ ***(A1)***

the direction of the normal of Π is $\begin{pmatrix}6\\-3\\5\end{pmatrix}$ (seen anywhere) ***(A1)***

attempt to use the scalar product with their normal and their direction vector and equate to 0 ***(M1)***

$$\begin{pmatrix}4\\1\\2-z\end{pmatrix}.\begin{pmatrix}6\\-3\\5\end{pmatrix}=0 \Rightarrow 24-3+10-5z=0$$

$z=\frac{31}{5}$ ***(A1)***

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}4\\1\\-\frac{21}{5}\end{pmatrix}$$

attempt to express equation in the form $\boldsymbol{s}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}p\\q\\r\end{pmatrix}$ ***(M1)***

$\boldsymbol{s}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}4\\1\\-\frac{21}{5}\end{pmatrix}$ OR $\boldsymbol{s}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\mu\begin{pmatrix}20\\5\\-21\end{pmatrix}$ ***A1***

continued…

Question 11 continued.

METHOD 2

let the equation of M be $M : \boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Note: Consideration of the z-axis intersection and consideration of direction may be done in either order and marks should be awarded independently.

recognition M intersects the z-axis where $x = y = 0$ ***(M1)***

$4 + \mu p = 0, 1 + \mu q = 0$

$\frac{-4}{p} = \frac{-1}{q} \Rightarrow p = 4q$ ***(A1)***

the direction of the normal of Π is $\begin{pmatrix} 6 \\ -3 \\ 5 \end{pmatrix}$ (seen anywhere) ***(A1)***

attempt to use the scalar product with their normal and their direction vector and equate to 0 ***(M1)***

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 6 \\ -3 \\ 5 \end{pmatrix} = 0 \Rightarrow \; 6p - 3q + 5r = 0$

$6(4q) - 3q + 5r = 0 \Rightarrow 21q = -5r$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \\ -21 \end{pmatrix}$ (or any multiple of $\begin{pmatrix} 20 \\ 5 \\ -21 \end{pmatrix}$) ***(A1)***

attempt to express equation in the form $\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ ***(M1)***

$\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -\frac{21}{5} \end{pmatrix}$ OR $\boldsymbol{s} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 20 \\ 5 \\ -21 \end{pmatrix}$ ***A1***

[7 marks]

Total [18 marks]

6. direction vectors are $a\boldsymbol{j}+\boldsymbol{k}\left(=\begin{pmatrix}a\\1\end{pmatrix}\right)$ and $\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}\left(=\begin{pmatrix}2\\3\end{pmatrix}\right)$ ***(A1)***

recognition that the scalar product of the direction vectors is 0 ***(M1)***

$$\begin{pmatrix}0\\a\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\3\end{pmatrix}\left(=2a+3\right)=0$$

$a=-\frac{3}{2}$ ***A1***

at point of intersection $4=1+\mu$, $-\frac{3}{2}\lambda=2\mu$ and $-1+\lambda=-b+3\mu$ ***(A1)***

attempt to solve 3 equations in μ, λ and b, derived from the point of intersection, to find μ, λ and b ***(M1)***

$\mu=3,\ \lambda=-4$

$b=14$ ***A1***

Total [6 marks]

11. (a) normals of Π_1 and Π_2 are $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\-2\end{pmatrix}$ ***(A1)***

attempt to use the scalar product for the angle between two vectors ***M1***

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\left\|\begin{pmatrix}1\\2\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right\|\cos\theta$$

$-3=\sqrt{6}\sqrt{6}\cos\theta\ \ (\text{so}\ -3=6\cos\theta)$ ***A1A1***

Note: Award ***A1*** for correct scalar product and ***A1*** for correct magnitudes of normals and $\cos\theta$.

$\cos\theta=-\frac{1}{2}$ ***A1***

$\theta=120^\circ$ OR $\cos(180^\circ-\theta)=\frac{1}{2}$ ***A1***

acute angle is 60° ***AG***

[6 marks]

(b) attempt to find vector product of their normal vectors ***M1***

$$\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}-4+1\\1+2\\-1-2\end{pmatrix}$$

$$=\begin{pmatrix}-3\\3\\-3\end{pmatrix}\left(=3\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right)$$ ***(A1)***

equation of Π_3 is $-3x+3y-3z=d$ OR $-x+y-z=d$ or equivalent

attempt to substitute $x=5,\ y=-5,\ z=5$ into their equation ***(M1)***

equation of Π_3 is $-3x+3y-3z=-45\ \ (\text{so}\ -x+y-z=-15)$ ***A1***

[4 marks]

continued…

Question 11 continued.

(c) (i) attempt to use the identity for $\sin(30^\circ + 45^\circ)$ ***(M1)***

$$\sin(30^\circ + 45^\circ) = \sin(30^\circ)\cos(45^\circ) + \cos(30^\circ)\sin(45^\circ)$$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \quad \textbf{A1A1}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4} \quad \textbf{AG}$$

Note: ***A1*** *for each term.* Award ***A1 A0*** for correct answers where the denominator has not been rationalized such as $= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$.

(ii) let $x =$ QR

attempt to use the sine rule with the angles $75^\circ, 45^\circ$ ***(M1)***

$$\frac{5}{\sin 75^\circ} = \frac{x}{\sin 45^\circ}$$

$x = \frac{20}{\sqrt{2}\left(\sqrt{2}+\sqrt{6}\right)}\left(= \frac{20}{2\left(1+\sqrt{3}\right)}\right)$ or equivalent ***A1***

attempt to rationalise a denominator of the form $\left(\sqrt{a}+\sqrt{b}\right)$ by multiplying numerator and denominator by $\pm\left(\sqrt{a}-\sqrt{b}\right)$ or equivalent ***(M1)***

$$x = \frac{20\left(\sqrt{2}-\sqrt{6}\right)}{\sqrt{2}\left(\sqrt{2}+\sqrt{6}\right)\left(\sqrt{2}-\sqrt{6}\right)}\left(= \frac{20\left(1-\sqrt{3}\right)}{2\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}\right)$$

$x = 5\left(\sqrt{3}-1\right)$ (cm) ($p = 5$, $q = 3$) ***A1***

[7 marks]

Total [17 marks]

2. (a) attempt to set equal to a parameter or to add detail to cartesian form ***(M1)***

$\frac{x-1}{2}=\frac{y+2}{3}=z=\lambda \Rightarrow x=2\lambda+1, y=3\lambda-2, z=\lambda$ OR $\frac{x-1}{2}=\frac{y-(-2)}{3}=\frac{z-0}{1}$

$\boldsymbol{r}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ OR $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $\boldsymbol{r}=$ OR $\begin{pmatrix}x\\y\\z\end{pmatrix}=$ is omitted.

[2 marks]

(b) attempt to equate at least one component ***(M1)***

$1+2\lambda=t$ OR $-2+3\lambda=4$ OR $\lambda=-8+2t$

$\lambda=2$ OR $t=5$ ***(A1)***

intersection point = $(5,4,2)$ ***A1***

Note: Condone coordinates given in vector form.

[3 marks]

Total [5 marks]

3.

(a) correct substitution in sine rule ***(A1)***

$\frac{\sin\theta}{5} = \frac{\sin 2\theta}{6\sqrt{2}}$ (or equivalent)

attempt to use double angle rule for $\sin 2\theta$ ***(M1)***

$\frac{\sin\theta}{5} = \frac{2\sin\theta\cos\theta}{6\sqrt{2}}$

$6\sqrt{2}\sin\theta = 10\sin\theta\cos\theta$ OR $\frac{1}{5} = \frac{2\cos\theta}{6\sqrt{2}}$ OR equivalent ***A1***

$\cos\theta = \frac{3\sqrt{2}}{5}$ ***AG***

[3 marks]

(b) valid attempt to find $\sin\theta$ ***(M1)***

$\sin^2\theta + \left(\frac{3\sqrt{2}}{5}\right)^2 = 1$ OR right triangle with adjacent side and hypotenuse labelled

$\sin\theta = \frac{\sqrt{7}}{5}$ ***A1***

[2 marks]

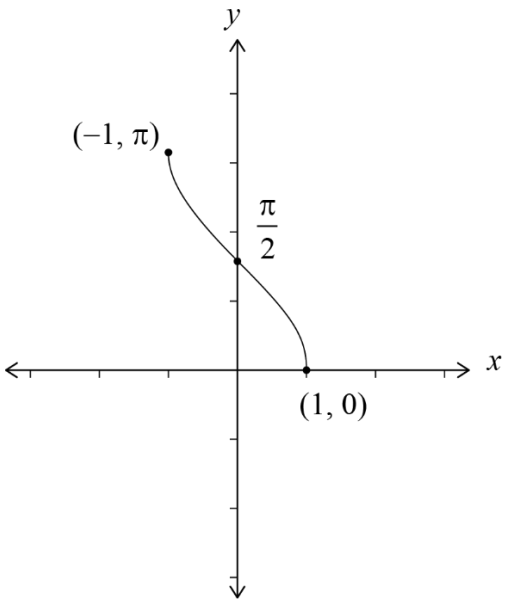
(c) $\frac{1}{2} \times 6\sqrt{2} \times \mathrm{DC} \times \frac{\sqrt{7}}{5} = 2\sqrt{14}$ ***(A1)***

$\mathrm{DC} = \frac{10}{3}$ ***A1***

[2 marks]

Total [7 marks]

8. (a)



For a curve with correct shape within the domain, correct labelled y-intercept and end points. ***A2***

Note: Award ***A1*** for any two of the following provided a curve is seen:

correct shape within the domain

correctly labelled y - intercept

correctly labelled end points.

[2 marks]

continued…

Question 8 continued.

(b) **METHOD 1**

recognizing to use compound angle rule for $\cos(A+B)$ ***(M1)***

$\cos(\arccos x)\cos\left(\arccos x\sqrt{3}\right)-\sin(\arccos x)\sin\left(\arccos x\sqrt{3}\right)$

valid attempt to find $\sin(\arccos x)$ or $\sin\left(\arccos x\sqrt{3}\right)$ in terms of x ***(M1)***

use of the identity $\sin\theta=\sqrt{1-\cos^2\theta}$ OR use of a right-angled triangle

($\sin A$ and $\sin B$ are both positive)

$x\times x\sqrt{3}-\sqrt{1-x^2}\times\sqrt{1-3x^2}\left(=\cos\frac{3\pi}{2}\right)$ ***A1***

$\cos\frac{3\pi}{2}=0$ (seen anywhere) ***(A1)***

attempt to write an equation in x without the square roots ***(M1)***

$3x^4=1-4x^2+3x^4$

$x=\pm\frac{1}{2}$

(as seen in part (a) graph, when $x=\frac{1}{2}$, $\arccos(x)+\arccos\left(x\sqrt{3}\right)<\pi\Rightarrow$ no solution)

$x=-\frac{1}{2}$ ***A1***

continued…

Question 8 continued.

METHOD 2

recognizing to use compound angle rule for $\sin(A+B)$ ***(M1)***

$\sin(\arccos x)\cos\left(\arccos x\sqrt{3}\right)+\cos(\arccos x)\sin\left(\arccos x\sqrt{3}\right)$

valid attempt to find $\sin(\arccos x)$ or $\sin\left(\arccos x\sqrt{3}\right)$ in terms of x ***(M1)***

use of the identity $\sin\theta=\sqrt{1-\cos^2\theta}$ OR use of a right-angled triangle

($\sin A$ and $\sin B$ are both positive)

$\sqrt{1-x^2}\times x\sqrt{3}+x\times\sqrt{1-3x^2}\left(=\sin\frac{3\pi}{2}\right)$ ***A1***

$\sin\frac{3\pi}{2}=-1$ (seen anywhere) ***(A1)***

attempt to write an equation in x without the square roots ***(M1)***

$4x^4-4x^2+1=4x^2-12x^4\left(\Rightarrow 16x^4-8x^2+1=0\Rightarrow\left(4x^2-1\right)^2=0\right)$

$x=\pm\frac{1}{2}$

(as seen in part (a) graph, when $x=\frac{1}{2}$, $\arccos(x)+\arccos\left(x\sqrt{3}\right)<\pi\Rightarrow$ no solution)

$x=-\frac{1}{2}$ ***A1***

[6 marks]

Total [8 marks]

3. recognizing to use $\cos 2\theta = 2\cos^2\theta - 1$ ***(M1)***

$2\left(2\cos^2\theta - 1\right) - 5\cos\theta + 2 \ \ (= 0)$ ***A1***

$4\cos^2\theta - 5\cos\theta \ \ (= 0)$

choosing an appropriate method to solve their quadratic equation ***(M1)***

$\cos\theta(4\cos\theta - 5)$ **OR** $\frac{5 \pm \sqrt{(-5)^2 - 4\times 4\times 0}}{2\times 4}$ ***(A1)***

$\cos\theta = 0$

$\theta = \frac{3\pi}{2}$ ***A1***

Note: Do not award final ***A1*** if any extra solutions given.

[5 marks]

11. (a) $\vec{BC} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **OR** $\vec{BA} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ ***(A1)***

$(3, 0, 2)$ ***A1***

[2 marks]

(b) **EITHER**

recognizing to find midpoint of $[AC]$ or $[BD]$ ***(M1)***

$\left(\frac{-1+1}{2}, \frac{-2-4}{2}, \frac{4-0}{2}\right)$ **OR** $\left(\frac{-3+3}{2}, \frac{-6+0}{2}, \frac{2+2}{2}\right)$

OR

Let $E(a, b, c)$

$\vec{AE} = \vec{EC} \Rightarrow \begin{pmatrix} a-1 \\ b+4 \\ c \end{pmatrix} = \begin{pmatrix} -1-a \\ -2-b \\ 4-c \end{pmatrix}$ **OR** $\vec{AE} = \frac{1}{2}\vec{AC} \Rightarrow \begin{pmatrix} a-1 \\ b+4 \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$ ***(M1)***

THEN

$(0, -3, 2)$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) (i) $\vec{AB} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}$ **AND** $\vec{AD} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ ***A1***

Note: Award ***A1*** for finding both vectors, one of which may have been seen in (a).

$$\begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \\ -12 \end{pmatrix}$$

$m = 12$ ***A1***

(ii) $\sqrt{(-12)^2 + 12^2 + (-12)^2}$ **OR** $12\sqrt{(-1)^2 + 1^2 + (-1)^2}$ ***(A1)***

$= 12\sqrt{3}$ ***A1***

[4 marks]

(d) substituting cross product and a point into the equation of a plane ***(M1)***

$-x + y - z = -5$ ***A1***

[2 marks]

continued…

Question 11 continued

(e) let a normal to Π_1 be $\boldsymbol{n}_1$ and a normal to Π_2 be $\boldsymbol{n}_2$

Eg. $\boldsymbol{n}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$

recognizing angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ is required ***(M1)***

$|\boldsymbol{n}_2| = \sqrt{75}$ ***A1***

$\cos\theta = \dfrac{-1\times5+1\times1+(-1)\times(-7)}{\sqrt{3}\times\sqrt{75}} \left(= \dfrac{3}{\sqrt{3}\times\sqrt{75}}\right)$ ***A1***

Note: The negative of this may be found but to obtain the final ***A1*** there needs to be a recognition that this obtains the obtuse angle for the minus to be dropped.

$\cos\theta = \dfrac{1}{5}$ ***AG***

[3 marks]

(f) attempt to substitute normal to Π_1 and their E into vector equation of line ***(M1)***

$(\boldsymbol{r} =) \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ ***A1***

substitutes for λ into Π_2 ***(M1)***

$5(-\lambda) + (-3+\lambda) - 7(2-\lambda) = 1$

$\lambda = 6$ ***A1***

$(-6, 3, -4)$ ***A1***

[5 marks]

Total [18 marks]

1. (a) **EITHER**

$210^\circ = \frac{210\pi}{180}\left(= \frac{7\pi}{6} = 3.66519...\right)$ radians ***(A1)***

attempt to use radian formula for area of sector ***(M1)***

area $= \frac{1}{2}(19.5)^2\left(\frac{7\pi}{6}\right)$

OR

attempt to use degree formula for area of sector ***(M1)***

area $= \frac{210}{360}\pi(19.5)^2$ ***(A1)***

THEN

area $= \frac{3549\pi}{16} = 696.844...$

$= 697\left(= \frac{3549\pi}{16}\right)$ (cm^2) ***A1***

[3 marks]

(b) **EITHER**

arc length $= 19.5\left(\frac{7\pi}{6}\right)$ OR $= \frac{210}{360}(2\pi(19.5))\left(= \frac{91\pi}{4} = 71.4712...\right)$ ***(A1)***

attempt to set $2\pi r$ equal to arc length ***(M1)***

$2\pi r = 71.4712...$

OR

attempt to set πrl equal to their area from (a) ***(M1)***

$19.5\pi r = 696.844...$ ***(A1)***

THEN

$r = 11.4\left(= \frac{91}{8} = 11.375\right)$(cm) ***A1***

[3 marks]

Total[6 marks]

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2. (a) period is $\frac{\pi}{2}\left(=1.57079...=1.57\right)$ ***A1***

[1 mark]

(b) attempt to substitute $x=\frac{\pi}{12}, f(x)=5$ and $x=\frac{\pi}{3}, f(x)=7$ to obtain two equations ***(M1)***

Note: accept work where x values have been converted into degrees

$a\tan\left(\frac{\pi}{6}\right)+b=5$ and $a\tan\left(\frac{2\pi}{3}\right)+b=7$ $\left(\Rightarrow \frac{a}{\sqrt{3}}+b=5 \text{ and } -a\sqrt{3}+b=7\right)$

$a=-\frac{\sqrt{3}}{2}\left(=-0.866025...=-0.866\right)$ ***A1***

$b=\frac{11}{2}\left(=5.5\right)$ ***A1***

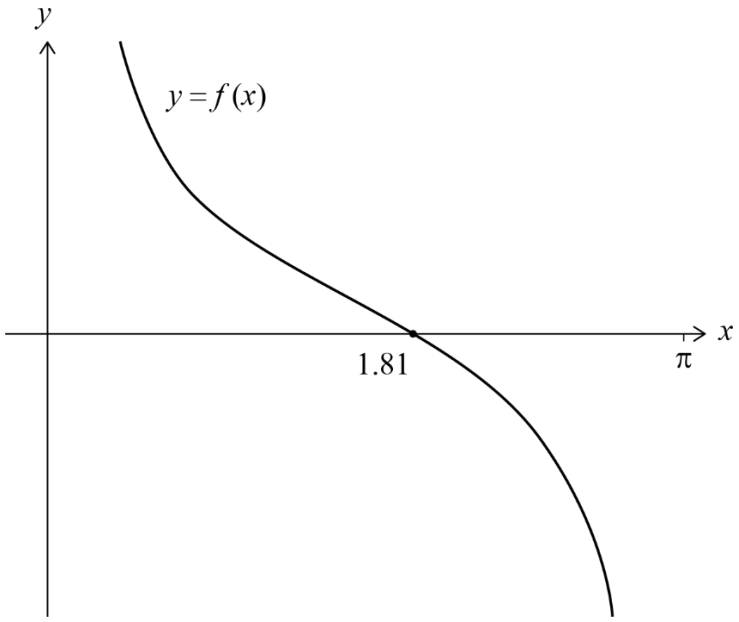
Note: These ***A1*** marks may be awarded independently.

[3 marks]

Total [4 marks]

6. (a) (i) $\left(f(x)=\right)\dfrac{4\cos x}{\sin x}+\sin x$ ***A1***

(ii)



correct domain and curvature ***A1***

Note: The correct domain may be implied by a single branch of the function.
Condone the absence of π on the x-axis.

Award ***A0*** if the graph is not a function.

correct x-intercept at $x=1.80911...=1.81$ ***A1***

Note: Accept coordinates $(1.81, 0)$. This mark is independent of the first ***A1***.

[3 marks]

(b) $f^{-1}(2)=1.31837...$

$=1.32$ ***A1***

[1 mark]

continued…

Question 6 continued.

(c) **METHOD 1**

EITHER

recognition that $\sec x = \frac{1}{\cos x}$ ***(M1)***

$\cos\alpha = \frac{2}{3}\left(= 0.666666...\right)$

$\alpha = \arccos\frac{2}{3}\left(= 0.841068...\right)$ ***(A1)***

OR

$\alpha = \operatorname{arcsec} 1.5\left(= 0.841068...\right)$ ***(A2)***

THEN

$f(\alpha) = f(0.841068...) = 4.32306...$

$= 4.32\left(= \frac{29\sqrt{5}}{15}\right)$ ***A1***

METHOD 2

recognition that $\sec x = \frac{1}{\cos x}$ ***(M1)***

$\cos\alpha = \frac{2}{3}, \sin\alpha = \frac{\sqrt{5}}{3}\left(\cot x = \frac{2}{\sqrt{5}}\right)$ ***(A1)***

$f(\alpha) = \frac{4\left(\frac{2}{3}\right)}{\frac{\sqrt{5}}{3}} + \frac{\sqrt{5}}{3}\left(= 4\left(\frac{2}{\sqrt{5}}\right) + \frac{\sqrt{5}}{3}\right)$

$= \frac{29\sqrt{5}}{15}\left(= 4.32306... = 4.32\right)$ ***A1***

[3 marks]

Total [7 marks]

7. (a) recognition that the speed is the magnitude of the velocity ***(M1)***

$\text{speed} = 4\sqrt{10^2 + (-25)^2}$

$= 107.703...$

$= 108 \quad (= 20\sqrt{29})$ (km/h) ***A1***

[2 marks]

(b) **METHOD 1**

attempt to use right-angled triangle with the horizontal speed found in (a) ***(M1)***

$\tan\beta = \frac{16}{20\sqrt{29}} \left(= \frac{4}{5\sqrt{29}} \right)$ ***(A1)***

$\beta = 8.44984(^\circ)$

$\beta = 8.45(^\circ)$ ***A1***

Note: Award ***M1A1A0*** for answer of 0.147 radians.

METHOD 2

attempt to find angle between $\begin{pmatrix} 10 \\ -25 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ -25 \\ 0 \end{pmatrix}$ ***(M1)***

$\cos\beta = \frac{\begin{pmatrix} 10 \\ -25 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 10 \\ -25 \\ 0 \end{pmatrix}}{\sqrt{10^2 + (-25)^2 + (-4)^2}\sqrt{10^2 + (-25)^2}} \left(= \frac{725}{\sqrt{741}\sqrt{725}} = 0.989144... \right)$ OR

$\sin\beta = \frac{\left| \begin{pmatrix} 10 \\ -25 \\ -4 \end{pmatrix} \times \begin{pmatrix} 10 \\ -25 \\ 0 \end{pmatrix} \right|}{\sqrt{10^2 + (-25)^2 + (-4)^2}\sqrt{10^2 + (-25)^2}} = \frac{\left| \begin{pmatrix} -100 \\ -40 \\ 0 \end{pmatrix} \right|}{\sqrt{741}\sqrt{725}} \left(= \frac{4}{\sqrt{741}} = 0.146943... \right)$ ***(A1)***

$\beta = 8.44984(^\circ)$

$\beta = 8.45^\circ$ ***A1***

continued…

Question 7 continued.

METHOD 3

attempt to find angle between $\begin{pmatrix} 10 \\ -25 \\ -4 \end{pmatrix}$ and a plane parallel to $z = 0$ ***(M1)***

$$\sin\beta = \frac{\left|\begin{pmatrix} 10 \\ -25 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}{\sqrt{10^2 + (-25)^2 + (-4)^2}} \left(= \frac{4}{\sqrt{741}} = 0.146943... \right)$$ ***(A1)***

Note: This could also be written as $\cos(90^\circ - \beta) = ...$

$\beta = 8.44984(^\circ)$

$\beta = 8.45^\circ$ ***A1***

[3 marks]

Total [5 marks]

2.

(a) attempt to find distance between two points ***(M1)***

$\sqrt{(-3-1)^2+(4-7)^2+(2-0)^2}\ \ (=5.38516...)$

$\mathrm{VX}=5.39\ \left(=\sqrt{29}\right)$ ***A1***

[2 marks]

(b) attempt to use Pythagoras' theorem ***(M1)***

$\sqrt{5^2+5^2}\ \ (=7.07106...)$

$\mathrm{AC}=7.07\ \left(=5\sqrt{2}\right)$ ***A1***

[2 marks]

(c) valid attempt to use trig ratio in triangle VXC ***(M1)***

$\tan\theta=\dfrac{\sqrt{29}}{\frac{5\sqrt{2}}{2}}\left(=\dfrac{5.38516...}{3.53553...}\right)$

$56.7138...^\circ$ OR $0.989842...$ rad

$\theta=56.7^\circ$ OR $\theta=0.990$ rad (accept $\theta=0.99$) ***A1***

[2 marks]

Total [6 marks]

6.

METHOD 1

recognising $\frac{\text{area}_{\text{segment}}}{\text{area}_{\text{triangle}}} = \frac{3}{5}$ (or equivalent) (seen anywhere) ***(M1)***

$\text{area}_{\text{triangle}} = 21.3333...$ OR $\text{area}_{\text{sec tor}} = 34.1333...$ ***A1***

correct equation in r and θ ***(A1)***

$\frac{1}{2}r^2\theta = 34.1333...$ OR $\frac{1}{2}r^2\sin\theta = 21.3333...$ OR $\frac{1}{2}r^2(\theta - \sin\theta) = 12.8$ (seen anywhere)

correct equation in one variable ***A1***

$\frac{1}{2}r^2\left(\frac{68.2666...}{r^2} - \sin\left(\frac{68.2666...}{r^2}\right)\right) = 12.8$ OR $\frac{1}{2}\left(\frac{68.2666...}{\theta}\right)\sin\theta = 21.3333...$ OR

$\frac{1}{2}\left(\frac{68.2666...}{\theta}\right)(\theta - \sin\theta) = 12.8$

attempt to solve their equation or use of graph ***(M1)***

$\theta = 1.59934...$

$6.53330...$

$r = 6.53$ ***A1***

METHOD 2

recognising $\frac{\text{area}_{\text{segment}}}{\text{area}_{\text{triangle}}} = \frac{3}{5}$ (or equivalent) (seen anywhere) ***(M1)***

$\text{area}_{\text{segment}} = \frac{1}{2}r^2(\theta - \sin\theta)$ (seen anywhere) ***(A1)***

$$\frac{\frac{1}{2}r^2(\theta - \sin\theta)}{\frac{1}{2}r^2\sin\theta} = \frac{3}{5}$$

correct equation without r ***A1***

$$\frac{(\theta - \sin\theta)}{\sin\theta} = \frac{3}{5}$$

$\theta = 1.59934...$ ***(A1)***

continued…

Question 6 continued

attempt to solve for r using their θ ***(M1)***

$\frac{1}{2}r^2\left(1.59... - \sin 1.59...\right) = 12.8$

$6.53330...$

$r = 6.53$ ***A1***

METHOD 3

recognising $\text{area}_{\text{segment}} = \frac{3}{8} \times \text{area}_{\text{sector}}$ (seen anywhere) ***(M1)***

$\text{area}_{\text{segment}} = \frac{1}{2}r^2\left(\theta - \sin\theta\right)$ (seen anywhere) ***(A1)***

$\frac{1}{2}r^2\left(\theta - \sin\theta\right) = \frac{3}{8} \times \frac{1}{2}r^2\theta$

correct equation without r ***A1***

$\frac{1}{2}\left(\theta - \sin\theta\right) = \frac{3}{8} \times \frac{1}{2}\theta$

$\theta = 1.59934...$ ***(A1)***

attempt to solve for r using their θ ***(M1)***

$\frac{1}{2}r^2\left(1.59... - \sin 1.59...\right) = 12.8$

$6.53330...$

$r = 6.53$ ***A1***

[6 marks]

9. (a) **METHOD 1**

correct direction vector, **e.g.** $\begin{pmatrix}2\\1\\5\end{pmatrix}-\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}$ (or equivalent) ***A1***

attempt to find direction normal using vector product ***(M1)***

$\begin{pmatrix}2\\1\\3\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}$

$\begin{pmatrix}1\\1\\-1\end{pmatrix}$ (or equivalent) ***A1***

correct substitution of a point on the line to obtain d ***A1***

$\begin{pmatrix}2\\1\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=d\Rightarrow d=-2$ OR $\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=d\Rightarrow d=-2$ OR

$\begin{pmatrix}2\\1\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\\1\end{pmatrix}=d\Rightarrow d=2$ (or equivalent)

hence equation of the plane $\prod_1$ is $x+y-z=-2$ ***AG***

METHOD 2

attempt to form parametric equations of a plane through point $(0,0,2)$ and two vectors in the plane $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\3\end{pmatrix}$ ***M1***

$\underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$ ***A1***

$x=\lambda+2\mu$

$y=\mu$

$z=2+\lambda+3\mu$

continued…

Question 9 continued

$\lambda = x - 2y$

$z = 2 + \lambda + 3y$ ***A1***

$z = 2 + x - 2y + 3y$ ***A1***

$x + y - z = -2$ ***AG***

[4 marks]

(b) **METHOD 1**

attempt to form an equation to eliminate one of the variables ***M1***

EITHER

$(b-2)y + z = 7$ ***A1***

$2y - 3z = -2 - d$ ***A1***

OR

$x + (b-1)y = 5$ ***A1***

$3x + y = d - 4$ ***A1***

THEN

$(3b-4)y = 19 - d$

$y = \frac{19-d}{3b-4}$

$b = \frac{4}{3}$ and $d = 19$ ***A1***

Note: Award ***M1A1*** for $b = \frac{4}{3}$ seen anywhere

continued…

Question 9 continued

METHOD 2

$$\left.\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & b & -1 & 3 \\ 1 & -1 & 2 & d \end{array}\right.$$

attempt to use row reduction ***M1***

$$\left.\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & b-2 & 1 & 7 \\ 0 & -2 & 3 & d+2 \end{array}\right.$$ or equivalent ***A1***

$$\left.\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & b-2 & 1 & 7 \\ 0 & 0 & 3+\frac{2}{b-2} & d+2+\frac{14}{b-2} \end{array}\right.$$ or equivalent ***(A1)***

$3+\frac{2}{b-2}=0$ and $d+2+\frac{14}{\frac{4}{3}-2}=0$

$b=\frac{4}{3}$ and $d=19$ ***A1***

Note: Award ***M1A1*** for $b=\frac{4}{3}$ seen anywhere

continued..

Question 9 continued

METHOD 3

attempt to use determinant and equate it to zero ***(M1)***

$$\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & b & -1 & 3 \\ 1 & -1 & 2 & d \end{array}$$

$$\det\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & b & -1 \\ 1 & -1 & 2 \end{bmatrix}\right) = 1\times\begin{vmatrix} b & -1 \\ -1 & 2 \end{vmatrix} - 1\times\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} + (-1)\times\begin{vmatrix} 2 & b \\ 1 & -1 \end{vmatrix}$$

$\det = 3b - 4$

$3b - 4 = 0$

$b = \frac{4}{3}$ ***A1***

$$\det\left(\begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & 3 \\ 1 & 2 & d \end{bmatrix}\right) = 0$$

(A1)

$d = 19$

A1

Note: Award ***(M1)A1*** for $b = \frac{4}{3}$ seen anywhere

[4 marks]

Total [8 marks]

SECTION B

1. (a) substituting into cosine rule ***(M1)***

$\left(\mathrm{BC}^2 =\right) 12^2 + 7^2 - 2\times 12\times 7\times \cos\left(116^\circ\right)$ ***(A1)***

$\sqrt{266.646...} = 16.3293...$

$\left(\mathrm{BC} =\right) 16.3$ ***A1***

[3 marks]

(b) substituting into sine rule or cosine rule for angle $\mathrm{A\hat{C}B}$ or angle $\mathrm{A\hat{B}C}$ ***(M1)***

$\frac{\sin\left(\mathrm{A\hat{C}B}\right)}{7} = \frac{\sin\left(116^\circ\right)}{16.3293...}$ **OR** $7^2 = 12^2 + 16.3293...^2 - 2\times 12\times 16.3293...\times\cos(\mathrm{A\hat{C}B})$

OR $180° - 116° - 41.3°$ from finding angle $\mathrm{A\hat{B}C}$ ***(A1)***

$22.6618\ldots$

$\left(\mathrm{A\hat{C}B} =\right) 22.7\,(^\circ)$ (accept $22.8(^\circ)$ if using previous 3sf AND the cosine rule) ***A1***

[3 marks]

Total [6 marks]

4. Attempts to use the area formula for triangle AOB **AND** equate it to 26 ***(M1)***

$\frac{1}{2}a \times b \times \sin(2.51) = 26$ **OR** $\frac{1}{2}\mathrm{OA} \times \mathrm{OB} \times \sin \mathrm{AOB} = 26$

$\frac{1}{2} \times r^2 \times \sin(2.51) = 26$ or equivalent ***(A1)***

$r = 9.38462...$ ***(A1)***

use of arc length formula to find the major arc ACB ***(M1)***

$9.38462... \times (2\pi - 2.51)$ **OR** $2\pi \times 9.38462... - 9.38462... \times 2.51$ **OR** $9.38462... \times 3.773185...$

$35.4099\ldots$

(arc length =) $35.4\ (\mathrm{cm})$ ***A1***

[5 marks]

6. (a) substituting into scalar product formula ***(M1)***

(i) $(\boldsymbol{a}\bullet\boldsymbol{c}=)-5p-42$ ***A1***

(ii) $(\boldsymbol{b}\bullet\boldsymbol{c}=)-8p-54$ ***A1***

[3 marks]

(b) substituting into the angle between two vectors formula to find either $\boldsymbol{a}\bullet\boldsymbol{c}$ or $\boldsymbol{b}\bullet\boldsymbol{c}$ ***(M1)***

$\cos\theta=\frac{-5p-42}{\sqrt{74}\sqrt{p^2+36}}, \cos\theta=\frac{-8p-54}{\sqrt{145}\sqrt{p^2+36}}$ ***(A1)(A1)***

equating their two expressions for $\cos\theta$ ***(M1)***

$\frac{-5p-42}{\sqrt{74}}=\frac{-8p-54}{\sqrt{145}}$

$p=4.78727...$

$(p=)\ 4.79$ ***A1***

[5 marks]

Total [8 marks]

2.

(a) (i) attempt to use Pythagoras' theorem ***(M1)***

$\sqrt{12500^2 + 6000^2}$

$= 13900$ (km) $(13865.4...)$ ***A1***

[2 marks]

(ii) attempt to use an appropriate inverse trig ratio ***(M1)***

e.g. $\arctan\left(\frac{12500}{6000}\right)$ or $\arctan\left(\frac{6000}{12500}\right)$

identifying the correct angle ***(M1)***

$= (0)64.4^\circ$ OR $064^\circ (64.3589...)$ ***A1***

Note: The two method marks can be awarded independently.

The first ***(M1)*** can come from any geometry leading to an inverse trigonometric ratio.

The second ***(M1)*** requires recognition of which angle is the bearing, possibly on a diagram. 64.4 appearing in intermediate working is not sufficient.

[3 marks]

(b) (i) $\boldsymbol{p.n} = 0$ ***M1***

so the angle is 90° OR $\frac{\pi}{2}$ rads ***A1***

Note: Award ***M0A0*** for an answer of 90 degrees without any mention that the scalar product is zero.

[2 marks]

(ii) attempt to use arc length formula $s = r\theta$ OR $\frac{90^\circ}{360}$ of circumference ***(M1)***

$= 6000\left(\frac{\pi}{2}\right)$ OR $\frac{1}{4} \times 2\pi \times 6000$ ***A1***

$= 3\pi$ thousand km OR 3000π km ***AG***

[2 marks]

continued

Question 2 continued.

(c) (i) attempt to find cross product (e.g. correct except for sign error) ***(M1)***

$$\boldsymbol{a}\times\boldsymbol{p}=\begin{pmatrix}0\\-36\\0\end{pmatrix}$$ ***A1***

[2 marks]

(ii) **METHOD 1**

$$\boldsymbol{a}\times\boldsymbol{n}=\begin{pmatrix}0\\0\\36\end{pmatrix}$$ ***A1***

choosing to use scalar product for their $\boldsymbol{a}\times\boldsymbol{p}$ and $\boldsymbol{a}\times\boldsymbol{n}$ ***M1***

$(\boldsymbol{a}\times\boldsymbol{p}).(\boldsymbol{a}\times\boldsymbol{n})=\mathbf{0}$ ***A1***

Note: The final ***A1*** can be awarded for seeing the dot product equals zero without incorrect working. Follow through from (i) is permitted and if they explicitly state that they are using $\boldsymbol{a}\times\boldsymbol{n}$ then allow internal follow through.

so angle is 90° ***AG***

METHOD 2

$$\boldsymbol{a}\times\boldsymbol{n}=\begin{pmatrix}0\\0\\36\end{pmatrix}$$ ***A1***

attempt to use cross product for their $\boldsymbol{a}\times\boldsymbol{p}$ and $\boldsymbol{a}\times\boldsymbol{n}$. This must include both an attempt to find the cross product and knowledge that $\sin\theta=\frac{|x\times y|}{|x||y|}$ ***M1***

$$(\boldsymbol{a}\times\boldsymbol{n})\times(\boldsymbol{a}\times\boldsymbol{p})=\begin{pmatrix}1296\\0\\0\end{pmatrix}\text{ OR }(\boldsymbol{a}\times\boldsymbol{p})\times(\boldsymbol{a}\times\boldsymbol{n})=\begin{pmatrix}-1296\\0\\0\end{pmatrix}$$

$\sin\theta=\frac{1296}{36\times36}=1$ ***A1***

so angle is 90° ***AG***

[3 marks]

continued

Question 2 continued.

(d) attempt to use $\frac{\theta}{360} \times 2\pi r = 6$ OR $s = r\theta$ ***(M1)***

$\frac{\theta}{360} \times 2\pi \times 6 = 6$ OR $6 = 6\theta \Rightarrow \theta = 1$ radian

$\theta = 57.2957...^\circ$ ***A1***

Note: The fourth significant figure must be seen somewhere to award the ***A1***.

$\theta = 57.3^\circ$ to 3 significant figures ***AG***

[2 marks]

continued

Question 2 continued.

(e) **METHOD 1**

EITHER (Scalar product)

attempt to use scalar product to find $\mathrm{B\hat{O}M} = \arccos\left(\frac{\boldsymbol{b}.\boldsymbol{m}}{|\boldsymbol{b}||\boldsymbol{m}|}\right)$ ***(M1)***

Note: This may be written as $\cos\left(\mathrm{B\hat{O}M}\right) = \frac{\boldsymbol{b}.\boldsymbol{m}}{|\boldsymbol{b}||\boldsymbol{m}|}$.

$= \arccos\left(\frac{\left(6\cos 120^\circ\right)\left(6\cos 57.3^\circ\right)}{6^2}\right)$ ***(A1)***

Note: This ***(A1)*** can be awarded for $\cos\theta = \frac{6\cos 120^\circ \times 6\cos 57.3^\circ}{6^2}$.

$= 106^\circ \left(= 105.673\ldots^\circ\right)$ OR 1.84 radians $(1.84434\ldots)$ ***(A1)***

OR (Cosine Rule)

$|\mathrm{BM}| = \sqrt{(6\sin 120 - 0)^2 + (6\cos 120 - 6\sin 57.3)^2 + (0 - 6\sin 57.3)^2}$

$= 9.56\left(9.56299...\right)$ OR $9560\mathrm{km}$ ***(A1)***

attempt to use the cosine rule to find angle BOM ***(M1)***

$|\mathrm{BM}|^2 = 6^2 + 6^2 - 2\times 6\times 6\times \cos \mathrm{BOM}$

$\mathrm{BOM} = 106^\circ \left(= 105.673\ldots^\circ\right)$ OR 1.84 radians $(1.84434\ldots)$ ***(A1)***

THEN

attempt to use $\frac{\theta}{360}\times 2\pi r$ OR $s = r\theta$ ***(M1)***

$= \frac{105.671^\circ}{360}\times 2\pi \times 6$

$= 11.1\ \left(11.0660\ldots \text{thousand km}\right)$ ***A1***

Note: Accept an answer of 11100 OR $11066.0\ldots$ without units (i.e. “km” is implicit).

continued

Question 2 continued.

METHOD 2 (Cross product)

attempt to use cross product to find $\mathrm{B\hat{O}M} = \arcsin\dfrac{|\boldsymbol{b}\times\boldsymbol{m}|}{|\boldsymbol{b}||\boldsymbol{m}|}$ ***(M1)***

Note: This may be written as $\sin\mathrm{B\hat{O}M} = \dfrac{|\boldsymbol{b}\times\boldsymbol{m}|}{|\boldsymbol{b}||\boldsymbol{m}|}$.

$$= \frac{\left|\begin{pmatrix} 36\cos 120\sin\theta \\ -36\sin 120\sin\theta \\ 36\sin 120\cos\theta \end{pmatrix}\right|}{6\times 6}$$

$$= \frac{\sqrt{\sin^2\theta + 3}}{2} = 0.962818\ldots$$ ***(A1)***

$$\mathrm{B\hat{O}M} = 180 - \arcsin 0.962818\ldots$$

$$= 106°\left(105.673\ldots,\ 1.844346\,\text{rads}\right)$$ ***(A1)***

attempt to use $\dfrac{\theta}{360}\times 2\pi r$ OR $s = r\theta$ ***(M1)***

$$= \frac{105.671}{360}\times 2\pi\times 6$$

$= 11.1$ (11.0660 thousand km) ***A1***

Note: Award at most ***(M1)(A1)(A0)(M1)A0*** for an answer of 7.78 thousand km from finding the principal root of the arcsin.

[5 marks]

continued

Question 2 continued.

(f) attempt to calculate $\boldsymbol{b}\times\boldsymbol{m}$ ***(M1)***

$$\boldsymbol{b}\times\boldsymbol{m}=\begin{pmatrix}36\cos 120^\circ \sin 57.3^\circ\\ -36\sin 120^\circ \sin 57.3^\circ\\ 36\sin 120^\circ \cos 57.3^\circ\end{pmatrix}=\begin{pmatrix}-15.1464\ldots\\ -26.2344\ldots\\ 16.8449\ldots\end{pmatrix}$$ ***(A1)***

attempt to use scalar product to find angle between vectors ***(M1)***

$$\left(\cos\theta=\right)\left(\frac{(\boldsymbol{b}\times\boldsymbol{m}).(\boldsymbol{b}\times\boldsymbol{p})}{|\boldsymbol{b}\times\boldsymbol{m}||\boldsymbol{b}\times\boldsymbol{p}|}\right)$$

$$\left(\cos\theta=\right)\left(\frac{1090.54...}{34.6614...\times 36}\right)$$ ***(A1)(A1)***

Note: Award ***(A1)*** for numerator, ***(A1)*** for denominator.

bearing is $(0)29.1^\circ\left(29.0770...^\circ\right)$ ***A1***

Note: Do not penalise absence of degree symbol.

Award full marks for answer in radians $=0.507$.

Award full marks for 029°.

Special Case: If a candidate calculates $\boldsymbol{m}\times\boldsymbol{b}$ instead of $\boldsymbol{b}\times\boldsymbol{m}$ then you can award at most ***(M1)(A0)(M1)(A1)(A1)A0***.

[6 marks]

Total[27 marks]

8. (a) **METHOD 1 Using the angle sum formula**

attempt to use angle sum formula ***M1***

$\sin 2\theta \cos\theta + \cos 2\theta \sin\theta$ ***A1***

attempt to use a double angle identity or the Pythagorean identity ***M1***

$2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta$ OR $2\sin\theta(1-\sin^2\theta) + \cos 2\theta \sin\theta$

$2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta$ (or equivalent) ***A1***

$3\sin\theta - 4\sin^3\theta$ ***AG***

METHOD 2 Using de Moivre's Theorem and comparing coefficients

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ ***A1***

attempt to expand LHS ***M1***

$\cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$

comparing imaginary parts ***M1***

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ ***A1***

$(\sin 3\theta =) 3\sin\theta - 4\sin^3\theta$ ***AG***

[4 marks]

(b) attempt to rewrite expression using result of part (a) ***(M1)***

$y = 2\sin 3\left(\theta + \frac{\pi}{6}\right)$, $y = 2\sin\left(3\theta + \frac{\pi}{2}\right)$

vertical stretch factor 2 (seen anywhere) ***A1***

horizontal stretch factor $\frac{1}{3}$ followed by horizontal translation $\frac{\pi}{6}$ to the left ***A2***

OR

horizontal translation $\frac{\pi}{2}$ to the left followed by horizontal stretch factor $\frac{1}{3}$ ***A2***

Note: Award ***A1A0*** for correct horizontal transformations specified in the wrong order.

[4 marks]

Total [8 marks]

Section B

10. (a) $L_1 : \boldsymbol{r} = \begin{pmatrix} 1 \\ -13 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ***A1***

Note: The $\boldsymbol{r} =$ must be seen.

[1 mark]

(b) $L_2 : \boldsymbol{s} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$ OR $\boldsymbol{s} = \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$ (or equivalent) ***A1A1***

Note: The $\boldsymbol{s} =$ must be seen, but do not penalise if same error seen in part (a).
Allow $\boldsymbol{r} =$ instead of $\boldsymbol{s} =$.

[2 marks]

(c) $\begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} \neq k \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$ so L_1, L_2 are not parallel (may be seen anywhere) ***R1***

Note: Not parallel may be shown using cross product.

equating any two components from each line ***M1***

$-1 + 7\lambda = 2 + 5\mu \ (7\lambda - 5\mu = 3)$

$1 + \lambda = -4 - 2\mu \ (\lambda + 2\mu = -5)$

$-13 + 2\lambda = 2 - \mu \ (2\lambda + \mu = 15)$

solve simultaneously: (1st/2nd compts OR 1st/3rd compts OR 2nd /3rd compts) ***M1***

$\lambda = -1, \mu = -2$ OR $\lambda = \frac{78}{17}, \mu = \frac{99}{17}$ OR $\lambda = \frac{35}{3}, \mu = -\frac{25}{3}$ ***A1***

check for remaining component, e.g. 3rd component: $-13 + (-1)2 \neq 2 + (-2)(-1)$ ***R1***

$(-15 \neq 4)$

hence lines are not parallel and do not intersect, so are skew. ***AG***

Note: Condone omission of ***AG*** line.

[5 marks]

continued…

Question 10 continued

(d) attempt to calculate $\vec{PN} = \vec{ON} - \vec{OP}$ ***(M1)***

$$\begin{pmatrix} 2+5\mu \\ -4-2\mu \\ 2-\mu \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -13 \end{pmatrix}$$

$$= \begin{pmatrix} 3+5\mu \\ -5-2\mu \\ 15-\mu \end{pmatrix}$$ ***A1***

attempt to calculate $\vec{PN} \bullet \vec{AB}$ ***(M1)***

$$= \begin{pmatrix} 3+5\mu \\ -5-2\mu \\ 15-\mu \end{pmatrix} . \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$= 10 + 30\mu$ ***A1***

[4 marks]

(e) setting and solving $10 + 30\mu = 0$ ***(M1)***

$\mu = -\frac{1}{3}$ ***A1***

so $N\left(\frac{1}{3}, -\frac{10}{3}, \frac{7}{3}\right)$ ***A1***

[3 marks]

continued…

Question 10 continued

(f) considers the vectors $k\begin{pmatrix}-1\\1\\-13\end{pmatrix}$ and $s\begin{pmatrix}1\\-10\\7\end{pmatrix}$ or equivalent ***(A1)***

attempt to find cross product in either order: ***(M1)***

e.g. $\begin{pmatrix}-1\\1\\-13\end{pmatrix}\times\begin{pmatrix}1\\-10\\7\end{pmatrix}$

$=\begin{pmatrix}-123\\-6\\9\end{pmatrix}$ ***A1***

equation of plane is $-123x-6y+9z=0$ (or equivalent) ***A1***

$(\alpha=-123,\ \beta=-6,\ \gamma=9,\ \delta=0)$

[4 marks]

Total [19 marks]

1. (a) (i) $1.25663\ldots$

$\theta = 1.26 \left(= \frac{2\pi}{5}\right)$ ***A1***

(ii) $\frac{2\pi}{5} \times r = 12$ or equivalent ***(A1)***

$9.54929...$

$r = 9.55$ (cm) $\left(= \frac{60}{2\pi} = \frac{30}{\pi}\right)$ ***A1***

[3 marks]

(b) **METHOD 1**

attempt to find area of at least one shaded segment ***(M1)***

$\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$ OR $\frac{1}{2}r^2(\theta - \sin\theta)$

$\frac{1}{2}(9.54929...)^2 \times 1.25663\ldots - \frac{1}{2}(9.54929...)^2 \sin 1.25663\ldots$ (or equivalent) ***(A1)***

shaded area $= 5 \times (57.2957\ldots - 43.3629\ldots) (= 5 \times 13.9328...)$

$69.6640...$

$= 69.7\ (\text{cm}^2)$ ***A1***

[3 marks]

METHOD 2

attempt to find area of one triangle ***(M1)***

$\frac{1}{2} \times (9.54929...)^2 \times \sin 1.25663...$

attempt to subtract five times their triangle area from their circle area ***(M1)***

$\pi \times (9.54929...)^2 - 5 \times \frac{1}{2} \times (9.54929...)^2 \times \sin 1.25663...$

$69.6640...$

$= 69.7\ (\text{cm}^2)$ ***A1***

[3 marks]

Total [6 marks]

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4. (a) (i) $a = -5$ ***A1***

(ii) **EITHER**

period $= 32$ ***(A1)***

OR

correct substitution of a point on the graph of g and their a ***(A1)***

$2 = -5\cos(b \times 16) - 3$ **OR** $-3 = -5\cos(b \times 8) - 3$

Note: Award ***A0*** if candidate substitutes $x = 0, y = -8$.

THEN

$0.196349...$

$b = 0.196 \left(= \frac{2\pi}{32} = \frac{\pi}{16}\right)$ ***A1***

[3 marks]

(b) $k = 32$ ***A1***

[1 mark]

(c) (i) **EITHER**

attempt to use a horizontal translation ***(M1)***

$24 - q = 16$ OR $16 - q = 8$

OR

substituting a point on the graph of g into $f(x) = p\sin(b(x-q)) - 3$ ***(M1)***

$-8 = p\sin\left(\frac{\pi}{16}(0-q)\right) - 3$ OR $-3 = p\sin\left(\frac{\pi}{16}(8-q)\right) - 3$ OR $2 = p\sin\left(\frac{\pi}{16}(16-q)\right) - 3$

THEN

$q = 8$ ***A1***

(ii) $p = 5$ ***A1***

[3 marks]

Total [7 marks]

12. (a) attempt to set up a system of equations with at least one correct equation ***M1***

$5+s=-1+2t$, $4-s=7+t$, $2+s=-5+3t$

$t=1$ **OR** $s=-4$ ***A1***

$\begin{pmatrix}5\\4\\2\end{pmatrix}-4\begin{pmatrix}1\\-1\\1\end{pmatrix}$ **OR** $\begin{pmatrix}-1\\7\\-5\end{pmatrix}+\begin{pmatrix}2\\1\\3\end{pmatrix}$ ***A1***

point of intersection $\begin{pmatrix}1\\8\\-2\end{pmatrix}$ ***AG***

[3 marks]

(b) (i) $\boldsymbol{r}=\begin{pmatrix}5\\4\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$ **OR** $\boldsymbol{r}=\begin{pmatrix}1\\8\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$

OR $\boldsymbol{r}=\begin{pmatrix}-1\\7\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$ ***A1***

Note: $\boldsymbol{r}=$ must be seen.

continued…

Question 12 continued.

(ii) **METHOD 1**

attempt to write equation of the plane in the form $\boldsymbol{r.n} = \boldsymbol{a.n}$ ***M1***

$$\boldsymbol{r} \cdot \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} (= -18)$$

$-4x - y + 3z = -18$ ***A1***

$4x + y - 3z = 18$ ***AG***

Note: Students may use $\begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix}$ in place of $\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$ or other points from lines L_1 or L_2.

METHOD 2

recognizing that $\boldsymbol{b} \times \boldsymbol{c}$ is normal to the plane. ***M1***

$-4x - y + 3z = k$

correct substitution of one of the points to find k ***A1***

$k = 18$

$4x + y - 3z = 18$ ***AG***

[3 marks]

(c) (i) $q = 18$ ***A1***

(ii) $r = -6$ ***A1***

[2 marks]

continued…

Question 12 continued.

(d) $\vec{PQ} = \begin{pmatrix} -4.5 \\ 18 \\ 0 \end{pmatrix}$ **OR** $\vec{PR} = \begin{pmatrix} -4.5 \\ 0 \\ -6 \end{pmatrix}$ seen anywhere ***(A1)***

attempt to find a vector product ***(M1)***

$\vec{PQ} \times \vec{PR}$

$\vec{PQ} \times \vec{PR} = \begin{pmatrix} -108 \\ -27 \\ 81 \end{pmatrix}$ ***A1***

attempt to find magnitude of their cross product ***(M1)***

$\sqrt{(-108)^2 + (-27)^2 + 81^2}$ **,** $\sqrt{18954}, 137.673$

Area= 68.8 $\left(= \frac{1}{2}\sqrt{18954}\right)\left(= \frac{27\sqrt{26}}{2}\right)$ ***A1***

[5 marks]

(e) $\boldsymbol{r}_3 = \begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}$ ***A2***

Note: Accept any multiple of the direction vector.
Award ***A1A0*** if $\boldsymbol{r}_3 =$ is not seen.
Accept $\boldsymbol{r} =$

[2 marks]

continued…

Question 12 continued.

(f) attempt to set their $\boldsymbol{r}_3$ equal to the vector $\begin{pmatrix} -11 \\ 5 \\ 7 \end{pmatrix}$ ***(M1)***

$-11 = 1 - 4\gamma, \quad 5 = 8 - \gamma, \quad 7 = -2 + 3\gamma$

$\gamma = 3$ ***A1***

Note: Accept any correct value of γ for their direction vector. For example accept $\gamma = -3$ for a direction vector of $\begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$.

[2 marks]

(g) attempt to subtract appropriate vectors (in any order) OR consider $3\boldsymbol{n}$ ***(M1)***

$\begin{pmatrix} -11 \\ 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -12 \\ -3 \\ 9 \end{pmatrix}$

(height of pyramid =) $\sqrt{(-12)^2 + (-3)^2 + 9^2}, \quad \sqrt{234}, \ 3\sqrt{26}\ , 15.2970...$ ***A1***

attempt to substitute their values into the formula for the volume of a pyramid ***(M1)***

$\frac{1}{3} \times \frac{1}{2}\sqrt{18954} \times \sqrt{234}$

$V = 351$ (exact) (cubic units) ***A1***

[4 marks]

Total [21 marks]

1. (a) attempt to use double angle formula for $\cos 2x$ (seen anywhere) ***M1***

Note: Only award the ***M1*** if a double angle formula for $\cos 2x$ is used, and not only for stating it.

$3\left(1-2\sin^2 x\right)=3-6\sin^2 x$ (seen anywhere) ***A1***

$3\cos 2x+11\sin x=3+11\sin x-6\sin^2 x$ ***AG***

[2 marks]

(b) **METHOD 1**

attempt to solve a quadratic equation in $\sin x$ (***M1)***

$\sin x=0.333333...\left(=\frac{1}{3}\right)$

$x=19.4712....,160.528...$

$x=19.5,161$ ***A1A1***

METHOD 2

attempt to solve $3\cos 2x+11\sin x-6=0$ (may be seen in a sketch) ***(M1)***

$x=19.4712....,160.528...$

$x=19.5,161$ ***A1A1***

[3 marks]

Total [5 marks]

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3. (a) perimeter of sector is $1.5 \times \text{circumference of circle}$ (seen anywhere) ***A1***

$1.5 \times 16\pi$ OR 24π OR $75.3982...$

perimeter of sector $r\theta + 2r$ ***(M1)***

$24\pi = r\theta + 2r$ ***A1***

$r = \dfrac{24\pi}{2+\theta}$ ***AG***

[3 marks]

(b) correct substitution of $r = \dfrac{24\pi}{2+\theta}$ into $\dfrac{1}{2}r^2\theta$ (***A1)***

$64\pi = \dfrac{1}{2} \times \theta \times \left(\dfrac{24\pi}{2+\theta}\right)^2$

attempt to solve their equation (must be quadratic) ***(M1)***

$\theta = 0.411273...$

$\theta = 0.411$ ***A1***

Note: Award ***(A1)(M1)A0*** if additional answers are given.

[3 marks]

Total [6 marks]

APOLLO

Statistics and probability

87 questions

3. (a) attempt to use definition of outlier

$1.5\times 20+Q_3$ ***(M1)***

$1.5\times 20+U\geq 75$ $\left(\Rightarrow U\geq 45\text{, accept } U>45\right)$ OR $1.5\times 20+Q_3=75$ ***A1***

minimum value of $U=45$ ***A1***

[3 marks]

(b) attempt to use interquartile range ***(M1)***

$U-L=20$ (may be seen in part (a)) OR $L\geq 25$ (accept $L>25$)

minimum value of $L=25$ ***A1***

[2 marks]

Total [5 marks]

10. (a) recognising probabilities sum to 1 ***(M1)***

$p + p + p + \frac{1}{2}p = 1$

$p = \frac{2}{7}$ ***A1***

[2 marks]

(b) valid attempt to find $\mathrm{E}(X)$ ***(M1)***

$1\times p + 2\times p + 3\times p + 4\times \frac{1}{2}p \left(= 8p\right)$

$\mathrm{E}(X) = \frac{16}{7}$ ***A1***

[2 marks]

(c) (i) $0 \le r \le 1$ ***A1***

(ii) Attempt to find a value of q ***(M1)***

$0 \le 1 - 3q \le 1$ OR $r = 0 \Rightarrow q = \frac{1}{3}$ OR $r = 1 \Rightarrow q = 0$

$0 \le q \le \frac{1}{3}$ ***A1***

[3 marks]

(d) $\mathrm{E}(Y) = 1\times q + 2\times q + 3\times q + 4\times r \left(= 2 + 2r \text{ OR } 4 - 6q\right)$ ***(A1)***

one correct boundary value ***A1***

$1\times\frac{1}{3} + 2\times\frac{1}{3} + 3\times\frac{1}{3} + 4\times 0 \left(= 2\right)$ OR

$1\times 0 + 2\times 0 + 3\times 0 + 4\times 1 \left(= 4\right)$ OR

$2 + 2(0) \left(= 2\right)$ OR

$2 + 2(1) \left(= 4\right)$ OR

$4 - 6(0) \left(= 4\right)$ OR $4 - 6\left(\frac{1}{3}\right) \left(= 2\right)$

$2 \le \mathrm{E}(Y) \le 4$ ***A1***

[3 marks]

continued…

Question 10 continued

(e) **METHOD 1**

evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

solving for either q or r ***M1***

$\frac{6}{7}(q+1-3q)=\frac{1}{2}$ OR $\frac{6}{7}\left(\frac{1-r}{3}+r\right)=\frac{1}{2}$ OR $3pq+3p(1-3q)=\frac{1}{2}$

OR $3p\left(\frac{1-r}{3}\right)+3pr=\frac{1}{2}$

EITHER two correct values

$q=\frac{5}{24}$ and $r=\frac{3}{8}$ ***A1A1***

OR one correct value

$q=\frac{5}{24}$ OR $r=\frac{3}{8}$ ***A1***

substituting their value for q or r ***A1***

$4-6\left(\frac{5}{24}\right)$ OR $2+2\left(\frac{3}{8}\right)$

THEN

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]

continued…

Question 10 continued

METHOD 2 (solving for $\mathrm{E}(Y)$)

evidence of choosing at least four correct outcomes from
1&2, 1&3, 1&4, 2&3, 2&4, 3&4 ***(M1)***

$\frac{6}{7}q+\frac{6}{7}r$ OR $3pq+3pr$ OR $pq+pq+p(1-3q)+pq+p(1-3q)+p(1-3q)$ ***(A1)***

rearranging to make q the subject ***M1***

$$q=\frac{4-\mathrm{E}(Y)}{6}$$

$3pq+3p(1-3q)=\frac{1}{2}$ ***M1***

$$\frac{6}{7}\times\left(\frac{4-\mathrm{E}(Y)}{6}\right)+\frac{6}{7}\left(1-3\left(\frac{4-\mathrm{E}(Y)}{6}\right)\right)=\frac{1}{2}$$ ***A1***

$$\frac{2(\mathrm{E}(Y)-1)}{7}=\frac{1}{2}$$

$\mathrm{E}(Y)=\frac{11}{4}$ ***A1***

[6 marks]
Total [16 marks]

1. (a) $a = 0.433156...,\ b = 4.50265...$

$a = 0.433,\ b = 4.50$ ***A1A1***

[2 marks]

(b) attempt to substitute $x = 18$ into their equation ***(M1)***

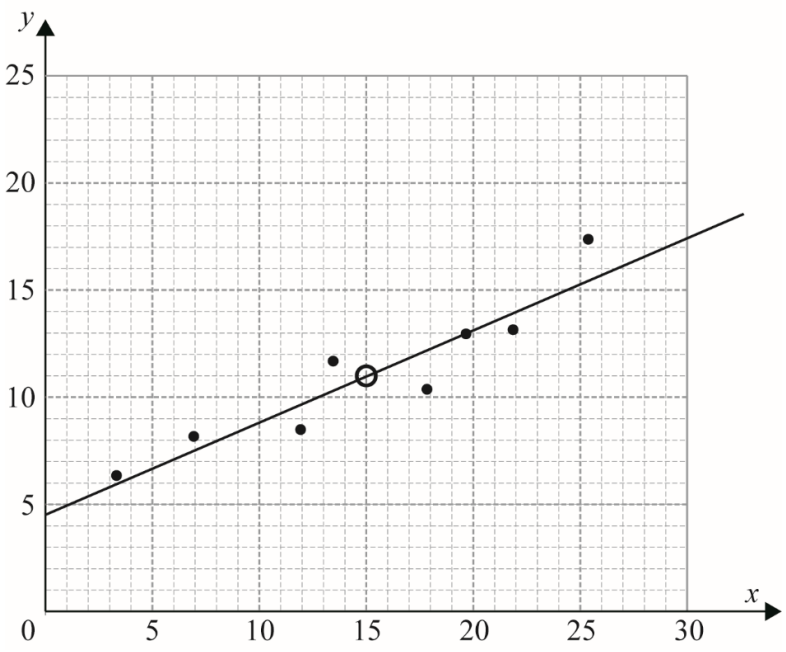
$y = 0.433 \times 18 + 4.50$
$= 12.2994...$
$= 12.3$ ***A1***

[2 marks]

(c) $\bar{x} = 15\ ,\ \bar{y} = 11$ ***A1***

[1 mark]

(d)



A1A1

Note: Award marks as follows:

A1 for a straight line going through $(15, 11)$

A1 for intercepting the y-axis between their $b \pm 1.5$ (when their line is extended), which includes all the data for $3.3 \leq x \leq 25.3$.

If the candidate does not use a ruler, award ***A0A1*** where appropriate.

[2 marks]
Total [7 marks]

2.

Note: In this question, do not penalise incorrect use of strict inequality signs.

Let X = mass of a bag of sugar

(a) evidence of identifying the correct area ***(M1)***

$\mathrm{P}(X < 995) = 0.0765637...$

$= 0.0766$ ***A1***

[2 marks]

(b) 0.0766×100

≈ 8 ***A1***

[1 mark]

Note: Accept 7.66 .

(c) recognition that $\mathrm{P}(X > 1005 | X \geq 995)$ is required ***(M1)***

$\dfrac{\mathrm{P}(X \geq 995 \cap X > 1005)}{\mathrm{P}(X \geq 995)}$

$\dfrac{\mathrm{P}(X > 1005)}{\mathrm{P}(X \geq 995)}$ ***(A1)***

$\dfrac{0.07656\ldots}{1 - 0.07656\ldots}\left(= \dfrac{0.07656\ldots}{0.9234\ldots}\right)$

$= 0.0829$ ***A1***

[3 marks]

Total [6 marks]

7. (a) recognition of the need to integrate $\frac{x}{\sqrt{\left(x^2+k\right)^3}}$ ***(M1)***

$$\int \frac{x}{\sqrt{\left(x^2+k\right)^3}}\mathrm{d}x\left(=1\right)$$

EITHER

$u = x^2 + k \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ (or equivalent) ***(A1)***

$$\int \frac{x}{\sqrt{\left(x^2+k\right)^3}}\mathrm{d}x = \frac{1}{2}\int u^{-\frac{3}{2}}\mathrm{d}u$$

$= -u^{-\frac{1}{2}}(+c)\left(= -\left(x^2+k\right)^{-\frac{1}{2}}(+c)\right)$ ***A1***

continued…

Question 7 continued

OR

$\int \frac{x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{\sqrt{\left(x^2+k\right)^3}}\,\mathrm{d}x$ ***(A1)***

$= -\left(x^2+k\right)^{-\frac{1}{2}}(+c)$ ***A1***

THEN

attempt to use correct limits for their integrand and set equal to 1 ***M1***

$\left[-u^{-\frac{1}{2}}\right]_k^{16+k} = 1$ OR $\left[-\left(x^2+k\right)^{-\frac{1}{2}}\right]_0^4 = 1$

$-\left(16+k\right)^{-\frac{1}{2}} + k^{-\frac{1}{2}} = 1 \left(\Rightarrow \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{16+k}} = 1\right)$ ***A1***

$\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***AG***

[5 marks]

(b) attempt to solve $\sqrt{16+k} - \sqrt{k} = \sqrt{k}\sqrt{16+k}$ ***(M1)***

$k = 0.645038...$

$= 0.645$ ***A1***

[2 marks]

Total [7 marks]

1. (a) (i) $a = 0.805084...$ and $b = 2.88135...$
$a = 0.805$ and $b = 2.88$ ***A1A1***

(ii) $r = 0.97777...$
$r = 0.978$ ***A1***

[3 marks]

(b) a represents the (average) increase in waiting time (0.805 mins) per additional customer (waiting to receive their coffee) ***R1***

[1 mark]

(c) attempt to substitute $x = 7$ into their equation (***M1)***

$8.51693...$
8.52 (mins) ***A1***

[2 marks]
Total [6 marks]

3. (a) **EITHER**

$\mathrm{P}(S)+\mathrm{P}(T)+\mathrm{P}(S'\cap T')-\mathrm{P}(S\cap T)=1$ OR $\mathrm{P}(S\cup T)=\mathrm{P}((S'\cap T')')$ ***(M1)***

$0.7+0.2+0.18-\mathrm{P}(S\cap T)=1$ ***OR*** $\mathrm{P}(S\cup T)=1-0.18$

OR

a clearly labelled Venn diagram ***(M1)***

THEN

$\mathrm{P}(S\cap T)=0.08$ (accept 8%) ***A1***

Note: To obtain the ***M1*** for the Venn diagram all labels must be correct and in the correct sections. For example, do not accept 0.7 in the area corresponding to $S\cap T'$.

[2 marks]

(b) **EITHER**

$\mathrm{P}(T\cap S')=\mathrm{P}(T)-\mathrm{P}(T\cap S)(=0.2-0.08)$ OR

$\mathrm{P}(T\cap S')=\mathrm{P}(T\cup S)-\mathrm{P}(S)(=0.82-0.7)$ ***(M1)***

OR

a clearly labelled Venn diagram including $\mathrm{P}(S)$, $\mathrm{P}(T)$ and $\mathrm{P}(S\cap T)$ ***(M1)***

THEN

$=0.12$ (accept 12%) ***A1***

[2 marks]

(c) $\mathrm{P}(G\cap T)=\mathrm{P}(T|G)\mathrm{P}(G)$ (0.25×0.48) ***(M1)***

$=0.12$ ***A1***

[2 marks]

continued…

Question 3 continued

(d) **METHOD 1**

$\mathrm{P}(G)\times\mathrm{P}(T)\left(=0.48\times0.2\right)=0.096$ ***A1***

$\mathrm{P}(G)\times\mathrm{P}(T)\neq\mathrm{P}(G\cap T)\Rightarrow$ G and T are not independent ***R1***

METHOD 2

$\mathrm{P}(T\,|\,G)=0.25$ ***A1***

$\mathrm{P}(T\,|\,G)\neq\mathrm{P}(T)\Rightarrow$ G and T are not independent ***R1***

Note: Do not award ***A0R1.***

[2 marks]
Total [8 marks]

6. (a) $\mathrm{E}(X) = (n+1)\int_0^1 x^{n+1}\mathrm{d}x$ ***M1***

$= (n+1)\left[\frac{x^{n+2}}{n+2}\right]_0^1$ ***A1***

leading to $\mathrm{E}(X) = \frac{n+1}{n+2}$ ***AG***

[2 marks]

(b) **METHOD 1**

use of $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ ***M1***

$\mathrm{Var}(X) = (n+1)\int_0^1 x^{n+2}\mathrm{d}x - \left(\frac{n+1}{n+2}\right)^2$

$= (n+1)\left[\frac{1}{n+3}x^{n+3}\right]_0^1 - \left(\frac{n+1}{n+2}\right)^2$

$= \frac{n+1}{n+3} - \left(\frac{n+1}{n+2}\right)^2$ ***A1***

$= \frac{(n+1)(n+2)^2 - (n+1)^2(n+3)}{(n+2)^2(n+3)}$ ***M1***

EITHER

$= \frac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$ ***A1***

OR

$= \frac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$ ***A1***

THEN

so $\mathrm{Var}(X) = \frac{n+1}{(n+2)^2(n+3)}$ ***AG***

continued…

Question 6 continued

METHOD 2

use of $\mathrm{Var}(X) = \mathrm{E}\left(X - \mathrm{E}(X)\right)^2$ ***M1***

$$\mathrm{Var}(X) = (n+1)\int_0^1 \left(x - \frac{n+1}{n+2}\right)^2 x^n \mathrm{d}x$$

$$= (n+1)\left[\frac{1}{n+3}x^{n+3} - \frac{2(n+1)}{(n+2)^2}x^{n+2} + \frac{n+1}{(n+2)^2}x^{n+1}\right]_0^1$$

$$= \frac{n+1}{n+3} - \left(\frac{n+1}{n+2}\right)^2$$ ***A1***

$$= \frac{(n+1)\left((n+2)^2 - (n+1)(n+3)\right)}{(n+2)^2(n+3)}$$ ***M1***

EITHER

$$= \frac{(n+1)\left(n^2+4n+4-\left(n^2+4n+3\right)\right)}{(n+2)^2(n+3)}$$ ***A1***

OR

$$= \frac{\left(n^3+5n^2+8n+4\right)-\left(n^3+5n^2+7n+3\right)}{(n+2)^2(n+3)}$$ ***A1***

THEN

so $\mathrm{Var}(X) = \frac{n+1}{(n+2)^2(n+3)}$ ***AG***

[4 marks]
Total [6 marks]

10. (a) use of inverse normal to find z-score ***(M1)***

$z = 2.0537...$

$2.0537... = \frac{82 - 75}{\sigma}$ ***(A1)***

$\sigma = 3.408401...$

$\sigma = 3.41$ ***A1***

[3 marks]

(b) evidence of identifying the correct area under the normal curve ***(M1)***

$\mathrm{P}(T > 80) = 0.071193...$

$\mathrm{P}(T > 80) = 0.0712$ ***A1***

[2 marks]

(c) recognition that $\mathrm{P}(80 < T < 82)$ is required ***(M1)***

$P(T < 82 | T > 80) = \frac{\mathrm{P}(80 < T < 82)}{\mathrm{P}(T > 80)} = \left(\frac{0.051193...}{0.071193...}\right)$ ***(M1)(A1)***

$= 0.719075...$

$= 0.719$ ***A1***

[4 marks]

continued…

Question 10 continued

(d) recognition of binomial probability ***(M1)***

$X \sim \mathrm{B}(64, 0.071193...)$ or $\mathrm{E}(X) = 64 \times 0.071193...$ ***(A1)***

$\mathrm{E}(X) = 4.556353...$

$\mathrm{E}(X) = 4.56$ (flights) ***A1***

[3 marks]

(e) $\mathrm{P}(X > 6) = P(X \geq 7) = 1 - \mathrm{P}(X \leq 6)$ ***(M1)***

$= 1 - 0.83088...$ ***(A1)***

$= 0.1691196...$

$= 0.169$ ***A1***

[3 marks]

Total [15 marks]

4. (a) valid approach to find $\mathrm{P}(R)$ ***(M1)***

tree diagram (must include probabilty of picking box) with correct required probabilities

OR $\mathrm{P}(R \cap B_1) + \mathrm{P}(R \cap B_2)$ OR $\mathrm{P}(R \mid B_1)\mathrm{P}(B_1) + \mathrm{P}(R \mid B_2)\mathrm{P}(B_2)$

$\frac{5}{7} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2}$ ***(A1)***

$\mathrm{P}(R) = \frac{9}{14}$ ***A1***

[3 marks]

(b) events A and R are not independent, since $\frac{9}{14} \cdot \frac{1}{2} \neq \frac{5}{14}$ OR $\frac{5}{7} \neq \frac{9}{14}$ OR $\frac{5}{9} \neq \frac{1}{2}$

OR an explanation e.g. different number of red balls in each box ***A2***

Note: Both conclusion and reasoning are required. Do not split the ***A2***.

[2 marks]

Total [5 marks]

1. (a) use of GDC to give ***(M1)***

$r = 0.883529...$

$r = 0.884$ ***A1***

Note: Award the ***(M1)*** for any correct value of r, a, b or $r^2 = 0.780624...$ seen in part (a) or part (b).

[2 marks]

(b) $a = 1.36609...$, $b = 64.5171...$

$a = 1.37$, $b = 64.5$ ***A1***

[1 mark]

(c) attempt to find their difference ***(M1)***

$5 \times 1.36609...$ OR $1.36609...(h+5) + 64.5171... - (1.36609...h + 64.5171...)$

$6.83045...$

$= 6.83 (6.85 \text{ from } 1.37)$

the student could have expected her score to increase by 7 marks. ***A1***

Note: Accept an increase of 6, 6.83 or 6.85.

[2 marks]

(d) Lucy is incorrect in suggesting there is a causal relationship.
This might be true, but the data can only indicate a correlation. ***R1***

Note: Accept 'Lucy is incorrect as correlation does not imply causation' or equivalent.

[1 mark]

(e) no effect ***A1***

[1 mark]

Total [7 marks]

3. (a) recognize that the variable has a Binomial distribution ***(M1)***

$X \sim \mathrm{B}(30, 0.05)$

attempt to find $\mathrm{P}(X \geq 1)$ ***(M1)***

$1 - \mathrm{P}(X = 0)$ OR $1 - 0.95^{30}$ OR $1 - 0.214638...$ OR $0.785361...$

Note: The two ***M*** marks are independent of each other.

$\mathrm{P}(X \geq 1) = 0.785$ ***A1***

[3 marks]

(b) recognition of conditional probability ***(M1)***

$\mathrm{P}(X \leq 2 \mid X \geq 1)$ OR $\mathrm{P}(\text{at most 2 defective} \mid \text{at least 1 defective})$

Note: Recognition must be shown in context either in words or symbols but not just $\mathrm{P}(A \mid B)$.

$\frac{\mathrm{P}(1 \leq X \leq 2)}{\mathrm{P}(X \geq 1)}$ OR $\frac{\mathrm{P}(X = 1) + \mathrm{P}(X = 2)}{\mathrm{P}(X \geq 1)}$ ***(A1)***

$\frac{0.597540...}{0.785361...}$ OR $\frac{0.812178... - 0.214638...}{0.785361...}$ OR $\frac{0.338903... + 0.258636...}{0.785361...}$ ***(A1)***

$= 0.760847...$

$\mathrm{P}(X \leq 2 \mid X \geq 1) = 0.761$ ***A1***

[4 marks]

Total [7 marks]

7. (a) recognises that $\int_0^m \arccos x \, \mathrm{d}x = 0.5$ ***(M1)***

$m \arccos m - \sqrt{1-m^2} - \left(0 - \sqrt{1}\right) = 0.5$

$m = 0.360034...$

$m = 0.360$ ***A1***

[2 marks]

(b) **METHOD 1**

attempts to find at least one endpoint (limit) both in terms of m (or their m) and a ***(M1)***

$\mathrm{P}\left(m - a \leq X \leq m + a\right) = 0.3$

$\int_{0.360034...-a}^{0.360034...+a} \arccos x \, \mathrm{d}x = 0.3$ ***(A1)***

Note*:* Award ***(A1)*** for $\int_{m-a}^{m+a} \arccos x \, \mathrm{d}x = 0.3$.

$\left[x \arccos x - \sqrt{1-x^2} \right]_{0.360034...-a}^{0.360034...+a}$

attempts to solve their equation for a ***(M1)***

Note*:* The above ***(M1)*** is dependent on the first ***(M1)***.

$a = 0.124861...$

$a = 0.125$ ***A1***

METHOD 2

$\int_{-a}^{a} \arccos\left|x - 0.360034...\right| \, \mathrm{d}x \ \left(= 0.3\right)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{-a}^{a} \arccos\left|x - m\right| \, \mathrm{d}x \ \left(= 0.3\right)$.

attempts to solve their equation for a ***(M1)***

$a = 0.124861...$

$a = 0.125$ ***A1***

METHOD 3

EITHER

$\int_{-a}^{a} \arccos(x + 0.360034...)\,\mathrm{d}x \ (= 0.3)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{-a}^{a} \arccos(x+m)\,\mathrm{d}x \ (= 0.3)$.

OR

$\int_{2(0.360034...)-a}^{2(0.360034...)+a} \arccos(x - 0.360034...)\,\mathrm{d}x \ (= 0.3)$ ***(M1)(A1)***

Note: Only award ***(M1)*** if at least one limit has been translated correctly.

Note: Award ***(M1)(A1)*** for $\int_{2m-a}^{2m+a} \arccos(x-m)\,\mathrm{d}x \ (= 0.3)$.

THEN

attempts to solve their equation for a ***(M1)***

Note*:* The above ***(M1)*** is dependent on the first ***(M1)***.

$a = 0.124861...$

$a = 0.125$ ***A1***

[4 marks]

Total [6 marks]

1. attempt to substitute into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

Note: Accept use of Venn diagram or other valid method.

$0.6 = 0.5 + 0.4 - \mathrm{P}(A \cap B)$ ***(A1)***

$\mathrm{P}(A \cap B) = 0.3$ (seen anywhere) ***A1***

attempt to substitute into $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ ***(M1)***

$= \dfrac{0.3}{0.4}$

$\mathrm{P}(A \mid B) = 0.75 \left(= \dfrac{3}{4} \right)$ ***A1***

Total [5 marks]

7. attempting integration by parts, *eg*

$u = \frac{\pi x}{36}, \mathrm{d}u = \frac{\pi}{36}\mathrm{d}x, \mathrm{d}v = \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x, v = -\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)$ ***(M1)***

$\mathrm{P}(0 \le X \le 3) = \frac{\pi}{36}\left(\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 + \frac{6}{\pi}\int_0^3 \cos\left(\frac{\pi x}{6}\right)\mathrm{d}x\right)$ (or equivalent) ***A1A1***

Note: Award ***A1*** for a correct uv and ***A1*** for a correct $\int v\,\mathrm{d}u$.

attempting to substitute limits ***M1***

$\frac{\pi}{36}\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 = 0$ ***(A1)***

so $\mathrm{P}(0 \le X \le 3) = \frac{1}{\pi}\left[\sin\left(\frac{\pi x}{6}\right)\right]_0^3$ (or equivalent) ***A1***

$= \frac{1}{\pi}$ ***A1***

Total [7 marks]

3. (a) recognition of binomial ***(M1)***

$X \sim \mathrm{B}(5,\ 0.7)$

attempt to find $\mathrm{P}(X \leq 3)$ ***M1***

$= 0.472 (= 0.47178)$ ***A1***

[3 marks]

(b) recognition of 2 sixes in 4 tosses ***(M1)***

$\mathrm{P}(\text{3rd six on the 5th toss}) = \left[\binom{4}{2} \times (0.7)^2 \times (0.3)^2 \right] \times 0.7 (= 0.2646 \times 0.7)$ ***A1***

$= 0.185 (= 0.18522)$ ***A1***

[3 marks]

Total [6 marks]

4. (a) $a = 1.29$ and $b = -10.4$ ***A1A1***

[2 marks]

(b) recognising both lines pass through the mean point ***(M1)***

$p = 28.7,\ q = 30.3$ ***A2***

[3 marks]

(c) substitution into **their** x on y equation ***(M1)***

$x = 1.29082(29) - 10.3793$

$x = 27.1$ ***A1***

Note: Accept 27.

[2 marks]

Total [7 marks]

6. (a) ***METHOD 1***

attempting to use the expected value formula ***(M1)***

$\mathrm{E}(X) = (1\times 0.60) + (2\times 0.30) + (3\times 0.03) + (4\times 0.05) + (5\times 0.02)$

$\mathrm{E}(X) = 1.59(\$)$ ***(A1)***

use of $\mathrm{E}(1.20X + 2.40) = 1.20\mathrm{E}(X) + 2.40$ ***(M1)***

$\mathrm{E}(T) = 1.20(1.59) + 2.40$

$= 4.31(\$)$ ***A1***

METHOD 2

attempting to find the probability distribution for T ***(M1)***

t	3.60	4.80	6.00	7.20	8.40
$\mathrm{P}(T=t)$	0.60	0.30	0.03	0.05	0.02

(A1)

attempting to use the expected value formula ***(M1)***

$\mathrm{E}(T) = (3.60\times 0.60) + (4.80\times 0.30) + (6.00\times 0.03) + (7.20\times 0.05) + (8.40\times 0.02)$

$= 4.31(\$)$ ***A1***

[4 marks]

(b) ***METHOD 1***

using $\mathrm{Var}(1.20X + 2.40) = (1.20)^2\,\mathrm{Var}(X)$ with $\mathrm{Var}(X) = 0.8419$ ***(M1)***

$\mathrm{Var}(T) = 1.21$ ***A1***

METHOD 2

finding the standard deviation for **their** probability distribution found in part (a) ***(M1)***

$\mathrm{Var}(T) = (1.101...)^2$

$= 1.21$ ***A1***

Note: Award ***M1A1*** for $\mathrm{Var}(T) = (1.093...)^2 = 1.20$.

[2 marks]

Total [6 marks]

10. (a) attempt to use the symmetry of the normal curve ***(M1)***
eg diagram, $0.5-0.1446$

$\mathrm{P}(24.15<X<25)=0.3554$ ***A1***

[2 marks]

(b) (i) use of inverse normal to find z score ***(M1)***
$z=-1.0598$

correct substitution $\frac{24.15-25}{\sigma}=-1.0598$ ***(A1)***

$\sigma=0.802$ ***A1***

(ii) $\mathrm{P}(X>26)=0.106$ ***(M1)A1***

[5 marks]

(c) recognizing binomial probability ***(M1)***
$\mathrm{E}(Y)=10\times 0.10621$ ***(A1)***
$=1.06$ ***A1***

[3 marks]

(d) $\mathrm{P}(Y=3)$ ***(M1)***
$=0.0655$ ***A1***

[2 marks]

(e) recognizing conditional probability ***(M1)***
correct substitution ***A1***

$$\frac{0.3554}{1-0.10621}$$
$=0.398$ ***A1***

[3 marks]

Total [15 marks]

2. (a) $\text{IQR} = 10 - 6\ (= 4)$ ***(A1)***

attempt to find $Q_3 + 1.5 \times \text{IQR}$ ***(M1)***

$10 + 6$

16 ***A1***

[3 marks]

(b) (i) choosing $c = \frac{1}{2}a - 9$ ***(M1)***

$\frac{1}{2} \times 42 - 9$

$= 12$ (years old) ***A1***

(ii) attempt to solve system by substitution or elimination ***(M1)***

34 (years old) ***A1***

[4 marks]

Total [7 marks]

7. (a) attempt to integrate $\frac{k}{\sqrt{4-3x^2}}$ ***(M1)***

$= k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]$ ***A1***

Note: Award ***(M1)A0*** for $\arcsin\left(\frac{\sqrt{3}}{2}x\right)$.

Condone absence of k up to this stage.

equating their integrand to 1 ***M1***

$k\left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}x\right)\right]_0^1 = 1$

$k = \frac{3\sqrt{3}}{\pi}$ ***A1***

[4 marks]

continued…

Question 7 continued

(b) $\mathrm{E}(X) = \frac{3\sqrt{3}}{\pi}\int_0^1 \frac{x}{\sqrt{4-3x^2}}\,\mathrm{d}x$ ***A1***

Note: Condone absence of limits if seen at a later stage.

EITHER

attempt to integrate by inspection ***(M1)***

$= \frac{3\sqrt{3}}{\pi} \times -\frac{1}{6}\int -6x\left(4-3x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$

$= \frac{3\sqrt{3}}{\pi}\left[-\frac{1}{3}\sqrt{4-3x^2}\right]_0^1$ ***A1***

Note: Condone the use of k up to this stage.

OR

for example, $u = 4-3x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -6x$

Note: Other substitutions may be used. For example, $u = -3x^2$.

$= -\frac{\sqrt{3}}{2\pi}\int_4^1 u^{-\frac{1}{2}}\,\mathrm{d}u$ ***M1***

Note: Condone absence of limits up to this stage.

$= -\frac{\sqrt{3}}{2\pi}\left[2\sqrt{u}\right]_4^1$ ***A1***

Note: Condone the use of k up to this stage.

THEN

$= \frac{\sqrt{3}}{\pi}$ ***A1***

Note: Award ***A0M1A1A0*** for their $k\left[-\frac{1}{3}\sqrt{4-3x^2}\right]$ or $k\left[-2\sqrt{u}\right]$ for working with incorrect or no limits.

[4 marks]

Total [8 marks]

8. let m be the median

EITHER

attempts to find the area of the required triangle ***M1***

base is $(m-a)$ ***(A1)***

and height is $\frac{2}{(b-a)(c-a)}(m-a)$

area $=\frac{1}{2}(m-a)\times\frac{2}{(b-a)(c-a)}(m-a)$ $\left(=\frac{(m-a)^2}{(b-a)(c-a)}\right)$ ***A1***

OR

attempts to integrate the correct function ***M1***

$\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$

$= \frac{2}{(b-a)(c-a)}\left[\frac{1}{2}(x-a)^2\right]_a^m$ OR $\frac{2}{(b-a)(c-a)}\left[\frac{x^2}{2}-ax\right]_a^m$ ***A1A1***

Note: Award ***A1*** for correct integration and ***A1*** for correct limits.

THEN

sets up (their) $\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$ or area $=\frac{1}{2}$ ***M1***

Note: Award ***M0A0A0M1A0A0*** if candidates conclude that $m>c$ and set up their area or sum of integrals $=\frac{1}{2}$.

$\frac{(m-a)^2}{(b-a)(c-a)}=\frac{1}{2}$

$m=a\pm\sqrt{\frac{(b-a)(c-a)}{2}}$ ***(A1)***

as $m>a$, rejects $m=a-\sqrt{\frac{(b-a)(c-a)}{2}}$

so $m=a+\sqrt{\frac{(b-a)(c-a)}{2}}$ ***A1***

[6 marks]

10. (a) uses $\sum \mathrm{P}(X=x)=1$ to form a linear equation in p and q ***(M1)***

correct equation in terms of p and q from summing to 1 ***A1***

$p+0.3+q+0.1=1$ OR $p+q=0.6$ (or equivalent)

uses $\mathrm{E}(X)=2$ to form a linear equation in p and q ***(M1)***

correct equation in terms of p and q from $\mathrm{E}(X)=2$ ***A1***

$p+0.6+3q+0.4=2$ OR $p+3q=1$ (or equivalent)

Note: The marks for using $\sum \mathrm{P}(X=x)=1$ and the marks for using $\mathrm{E}(X)=2$ may be awarded independently of each other.

evidence of correctly solving these equations simultaneously ***A1***

for example, $2q=0.4 \Rightarrow q=0.2$ or $p+3\times(0.6-p)=1 \Rightarrow p=0.4$

so $p=0.4$ and $q=0.2$ ***AG***

[5 marks]

(b) valid approach ***(M1)***

$\mathrm{P}(X>2)=\mathrm{P}(X=3)+\mathrm{P}(X=4)$ OR $\mathrm{P}(X>2)=1-\mathrm{P}(X=1)-\mathrm{P}(X=2)$

$=0.3$ ***A1***

[2 marks]

continued…

Question 10 continued

(c) recognises at least one of the valid scores (6, 7, or 8) required to win the game ***(M1)***

Note: Award ***M0*** if candidate also considers scores other than 6, 7, or 8 (such as 5).

let T represent the score on the last two rolls

a score of 6 is obtained by rolling $(2,4),(4,2)$ or $(3,3)$

$\mathrm{P}(T=6)=2(0.3)(0.1)+(0.2)^2 \ (=0.1)$ ***A1***

a score of 7 is obtained by rolling $(3,4)$ or $(4,3)$

$\mathrm{P}(T=7)=2(0.2)(0.1) \ (=0.04)$ ***A1***

a score of 8 is obtained by rolling $(4,4)$

$\mathrm{P}(T=8)=(0.1)^2 \ (=0.01)$ ***A1***

Note: The above 3 ***A1*** marks are independent of each other.

$\mathrm{P}(\text{Nicky wins})=0.1+0.04+0.01$

$=0.15$ ***A1***

[5 marks]

(d) $3+b=8$ ***(M1)***

$b=5$ ***A1***

[2 marks]

continued…

Question 10 continued

(e) **METHOD 1**

EITHER

$\mathrm{P}(S=5)=\frac{4}{16}$

$\mathrm{P}(S=a+2)=\frac{4}{16}$ ***A1***

$\Rightarrow a+2=5$

OR

$\mathrm{P}(S=6)=\frac{3}{16}$

$\mathrm{P}(S=a+3)=\frac{2}{16}$ and $\mathrm{P}(S=5+1)=\frac{1}{16}$ ***A1***

$\Rightarrow a+3=6$

OR

$\mathrm{P}(S=4)=\frac{3}{16}$

$\mathrm{P}(S=a+1)=\frac{2}{16}$ and $\mathrm{P}(S=1+3)=\frac{1}{16}$ ***A1***

$\Rightarrow a+1=4$

THEN

$\Rightarrow a=3$ ***A1***

Note: Award ***A0A0*** for $a=3$ obtained without working/reasoning/justification.

[2 marks]

continued…

Question 10 continued

METHOD 2

EITHER

correctly lists a relevant part of the sample space ***A1***

for example, $\{S=4\}=\{(3,1),(1,a),(1,a)\}$ or $\{S=5\}=\{(2,a),(2,a),(2,a),(2,a)\}$

or $\{S=6\}=\{(3,a),(3,a),(1,5)\}$

$a+3=6$

OR

eliminates possibilities (exhaustion) for $a<5$

convincingly shows that $a\neq 2,4$ ***A1***

$a\neq 4$, for example, $\mathrm{P}(S=7)=\frac{2}{16}$ from $(2,5),(2,5)$ and so $(3,a),(3,a)\Rightarrow a+3\neq 7$

THEN

$\Rightarrow a=3$ ***A1***

[2 marks]

Total [16 marks]

1. (a) **EITHER**

recognising that half the total frequency is 10 (may be seen in an ordered list or indicated on the frequency table) ***(A1)***

OR

$5+1+4=3+x$ ***(A1)***

OR

$\sum f = 20$ ***(A1)***

THEN

$x = 7$ ***A1***

[2 marks]

continued…

Question 1 continued

(b) **METHOD 1**

$1.58429\ldots$

1.58 ***A2***

METHOD 2

EITHER

$$\sigma^2 = \frac{5\times(2-4.3)^2+1\times(3-4.3)^2+4\times(4-4.3)^2+3\times(5-4.3)^2+7\times(6-4.3)^2}{20} \ (=2.51) \quad \textbf{(A1)}$$

OR

$$\sigma^2 = \frac{5\times 2^2+1\times 3^2+4\times 4^2+3\times 5^2+7\times 6^2}{20} - 4.3^2 \ (=2.51) \quad \textbf{(A1)}$$

THEN

$\sigma = \sqrt{2.51} = 1.58429...$

$= 1.58$ ***A1***

[2 marks]

Total [4 marks]

3. (a) $0.41+k-0.28+0.46+0.29-2k^2=1$ OR $k-2k^2+0.01=0.13$ (or equivalent) ***A1***

$2k^2-k+0.12=0$ ***AG***

[1 mark]

(b) one of 0.2 OR 0.3 ***(M1)***

$k=0.3$ ***A1***

reasoning to reject $k=0.2$ eg $\mathrm{P}(1)=k-0.28\geq 0$ therefore $k\neq 0.2$ ***R1***

[3 marks]

(c) attempting to use the expected value formula ***(M1)***

$\mathrm{E}(X)=0\times 0.41+1\times(0.3-0.28)+2\times 0.46+3\times(0.29-2\times 0.3^2)$

$=1.27$ ***A1***

Note: Award ***M1A0*** if additional values are given.

[2 marks]

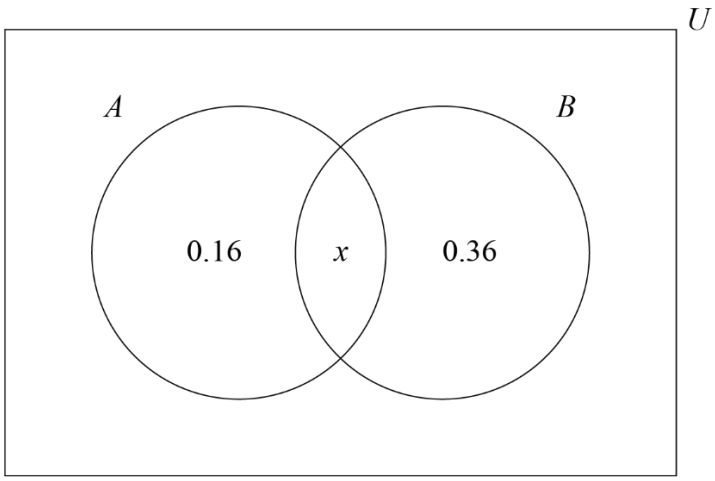
Total [6 marks]

5. (a) **METHOD 1**

EITHER

one of $\mathrm{P}(A) = x + 0.16$ OR $\mathrm{P}(B) = x + 0.36$ ***A1***

OR



A1

THEN

attempt to equate their $\mathrm{P}(A \cap B)$ with their expression for $\mathrm{P}(A) \times \mathrm{P}(B)$ ***M1***

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) \Rightarrow x = (x + 0.16) \times (x + 0.36)$ ***A1***

$x = 0.24$ ***A1***

METHOD 2

attempt to form at least one equation in $\mathrm{P}(A)$ and $\mathrm{P}(B)$ using independence ***M1***

$\left(\mathrm{P}(A \cap B') = \mathrm{P}(A) \times \mathrm{P}(B') \Rightarrow\right)\ \mathrm{P}(A) \times \left(1 - \mathrm{P}(B)\right) = 0.16$ OR

$\left(\mathrm{P}(A' \cap B) = \mathrm{P}(A') \times \mathrm{P}(B) \Rightarrow\right)\ \left(1 - \mathrm{P}(A)\right) \times \mathrm{P}(B) = 0.36$

$\mathrm{P}(A) = 0.4$ AND $\mathrm{P}(B) = 0.6$ ***A1***

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) = 0.4 \times 0.6$ ***(A1)***

$x = 0.24$ ***A1***

[4 marks]

continued…

Question 5 continued

(b) **METHOD 1**

recognising $\mathrm{P}(A' \mid B') = \mathrm{P}(A')$ ***(M1)***

$= 1 - 0.16 - 0.24$

$= 0.6$ ***A1***

METHOD 2

$\mathrm{P}(B) = 0.36 + 0.24 \left(= 0.6\right)$

$\mathrm{P}(A' \mid B') = \dfrac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')} \quad \left(= \dfrac{0.24}{0.4}\right)$ ***(A1)***

$= 0.6$ ***A1***

[2 marks]

Total [6 marks]

11. (a) $\mathrm{P}(C < 61)$ ***(M1)***

$= 0.365112...$

$= 0.365$ ***A1***

[2 marks]

(b) recognition of binomial eg $X \sim \mathrm{B}(12, 0.365...)$ ***(M1)***

$\mathrm{P}(X = 5) = 0.213666...$

$= 0.214$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) (i) Let CM represent 'chocolate muffin' and BM represent 'banana muffin'

$\mathrm{P}(B < 61) = 0.0197555...$ ***(A1)***

EITHER

$\mathrm{P}(CM) \times \mathrm{P}(C < 61 | CM) + \mathrm{P}(BM) \times \mathrm{P}(B < 61 | BM)$ (or equivalent in words) ***(M1)***

OR

tree diagram showing two ways to have a muffin weigh < 61 ***(M1)***

THEN

$(0.6 \times 0.365...) + (0.4 \times 0.0197...)$ ***(A1)***

$= 0.226969...$

$= 0.227$ ***A1***

(ii) recognizing conditional probability ***(M1)***

Note: Recognition must be shown in context either in words or symbols, not just $\mathrm{P}(A | B)$.

$\frac{0.6 \times 0.365112...}{0.226969...}$ ***(A1)***

$= 0.965183...$

$= 0.965$ ***A1***

[7 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$\mathrm{P}(CM)\times\mathrm{P}(C<61|CM)+\mathrm{P}(BM)\times\mathrm{P}(B<61|BM)=0.157$ ***(M1)***

$\left(0.6\times\mathrm{P}(C<61)\right)+(0.4\times0.0197555...)=0.157$

$\mathrm{P}(C<61)=0.248496...$ ***(A1)***

attempt to solve for σ using GDC ***(M1)***

Note: Award ***(M1)*** for a graph or table of values to show their $\mathrm{P}(C<61)$ with a variable standard deviation.

$\sigma=1.47225...$

$\sigma=1.47$ (g) ***A2***

continued…

Question 11 continued

METHOD 2

$\mathrm{P}(CM)\times\mathrm{P}(C<61|CM)+\mathrm{P}(BM)\times\mathrm{P}(B<61|BM)=0.157$ ***(M1)***

$\left(0.6\times\mathrm{P}(C<61)\right)+(0.4\times0.0197555...)=0.157$

$\mathrm{P}(C<61)=0.248496...$ ***(A1)***

use of inverse normal to find z score of their $\mathrm{P}(C<61)$ ***(M1)***

$z=-0.679229...$

correct substitution ***(A1)***

$\frac{61-62}{\sigma}=-0.679229...$

$\sigma=1.47225...$

$\sigma=1.47$ (g) ***A1***

[5 marks]

Total [16 marks]

3. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.68$

substitution of $\mathrm{P}(A) \cdot \mathrm{P}(B)$ for $\mathrm{P}(A \cap B)$ in $\mathrm{P}(A \cup B)$ ***(M1)***

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$ $(= 0.68)$

substitution of $3\mathrm{P}(B)$ for $\mathrm{P}(A)$ ***(M1)***

$3\mathrm{P}(B) + \mathrm{P}(B) - 3\mathrm{P}(B)\mathrm{P}(B) = 0.68$ (or equivalent) ***(A1)***

Note: The first two marks are independent of each other.

attempts to solve their quadratic equation ***(M1)***

$\mathrm{P}(B) = 0.2, 1.133... \left(\frac{1}{5}, \frac{17}{15}\right)$

$\mathrm{P}(B) = 0.2 \left(= \frac{1}{5}\right)$ ***A2***

Note: Award ***A1*** if both answers are given as final answers for $\mathrm{P}(B)$.

[6 marks]

4. (a) 0.28 (s) ***A1***

[1 mark]

(b) $\text{IQR} = 0.35 - 0.27 \ (= 0.08)$ (s) ***(A1)***

substituting **their** IQR into correct expression for upper fence ***(A1)***

$0.35 + 1.5 \times 0.08 \ (= 0.47)$ (s)

$0.46 < 0.47$ ***R1***

so 0.46 (s) is not an outlier ***AG***

[3 marks]

(c) **EITHER**

the median is closer to the lower quartile (positively skewed) ***R1***

OR

the distribution is positively skewed ***R1***

OR

the range of reaction times below the median is smaller than the range of reaction times above the median ***R1***

Note: These are sample answers from a range of acceptable correct answers. Award ***R1*** for any correct statement that explains this.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct. Accept a correctly and clearly labelled diagram.

[1 mark]

continued…

Question 4 continued

(d) **EITHER**

the distribution for ‘not sleeping well’ is centred at a higher reaction time ***R1***

OR

the median reaction time after not sleeping well is equal to the upper quartile reaction time after sleeping well ***R1***

OR

75% of reaction times are <0.35 seconds after sleeping well, compared with 50% after not sleeping well ***R1***

OR

the sample size of 9 is too small to draw any conclusions ***R1***

Note: These are sample answers from a range of acceptable correct answers. Accept any relevant correct statement t**hat relates to the median and/or quartiles shown in the box plots**. Do not accept a comparison of means.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct

Award ***R0*** to “correlation does not imply causation”.

[1 mark]

Total [6 marks]

8. Rachel: $R \sim \mathrm{N}(56.5, 3^2)$

$\mathrm{P}(R \geq 60) = 0.1216...$ ***(A1)***

Sophia: $S \sim \mathrm{N}(57.5, 1.8^2)$

$\mathrm{P}(S \geq 60) = 0.0824...$ ***(A1)***

recognises binomial distribution with $n = 5$ ***(M1)***

let N_R represent the number of Rachel's throws that are longer than 60 metres

$N_R \sim \mathrm{B}(5, 0.1216...)$

either $\mathrm{P}(N_R \geq 1) = 0.4772...$ or $\mathrm{P}(N_R = 0) = 0.5227...$ ***(A1)***

let N_S represent the number of Sophia's throws that are longer than 60 metres

$N_S \sim \mathrm{B}(5, 0.0824...)$

either $\mathrm{P}(N_S \geq 1) = 0.3495...$ or $\mathrm{P}(N_S = 0) = 0.6504...$ ***(A1)***

EITHER

uses $\mathrm{P}(N_R \geq 1)\mathrm{P}(N_S = 0) + \mathrm{P}(N_S \geq 1)\mathrm{P}(N_R = 0)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = (0.4772... \times 0.6504...) + (0.3495... \times 0.5227...)$

OR

uses $\mathrm{P}(N_R \geq 1) + \mathrm{P}(N_S \geq 1) - 2 \times \mathrm{P}(N_R \geq 1) \times \mathrm{P}(N_S \geq 1)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = 0.4772... + 0.3495... - 2 \times 0.4772... \times 0.3495...$

THEN

$= 0.4931...$

$= 0.493$ ***A1***

[7 marks]

Note: ***M*** marks are not dependent on the previous ***A*** marks.

6. (a) $\mathrm{P}(A \cap B) = 0.24$ ***A1***

[1 mark]

(b) $\mathrm{P}(A \cup B) = 1.1 - \mathrm{P}(A \cap B)$ ***(A1)***

$(0 \leq)\mathrm{P}(A \cup B) \leq 1$ ***(M1)***

Note: This may be conveyed in a clearly labelled diagram or written explanation where $\mathrm{P}(A \cup B) = 1$

the minimum value of $\mathrm{P}(A \cap B)$ is 0.1 ***A1***

[3 marks]

(c) A is a subset of B (so $\mathrm{P}(A \cap B) = \mathrm{P}(A)$). ***R1***

Note: This may be conveyed in a clearly labelled diagram where A is completely inside B, or in a written explanation indicating that $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

so the maximum value of $\mathrm{P}(A \cap B)$ is 0.3 ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [6 marks]

1. (a) $1.01206\ldots, 2.45230\ldots$

$a = 1.01$, $b = 2.45$ $(1.01x + 2.45)$ ***A1A1***

[2 marks]

(b) $0.981464\ldots$

$r = 0.981$ ***A1***

Note: A common error is to enter the data incorrectly into the GDC, and obtain the answers $a = 1.01700...$, $b = 2.09814...$ and $r = 0.980888...$ Some candidates may write the 3 sf answers, ie. $a = 1.02$, $b = 2.10$ and $r = 0.981$ or 2 sf answers, ie. $a = 1.0$, $b = 2.1$ and $r = 0.98$. In these cases award ***A0A0*** for part (a) and ***A0*** for part (b). Even though some values round to an accepted answer, they come from incorrect working.

[1 mark]

(c) correct substitution of 78 into **their** regression equation ***(M1)***

$81.3930\ldots$, 81.23 from 3 sf answer

81 ***A1***

[2 marks]

Total [5 marks]

6. (a) $\int_0^b axe^x dx = 1$ (seen anywhere) ***M1***

attempt to use integration by parts (either way around) (***M1)***

$\left[axe^x\right]_0^b - \int_0^b ae^x dx \left(=1\right)$ ***(A1)***

$\left[axe^x\right]_0^b - \left[ae^x\right]_0^b \left(=1\right)$ ***A1***

Note: Condone incorrect or absent limits up to this point.

$abe^b - ae^b + a = 1$

$a = \frac{1}{be^b - e^b + 1}$ ***A1***

[5 marks]

(b) $\int_0^m xe^x dx = \frac{1}{2}$ ***(M1)***

$\left[xe^x\right]_0^m - \left[e^x\right]_0^m = \frac{1}{2}$

$me^m - e^m + 1 = \frac{1}{2}$ ***(A1)***

$m = 0.768039...$

$m = 0.768$ ***A1***

[3 marks]

Total [8 marks]

10.

Note: Do not penalize for inclusion or non-inclusion of endpoints for probabilities using a normal distribution. For example, for $\mathrm{P}(T < 55 \mid T > 40)$ accept $\mathrm{P}(T \leq 55 \mid T > 40)$, $\mathrm{P}(T \leq 55 \mid T \geq 40)$, etc.

(a) recognising to find $\mathrm{P}(T > 40)$ ***(M1)***

$\mathrm{P}(T > 40) = 0.574136...$

$\mathrm{P}(T > 40) = 0.574$ ***A1***

[2 marks]

(b) attempt to multiply four independent probabilities using their $\mathrm{P}(T > 40)$ and $\mathrm{P}(T < 40)$ ***(M1)***

$(1-p)^3 \cdot p$ OR $(1-0.574136...)^3 \cdot 0.574136...$ OR $(0.425863...)^3 \cdot 0.574136...$ ***(A1)***

$0.0443430...$

0.0443 , 0.0444 from 3 sf values ***A1***

[3 marks]

continued…

Question 10 continued

(c) (i) recognizing conditional probability ***(M1)***

$\mathrm{P}(T < 55 \mid T > 40)$

Note: Award ***(M1)*** for an expression or description in context. Accept $\mathrm{P}(T > 40 \mid T < 55)$ but do not accept just $\mathrm{P}(A \mid B)$.

$\frac{\mathrm{P}(40 < T < 55)}{\mathrm{P}(T > 40)}$ ***(A1)***

$\frac{0.461944...}{0.574136...}$ ***(A1)***

$\mathrm{P}(T < 55 \mid T > 40) = 0.804590...$

$= 0.805$ ***A1***

(ii) recognizing binomial probability ***(M1)***

$X \sim \mathrm{B}(n, p)$

$n = 10$ and $p = 0.804589...$ ***(A1)***

$0.0242111..., 0.0240188...$ using $p = 0.805$

$\mathrm{P}(X = 5) = 0.0242$ ***A1***

[7 marks]

continued…

Question 10 continued

(d) Let $\mathrm{P}(T < a) = x$

recognition that probabilities sum to 1 (seen anywhere) ***(M1)***

EITHER

expressing the three regions in one variable ***(M1)***

$x + 0.904 + 2x$ OR $\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a)$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b)$

OR x and $2x$ correctly indicated on labelled bell diagram

$\mathrm{P}(T < a) + 0.904 + 2\mathrm{P}(T < a) = 1$ OR $\frac{1}{2}\mathrm{P}(T > b) + 0.904 + \mathrm{P}(T > b) = 1$ (or equivalent) ***(A1)***

OR

expressing either $\mathrm{P}(T < a)$ or $\mathrm{P}(T > b)$ only in terms of $\mathrm{P}(a \le T \le b)$ ***(M1)***

$\left(\mathrm{P}(T < a) =\right)\frac{1}{3}\left(1 - \mathrm{P}(a \le T \le b)\right)$ OR $\left(\mathrm{P}(T > b) =\right)\frac{2}{3} \cdot \left(1 - \mathrm{P}(a \le T \le b)\right)$

$x = \frac{1}{3}(1 - 0.904)\left(= 0.032\right)$ OR $\mathrm{P}(T > b) = \frac{2}{3}(1 - 0.904)\left(= 0.064\right)$ ***(A1)***

THEN

$\mathrm{P}(T < a) = 0.032$

$a = 22.18167...$

$a = 22.2$ (accept 22.1) ***A1***

[4 marks]

Total [16 marks]

2. (a) (i) summing frequencies of riders or finding complement ***(M1)***

probability $= \frac{34}{40}$ ***A1***

(ii) attempt to find expected value ***(M1)***

$\frac{16}{40}+\left(2\times\frac{13}{40}\right)+\left(3\times\frac{2}{40}\right)+\left(4\times\frac{3}{40}\right)$

$\frac{60}{40}\left(=1.5\right)$ ***A1***

[4 marks]

(b) evidence of **their** rides/visitor $\times 1000$ or $\div 10$ ***(M1)***

1500 OR 0.15

150 (times) ***A1***

[2 marks]

Total [6 marks]

3. substitutes into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ to form

$0.55 = 0.4 + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ (or equivalent) ***(A1)***

substitutes into $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ to form $0.25 = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ (or equivalent) ***(A1)***

attempts to combine their two probability equations to form an equation in $\mathrm{P}(B)$ ***(M1)***

Note: The above two ***A*** marks are awarded independently.

correct equation in $\mathrm{P}(B)$ ***A1***

$0.55 = 0.4 + \mathrm{P}(B) - 0.25\mathrm{P}(B)$ OR $\dfrac{\mathrm{P}(B) - 0.15}{\mathrm{P}(B)} = 0.25$ OR $\mathrm{P}(B) - 0.15 = 0.25\mathrm{P}(B)$

(or equivalent)

$\mathrm{P}(B) = \dfrac{15}{75} \left(= \dfrac{1}{5} = 0.2 \right)$ ***A1***

Total [5 marks]

6. (a) $\mathrm{E}(X) = 2a$ (by symmetry) ***A1***

[1 mark]

(b) **METHOD 1**

uses $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ ***(M1)***

$$\mathrm{Var}(X) = \int_a^{3a} \frac{x^2}{2a}\mathrm{d}x - (2a)^2$$

$$= \left[\frac{x^3}{6a}\right]_a^{3a} - (2a)^2$$ ***A1***

$$= \frac{13a^2}{3} - (2a)^2$$ ***(A1)***

$$= \frac{a^2}{3}$$ ***A1***

Note: Award as above if $\mathrm{E}(X^2)$ and $[\mathrm{E}(X)]^2$ are calculated separately leading to $\mathrm{Var}(X) = \frac{a^2}{3}$. Award ***(M1)A0A0A0*** for $\mathrm{Var}(X) = \frac{13a^2}{3}$.

continued…

Question 6 continued

METHOD 2

uses $\mathrm{Var}(X) = \mathrm{E}\left(X - \mathrm{E}(X)\right)^2$ ***(M1)***

$$\mathrm{Var}(X) = \int_a^{3a} \frac{(x-2a)^2}{2a}\,\mathrm{d}x$$

$$= \left[\frac{(x-2a)^3}{6a}\right]_a^{3a}$$ ***A1***

$$= \frac{a^3 - (-a^3)}{6a}$$ or equivalent ***(A1)***

$$= \frac{a^2}{3}$$ ***A1***

[4 marks]

Total [5 marks]

3. (a) recognising to find $y(25)$ ***(M1)***

$y(25) = -0.6 \times 25^2 + 23 \times 25 + 110$

$= 310$ (children) ***A1***

[2 marks]

(b) recognizing x on y is required ***(M1)***

$0.0935114\ldots$ and $7.43053\ldots$ ***(A1)***

$x = 0.0935y + 7.43$ ***A1***

[3 marks]

(c) attempt to substitute their answer to part (a) into their regression equation for either x or y ***(M1)***

$x = 0.0935114... \times 310 + 7.43053... (= 36.4190...)$

36 (accept 37 or 36.4) ***A1***

Note: Award ***(M1)A1FT*** for $x = 37$ found from using $y = 9.39x - 41.5$.

Award ***(M1)A0FT*** for a correct ***FT*** answer that lies outside $[15, 46]$.

[2 marks]

Total [7 marks]

4. METHOD 1

$Q_1 = 31.86$ OR $Q_3 = 32.14$ ***(A1)***

recognition that the area under the normal curve below Q_1 or above Q_3 is 0.25 OR the area between Q_1 and Q_3 is 0.5 (seen anywhere including on a diagram) ***(M1)***

EITHER

equating an appropriate correct normal CDF function to its correct probability (0.25 or 0.5 or 0.75) ***(A2)***

OR

$z = -0.674489...$ OR $z = 0.674489...$ (seen anywhere) ***(A1)***

$-0.674489... = \frac{31.86 - 32}{\sigma}$ OR $0.674489... = \frac{32.14 - 32}{\sigma}$ ***(A1)***

THEN

$0.207564...$

$\sigma = 0.208$ (mm) ***A1***

METHOD 2

recognition that the area under the normal curve below Q_1 or above Q_3 is 0.25 OR the area between Q_1 and Q_3 is 0.5 (seen anywhere including on a diagram) ***(M1)***

$z = -0.674489...$ OR $z = 0.674489...$ ***(A1)***

$(Q_1 =) 32 - 0.674489...\sigma$ OR $(Q_3 =) 32 + 0.674489...\sigma$ ***(A1)***

$(Q_3 - Q_1 =) 2 \times 0.674489...\sigma$

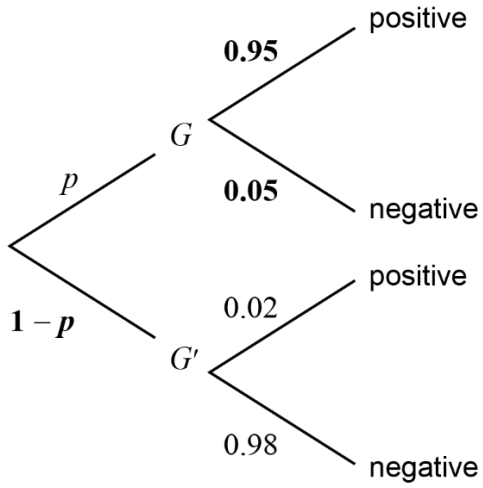
$2 \times 0.674489...\sigma = 0.28$ ***(A1)***

$0.207564...$

$\sigma = 0.208$ (mm) ***A1***

Total [5 marks]

7. (a)



A1A1

Note: award ***A1*** for branch correctly labelled $1-p$
award ***A1*** for branches correctly labelled 0.95 and 0.05
award ***A0*** for G' branch labelled p'
award ***A0*** for G' branch labelled q unless explicitly defined as $1-p$

[2 marks]

continued…

Question 7 continued

(b) **METHOD 1**

recognizing conditional probability ***(M1)***

$\mathrm{P}(G'|pos)$ OR $\mathrm{P}(G|pos)$

$\frac{0.02(1-p)}{0.95p+0.02(1-p)}\left(=\frac{18}{150}\right)$ OR $\frac{0.95p}{0.95p+0.02(1-p)}\left(=\frac{132}{150}\right)$ ***(A1)(A1)***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

$p=0.133738$

$p=0.134$ ***A1***

METHOD 2

attempt to set up a system of equations (S = sample size) ***(M1)***

$p(0.95S)=132$ and $(1-p)(0.02S)=18$ ***(A1)***

attempt to solve for p or S ***(M1)***

$\frac{0.95p}{0.02(1-p)}=\frac{132}{18}$

OR $S=pS+(1-p)S=\frac{132}{0.95}+\frac{18}{0.02}=138.947...+900=1038.94...$

$p=0.133738...$

$p=0.134$ ***A1***

METHOD 3

attempt to find the number of parrots with the gene and the number without ***(M1)***

number of parrots with the gene $\approx \frac{132}{0.95} = 138.947...$ AND

number of parrots without the gene $\approx \frac{18}{0.02} = 900$ ***(A1)***

number of parrots in the sample $\approx 138.947... + 900 = 1038.94...$

attempt to find proportion of sample with the gene ***(M1)***

$p \approx \frac{138.947...}{1038.94...} = 0.133738...$

$p = 0.134$ ***A1***

[4 marks]

Total [6 marks]

(3)

11. (a) $\mathrm{E}(X) = \int_0^2 \frac{6x}{\pi\sqrt{16-x^2}}\,\mathrm{d}x$ ***(A1)***

Note: Condone the absence of $\mathrm{d}x$.

Accept $\int_0^2 xf(x)\,\mathrm{d}x$

attempt to integrate $\frac{6x}{\pi\sqrt{16-x^2}}$ using inspection/substitution ***(M1)***

$-\frac{6}{2\pi}\int -2x\left(16-x^2\right)^{-\frac{1}{2}}\,\mathrm{d}x$ or let $u = 16-x^2$

$-\frac{6}{2\pi}\left[2\left(16-x^2\right)^{\frac{1}{2}}\right]_0^2$ OR $\frac{6}{\pi}\left[-u^{\frac{1}{2}}\right]_{16}^{12}$ ***A1***

Note: For this ***A1*** condone absent or incorrect limits.

attempt to substitute their limits and evaluate ***(M1)***

$\frac{24}{\pi} - \frac{6}{\pi}\sqrt{12}\left(= \frac{12}{\pi}\left(2-\sqrt{3}\right)\right)$ ***A1***

Note: The substitution $\sin\theta = \frac{x}{4}$ may also be used, leading to $\frac{24}{\pi}\int_0^{\frac{\pi}{6}} \sin\theta\mathrm{d}\theta = \frac{24}{\pi}\left[-\cos\theta\right]_0^{\frac{\pi}{6}} = \frac{24}{\pi}\left(1-\cos\frac{\pi}{6}\right)$. Award marks as appropriate and accept $\frac{24}{\pi}\left(1-\cos\frac{\pi}{6}\right)$ for the final ***A1***.

[5 marks]

(b) $\int_0^{0.5} f(x)\,\mathrm{d}x \left(= \int_0^{0.5} \frac{6}{\pi\sqrt{16-x^2}}\,\mathrm{d}x\right)$ ***(M1)***

$P(X < 0.5) = 0.239358...$

$= 0.239$ ***A1***

[2 marks]

continued…

Question 11 continued

(c) **EITHER**

recognition P(at least one success after n trials) = 1 – P(no successes after n trials) ***(M1)***

$1-(1-0.239...)^n \geq 0.99$ ***(A1)***

$n = 16.8321...$

Note: Use of 0.239 results in $n = 16.8612...$

OR

recognition that $Y \sim \mathrm{B}(n, 0.239...)$ ***(M1)***

If $n = 16$ P(at least one success after n trials) $= 0.987443...$

and if $n = 17$ P(at least one success after n trials) $= 0.990448...$ ***(A1)***

Note: Use of 0.239 results in the values $0.987348...$ and $0.990371...$

THEN

17 trials ***A1***

[3 marks]

(d) recognition that $Y \sim \mathrm{B}(10, \text{their part b})$ ***(M1)***

$\mathrm{B}(10, 0.239...)$

$\mathrm{P}(X = 3) = 0.242430...$

$= 0.242$ ***A1***

[2 marks]

continued…

Question 11 continued

(e) 8 ***A1***

[1 mark]

(f) (i) $n-2$ ***A1***

(ii) ${}^nC_3$ (ways of 3 successes in n trials) ***(A1)***

$\frac{n-2}{{}^nC_3}$ ***(A1)***

Attempt to solve their $\frac{n-2}{{}^nC_3} > 0.05$ OR $\frac{6}{n(n-1)} > 0.05$ (or equivalent) ***(M1)***

Note: Accept an equation.

$n = 11.4658...$ OR

table values $n = 11, \frac{n-2}{{}^nC_3} = 0.0545454...$ and $n = 12, \frac{n-2}{{}^nC_3} = 0.0454545...$ ***(A1)***

$n = 11$ ***A1***

[6 marks]

Total [19 marks]

1. (a) $a = 1.93258...,\ b = 7.21662...$

$a = 1.93,\ b = 7.22$ ***A1A1***

[2 marks]

(b) attempt to substitute $d = 20$ into their equation ***(M1)***

$\text{height} = 45.8683...$

$\text{height} = 45.9$ (cm) ***A1***

[2 marks]

Total [4 marks]

3. (a) evidence of attempting to find correct area under normal curve ***(M1)***

$\mathrm{P}(W > 210)$ OR sketch

$\mathrm{P}(W > 210) = 0.115069...$

$\mathrm{P}(W > 210) = 0.115$ ***A1***

[2 marks]

(b) recognizing $\mathrm{P}(W < w) = 1 - \mathrm{P}(w < W < 210) - \mathrm{P}(W > 210)$ ***(M1)***

$\mathrm{P}(W < w) = 1 - 0.8 - 0.115069...$

$\mathrm{P}(W < w) = 0.084930...$

$\mathrm{P}(W < w) = 0.0849$ ***A1***

[2 marks]

(c) evidence of attempting to use inverse normal function ***(M1)***

$w = 197.136...$

$w = 197$ (grams) ***A1***

[2 marks]

(d) recognition of binomial distribution ***(M1)***

$X \sim \mathrm{B}(10, 0.0849302...)$

$\mathrm{P}(X = 1) = 0.382076...$

$\mathrm{P}(X = 1) = 0.382$ ***A1***

[2 marks]

Total [8 marks]

11. (a) attempts to form a numerator involving a product of two terms involving y

and a denominator involving a product of two terms involving $r+y$ ***(M1)***

$$\frac{y(y-1)}{(r+y)(r+y-1)}=\frac{1}{3}$$ ***A1***

attempts to remove the fractions and expand the brackets ***(M1)***

$3y^2-3y=y^2+2ry-y+r^2-r$ ***A1***

$2y^2-2ry-2y+r-r^2=0$

$2y^2-2(r+1)y+r-r^2=0$ ***AG***

[4 marks]

(b) attempts to solve for y ***(M1)***

$$y=\frac{2(r+1)\pm\sqrt{4(r+1)^2-8(r-r^2)}}{4}$$ ***A1***

$$y=\frac{2(r+1)\pm\sqrt{12r^2+4}}{4}$$ ***A1***

$$y=\frac{(r+1)\pm\sqrt{3r^2+1}}{2}$$

(since $r, y\in\mathbb{Z}^+$) and $\frac{(r+1)-\sqrt{3r^2+1}}{2}<0$ for $r>1$ ***R1***

Note: Award the ***R1*** for stating that number of balls cannot be negative, or similar.

Note: Accept $y>0$

so $y=\frac{(r+1)+\sqrt{3r^2+1}}{2}$ ***AG***

[4 marks]

continued…

Question 11 continued

(c) attempts to find a pair of positive integer values *eg* by using a table ***(M1)***

Note: Award ***M0*** if numbers are not positive integers.

1 red ball and 2 yellow balls ($r = 1$ and $y = 2$) ***A1***

4 red balls and 6 yellow balls ($r = 4$ and $y = 6$) ***A2***

Note: Award ***A1*** for one solution and ***A2*** for another.
15 red balls and 21 yellow balls ($r = 15$ and $y = 21$) is the next solution.

[4 marks]

(d) attempts to form a numerator involving a product of three terms involving y and a denominator involving a product of three terms that includes a $(y+10)$ term ***(M1)***

$$\mathrm{P}(YYY) = \frac{y(y-1)(y-2)}{(y+10)(y-1+10)(y-2+10)}\left(= \frac{y(y-1)(y-2)}{(y+10)(y+9)(y+8)}\right)$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

[3 marks]

(e) $$\mathrm{P}(\text{new } YYY) = \frac{(y+1)(y)(y-1)}{(y+1+10)(y+10)(y-1+10)}\left(= \frac{(y+1)(y)(y-1)}{(y+11)(y+10)(y+9)}\right)$$ ***(A1)***

equates their answer for $\mathrm{P}(\text{new } YYY)$ to $2\times$ their answer for part (d) ***M1***

$$\frac{2y(y-1)(y-2)}{(y+10)(y+9)(y+8)} = \frac{(y+1)(y)(y-1)}{(y+11)(y+10)(y+9)}$$

attempts to solve for y ***(M1)***

Note: Award ***(M1)*** for attempting to write the above expression as $\frac{2(y-2)}{y+8} = \frac{y+1}{y+11}$.

$y = 4$ ***A2***

[5 marks]

Total [20 marks]

2. (a) $\left(\mathrm{P}\left(A \cup B\right) =\right) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

$= 0.8$ ***A1***

[2 marks]

(b) recognition that $A' \cap B' = \left(A \cup B\right)'$ OR $A' \cap B' = 1 - A \cup B$
(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$\left(= 1 - 0.8\right)$

$= 0.2$ ***A1***

Note: For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}\left(A' \cap B'\right)$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

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2. (a) $\left(\mathrm{P}\left(A \cup B\right) =\right) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

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(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$\left(= 1 - 0.8\right)$

$= 0.2$ ***A1***

Note: For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}\left(A' \cap B'\right)$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

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5. $\mathrm{E}(X) = k + 2k^2 + 3a + 4k^3 = 2.3$ ***(A1)***

$k + k^2 + a + k^3 = 1$ ***(A1)***

Note: The first two ***A*** marks are independent of each other.

EITHER (finding intersections of functions)

attempt to make a the subject in both of their equations ***(M1)***

$a = 1 - k - k^2 - k^3$ and $a = \frac{1}{3}\left(2.3 - k - 2k^2 - 4k^3\right)$

use of graph or table to attempt to find intersection ***(M1)***

OR (solving algebraically)

attempt to solve their equations algebraically to find a cubic in k ***(M1)***

$k^3 - k^2 - 2k + 0.7 = 0$ OR $3\left(1 - k - k^2 - k^3\right) = 2.3 - k - 2k^2 - 4k^3$ (or equivalent)

attempt to solve their cubic in k ***(M1)***

THEN

$a = 0.552839...$ OR $k = 0.315870...$ (other solutions to cubic are $k = -1.18538..., 1.86951...$)

$a = 0.553$ ***A1***

Note: If no working shown, award ***(A1)(A1)(M1)(M1)A0*** for $a = 2.44587...$ OR $a = -10.8987...$ and award ***(A0)(A0)(M1)(M1)A0*** for $a = 0.55$ (2sf) .

Total [5 marks]

6. (a) $5.75 = 25p(1-p)$ ***(A1)***

$p = 0.641421..., 0.358578...$

$p = 0.641, 0.359 \left(= \frac{5 \pm \sqrt{2}}{10} \right)$ ***A1A1***

[3 marks]

(b) $\text{Var}(Y) = (-2)^2 \text{Var}(X) \left(= 4\text{Var}(X) \right)$ ***(A1)***

$= 23$ ***A1***

[2 marks]

Total [5 marks]

10. (a) recognizing probabilities sum to 1 ***(M1)***

$0.288 + \mathrm{P}(94.6 < X < 98.1) + 0.434 = 1$

$\mathrm{P}(94.6 < X < 98.1) = 0.278$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(94.6 < X < 98.1) = 0.28$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing the need to use inverse normal with 0.288, $(1-0.434)$ or 0.434 ***(M1)***

Note: Accept use of calculator notation eg $\mathrm{invNorm}(0.288)\,(= -0.559236...)$.

$\mu + \mathrm{invNorm}(0.288)\sigma = 94.6, \mu + \mathrm{invNorm}(1-0.434)\sigma = 98.1$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations in two variables using the GDC (that involve either z-values or 'invNorm' rather than probabilities) ***(M1)***

$\mu = 97.2981..., \sigma = 4.82468...$

$\mu = 97.3, \sigma = 4.82$ ***A1***

Note: Condone use of different variables throughout, but do not award the final ***A1*** if they do not clearly identify which variable is their mean and standard deviation.

METHOD 2

use of inverse normal to find at least one z-score for $\mathrm{P}(Z < z) = 0.288$ or $\mathrm{P}(Z < z) = 1 - 0.434$ ***(M1)***

$z_1 = -0.559236...$ OR $z_2 = 0.166199...$

$\frac{94.6-\mu}{\sigma} = -0.559236...$, $\frac{98.1-\mu}{\sigma} = 0.166199...$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations (that involve z-values rather than probabilities) ***(M1)***

$\mu = 97.2981...,\ \sigma = 4.82468...$

$\mu = 97.3,\ \sigma = 4.82$ ***A1***

Note: Award marks as appropriate for work seen in part (a).

Note: If no working shown, award ***(M1)(A0)(A0)(M1)A0*** for $\mu = 97, \sigma = 4.8$ (2sf).

[5 marks]

Question 10 continued

(c) (i) recognition of Binomial distribution ***(M1)***

$X \sim \mathrm{B}(100, 0.434)$

$\mathrm{P}(X = 34) = 0.0133198...$

$= 0.0133$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(X = 34) = 0.013$ (2sf).

(ii) $\mathrm{P}(X < 49) = 0.848218...$ (seen anywhere) ***(A1)***

recognition of conditional probability ***(M1)***

Note: recognition must be shown in context, either in symbols eg $\mathrm{P}(X = 34 \mid X < 49)$, or in words eg $\mathrm{P}(34 \text{ plants} \mid \text{less than } 49 \text{ plants})$, not only as $\mathrm{P}(A \mid B)$.

$\left(\mathrm{P}(X = 34 \mid X < 49) =\right) \dfrac{\mathrm{P}(X = 34)}{\mathrm{P}(X < 49)}$ OR $\dfrac{\mathrm{P}(X = 34)}{\mathrm{P}(X \leq 48)} \left(= \dfrac{0.0133198...}{0.848218...}\right)$ ***(A1)***

$= 0.0157033...$

$\mathrm{P}(X = 34 \mid X < 49) = 0.0157$ ***A1***

Note: Exception to ***FT***: If the candidate finds $\mathrm{P}(X \leq 49) (= 0.890474...)$ and uses that to calculate $\mathrm{P}(X = 34 \mid X \leq 49) = 0.0149581...$ award ***(A0)(M1)(A1)A0***.

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{P}(X = 34 \mid X < 49) = 0.016$ (2sf).

[6 marks]

continued…

Question 10 continued

(d) $Q_1 = 96.19$ OR $Q_3 = 101.01$ (may be seen on a labelled diagram with areas indicated) ***(A1)***

$\mathrm{P}(96.19 < F < 101.01) = 0.5$ OR $\mathrm{P}(F < 96.19) = 0.25$ OR $\mathrm{P}(F < 101.01) = 0.75$ (or equivalent)

EITHER

attempt to find d using graph or table ***(M1)***

OR

$1 - 2\mathrm{P}\left(Z < -\frac{2.41}{d}\right) = 0.5$ OR $\mathrm{P}(Z < -\frac{2.41}{d}) = 0.25$ OR $\mathrm{P}(Z < \frac{2.41}{d}) = 0.75$

OR $\mathrm{P}(-\frac{2.41}{d} < Z < \frac{2.41}{d}) = 0.5$ (or equivalent) ***(M1)***

$-\frac{2.41}{d} = -0.674489...$ OR $\frac{2.41}{d} = 0.674489...$

THEN

$3.57307...$

$d = 3.57$ ***A1***

Note: Accept 3.56 using 96.2 or 101.

Note: If no working shown, award ***(A0)(M1)A0*** for $d = 3.6$ (2sf).

[3 marks]

Total [16 marks]

5. $\mathrm{E}(X) = k + 2k^2 + 3a + 4k^3 = 2.3$ ***(A1)***

$k + k^2 + a + k^3 = 1$ ***(A1)***

Note: The first two ***A*** marks are independent of each other.

EITHER (finding intersections of functions)

attempt to make a the subject in both of their equations ***(M1)***

$a = 1 - k - k^2 - k^3$ and $a = \frac{1}{3}\left(2.3 - k - 2k^2 - 4k^3\right)$

use of graph or table to attempt to find intersection ***(M1)***

OR (solving algebraically)

attempt to solve their equations algebraically to find a cubic in k ***(M1)***

$k^3 - k^2 - 2k + 0.7 = 0$ OR $3\left(1 - k - k^2 - k^3\right) = 2.3 - k - 2k^2 - 4k^3$ (or equivalent)

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THEN

$a = 0.552839...$ OR $k = 0.315870...$ (other solutions to cubic are $k = -1.18538..., 1.86951...$)

$a = 0.553$ ***A1***

Note: If no working shown, award ***(A1)(A1)(M1)(M1)A0*** for $a = 2.44587...$ OR $a = -10.8987...$ and award ***(A0)(A0)(M1)(M1)A0*** for $a = 0.55$ (2sf) .

Total [5 marks]

6. (a) $5.75 = 25p(1-p)$ ***(A1)***

$p = 0.641421..., 0.358578...$

$p = 0.641, 0.359 \left(= \frac{5 \pm \sqrt{2}}{10} \right)$ ***A1A1***

[3 marks]

(b) $\text{Var}(Y) = (-2)^2 \text{Var}(X) \left(= 4\text{Var}(X) \right)$ ***(A1)***

$= 23$ ***A1***

[2 marks]

Total [5 marks]

10. (a) recognizing probabilities sum to 1 ***(M1)***

$0.288 + \mathrm{P}(94.6 < X < 98.1) + 0.434 = 1$

$\mathrm{P}(94.6 < X < 98.1) = 0.278$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(94.6 < X < 98.1) = 0.28$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing the need to use inverse normal with 0.288, $(1-0.434)$ or 0.434 ***(M1)***

Note: Accept use of calculator notation eg $\mathrm{invNorm}(0.288)\,(= -0.559236...)$.

$\mu + \mathrm{invNorm}(0.288)\sigma = 94.6,\ \mu + \mathrm{invNorm}(1-0.434)\sigma = 98.1$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations in two variables using the GDC (that involve either z-values or 'invNorm' rather than probabilities) ***(M1)***

$\mu = 97.2981...,\ \sigma = 4.82468...$

$\mu = 97.3,\ \sigma = 4.82$ ***A1***

Note: Condone use of different variables throughout, but do not award the final ***A1*** if they do not clearly identify which variable is their mean and standard deviation.

METHOD 2

use of inverse normal to find at least one z-score for $\mathrm{P}(Z < z) = 0.288$ or $\mathrm{P}(Z < z) = 1 - 0.434$ ***(M1)***

$z_1 = -0.559236...$ OR $z_2 = 0.166199...$

$\frac{94.6-\mu}{\sigma} = -0.559236...$, $\frac{98.1-\mu}{\sigma} = 0.166199...$ (or equivalent) ***(A1)(A1)***

attempt to solve their equations (that involve z-values rather than probabilities) ***(M1)***

$\mu = 97.2981...,\ \sigma = 4.82468...$

$\mu = 97.3,\ \sigma = 4.82$ ***A1***

Note: Award marks as appropriate for work seen in part (a).

Note: If no working shown, award ***(M1)(A0)(A0)(M1)A0*** for $\mu = 97,\ \sigma = 4.8$ (2sf).

[5 marks]

Question 10 continued

(c) (i) recognition of Binomial distribution ***(M1)***

$X \sim \mathrm{B}(100, 0.434)$

$\mathrm{P}(X = 34) = 0.0133198...$

$= 0.0133$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $\mathrm{P}(X = 34) = 0.013$ (2sf).

(ii) $\mathrm{P}(X < 49) = 0.848218...$ (seen anywhere) ***(A1)***

recognition of conditional probability ***(M1)***

Note: recognition must be shown in context, either in symbols eg $\mathrm{P}(X = 34 \mid X < 49)$, or in words eg $\mathrm{P}(34 \text{ plants} \mid \text{less than } 49 \text{ plants})$, not only as $\mathrm{P}(A \mid B)$.

$\left(\mathrm{P}(X = 34 \mid X < 49) =\right) \dfrac{\mathrm{P}(X = 34)}{\mathrm{P}(X < 49)}$ OR $\dfrac{\mathrm{P}(X = 34)}{\mathrm{P}(X \leq 48)} \left(= \dfrac{0.0133198...}{0.848218...}\right)$ ***(A1)***

$= 0.0157033...$

$\mathrm{P}(X = 34 \mid X < 49) = 0.0157$ ***A1***

Note: Exception to ***FT***: If the candidate finds $\mathrm{P}(X \leq 49) (= 0.890474...)$ and uses that to calculate $\mathrm{P}(X = 34 \mid X \leq 49) = 0.0149581...$ award ***(A0)(M1)(A1)A0***.

Note: If no working shown, award ***(A0)(M1)(A0)A0*** for $\mathrm{P}(X = 34 \mid X < 49) = 0.016$ (2sf).

[6 marks]

continued…

Question 10 continued

(d) $Q_1 = 96.19$ OR $Q_3 = 101.01$ (may be seen on a labelled diagram with areas indicated) ***(A1)***

$\mathrm{P}(96.19 < F < 101.01) = 0.5$ OR $\mathrm{P}(F < 96.19) = 0.25$ OR $\mathrm{P}(F < 101.01) = 0.75$ (or equivalent)

EITHER

attempt to find d using graph or table ***(M1)***

OR

$1 - 2\mathrm{P}\left(Z < -\frac{2.41}{d}\right) = 0.5$ OR $\mathrm{P}(Z < -\frac{2.41}{d}) = 0.25$ OR $\mathrm{P}(Z < \frac{2.41}{d}) = 0.75$

OR $\mathrm{P}(-\frac{2.41}{d} < Z < \frac{2.41}{d}) = 0.5$ (or equivalent) ***(M1)***

$-\frac{2.41}{d} = -0.674489...$ OR $\frac{2.41}{d} = 0.674489...$

THEN

$3.57307...$

$d = 3.57$ ***A1***

Note: Accept 3.56 using 96.2 or 101.

Note: If no working shown, award ***(A0)(M1)A0*** for $d = 3.6$ (2sf).

[3 marks]

Total [16 marks]

1. (a) attempt to form equation for the sum of frequencies$=16$ or mean$=3$ ***(M1)***

$p+q+4+2+3=16\left(\Rightarrow p+q=7\right)$ ***A1***

$\frac{p+2q+12+8+18}{16}=3\left(\Rightarrow p+2q=10\right)$ OR $\frac{p+2q+12+8+18}{9+p+q}=3\left(\Rightarrow 2p+q=11\right)$ ***A1***

attempt to eliminate one variable from their equations ***(M1)***

$p+2\left(7-p\right)+38=48$ OR $2\left(7-q\right)+q=11$

$p=4$ and $q=3$ ***A1***

Note: Award ***M1A0A0M0A1*** for $p=4, q=3$ with no working.

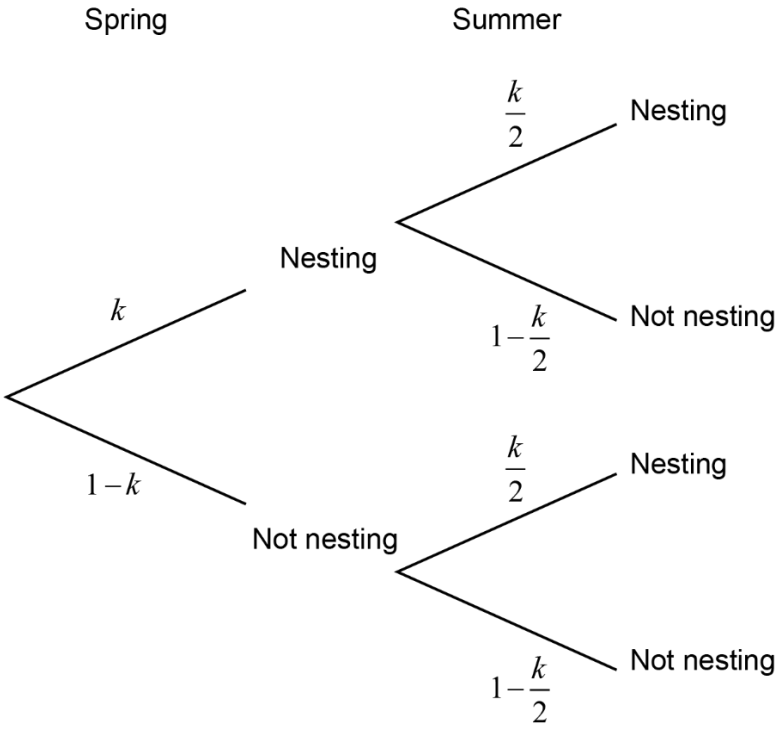
[5 marks]

(b) mean final score $=30$ ***A1***

[1 mark]

Total [6 marks]

4. (a)



$1-k$ for Spring ***A1***

$1-\frac{k}{2}$ for both Summers ***A1***

[2 marks]

continued…

Question 4 continued.

(b) (i) multiplying the two correct branches ***(A1)***

$(1-k)\left(1-\frac{k}{2}\right)$

attempt to expand and equate to $\frac{5}{9}$ ***(M1)***

$1-k-\frac{k}{2}+\frac{k^2}{2}=\frac{5}{9}$

$18-18k-9k+9k^2=10$ OR $\frac{k^2}{2}-\frac{3k}{2}+\frac{4}{9}=0$ OR $\frac{9k^2}{2}-\frac{27k}{2}+4=0$ ***A1***

$9k^2-27k+8=0$ ***AG***

(ii) ($k=\frac{1}{3}$ is the only valid solution as) $\frac{8}{3}>1$ ***R1***

Note: Accept any valid reasoning indicating that any probability cannot be greater than 1 and/or probability cannot be less than 0.

[4 marks]

Total [6 marks]

6. (a) $2.5(\%)$ ***A1***

[1 mark]

(b) $\mathrm{P}(\text{weight of cooking apples} > 140) = 0.5\,(50\%)$ (seen anywhere) ***(A1)***

recognition of conditional probability in context ***(M1)***

$$\mathrm{P}(\text{eating apple} \mid \text{weight} > 140) = \frac{\mathrm{P}(\text{eating apple and weight of eating apple} > 140)}{\mathrm{P}(\text{weight of apple} > 140)}$$

OR

$$\frac{\mathrm{P}(\text{eating apple} \cap \text{weight of eating apple} > 140)}{\mathrm{P}(\text{eating apple} \cap \text{weight of eating apple} > 140) + \mathrm{P}(\text{cooking apple} \cap \text{weight of cooking apple} > 140)}$$

OR

$$\frac{\mathrm{P}(\text{eating apple})\mathrm{P}(\text{weight of eating apple} > 140)}{\mathrm{P}(\text{eating apple})\mathrm{P}(\text{weight of eating apple} > 140) + \mathrm{P}(\text{cooking apple})P(\text{weight of cooking apple} > 140)}$$

$$= \frac{0.8 \times 0.025}{0.8 \times 0.025 + 0.2 \times 0.5} \left(= \frac{80 \times 2.5}{80 \times 2.5 + 20 \times 50} \right)$$ ***(A1)***

$$= \frac{200}{1200} \left(= \frac{1}{6} \right)$$ ***A1***

Note: Accept any equivalent exact answer written as a fraction.

[4 marks]

Total [5 marks]

3. (a) recognition of $X > 13$ OR $Z > 1.5$ (could be seen in a diagram) ***(M1)***

$\left(\mathrm{P}\left(X > 13\right) =\right)\ 0.0668072...$

$= 0.0668$ ***A1***

[2 marks]

(b) **EITHER**

equating an appropriate correct normal CDF function to 0.1 or 0.9 ***(M1)***

$\mathrm{P}\left(X > 10 + 2k\right) = 0.1$ OR $\mathrm{P}\left(Z < k\right) = 0.9$ OR $\mathrm{P}\left(X < 10 - 2k\right) = 0.1$ OR $\mathrm{P}\left(Z < -k\right) = 0.1$

OR

recognising need to use inverse normal with 0.1 or 0.9 ***(M1)***

THEN

$1.28155...$

$k = 1.28$ ***A1***

[2 marks]

Total [4 marks]

5. (a) $r = 0.901017\ldots$

$r = 0.901$ ***A2***

[2 marks]

(b) Student 11 Test B: should not extrapolate ***R1***

[1 mark]

(c) (i) Student 12 Test A: should not use line of y on x to predict x from y (or equivalent) ***R1***

(ii) attempt to find the equation of the regression line of x on y ***(M1)***

$(x =)\ 0.987124\ldots y - 3.21970\ldots\ \ ((x =)\ 0.987y - 3.22)$ ***A1***

$(x =)\ 0.987124\ldots(90) - 3.21970\ldots\ \ (= 85.6214...)$ ***A1***

$= 86$ to nearest integer. ***AG***

Note: Condone notation for x and y switched if values are correct.

[4 marks]

Total [7 marks]

6. let X be the number of days of rain in May

(a) recognition of binomial distribution ***(M1)***

$X \sim \mathrm{B}(31, 0.2)$ or ${}^{31}C_{10}(0.2)^{10}(0.8)^{21}$ or $X \sim \mathrm{B}(n, p)$ or ${}^{n}C_{r}p^{r}(1-p)^{n-r}$

$\mathrm{P}(X = 10) = 0.0418894\ldots$

$= 0.0419$ ***A1***

Note: If no working shown, award ***(M1)A0*** for 0.042 (2 sf)

[2 marks]

(b) recognition of need to find $\mathrm{P}(X \geq 10)\left(= 1 - \mathrm{P}(X \leq 9)\right)$ ***(M1)***

$= 0.0745998\ldots.\left(= 1 - 0.925400\ldots\right)$

$= 0.0746$ ***A1***

Note: If no working shown, award ***(M1)A0*** for 0.075 (2 sf)

[2 marks]

(c) recognition of 9 days with no rain followed by a day of rain ***(M1)***

$0.8^{9} \times 0.2 = 0.0268435\ldots$

$= 0.0268$ ***A1***

Note: If no working shown, award ***(M1)A0*** for 0.027 (2 sf)

[2 marks]

Total [6 marks]

8. (a) $\int_0^k kx\mathrm{d}x + \int_k^{2k}\left(2kx - x^2\right)\mathrm{d}x = 1$ ***A1***

$\left[\frac{kx^2}{2}\right]_0^k + \left[kx^2 - \frac{x^3}{3}\right]_k^{2k} = 1$

$\frac{k^3}{2} + \left(4k^3 - \frac{8k^3}{3}\right) - \left(k^3 - \frac{k^3}{3}\right) = 1$ ***A1***

$7k^3 = 6$ ***AG***

[2 marks]

(b) recognition that the median m is the value such that $\int_0^m f(x)\mathrm{d}x = 0.5$ or

$\int_m^{2k} f(x)\mathrm{d}x = 0.5$ ***(M1)***

($k = 0.949914\ldots\left(= \sqrt[3]{\frac{6}{7}}\right)$ so) $\int_0^k kx\mathrm{d}x = 0.428571\ldots\left(= \frac{3}{7}\right)$ seen anywhere ***(A1)***

Note: This ***A1*** is independent of ***M1***.

EITHER

($m > k$ so) $\int_0^k kx\mathrm{d}x + \int_k^m\left(2kx - x^2\right)\mathrm{d}x = 0.5$ OR $\int_k^m\left(2kx - x^2\right)\mathrm{d}x = 0.0714285\ldots\left(= \frac{1}{14}\right)$ ***(A1)***

$\left[kx^2 - \frac{x^3}{3}\right]_k^m = 0.0714285\ldots$

$m = 1.02925\ldots$

continued…

Question 8 continued

OR

($m > k$ so) $\int_m^{2k} \left(2kx - x^2\right) \mathrm{d}x = 0.5$ ***(A1)***

$$\left[kx^2 - \frac{x^3}{3}\right]_m^{2k} = 0.5$$

$m = 1.02925\ldots$

THEN

$m = 1.03$ ***A1***

Note: The correct 3sf answer can be found by solving $\int_0^m kx \mathrm{d}x = 0.5$. This method is not valid since $m > k$. In this case award ***M1A0M0A0*.**

[4 marks]

Total [6 marks]

2. (a) **EITHER**

$\overline{y} = 2.1875 \times 7 + 0.6875$ ***A1***

OR

$\overline{y} = 15.3125 + 0.6875$ ***A1***

THEN

$\overline{y} = 16$ ***AG***

[1 mark]

(b) attempts to use $16 = \frac{\sum y}{n}$ to form a linear equation in p and q ***(M1)***

$16 = \frac{9 + 13 + p + q + 21}{5}$ $\left(80 = p + q + 43 \Rightarrow p + q = 37\right)$ ***(A1)***

attempts to solve two linear equations simultaneously for p and q (one of which is $q = p + 3$) ***(M1)***

$16 = \frac{9 + 13 + p + p + 3 + 21}{5}$ $\left(80 = 2p + 46\right)$

$p = 17$ and $q = 20$ ***A1***

[4 marks]

Total [5 marks]

5.

METHOD 1

correct inequality or equation involving $\mathrm{P}(X=0)$ ***(A1)***

$1-\mathrm{P}(X=0)>0.99$ OR $\mathrm{P}(X=0)<0.01$ OR $1-\mathrm{P}(X=0)=0.99$ OR $\mathrm{P}(X=0)=0.01$

attempts to solve their inequality (equality) involving 0.75^n for n ***(M1)***

$1-0.75^n>0.99$ OR $0.75^n<0.01$ OR $0.75^n=0.01$ OR $1-0.75^n=0.99$

Note: Valid solving attempts include graphical, use of logarithms, tabular or trial and error.

EITHER

$n>16.0078...$ OR $n=16.0078...$ ***(A2)***

the least value of n is 17 ***A1***

OR

$\mathrm{P}(X=0)=0.010022...\ (>0.01)$ (corresponding to $n=16$) ***(A1)***

$\mathrm{P}(X=0)=0.0075169...\ (<0.01)$ ***(A1)***

corresponding to $n=17$ (which is the least value of n) ***A1***

continued…

Question 5 continued.

METHOD 2 (TABLE ONLY APPROACH)

attempts to use binomial cdf to calculate a correct value of $\mathrm{P}(X \geq 1)$ for one value of n ***(M1)***

calculates correct values of $\mathrm{P}(X \geq 1)$ for at least one value of n ***(A1)***

$\mathrm{P}(X \geq 1) = 0.989977... \ (< 0.99)$ (corresponding to $n = 16$) ***(A1)***

$\mathrm{P}(X \geq 1) = 0.992483... \ (> 0.99)$ ***(A1)***

corresponding to $n = 17$ (which is the least value of n) ***A1***

[5 marks]

10. (a) recognizes that the mode is a value of x at which f has a maximum value ***(M1)***

a clearly labelled graph of f OR states $f'(x)=0$ OR considers the axis of symmetry

mode is 1.5 (kg) ***A1***

Note: Award ***M1A0*** for $(1.5, 0.441)$ or 0.441 stated as the final answer.

[2 marks]

(b) attempts to find $\int_1^2 f(x)\,\mathrm{d}x$ ***(M1)***

$=0.435294...$

$=0.435\left(=\frac{37}{85}\right)$ ***A1***

[2 marks]

continued…

Question 10 continued.

(c) **METHOD 1**

recognizes that $\int_{0.5}^{m} f(x)\,\mathrm{d}x = 0.5$ ***(M1)***

$m = 1.68701...$

$m = 1.69$ (kg) ***A2***

METHOD 2

recognizes that $\int_{0.5}^{m} f(x)\,\mathrm{d}x = 0.5$ ***(M1)***

$\frac{6}{85}\left(4m + \frac{3}{2}m^2 - \frac{1}{3}m^3\right) - \frac{6}{85}\left(2 + \frac{3}{8} - \frac{1}{24}\right) = 0.5$ ***A1***

$m = 1.68701...$

$m = 1.69$ (kg) ***A1***

[3 marks]

(d) $0.5 \le x \le 2$ (can be seen in a definite integral) ***(A1)***

attempts to evaluate their definite integral ***(M1)***

$\int_{0.5}^{2} f(x)\,\mathrm{d}x = 0.635294...$

$= 0.635$ ***A1***

[3 marks]

continued…

Question 10 continued.

(e) an attempt at forming an expected value integral $\int_{x_1}^{x_2} x f(x)\,dx$ (***M1)***

$\int_{0.5}^{0.75} x f(x)\mathrm{d}x \ (=0.060592...)$ OR $\int_{0.5}^{0.75} 25x f(x)\mathrm{d}x \ (=1.51482...)$ ***(A1)***

$\int_{0.75}^{3} x f(x)\mathrm{d}x \ (=1.64345...)$ OR $\int_{0.75}^{3} 24x f(x)\mathrm{d}x \ (=39.4428...)$ ***(A1)***

sums their two definite integrals ***(M1)***

(expected amount spent per customer is) $= \int_{0.5}^{0.75} 25x f(x)\mathrm{d}x + \int_{0.75}^{3} 24x f(x)\mathrm{d}x$

$=40.9576...$

(expected amount spent per customer is) $\$40.96$ ***A1***

[5 marks]

Total [15 marks]

1. (a) (i) $0.9 \times 0.9 \times 0.1$ ***(M1)***

Note: Award ***M1*** for 0.9 seen as part of a multiplication.

$= 0.081$ ***A1***

[2 marks]

(ii) let $A =$ number of boosts in the first 6 actions

METHOD 1

Recognition of binomial distribution ***(M1)***

with parameters $n = 6$ and $p = 0.1$ ***(A1)***

$\left(P\left(A \geq 1\right) =\right) 0.469 \ \ \left(0.468558...\right)$ ***A1***

Note: Award ***(M1)(A1)A0*** for an answer of 0.114 (from finding $P\left(A > 1\right)$).

METHOD 2

$\left(\mathrm{P}\left(A \geq 1\right) =\right)\ 1 - \mathrm{P}\left(\text{no boost in first } 6\right)$ ***(M1)***

$= 1 - 0.9^6$ ***(A1)***

$= 0.469 \ \left(0.468558...\right)$ ***A1***

METHOD 3

$0.1 + 0.9 \times 0.1 + 0.9^2 \times 0.1 + 0.9^3 \times 0.1 + 0.9^4 \times 0.1 + 0.9^5 \times 0.1$ ***(M1)(A1)***

Note: Award ***(M1)*** for at least 3 correct terms in a sum.

$= 0.469 \ \left(0.468558...\right)$ ***A1***

[3 marks]

continued…

Question 1 continued

(b) (i) **EITHER**

there will be $x-1$ actions not boosted, followed by one boosted action ***A1***

OR

The xth action is boosted and all previous actions are not boosted **A1**

OR

$\underbrace{(1-p)\times(1-p)\times\ldots\times(1-p)}_{(x-1)\text{ times}}\times p$ ***A1***

Note: For the **OR** method, the order of the operations must be clear (accept also a tree diagram etc.).

THEN

$p(1-p)^{x-1}$ ***AG***

[1 mark]

(ii) $\mathrm{E}(X)=\sum_{x=1}^{\infty} xp(1-p)^{x-1}$ ***A1***

[1 mark]

continued…

Question 1 continued

(c) (i) LHS:

$a + 2ar + 3ar^2 + ...$ ***A1***

Note: Award A1 for writing LHS $= \sum ar^n$ and differentiating to get $= \sum anr^{n-1}$.

RHS:

attempt to differentiate $a(1-r)^{-1}$ or use quotient rule ***(M1)***

$\frac{a}{(1-r)^2}$ ***A1***

$\sum_{n=1}^{\infty} nar^{n-1} = \frac{a}{(1-r)^2}$ ***A1***

Note: To award the final ***A1*** "$\sum_{n=1}^{\infty} nar^{n-1}$" must be seen either as part of the LHS working or as part of the final answer and all previous marks must be awarded.

[4 marks]

(ii) recognition that $a = p$ and $r = 1 - p$ (and $x = n$) ***(M1)***

$\mathrm{E}(X) = \frac{p}{(1-(1-p))^2}$ ***A1***

Note: Award ***A0 FT*** if their answer does not lead to the ***AG***.

$= \frac{1}{p}$ ***AG***

[2 marks]

(d) $\mathrm{E}(X) = 10,\ \mathrm{Var}(X) = 90$ ***A1A1***

[2 marks]

continued…

Question 1 continued

(e) $0.8 \times 0.6 \times 0.6$ ***(M1)A1***

Note: Award ***M1*** for evidence of P("not boosted") changing, e.g. a labelled tree diagram and/or 0.8 AND 0.6 seen.

$= 0.288$ ***AG***

[2 marks]

(f) after four actions that are not boosted, the probability that the next action is boosted is 1 (and $p = 1$ is a certainty) ***R1***

Note: Accept "when $Y = 5$, the probability of a boost is 1".
Do not accept "on the 5th action the probability of a boost is 1", unless there is reference to the previous actions being not boosted.

[1 mark]

(g) (i) $m = 0.32, n = 0.1536 \quad (0.154)$ ***A1A1***

[2 marks]

(ii) attempt to multiply outcomes by probabilities and find the sum ***(M1)***

$0.2 + 2 \times 0.32 + 3 \times 0.288 + 4 \times 0.1536 + 5 \times 0.0384$ ***A1***

Note: The ***A1*** can only be awarded if m and n are correct and the exact value of n is used.

$= 2.5104$ ***AG***

[2 marks]

(iii) $0.2 + 2^2 m + 3^2 \times 0.288 + 4^2 \times n + 5^2 \times 0.0384 - 2.5104^2$ ***(A1)***

$= 1.19 \ (1.18749\ldots)$ ***A1***

Note: Award ***A1A1*** for a correct answer found using the GDC.

[2 marks]

continued…

Question 1 continued

(h) (i) $\left(p = \frac{1}{2.5104} \right) = 0.398 \ (0.398342...)$ ***A1***

[1 mark]

(ii) $\text{Var}(X) = 3.79 \ (3.79170...)$ ***A1***

Note: Award ***A1*** for an answer of $3.80 \ (3.80040\ldots)$ (from using their 3sf answer from part (i).)
Condone 3.8.

[1 mark]

(iii) since $\text{Var}(X) > \text{Var}(Y)$, 2nd model provides more consistent experience ***R1***

Note: Only award ***R1 FT*** if both their variances are positive.

[1 mark]

Total [27 marks]

2. (a) 6^3 OR $6\times6\times6$ ***A1***

Note: Accept a labelled diagram that clearly illustrates correct application of the multiplication principle leading to 216.

$=216$ ***AG***

[1 mark]

(b) **EITHER**

attempts to find Δ ***(M1)***

$\Delta\left(=\left(4^2-4(1)(4)\right)\right)=0$ ***A1***

OR

attempts to solve $x^2+4x+4=0$ ***(M1)***

$\left((x+2)^2=0\Rightarrow\right)x=-2$ ***A1***

OR

attempts to express $x^2+4x+4(=0)$ as a perfect square ***(M1)***

$(x+2)^2(=0)$ is a perfect square ***A1***

OR

a graph of $y=x^2+4x+4$ with the vertex touching the x- axis at $x=-2$ ***A2***

THEN

graph of f has only one x-intercept ***AG***

[2 marks]

continued…

Question 2 continued.

Note: In parts (c) – (f), $(a,b,c)=(1,2,1)$, for example, represents an ordered 3-tuple $a=1, b=2$ and $c=1$.

(c) recognizes that $b^2-4ac=0$ (or equivalent) ***(M1)***

EITHER

attempts to use $\frac{b^2}{ac}=4\left(\frac{b^2}{4}=ac\right)$ ***(M1)***

determines one value of b from $b=2,4$ or 6 only (seen anywhere) OR one value of ac from $ac=1,4$ or 9 only (seen anywhere) ***(A1)***

OR

attempts to find a possible value of b ***(M1)***

determines one value of b from $b=2,4$ or 6 only (seen anywhere) ***(A1)***

OR

recognizes that b^2 must be a multiple of 4 OR b must be a multiple of 2 ***(M1)***

determines one value of b from $b=2,4$ or 6 only (seen anywhere) ***(A1)***

OR

attempts to find a possible value of ac ***(M1)***

determines one value of ac from $ac=1,4$ or 9 only (seen anywhere) ***(A1)***

THEN

$b=2$ and $ac=1$:

$(a,b,c)=(1,2,1)$ OR 1 possible way OR $\frac{1}{216}$ ***A1***

$b=4$ and $ac=4$:

$(a,b,c)=(1,4,4),(4,4,1),(2,4,2)$ OR 3 possible ways OR $\frac{3}{216}$ ***A1***

$b=6$ and $ac=9$:

$(a,b,c)=(3,6,3)$ OR 1 possible way OR $\frac{1}{216}$ ***A1***

therefore the required probability is $\frac{1}{216}+\frac{3}{216}+\frac{1}{216}$

$=\frac{5}{216}$ ***AG***

[6 marks]

continued…

Question 2 continued.

(d) recognizes that $b^2 - 4ac > 0$ (or equivalent eg. $\frac{b^2}{4} > ac$) ***(M1)***

maximum value of b^2 is 36 OR maximum value of ac is 8 ***(A1)***

Note: The above ***(A1)*** is independent of the ***(M1)***.

$ac = 1, 2, 3, 4, 5, 6, 8$ ***A1***

[3 marks]

(e) (i) $ac = 1 \ (b^2 > 4)$

$b = 3, 4, 5, 6$ OR 1×4 (quadratics) OR $6 - 2$ (quadratics) ***A1***

there are four quadratic functions ***AG***

[1 mark]

(ii)

$ac = 2 \ (b^2 > 8)$

$b = 3, 4, 5, 6$ ***(A1)***

Note: Award ***(A1)*** for referencing their result shown in part (e) (i).

EITHER

$(a, b, c) = (1, 3, 2), (1, 4, 2), (1, 5, 2), (1, 6, 2), (2, 3, 1), (2, 4, 1), (2, 5, 1), (2, 6, 1)$ ***A1***

Note: Award ***A1*** for listing the eight quadratic expressions.

OR

2×4 (quadratics) ***A1***

THEN

there are eight quadratics functions ***AG***

[2 marks]

continued…

Question 2 continued.

(f) **METHOD 1**

varies ac $(ac \neq 1, 2)$ and determines possible values of b such that $\Delta > 0$ ***(M1)***

correctly determines one of the following five cases ***(A1)***

correctly determines a further two of the following five cases ***(A1)***

correctly determines the remaining two cases ***(A1)***

case 1: $ac = 3$ $\left(b^2 > 12 \Rightarrow b = 4, 5, 6\right)$

$(a, b, c) = (1, 4, 3), (1, 5, 3), (1, 6, 3), (3, 4, 1), (3, 5, 1), (3, 6, 1)$ OR

6 possible ways OR $\frac{6}{216}$

case 2: $ac = 4$ $\left(b^2 > 16 \Rightarrow b = 5, 6\right)$

$(a, b, c) = (1, 5, 4), (1, 6, 4), (2, 5, 2), (2, 6, 2), (4, 5, 1), (4, 6, 1)$ OR

6 possible ways OR $\frac{6}{216}$

case 3: $ac = 5$ $\left(b^2 > 20 \Rightarrow b = 5, 6\right)$

$(a, b, c) = (1, 5, 5), (1, 6, 5), (5, 5, 1), (5, 6, 1)$ OR 4 possible ways OR $\frac{4}{216}$

case 4: $ac = 6$ $\left(b^2 > 24 \Rightarrow b = 5, 6\right)$

$(a, b, c) = (1, 5, 6), (2, 5, 3), (3, 5, 2), (6, 5, 1), (1, 6, 6), (2, 6, 3), (3, 6, 2), (6, 6, 1)$

OR 8 possible ways OR $\frac{8}{216}$

case 5: $ac = 8$ $\left(b^2 > 32 \Rightarrow b = 6\right)$

$(a, b, c) = (2, 6, 4), (4, 6, 2)$ OR 2 possible ways OR $\frac{2}{216}$

adds their probabilities ***(M1)***

Note: Award ***(M1)*** for adding at least 3 of their probabilities (denominator 216).

$$(p =) \ \frac{4}{216} + \frac{8}{216} + \frac{6}{216} + \frac{6}{216} + \frac{4}{216} + \frac{8}{216} + \frac{2}{216}$$

$$(= 0.0185... + 0.0370... + 0.0277... + 0.0277... + 0.0185... + 0.0370... + 0.0092...)$$

$$= \frac{38}{216} \ \left(= \frac{19}{108}, = 0.176\right)$$ ***A1***

continued…

Question 2 continued.

METHOD 2

varies $b^2 (\neq 1,4)$ OR $b(\neq 1,2)$ and determines possible values of ac such that $\Delta > 0$ ***(M1)***

correctly determines one of the following four cases ***(A1)***

correctly determines another case from the following four cases ***(A1)***

correctly determines the remaining two cases ***(A1)***

case 1: $b^2 = 9$ $(b = 3)$ $(ac = 1,2)$

$(a,b,c) = (1,3,1),(1,3,2),(2,3,1)$ OR 3 possible ways OR $\frac{3}{216}$

case 2: $b^2 = 16$ $(b = 4)$ $(ac = 1,2,3)$

$(a,b,c) = (1,4,1),(1,4,2),(2,4,1),(1,4,3),(3,4,1)$ OR 5 possible ways OR $\frac{5}{216}$

case 3: $b^2 = 25$ $(b = 5)$ $(ac = 1,2,3,4,5,6)$

$(a,b,c) = (1,5,1),(1,5,2),(2,5,1),(1,5,3),(3,5,1),(1,5,4),(2,5,2)$
$(4,5,1),(1,5,5),(5,5,1),(1,5,6),(2,5,3),(3,5,2),(6,5,1)$

OR 14 possible ways OR $\frac{14}{216}$

case 4: $b^2 = 36$ $(b = 6)$ $(ac = 1,2,3,4,5,6,8)$

$(a,b,c) = (1,6,1),(1,6,2),(2,6,1),(1,6,3),(3,6,1),(1,6,4),(2,6,2),(4,6,1)$
$(1,6,5),(5,6,1),(1,6,6),(2,6,3),(3,6,2),(6,6,1),(2,6,4),(4,6,2)$

OR 16 possible ways OR $\frac{16}{216}$

adds their probabilities ***(M1)***

Note: Award ***(M1)*** for adding at least 3 of their probabilities (denominator 216).

$(p =) \ \frac{3}{216} + \frac{5}{216} + \frac{14}{216} + \frac{16}{216}$

$(= 0.013889... + 0.023148... + 0.064815... + 0.074074...)$

$= \frac{38}{216} \ \ (= \frac{19}{108}, = 0.176)$ ***A1***

continued…

Question 2 continued.

METHOD 3

varies b^2 OR b and determines possible values of ac such that $\Delta < 0$ ***(M1)***

correctly determines two of the following six cases ***(A1)***

correctly determines a further two of the following six cases ***(A1)***

correctly determines the remaining two cases ***(A1)***

case 1: $b^2 = 1$ $(b = 1)$ 36 possible ways OR $\frac{36}{216}$

case 2: $b^2 = 4$ $(b = 2)$ 35 possible ways OR $\frac{35}{216}$

case 3: $b^2 = 9$ $(b = 3)$ 33 possible ways OR $\frac{33}{216}$

case 4: $b^2 = 16$ $(b = 4)$ 28 possible ways OR $\frac{28}{216}$

case 5: $b^2 = 25$ $(b = 5)$ 22 possible ways OR $\frac{22}{216}$

case 6: $b^2 = 36$ $(b = 6)$ 19 possible ways OR $\frac{19}{216}$

$(p =)\ 1 - \left(\frac{36}{216} + \frac{35}{216} + \frac{33}{216} + \frac{28}{216} + \frac{22}{216} + \frac{19}{216} + \frac{5}{216}\right)$ ***(M1)***

$\left(= 1 - \left(0.16666... + 0.16203... + 0.15277... + 0.12962... + 0.10185... + 0.087962... + 0.023148...\right)\right)$

Note: Award ***(M1)*** for adding at least 3 of their probabilities inside the above bracket (denominator 216).

$= \frac{38}{216}$ $\left(= \frac{19}{108}, = 0.176\right)$ ***A1***

[6 marks]

continued…

Question 2 continued.

(g) recognizes that $4Z^2 - 4 > 0$ $(Z^2 > 1)$ ***(M1)***

probability of two x-intercepts is

EITHER

$\mathrm{P}(|Z| > 1)$ ***(A1)***

OR

$\mathrm{P}(Z < -1)$ or $\mathrm{P}(Z > 1)$ (can be shown on a labelled diagram) ***(A1)***

$= 0.158655... + 0.158655...$

OR

$1 - \mathrm{P}(-1 \le Z \le 1)$ (can be shown on a labelled diagram) ***(A1)***

$= 1 - 0.682689...$

THEN

$= 0.317310...$

$= 0.317$ ***A1***

[3 marks]

(h) attempts to solve $X_1 > 0.5$ for Z ***(M1)***

$-1.25 < Z \le -1$ ***(A1)(A1)***

Note: Award ***(M1)(A1)*** for obtaining $Z = -1.25$ from solving $X_1 = 0.5$ and award ***(A1)*** for stating the correct inequality.
Award ***(M1)(A1)(A1)*** for $-1.25 < Z < -1$.
Award ***(M1)(A1)(A0)*** for $-1.25 < Z$.
Award ***(M1)*** for rearranging to form $-\sqrt{Z^2 - 1} = Z + 0.5$ and then attempting to square both sides $Z^2 - 1 = (Z + 0.5)^2$ $\left(= Z^2 + Z + 0.25\right)$.

attempts to calculate their $\mathrm{P}(X_1, X_2 \text{ both } > 0.5)$ ***(M1)***

$\mathrm{P}(-1.25 < Z \le -1) = 0.053005...$ ***(A1)***

attempts to calculate their $\mathrm{P}(X_1, X_2 \text{ both } > 0.5 \,|\, x\text{-intercepts})$ ***(M1)***

$= \dfrac{0.053005...}{0.317310...}$

$= 0.167$ ***A1***

[7 marks]

Total [31 marks]

2. (a) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

$\mathrm{P}(A \cap B) = 0.65 + 0.45 - 0.85$ (or equivalent) ***(A1)***

$= 0.25$ ***A1***

[3 marks]

(b) $\mathrm{P}(A' \cap B') = 0.15$ (may be seen in Venn diagram) ***(A1)***

attempt to substitute their values into $\mathrm{P}(A' \mid B') = \dfrac{\mathrm{P}(A' \cap B')}{\mathrm{P}(B')}$ ***(M1)***

$\mathrm{P}(A' \mid B') = \dfrac{0.15}{0.55}$

$= \dfrac{15}{55}\left(= \dfrac{3}{11}\right)$ ***A1***

[3 marks]

Total [6 marks]

3. (a) recognition of sum of probabilities equals 1 ***(M1)***

$\frac{3k}{20}+\frac{5k}{20}+\frac{8k}{20}+\frac{11k}{20}=1$

$k=0.740740$

$k=0.741\left(=\frac{20}{27}\right)$ ***A1***

[2 marks]

(b) correct probabilities: $\frac{3}{27},\frac{5}{27},\frac{8}{27},\frac{11}{27}$ OR $0.111,\ 0.185,\ 0.296,\ 0.407$ ***(A1)***

substitution of their probabilities into formula for expected value ***(M1)***

$3\times\frac{3}{27}+5\times\frac{5}{27}+8\times\frac{8}{27}+11\times\frac{11}{27}$ OR $\frac{219k}{20}$

$=8.11111...$

$\mathrm{E}(X)=8.11\left(=\frac{219}{27}=\frac{73}{9}\right)$ (same 3sf from previous 3sf answer) ***A1***

[3 marks]

Total [5 marks]

6. (a) $\mathrm{E}(X) = \int_0^2 \frac{x}{5}\,\mathrm{d}x + \int_2^8 \left(-\frac{x^2}{30} + \frac{4x}{15}\right)\mathrm{d}x$ ***(A1)(A1)***

Note: Award ***(A1)*** $\int_0^2 \frac{x}{5}\,(\mathrm{d}x)$ and ***(A1)*** for $\int_2^8 \left(-\frac{x^2}{30} + \frac{4x}{15}\right)(\mathrm{d}x)$

$= \frac{2}{5} + \frac{12}{5}$

$= \frac{14}{5} (= 2.8)$ ***A1***

[3 marks]

(b) attempt to use the expectation formula $\mathrm{E}(aX+b) = a\mathrm{E}(X) + b$ ***(M1)***

$\mathrm{E}(c - 2X) = c - 2\mathrm{E}(X) (= 0)$

$c = 2\mathrm{E}(X)$

$= \frac{28}{5} (= 5.6)$ ***A1***

[2 marks]

(c) recognition that median m lies between 2 and 8 e.g. using a diagram or integral ***(M1)***

$\int_0^2 \frac{1}{5}\mathrm{d}x + \int_2^m \left(-\frac{x}{30} + \frac{4}{15}\right)\mathrm{d}x = \frac{1}{2}$ OR $\int_m^8 \left(-\frac{x}{30} + \frac{4}{15}\right)\mathrm{d}x = \frac{1}{2}$ OR $\int_2^m \left(-\frac{x}{30} + \frac{4}{15}\right)\mathrm{d}x = \frac{1}{10}$ ***(A1)***

$m = 2.52277...$

$m = 2.52$ ***A1***

[3 marks]

Total [8 marks]

4. (a) $\mathrm{P}(A) = \frac{1}{4}$ ***(A1)***

attempt to use $\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$ ***(M1)***

$\frac{2}{3} = \frac{\mathrm{P}(B \cap A)}{\left(\frac{1}{4}\right)}$

$\mathrm{P}(A \cap B) = \frac{2}{3}\left(\frac{1}{4}\right)$

$= \frac{2}{12}\left(= \frac{1}{6}\right)$ ***A1***

[3 marks]

(b) attempt to use $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ OR a Venn diagram, with their values of $\mathrm{P}(A)$ and $\mathrm{P}(B \cap A)$ ***M1***

$\frac{3}{4} = \frac{1}{4} + \mathrm{P}(B) - \frac{1}{6}$

$\mathrm{P}(B) = \frac{1}{2} + \frac{1}{6}$

$= \frac{4}{6}\left(= \frac{2}{3}\right)$ ***A1***

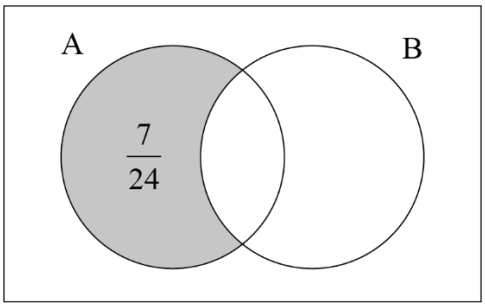
$\mathrm{P}(B|A) = \mathrm{P}(B)$ OR $\mathrm{P}(A)\mathrm{P}(B) = \frac{1}{6}$ so $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ (hence A and B are independent) ***R1***

Note: The ***R1*** is dependent on all previous marks

[3 marks]

Total [6 marks]

4. (a) attempt to use a Venn diagram OR law of addition ***(M1)***



$\mathrm{P}(A \cup B) = \mathrm{P}(A \cap B') + \mathrm{P}(B)$

$\frac{5}{8} = \frac{7}{24} + \mathrm{P}(B)$ ***(A1)***

$\mathrm{P}(B) = \frac{8}{24}\left(= \frac{1}{3}\right)$ ***A1***

[3 marks]

(b) **METHOD 1:** finding $\mathrm{P}(A)$

attempt to find $\mathrm{P}(A)$ ***M1***

$\mathrm{P}(A \cap B') = \mathrm{P}(A) \times \mathrm{P}(B')$ OR $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \times \mathrm{P}(B)$

$\mathrm{P}(A) \times \frac{2}{3} = \frac{7}{24}$ OR $\frac{5}{8} = \frac{2}{3}\mathrm{P}(A) + \frac{1}{3}$

$\mathrm{P}(A) = \frac{7}{16}$ ***(A1)***

EITHER

$\mathrm{P}(A' \mid B) = \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A') \times \mathrm{P}(B)}{\mathrm{P}(B)} = \mathrm{P}(A')$ (seen anywhere) OR $1 - \frac{7}{16}$ ***(A1)***

OR

$\mathrm{P}(A' \mid B)\left(= \frac{\mathrm{P}(A' \cap B)}{\mathrm{P}(B)}\right) = \frac{\frac{9}{48}}{\frac{16}{48}}\left(= \frac{\frac{3}{16}}{\frac{1}{3}}\right)$ ***(A1)***

THEN

$\mathrm{P}(A' \mid B) = \frac{9}{16}$ ***A1***

continued…

Question 4 continued.

METHOD 2: attempt to find $\mathrm{P}(A \cap B)$

attempt to use $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$ ***M1***

$x = \left(x + \frac{7}{24}\right) \times \frac{1}{3}$

$x = \frac{7}{48}$ ***A1***

$\mathrm{P}(A' \mid B) = \mathrm{P}(B) - \mathrm{P}(A \cap B) + 1 - \mathrm{P}(A \cup B)$ OR $= \frac{1}{3} - \frac{7}{48} + 1 - \frac{5}{8}$ ***(A1)***

$\mathrm{P}(A' \mid B) = \frac{27}{48} \left(= \frac{9}{16}\right)$ ***A1***

[4 marks]

Total [7 marks]

5. (a) recognizing constant change from Y to X and/or comparing the two distributions

$b-7=22-19$ **OR** $b=7+(22-19)$ **OR** $\frac{b-7}{a}=\frac{22-19}{a}$ ***(M1)***

$b=10$ ***A1***

[2 marks]

(b) $\left(\mathrm{P}(7-a<X<7+a)=\right)\ 0.68$ ***A1***

[1 mark]

(c) **EITHER**

recognizing that 22 is one standard deviation above the mean ***(M1)***

OR

recognizing symmetry of the normal curve and total area $=1$ ***(M1)***

THEN

0.16 or 0.34 (or equivalent) seen in correct sketch or probability statement ***(A1)***

$\left(\mathrm{P}(Y<22)=\right)1-\frac{0.32}{2}$ **OR** $0.5+0.34$ **OR** $0.68+0.16$

0.84 ***A1***

[3 marks]

Total [6 marks]

4. (a) attempt to substitute $F = 19.8$ into the regression line for A on F ***(M1)***

$A = 2.89(19.8) + 99.3$

$= 156.522 \text{ (cm)}$

arm span = 157 (cm) ***A1***

Note: Award ***M0A0*** for choosing the wrong regression line to get $A = 156.417...$ so $A = 156$.

[2 marks]

(b) recognition that the lines intersect at the mean point (may be seen on a sketch) ***(M1)***

$2.89F + 99.3 = \frac{F + 32.6}{0.335}$ OR $0.335A - 32.6 = \frac{A - 99.3}{2.89}$

$159.686...$ or $20.8948...$

the mean arm span $= 160 \text{ (cm)}$, the mean foot length $= 20.9 \text{ (cm)}$ ***A1A1***

[3 marks]

(c) **METHOD 1**

recognition of symmetry of interval around mean (may be seen on a sketch) ***(M1)***

$P(H < 153) = 0.06$ OR $P(H < 173) = 0.94$ OR equivalent

$\frac{153 - 163}{\sigma} = -1.55477...$ OR $\frac{173 - 163}{\sigma} = 1.55477...$ (***A1)***

$\sigma = 6.43181...$

$\sigma = 6.43 \text{ (cm)}$ ***A1***

METHOD 2

attempt to find σ by equating an appropriate correct normal CDF function to 0.88 (or e.g. 0.06 or 0.94) ***(M1)***

$\sigma = 6.43181...$

$\sigma = 6.43 \text{ (cm)}$ ***A2***

Note: Accept use of calculator notation eg $normcdf(153, 173, 163, \sigma) = 0.88$

[3 marks]

Total [8 marks]

11. (a) (i) $\int_{2.25}^{4.5} \frac{4}{21}\left(1-\cos\left(\frac{4\pi}{9}(t-2.25)\right)\right)\mathrm{d}t$

$= \frac{3}{7}\left(= 0.428571... = 0.429\right)$ ***A1***

(ii) mode of $T = 4.5$ ***A1***

(iii) the median is greater ***A1***

$\mathrm{P}(T < 4.5) < 0.5$ OR $\mathrm{P}(T > 4.5) > 0.5$ OR $\mathrm{P}\left(T < 4.5\right) < \mathrm{P}\left(T > 4.5\right)$ OR median $= 4.69$ ***R1***

Note: Accept reference to areas rather than probabilities.

[4 marks]

(b) recognition of the need to integrate f ***(M1)***

$\int_{2.25}^{3.5} f(t)\mathrm{d}t$

$= 0.103749\ldots$

$= 0.104$ ***A1***

[2 marks]

(c) attempt to use formula for conditional probability in context ***(M1)***

$\frac{\mathrm{P}(T \leq 3)}{\mathrm{P}(\text{fast})}$ OR $\frac{\mathrm{P}(T \leq 3)}{\mathrm{P}(T \leq 3.5)}$ OR $\frac{\mathrm{P}(\text{very fast})}{\mathrm{P}(\text{fast})}$ (accept strict inequality signs)

$= \frac{0.0247152...}{0.103749...}$ ***(A1)***

$= 0.238220...$

$= 0.238$ ***A1***

[3 marks]

continued…

Question 11 continued.

(d) recognition that the lower quartile q is the value such that $\int_{2.25}^{q} f(t)\mathrm{d}t = 0.25$ ***(M1)***

$$\int_{2.25}^{q} \frac{4}{21}\left(1-\cos\left(\frac{4\pi}{9}(t-2.25)\right)\right)\mathrm{d}t = 0.25$$ ***(A1)***

Note: Condone the absence of $\mathrm{d}t$ for this ***A1***.

$$\left[\frac{4}{21}\left(t-\frac{9}{4\pi}\sin\left(\frac{4\pi}{9}(t-2.25)\right)\right)\right]_{2.25}^{q} = 0.25$$

$q = 4.01290...$

$q = 4.01$ ***A1***

[3 marks]

(e) attempt to find the expected value for a transformed linear variable ***(M1)***

$\mathrm{E}(P) = \mathrm{E}(a-bT) = a - b\mathrm{E}(T)$

$a - 4.723b = 100$ ***(A1)***

recognition that max score is achieved with fastest time $t = 2.25$ ***(M1)***

maximum score $a - 2.25b = 150$ ***(A1)***

$a = 195.491...,\ b = 20.2183...$

$a = 195,\ b = 20$ ***A1***

Note: these values must be given to the nearest integer for the ***A1*** to be awarded.

[5 marks]

(f) attempt to find variance for a transformed linear variable ***(M1)***

$\mathrm{Var}(P) = \mathrm{Var}(a-bT) = b^2\mathrm{Var}(T)$

$0.906\left(20.2183...^2\right) = 370.356...$

$\mathrm{Var}(P) = 370$ ***A1***

Note: accept any answer which rounds to any value between 362 and 370 inclusive based on use of less accurate values of b.

[2 marks]

Total [19 marks]

1.

(a) $1.97651..., 40.2286...$

$a = 1.98,\ b = 40.2\ \ \left((y =)1.98x + 40.2\right)$ ***A1A1***

[2 marks]

(b) attempt to substitute $x = 20$ into their regression equation ***(M1)***

$y = 1.97651... \times 20 + 40.2286...$

$79.7589...$

79.8 (hours) ***A1***

[2 marks]

Total [4 marks]

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8. an attempt to set up an expression for mean or variance that includes total frequency and both p and q ***(M1)***

$$\bar{x} = \frac{20\times 12 + 35\times q + p\times 8}{12+q+8} \left(= \frac{240+35q+8p}{20+q}\right)$$

$$\frac{240+35q+8p}{20+q} = 31 \text{ OR } 4q+8p = 380 \left(q+2p=95\right) \quad \textbf{(A1)}$$

$$\sigma^2 = \frac{12\times 20^2 + q\times 35^2 + 8\times p^2}{12+q+8} - 31^2 = 124 \left(\frac{4800+1225q+8p^2}{20+q} - 31^2 = 124\right)$$

OR

$$\sigma^2 = \frac{12\times(20-31)^2 + q\times(35-31)^2 + 8\times(p-31)^2}{12+q+8} = 124 \left(\frac{1452+16q+8(p-31)^2}{20+q} = 124\right) \quad \textbf{(A1)}$$

EITHER

$q = 95-2p$ ***(A1)***

substitute their q into their expression for variance and solve for p ***(M1)***

$$\frac{4800+1225(95-2p)+8p^2}{20+(95-2p)} - 31^2 = 124 \text{ OR } \frac{1452+16(95-2p)+8(p-31)^2}{20+(95-2p)} = 124$$

$p = 45$ ***A1***

$q = 5$ ***A1***

OR

$p = \frac{95-q}{2}$ ***(A1)***

substitute their p into their expression for variance and solve for q ***M1***

$$\frac{4800+1225q+8\left(\frac{95-q}{2}\right)^2}{20+q} - 31^2 = 124 \text{ OR } \frac{1452+16q+8\left(\frac{95-q}{2}-31\right)^2}{20+q} = 124$$

$q = 5$ ***A1***

$p = 45$ ***A1***

Note: Award ***A1A0*** for $p = 45, q = 5$ and $p = -10, q = 115$.

Note: Award full marks for $p = 45, q = 5$ regardless of working.

[7 marks]

10.

(a) recognising to find $\mathrm{P}(W < 170)$ ***(M1)***

$\mathrm{P}(W < 170) = 0.265985\ldots$

$\mathrm{P}(W < 170) = 0.266$ (accept 26.6%) ***A1***

[2 marks]

(b) recognising $\mathrm{P}(W > w) = 0.2$ OR labelled sketch (or equivalent) ***(M1)***

$w = 181.732\ldots$

$w = 181.7$ (grams) (must be 4 sf) ***A1***

[2 marks]

(c) recognizing the need to find $\mathrm{P}(170 < W < 185)$ ***(M1)***

diagram, $\mathrm{P}(170 < W < 185) = 0.628(364\ldots)$

62.8% ***A1***

[2 marks]

(d) recognising binomial probability ***(M1)***

$X \sim \mathrm{B}(40, 0.628364\ldots)$ OR $n = 40$ and $p = 0.628364\ldots$ ***(A1)***

$\mathrm{P}(X \geq 30) = 0.073861\ldots$

$\mathrm{P}(X \geq 30) = 0.0739$ (accept 7.39%) ***A1***

[3 marks]

continued…

Question 10 continued.

(e) $Y \sim \mathrm{B}(10, 0.073861\ldots)$ ***(A1)***

$\mathrm{P}(Y = 4) = 0.003944\ldots$

$\mathrm{P}(Y = 4) = 0.00394$ (accept 0.394%) ***A1***

[2 marks]

(f) $\mathrm{P}(W < 170) = 0.12$ OR $\mathrm{P}(W > 185) = 0.06$ ***(A1)***

use of inverse normal to find z score for their probability ***(M1)***

$z = -1.17(4986\ldots)$, $z = 1.55(4773\ldots)$ ***(A1)(A1)***

attempt to solve their two equations ***(M1)***

$$-1.174986\ldots = \frac{170 - \mu}{\sigma}$$

$$1.554773\ldots = \frac{185 - \mu}{\sigma}$$

$\mu = 176.456\ldots$

$\mu = 176$ (grams) ***A1***

[6 marks]

Total [17 marks]

3. (a) (i) summing probabilities and equating to 1 ***(M1)***

$1.5a + 2a + 0.281 + a + 0.026 = 1$

$(a =)\ 0.154$ ***A1***

(ii) 2 (days) ***A1***

[3 marks]

(b) using expected value formula ***(M1)***

(mean =)2.44 (2.436 (exact)) ***A1***

[2 marks]

(c) convenience (sampling) ***A1***

[1 mark]

Total [6 marks]

11. (a) (i) recognizing binomial distribution ***(M1)***

$E \sim \mathrm{B}(50,\ 0.08)$, where E is the number of inaccurate surveys

$\mathrm{P}(E \leq 6) = 0.898128...$

$(\mathrm{P}(E \leq 6) =)\ 0.898$ (accept 89.8%) ***A1***

(ii) recognition of conditional probability ***(M1)***

Note: Recognition must be shown in context either in words or symbols, not just $\mathrm{P}(A \mid B)$.

$$\mathrm{P}(E = 4 \mid E \leq 6) = \frac{\mathrm{P}(E = 4)}{\mathrm{P}(E \leq 6)}$$

$$= \frac{0.203654...}{0.898128...}$$ ***(A1)***

$= 0.227$ (accept 22.7%) ***A1***

[5 marks]

(b) use of tables ***(M1)***

Note: Award ***(M1)*** for at least one correct value of $\mathrm{P}(E \leq 6)$ for a value of n, where $n \neq 50$.

$(n = 94 \Rightarrow)\ \mathrm{P}(E \leq 6) = 0.366746...$

$n = 94$ ***A2***

[3 marks]

(c) (i) $\mathrm{P}(I) = \mathrm{P}(I \cap A) + \mathrm{P}(I \cap B) + \mathrm{P}(I \cap C)$ ***(M1)***

$= 0.55 \times 0.08 + 0.25 \times 0.06 + 0.2 \times 0.11$ ***(A1)***

$= 0.081$ (accept 8.1%) ***A1***

(ii) $\mathrm{P}(A \mid I) = \frac{\mathrm{P}(A \cap I)}{\mathrm{P}(I)}$ (or use of Bayes' formula) ***(M1)***

$$\mathrm{P}(A \mid I) = \frac{0.55 \times 0.08}{0.081}$$ ***(A1)***

$0.543209...$

$= 0.543$ (accept 54.3%) ***A1***

[6 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$\mathrm{P}(I \mid C) = \frac{x}{100}$ (seen anywhere) ***(M1)***

$\mathrm{P}(I) = 0.55 \times 0.08 + 0.25 \times 0.06 + 0.2 \times \frac{x}{100}$ ***(A1)***

equating their expression for $\mathrm{P}(I)$ to $\frac{x}{100}$ ***(M1)***

$\frac{x}{100} = 0.55 \times 0.08 + 0.25 \times 0.06 + 0.2 \times \frac{x}{100}$

$x = 7.375$

$x = 7.38$ (accept 7.38%) ***A1***

METHOD 2

$\mathrm{P}(I) = 0.55 \times 0.08 + 0.25 \times 0.06 + 0.2 \times y$ ***(A1)***

equating their expression for $\mathrm{P}(I)$ to y ***(M1)***

$y = 0.55 \times 0.08 + 0.25 \times 0.06 + 0.2 \times y$

$y = 0.07375$

recognizing to multiply their y value by 100 ***(M1)***

$x = 7.38$ (accept 7.38%) ***A1***

[4 marks]

Total [18 marks]

1. (a) attempts to use $\mathrm{E}(M)=\sum m\mathrm{P}(M=m)$ ***(M1)***

$=1\times\frac{1}{16}+2\times\frac{3}{16}+3\times\frac{5}{16}+4\times\frac{7}{16}$ (or equivalent)

$=\frac{25}{8}\left(=3.125,=\frac{50}{16},=3.13\right)$ ***A1***

[2 marks]

(b) $G(1)=1$ ***A1***

[1 mark]

(c) (i) $G'(t)=\frac{1}{16}+\frac{3}{8}t+\frac{15}{16}t^2+\frac{7}{4}t^3\left(=\frac{1}{16}+\frac{6}{16}t+\frac{15}{16}t^2+\frac{28}{16}t^3\right)$ ***A2***

Note: Award ***A1A0*** for three correct terms.

[2 marks]

(ii) substitutes $t=1$ into their $G'(t)$ (***M1)***

$G'(1)=\frac{1}{16}+\frac{3}{8}+\frac{15}{16}+\frac{7}{4}\left(=\frac{1}{16}+\frac{6}{16}+\frac{15}{16}+\frac{28}{16}\right)$ ***A1***

$=\frac{25}{8}\left(=3.125,=\frac{50}{16}\right)$ ***A1***

$=\mathrm{E}(M)$ ***AG***

Note: Award a maximum of ***M1A1A0FT*** for an incorrect $\mathrm{E}(M)$ obtained with an incorrect $G'(t)$.

Award a maximum of ***M1A1A0FT*** for a correct $\mathrm{E}(M)$ obtained with an incorrect $G'(t)$.

[3 marks]

continued…

Question 1 continued.

(d) attempts to find at least one required probability ***(M1)***

$\mathrm{P}(X=0)=\frac{3}{5}\times\frac{2}{4}$ OR $\mathrm{P}(X=0)=\frac{{}^2C_0\times{}^3C_2}{{}^5C_2}\left(=\frac{{}^3C_2}{{}^5C_2}\right)$ ***A1***

$=\frac{3}{10}$

$\mathrm{P}(X=1)=\frac{2}{5}\times\frac{3}{4}+\frac{3}{5}\times\frac{2}{4}\ \left(=2\times\frac{6}{20}\right)$ ***A1A1***

Note: Award ***A1*** for $\frac{2}{5}\times\frac{3}{4}$ OR $\frac{6}{20}$ and ***A1*** for $\frac{3}{5}\times\frac{2}{4}$ OR $\frac{6}{20}$.

Award ***A2*** for $\mathrm{P}(X=1)=\frac{{}^2C_1\times{}^3C_1}{{}^5C_2}\left(=\frac{2\times{}^3C_1}{{}^5C_2}\right)$.

$=\frac{3}{5}$

EITHER

$\mathrm{P}(X=2)=\frac{2}{5}\times\frac{1}{4}$ OR $\mathrm{P}(X=2)=\frac{{}^2C_2\times{}^3C_0}{{}^5C_2}\left(=\frac{{}^2C_2}{{}^5C_2},=\frac{{}^3C_0}{{}^5C_2}\right)$ ***A1***

OR

$\mathrm{P}(X=2)=1-\left(\frac{3}{10}+\frac{3}{5}\right)$ ***A1***

THEN

$=\frac{1}{10}$

leading to $G_X(t)=\frac{3}{10}+\frac{3}{5}t+\frac{1}{10}t^2$ ***AG***

Note: Award ***A*** marks as appropriate for clearly labelled correct tree diagrams.

Do not award ***A*** marks for $\mathrm{P}(X=0)=\frac{3}{10}$, $\mathrm{P}(X=1)=\frac{3}{5}$ and $\mathrm{P}(X=2)=\frac{1}{10}$ stated without any working shown.

[5 marks]

continued…

Question 1 continued.

(e) (i) **EITHER**

$\mathrm{P}(Y=2)=\frac{1}{2}p$ ***(A1)***

OR

$\frac{1}{2}p=\frac{1}{3}$ ***(A1)***

THEN

$p=\frac{2}{3}\left(=0.667\right)$ ***A1***

[2 marks]

(ii) uses their $G_Y(t)=\mathrm{P}(Y=0)t^0+\mathrm{P}(Y=1)t^1+\frac{1}{3}t^2$ with their value of p ***(M1)***

$\mathrm{P}(Y=0)=\frac{1}{6}$ ***(A1)***

$\mathrm{P}(Y=1)=\frac{1}{2}$ ***(A1)***

$G_Y(t)=\frac{1}{6}+\frac{1}{2}t+\frac{1}{3}t^2$ ***A1***

Note: Award a maximum ***(M1)(A1)(A1)A0FT*** from an incorrect value of p.
Award ***(M1)(A1)(A1)A0FT*** from $p=\frac{1}{3}$ leading to $G_Y(t)=\frac{1}{6}+\frac{1}{2}t+\frac{1}{3}t^2$.

[4 marks]

continued…

Question 1 continued.

(f) **METHOD 1**

$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right) \left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4\right)$ ***(A1)***

attempts to evaluate their $G_Z'(1)$ ***(M1)***

$G_Z'(1) = 1.96666...$

$\mathrm{E}(Z) = 1.97 \left(= \frac{59}{30}\right)$ ***A2***

Note: Award a maximum of ***(A1)FT(M1)(M1)A0*** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).

Do not award ***(A1)FT*** if $G_Y(t)$ is not a quadratic of the required form.

METHOD 2

$G_Z(t) = \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right) \left(G_Z(t) = \frac{1}{20} + \frac{1}{4}t + \frac{5}{12}t^2 + \frac{1}{4}t^3 + \frac{1}{30}t^4\right)$ ***(A1)***

attempts to find $G_Z'(t)$ by differentiation ***M1***

$G_Z'(t) = \frac{1}{4} + \frac{5}{6}t + \frac{3}{4}t^2 + \frac{2}{15}t^3$ OR

$G_Z'(t) = \left(\frac{3}{5} + \frac{1}{5}t\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right) + \left(\frac{1}{2} + \frac{2}{3}t\right)\left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)$

substitutes $t = 1$ into their $G_Z'(t)$ ***(M1)***

$G_Z'(1) = 1.96666...$

$\mathrm{E}(Z) = 1.97 \left(= \frac{59}{30}\right)$ ***A1***

Note: Award a maximum of ***(A1)FT(M1)(M1)A0*** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).

Do not award ***(A1)FT*** if $G_Y(t)$ is not a quadratic of the required form.

continued…

Question 1 continued.

METHOD 3

$G_Z(t)=\left(\frac{3}{10}+\frac{3}{5}t+\frac{1}{10}t^2\right)\left(\frac{1}{6}+\frac{1}{2}t+\frac{1}{3}t^2\right)\ \left(G_Z(t)=\frac{1}{20}+\frac{1}{4}t+\frac{5}{12}t^2+\frac{1}{4}t^3+\frac{1}{30}t^4\right)$ ***(A1)***

recognizes that $G_Z'(1)=G_X'(1)+G_Y'(1)\Rightarrow \mathrm{E}(Z)=\mathrm{E}(X)+\mathrm{E}(Y)$ ***(M1)***

attempts to evaluate their $G_X'(1)$ and $G_Y'(1)$ (or their $\mathrm{E}(X)$ and $\mathrm{E}(Y)$) ***(M1)***

$G_X'(1)=\frac{3}{5}+\frac{1}{5}\ \left(=\frac{4}{5}\right)$ and $G_Y'(1)=\frac{1}{2}+\frac{2}{3}\ \left(=\frac{7}{6}\right)$

$\mathrm{E}(Z)=1.97\ \left(=\frac{59}{30}\right)$ ***A1***

Note: Award a maximum of ***(A1)FT(M1)(M1)A0*** for use of an incorrect quadratic $G_Y(t)$ from part (e) (ii).

Do not award ***(A1)FT*** if $G_Y(t)$ is not a quadratic of the required form.

[4 marks]

Total [23 marks]

2. (a) attempt to use $\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B)\mathrm{P}(B)\left(= \frac{1}{4} \times \frac{1}{3}\right)$ ***(M1)***

$= \frac{1}{12}$ ***A1***

[2 marks]

(b) $\mathrm{P}(B') = \frac{2}{3}$ or $\mathrm{P}(A \cap B') = \frac{5}{12}$ ***(A1)***

attempt to use $\mathrm{P}(A \mid B') = \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')}$ ***(M1)***

$= \frac{\left(\frac{5}{12}\right)}{\left(\frac{2}{3}\right)}$ ***(A1)***

$= \frac{5}{8}\left(= \frac{15}{24}\right)$ ***A1***

[4 marks]

Total [6 marks]

7. (a) considers at least two of $0.6-2a\geq 0$, $3a\geq 0$ and $0.4-a\geq 0$ ***(M1)***

$\Rightarrow 0\leq a\leq 0.3$ ***A1***

[2 marks]

(b) **METHOD 1**

recognition that $\text{Var}(2-X)=\text{Var}(X)$ (may be seen anywhere) ***R1***

$a=0.2\Rightarrow \text{E}(X)=2$ (by symmetry) ***A1***

attempt to use $\text{E}(X^2)=\sum x^2\text{P}(X=x)$ ***(M1)***

$\text{E}(X^2)=0.2+2.4+1.8$

$=4.4$ (***A1)***

attempt to use $\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2$ OR $\text{Var}(X)=\text{E}\left[(X-\mu)^2\right]$ ***(M1)***

$4.4-4$ OR $(-1)^2\times 0.2+(1)^2\times 0.2$

$\text{Var}(X)=0.4$ ***A1***

METHOD 2

attempt to find distribution for $Y=2-X$ ***M1***

$a=0.2\Rightarrow \text{E}(Y)=0$ (by symmetry) ***A1***

attempt to use $\text{E}(Y^2)=\sum y^2\text{P}(Y=y)$ ***(M1)***

$\text{E}(Y^2)=0.2+0+0.2$

$=0.4$ (***A1)***

attempt to use $\text{Var}(Y)=\text{E}(Y^2)-(\text{E}(Y))^2$ OR $\text{Var}(Y)=\text{E}\left[(Y-\mu)^2\right]$ ***(M1)***

$0.4-0$ OR $(1)^2\times 0.2+(-1)^2\times 0.2$

$\text{Var}(Y)=0.4$ ***A1***

[6 marks]

Total [8 marks]

3. (a) attempt to find mean of x-values ***(M1)***

$\overline{x} = 6.45$ ***A1***

[2 marks]

(b) (i) attempt to substitute their $\overline{x}$ into regression equation ***(M1)***

$\overline{y} = 0.014375 + 2.1625 \times 6.45$

$\overline{y} = 14.0 \ (13.9625 \text{ exact})$ ***A1***

(ii) attempt to use formula for the mean ***(M1)***

$\sum_{i=1}^{8} y_i = 8 \times 13.9625... \left(\Rightarrow p + 93.2 = 111.7\right)$

$p = 18.5$ ***A1***

Note: Accept $p = 18.8$ from the use of 3 sf in part (b)(i).
Some candidates are attempting part (b)(ii) before (b)(i), and substitute 8.9 into the given regression equation to find p. They then use their value of p to find $\overline{y}$. In this case, do not award any marks in this part. Allow ***FT*** from their incorrect p in part (c).

[4 marks]

(c) $0.953958...$

$r = 0.954$ ***A1***

[1 mark]

Total [7 marks]

5. (a) (i) 27.5 ***A1***

(ii) 6.26374...

6.26 ***A1***

[2 marks]

(b) 75th percentile = 31.7248... ***(A1)***

25th percentile = 23.2751... ***(A1)***

8.44965...

IQR = 8.4 ***A1***

[3 marks]

Total [5 marks]

7. (a) recognition of binomial distribution ***(M1)***

$\binom{10}{2}(p)^2(1-p)^8,\ \mathrm{B}(10, p)$

Note: The ***(M1)*** may be awarded for just stating 'binomial', parameters are not required, accept incorrect parameters. Do not award the ***(M1)*** if candidates only state np. Accept the use of calculator notation, such as binompdf.

attempt to consider both machines using $p = 0.1$ AND $p = 0.05$ ***(M1)***

$0.5\times(0.193710...)+0.5\times(0.0746347...)$ ***(A1)***

$0.134172...$

0.134 ***A1***

[4 marks]

continued…

Question 7 continued.

(b) recognition of conditional probability ***(M1)***

$\mathrm{P}(A \mid 2 \text{ defective})$ OR $\frac{\mathrm{P}(\mathrm{A} \cap 2 \text{ defective})}{\mathrm{P}(2 \text{ defective})}$

Note: recognition must be shown in context, either in words or symbols, not just $\mathrm{P}(A \mid B)$.

$\frac{0.0968...}{0.134...}$ ***(A1)***

$0.72187...$

0.722 ***A1***

Note: Accept 0.723 or 0.724 from using 3 sf values from part (a).

[3 marks]

Total [7 marks]

11. (a) **METHOD 1**

attempt to use integration by parts ***M1***

$u = \arccos x,\ \frac{\mathrm{d}v}{\mathrm{d}x} = 1,\ \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}},\ v = x$

$x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ ***(A1)***

attempt to integrate $\frac{x}{\sqrt{1-x^2}}$ by substitution or inspection ***(M1)***

$-\frac{1}{2}\times 2u^{\frac{1}{2}},\quad -\sqrt{1-x^2}$

$x\arccos x - \sqrt{1-x^2} + C$ ***A1***

Note: Do not award the final ***A1*** if $+C$ is missing.

METHOD 2

attempt to use substitution ***(M1)***

$\cos\theta = x$

$\theta = \arccos x$

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}},\ \sin\theta = \sqrt{1-x^2}$

$-\int \theta\sin\theta\mathrm{d}\theta$ ***(A1)***

attempt to use integration by parts ***M1***

$u = \theta,\ \frac{\mathrm{du}}{\mathrm{d}\theta} = 1,\ \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sin\theta,\ v = -\cos\theta$

$\theta\cos\theta - \sin\theta$

$x\arccos x - \sqrt{1-x^2} + C$ ***A1***

Note: Do not award the final ***A1*** if $+C$ is missing.

[4 marks]

continued…

Question 11 continued.

(b) (i) $\int_0^k 3x \arccos\left(x^2\right) \mathrm{d}x = 1$ (seen anywhere) ***M1***

attempt to integrate ***(M1)***

$\frac{3}{2}\int \arccos u \, \mathrm{d}u$

$\frac{3}{2}\left[u \arccos u - \sqrt{1-u^2}\right]_0^{k^2} (=1)$, $\frac{3}{2}\left[x^2 \arccos x^2 - \sqrt{1-x^4}\right]_0^{k} (=1)$ ***A1***

Note: Condone incorrect or absent limits up to this point.

$\frac{3}{2}\left(k^2 \arccos\left(k^2\right) - \sqrt{1-k^4} - (0-1)\right) = 1$ **OR** $k^2 \arccos\left(k^2\right) - \sqrt{1-k^4} - (0-1) = \frac{2}{3}$ ***A1***

$k^2 \arccos\left(k^2\right) - \sqrt{1-k^4} + \frac{1}{3} = 0$ ***AG***

(ii) $k = 0.713250$ ***A2***

[6 marks]

continued…

Question 11 continued.

(c) (i) attempt to substitute their value of k into the formula for $\mathrm{E}(X)$ ***(M1)***

$$\int_0^k 3x^2 \arccos\left(x^2\right) \mathrm{d}x$$

$0.456309...$

$\mathrm{E}(X) = 0.456$ ***A1***

(ii) **METHOD 1**

$\mathrm{E}\left(X^2\right) = 0.237 \quad (0.237198)$ ***(A1)***

attempt to use formula for variance ***(M1)***

$\mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$

$0.0289805...$

$\mathrm{Var}(X) = 0.0290$ ***A1***

[5 marks]

METHOD 2

$$\mathrm{Var}(X) = \int_0^{0.713250} \left(x - 0.456309...\right)^2 f(x)\,\mathrm{d}x$$ ***(M1)(A1)***

$\mathrm{Var}(X) = 0.0290$ ***(A1)***

[5 marks]

continued…

Question 11 continued.

(d) $\sigma = 0.170236...$ (seen anywhere) ***(A1)***

$\int_{0.28607...}^{0.626549...} f(x)\,\mathrm{d}x$ ***(A1)***

$0.620012\ldots$

0.620 ***A1***

[3 marks]

Total [18 marks]

2.

(a) the median is the middle value and as Aiden's score is below 10.5 and Brett's score is above 10.5, hence the middle value stays in the same place ***R1***

Note: Award the ***R1*** for any valid reasoning referring to the scores being on either side of the median.

[1 mark]

(b) attempt to use that the sum of scores is $n \times \text{mean}$ ***(M1)***

10.5×28 OR 10.6×30

sum of Aiden and Brett's scores = 24 ($24 = B + A$) (seen anywhere) ***(A1)***

difference between Aiden and Brett's scores = 14 ($14 = B - A$) (seen anywhere) ***(A1)***

attempt to solve their two simultaneous equations or by trial and improvement ***(M1)***

(Aiden's score =) 5, (Brett's score =) 19 ***A1***

Note: The first two ***A*** marks are independent of each other.

[5 marks]

Total [6 marks]

4. (a) recognition that $\mathrm{P}(\text{height of large sunflowers} > 180) = 0.25$. ***(M1)***

$\frac{180-\mu}{\sigma} = 0.674489...$ OR $\mu + 0.674\sigma = 180$ ***(A1)***

$A = 0.674489\ldots$

$A = 0.674$ ***A1***

[3 marks]

(b) correct parameters for the height of giant sunflowers ***(A1)***

$\mu + 35$ and 2σ OR $\mu + 35$ and $(2\sigma)^2$

$\frac{180-\mu-35}{2\sigma} = -2.05374...$ ***(A1)***

attempt to solve their two simultaneous equations in two unknowns, μ and σ. ***(M1)***

$\mu = 175.063..., \sigma = 7.31913....$

$\mu = 175,\ \sigma = 7.32$ ***A1***

[4 marks]

Total [7 marks]

11. (a) attempt to use $\int f(x)\,\mathrm{d}x = 1$ ***M1***

Note: The $= 1$ can be seen at any stage before ***AG*** last line. If the = 1 is not seen before ***AG*** last line award a maximum of ***M0M1A1A0*** .

recognition for the need to integrate $a(x-3)^3 + b(x-3)^2$ with limits of 3 and 9 ***M1***

$\int_3^9 a(x-3)^3 + b(x-3)^2\,\mathrm{d}x (= 1)$

$\left[\frac{a}{4}(x-3)^4 + \frac{b}{3}(x-3)^3\right]_3^9 (= 1)$ OR $\left[a\left(\frac{x^4}{4} - 3x^3 + \frac{27x^2}{2} - 27x\right) + b\left(\frac{x^3}{3} - 3x^2 + 9x\right)\right]_3^9 (= 1)$ ***(A1)***

$\frac{a}{4} \times 6^4 + \frac{b}{3} \times 6^3 (= 1)$ OR $\left(\frac{1215a}{4} + 81b\right) - \left(\frac{-81a}{4} + 9b\right)(= 1)$ ***A1***

$324a + 72b = 1$ ***AG***

[4 marks]

(b) (i) attempt to substitute $x = 9$ into $a(x-3)^3 + b(x-3)^2$ ***(M1)***

$a(9-3)^3 + b(9-3)^2 (= 0)$

$a(9-3)^3 + b(9-3)^2 = 0$ *OR* $a \times 6^3 + b \times 6^2 = 0$ OR $216a + 36b = 0$ ***A1***

$6a + b = 0$ ***AG***

(ii) attempt to solve two linear equations in a and b ***(M1)***

$a = -\frac{1}{108}, b = \frac{1}{18}$ OR

$a = -0.00926 (-0.00925925...),\ b = 0.0556 (0.0555555...)$ ***A1***

[4 marks]

continued…

Question 11 continued

(c) attempt to find the maximum point on the graph of f ***(M1)***

$(7, 0.296296...)$

mode $= 7$ ***A1***

EITHER

attempt to find the area below f between 3 and 7 ***(M1)***

area = $0.592592...$ ***A1***

area > 0.5 or area $> 0.407407\ldots$ (area below f between 7 and 9) ***R1***

OR

attempt to find the median by solving $\int_3^m f(x) = 0.5$ OR $\int_m^9 f(x) = 0.5$ ***(M1)***

$-\frac{1}{432}(m-3)^4 + \frac{1}{54}(m-3)^3 = 0.5$ *OR* $-\frac{6^4}{432} + \frac{6^3}{54} - \left(-\frac{1}{432}(m-3)^4 + \frac{1}{54}(m-3)^3\right) = 0.5$ ***(A1)***

median $= 6.68563\ldots$ ***A1***

Note: Award ***(M1)(A1)A0*** for 6.69 seen but not recognized as the median.

THEN

median < mode ***AG***

[5 marks]

(d) recognising conditional probability in context ***(M1)***

$\mathrm{P}(X > 8.52 \mid X > 8)$

recognition that $\mathrm{P}(X > 8.52 \cap X > 8) = \mathrm{P}(X > 8.52)$ ***(M1)***

$$\frac{\int_{8.52}^{9} f(x)dx}{\int_{8}^{9} f(x)dx} \left(= \frac{0.03442688...}{0.13194444...}\right)$$ ***(A1)***

$0.260919\ldots.$

0.261 (0.263 from using 3 sf values for a and b) ***A1***

[4 marks]

Total [17 marks]

APOLLO

Calculus

135 questions

4. (a) $f'(x) = -2(x-h)$ ***A1***

[1 mark]

(b) $g'(x) = e^{x-2}$ OR $g'(3) = e^{3-2}$ (may be seen anywhere) ***A1***

Note: The derivative of g must be explicitly seen, either in terms of x or 3.

recognizing $f'(3) = g'(3)$ ***(M1)***

$-2(3-h) = e^{3-2}$ $(= e)$

$-6+2h = e$ OR $3-h = -\frac{e}{2}$ ***A1***

Note: The final ***A1*** is dependent on one of the previous marks being awarded.

$h = \frac{e+6}{2}$ ***AG***

[3 marks]

(c) $f(3) = g(3)$ ***(M1)***

$-(3-h)^2 + 2k = e^{3-2} + k$

correct equation in k

EITHER

$-\left(3-\frac{e+6}{2}\right)^2 + 2k = e^{3-2} + k$ ***A1***

$k = e + \left(\frac{6-e-6}{2}\right)^2 \left(= e + \left(\frac{-e}{2}\right)^2\right)$ ***A1***

OR

$k = e + \left(3-\frac{e+6}{2}\right)^2$ ***A1***

$k = e + 9 - 3e - 18 + \frac{e^2+12e+36}{4}$ ***A1***

THEN

$k = e + \frac{e^2}{4}$ ***AG***

[3 marks]
Total [7 marks]

8. attempt to differentiate numerator and denominator ***M1***

$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$$ ***A1A1***

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x=0$ ***(M1)***

$=\frac{2}{3}$ ***A1***

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

12. (a) attempt to use the chain rule ***M1***

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ ***A1***

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ ***A1***

$= -\frac{1}{4\sqrt{(1+x)^3}}$ ***AG***

Note: Award ***M1A0A0*** for $f'(x) = \frac{1}{\sqrt{1+x}}$ or equivalent seen

[3 marks]

(b) let $n = 2$

$f''(x) = \left(-\frac{1}{4\sqrt{(1+x)^3}} = \right)\left(-\frac{1}{4}\right)^1 \frac{1!}{0!}(1+x)^{\frac{1}{2}-2}$ ***R1***

Note: Award ***R0*** for not starting at $n = 2$. Award subsequent marks as appropriate.

assume true for $n = k$, (so $f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$) ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks can still be awarded.

consider $n = k+1$

$\text{LHS} = f^{(k+1)}(x) = \frac{d\left(f^{(k)}(x)\right)}{dx}$ ***M1***

$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

EITHER

$\text{RHS} = f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ (or equivalent) ***A1***

$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

continued…

Question 12 continued

Note: Award ***A1*** for $\frac{(2k-1)!}{(k-1)!} = \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}$ $\left(= \frac{2(2k-1)(2k-3)!}{(k-2)!}\right)$

$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-2)(2k-3)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

$\left(= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}\right)$

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$= \left(\frac{1}{2}-k\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

OR

Note: The following ***A*** marks can be awarded in any order.

$= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-k-1}$

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for isolating $(2k-1)$ correctly.

$= \left(-\frac{1}{2}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(2k-2)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for multiplying top and bottom by $(k-1)$ or $2(k-1)$.

$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-1)!}{(k-1)(k-2)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

Note: Award ***A1*** for leading coefficient of $-\frac{1}{4}$.

$= \left(-\frac{1}{4}\right)^{k} \frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-k-1}$ ***A1***

continued…

Question 12 continued

$$=\left(-\frac{1}{4}\right)^{(k+1)-1}\frac{\left(2\left(k+1\right)-3\right)!}{\left(\left(k+1\right)-2\right)!}(1+x)^{\frac{1}{2}-(k+1)}=\text{RHS}$$

THEN

since true for $n=2$, and true for $n=k+1$ if true for $n=k$, the statement is true for all $n\in\mathbb{Z}, n\geq 2$ by mathematical induction ***R1***

Note: To obtain the final ***R1,*** at least four of the previous marks must have been awarded.

[9 marks]

(c) **METHOD 1**

$h(x)=\sqrt{1+x}\,\mathrm{e}^{mx}$

using product rule to find $h'(x)$ ***(M1)***

$h'(x)=\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}$ ***A1***

$h''(x)=m\left(\sqrt{1+x}\,m\mathrm{e}^{mx}+\frac{1}{2\sqrt{1+x}}\mathrm{e}^{mx}\right)+\frac{1}{2\sqrt{1+x}}m\mathrm{e}^{mx}-\frac{1}{4\sqrt{(1+x)^3}}\mathrm{e}^{mx}$ ***A1***

substituting $x=0$ into $h''(x)$ ***M1***

$h''(0)=m^2+\frac{1}{2}m+\frac{1}{2}m-\frac{1}{4}\left(=m^2+m-\frac{1}{4}\right)$ ***A1***

$h(x)=h(0)+xh'(0)+\frac{x^2}{2!}h''(0)+\ldots$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!}=\frac{7}{4}\ \left(\Rightarrow h''(0)=\frac{7}{2}\right)$

$4m^2+4m-15=0$ ***A1***

$(2m+5)(2m-3)=0$

$m=-\frac{5}{2}$ or $m=\frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 2

EITHER

attempt to find $f(0)$, $f'(0)$, $f''(0)$ ***(M1)***

$f(x) = (1+x)^{\frac{1}{2}}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$ $\quad f''(0) = -\frac{1}{4}$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

OR

attempt to apply binomial theorem for rational exponents ***(M1)***

$f(x) = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \ldots$

$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$ ***A1***

THEN

$g(x) = 1 + mx + \frac{m^2}{2}x^2 + \ldots$ ***(A1)***

$h(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + mx + \frac{m^2}{2}x^2 + \ldots\right)$ ***(M1)***

coefficient of x^2 is $\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8}$ ***A1***

attempt to set equal to $\frac{7}{4}$ and solve ***M1***

$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8} = \frac{7}{4}$

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m+5)(2m-3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

continued…

Question 12 continued

METHOD 3

$g'(x) = me^{mx}$ and $g''(x) = m^2e^{mx}$ ***(A1)***

$h(x) = h(0) + xh'(0) + \frac{x^2}{2!}h''(0) + ...$

equating x^2 coefficient to $\frac{7}{4}$ ***M1***

$\frac{h''(0)}{2!} = \frac{7}{4}$ $\left(\Rightarrow h''(0) = \frac{7}{2}\right)$

using product rule to find $h'(x)$ and $h''(x)$ ***(M1)***

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h''(x) = f(x)g''(x) + 2f'(x)g'(x) + f''(x)g(x)$ ***A1***

substituting $x = 0$ into $h''(x)$ ***M1***

$h''(0) = f(0)g''(0) + 2g'(0)f'(0) + g(0)f''(0)$

$= 1 \times m^2 + 2m \times \frac{1}{2} + 1 \times \left(-\frac{1}{4}\right)$ $\left(= m^2 + m - \frac{1}{4}\right)$ ***A1***

$4m^2 + 4m - 15 = 0$ ***A1***

$(2m + 5)(2m - 3) = 0$

$m = -\frac{5}{2}$ or $m = \frac{3}{2}$ ***A1***

[8 marks]

Total [20 marks]

4. (a) $\ln\left(x^2-16\right)=0$ ***(M1)***

$e^0=x^2-16\left(=1\right)$

$x^2=17$ OR $x=\pm\sqrt{17}$ ***(A1)***

$a=\sqrt{17}$ ***A1***

[3 marks]

(b) attempt to differentiate (must include $2x$ and/or $\frac{1}{x^2-16}$) ***(M1)***

$f'(x)=\frac{2x}{x^2-16}$ ***A1***

setting their derivative $=\frac{1}{3}$ ***M1***

$\frac{2x}{x^2-16}=\frac{1}{3}$

$x^2-16=6x$ ***OR*** $x^2-6x-16=0$ (or equivalent) ***A1***

valid attempt to solve their quadratic ***(M1)***

$x=8$ ***A1***

Note: Award ***A0*** if the candidate's final answer includes additional solutions (such as $x=-2, 8$).

[6 marks]
Total [9 marks]

9. $u = \sin x \Rightarrow \mathrm{d}u = \cos x \, \mathrm{d}x$ (or equivalent) ***A1***

$= \int \frac{u}{u^2 - u - 2} \mathrm{d}u$ ***A1***

attempt to use partial fractions ***M1***

$\left(\frac{u}{(u+1)(u-2)} \equiv \frac{A}{u+1} + \frac{B}{u-2} \Rightarrow u \equiv A(u-2) + B(u+1) \right)$

Valid attempt to solve for A and B ***(M1)***

$A = \frac{1}{3}$ and $B = \frac{2}{3}$ ***A1***

$\frac{u}{(u+1)(u-2)} \equiv \frac{1}{3(u+1)} + \frac{2}{3(u-2)}$

$\int \left(\frac{1}{3(u+1)} + \frac{2}{3(u-2)} \right) \mathrm{d}u = \frac{1}{3} \ln|u+1| + \frac{2}{3} \ln|u-2| (+C)$ (or equivalent) ***A1***

Note: Condone the absence of $+C$ or lack of moduli here but not in the final answer.

$= \frac{1}{3} \ln|\sin x + 1| + \frac{2}{3} \ln|\sin x - 2| + C$ ***A1***

Note: Condone further simplification of the correct answer.

[7 marks]

Section B

11. (a) $\frac{\mathrm{d}v}{\mathrm{d}t} = -(1+v)$ ***(A1)***

$\int 1\,\mathrm{d}t = \int -\frac{1}{1+v}\,\mathrm{d}v$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln(1+v)(+C)$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$C = \ln(1+v_0)$

$t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow \mathrm{e}^t = \frac{1+v_0}{1+v}$ ***A1***

$\mathrm{e}^t(1+v) = 1+v_0$

$1+v = (1+v_0)\mathrm{e}^{-t}$ ***A1***

$v(t) = (1+v_0)\mathrm{e}^{-t} - 1$ ***AG***

continued…

OR

$t - C = -\ln(1+v) \Rightarrow e^{t-C} = \frac{1}{(1+v)}$

Attempt to find C with initial conditions $t = 0, v = v_0$ ***M1***

$e^{-C} = \frac{1}{(1+v_0)} \Rightarrow C = \ln(1+v_0)$

$t - \ln(1+v_0) = -\ln(1+v) \Rightarrow t = \ln(1+v_0) - \ln(1+v)$

$t = \ln\left(\frac{1+v_0}{1+v}\right) \Rightarrow e^t = \frac{1+v_0}{1+v}$ ***A1***

$e^t(1+v) = 1+v_0$

$1+v = (1+v_0)e^{-t}$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

OR

$t - C = -\ln(1+v) \Rightarrow e^{-t+C} = 1+v$ ***A1***

$ke^{-t} - 1 = v$

Attempt to find k with initial conditions $t = 0, v = v_0$ ***M1***

$k = 1+v_0$

$e^{-t}(1+v_0) = 1+v$ ***A1***

$v(t) = (1+v_0)e^{-t} - 1$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

continued…

Question 11 continued

(b) (i) recognition that when $t = T, v = 0$ ***M1***

$\left(1+v_0\right)\mathrm{e}^{-T} - 1 = 0 \Rightarrow \mathrm{e}^{-T} = \dfrac{1}{1+v_0}$ ***A1***

$\mathrm{e}^{T} = 1+v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = \mathrm{e}^{T} - 1$ into v and showing that $v = 0$.

(ii) $s(t) = \int v(t)\ \mathrm{d}t\ \left(= \int \left(\left(1+v_0\right)\mathrm{e}^{-t} - 1\right)\mathrm{d}t\right)$ ***(M1)***

$= -\left(1+v_0\right)\mathrm{e}^{-t} - t\left(+D\right)$ ***A1***

$(t = 0, s = 0$ so$)\ D = 1+v_0$ ***A1***

$s(t) = -\left(1+v_0\right)\mathrm{e}^{-t} - t + 1 + v_0$

at $s_{\max}$, $\mathrm{e}^{T} = 1+v_0 \Rightarrow T = \ln\left(1+v_0\right)$

Substituting into $s(t)\left(= -\left(1+v_0\right)\mathrm{e}^{-t} - t + 1 + v_0\right)$ ***M1***

$s_{\max} = -\left(1+v_0\right)\left(\dfrac{1}{1+v_0}\right) - \ln\left(1+v_0\right) + v_0 + 1$ ***A1***

$\left(s_{\max} = v_0 - \ln\left(1+v_0\right)\right)$

[7 marks]

(c) **METHOD 1**

$v\left(T-k\right) = \left(1+v_0\right)\mathrm{e}^{-T}\mathrm{e}^{k} - 1$ ***(M1)***

$= \left(1+v_0\right)\left(\dfrac{1}{1+v_0}\right)\mathrm{e}^{k} - 1$ ***A1***

$= \mathrm{e}^{k} - 1$ ***AG***

[2 marks]

METHOD 2

$v\left(T-k\right) = \left(1+v_0\right)\mathrm{e}^{-(T-k)} - 1$

$= e^{T}\mathrm{e}^{-(T-k)} - 1$ ***M1***

$= e^{T-T+k} - 1$ ***A1***

$= \mathrm{e}^{k} - 1$ ***AG***

[2 marks]

continued…

Question 11 continued

(d) **METHOD 1**

$v(T+k) = (1+v_0)e^{-T}e^{-k} - 1$ ***(A1)***

$= e^{-k} - 1$ ***A1***

[2 marks]

METHOD 2

$v(T+k) = (1+v_0)e^{-(T+k)} - 1$ ***(A1)***

$= e^{T}e^{-(T+k)} - 1$

$= e^{T-T-k} - 1$

$= e^{-k} - 1$ ***A1***

[2 marks]

(e) **METHOD 1**

$v(T-k) + v(T+k) = e^{k} + e^{-k} - 2$ ***A1***

attempt to express as a square ***M1***

$= \left(e^{\frac{k}{2}} - e^{-\frac{k}{2}}\right)^2 (\geq 0)$ ***A1***

so $v(T-k) + v(T+k) \geq 0$ ***AG***

[3 marks]

METHOD 2

$v(T-k) + v(T+k) = e^{k} + e^{-k} - 2$ ***A1***

Attempt to solve $\frac{d}{dk}(e^{k} + e^{-k}) = 0 \ (\Rightarrow k = 0)$ ***M1***

minimum value of 2, (when $k = 0$), hence $e^{k} + e^{-k} \geq 2$ ***R1***

so $v(T-k) + v(T+k) \geq 0$ ***AG***

[3 marks]
Total [20 marks]

4. (a) recognising $v = 0$ ***(M1)***

$t = 6.74416...$

$= 6.74$ (sec) ***A1***

Note: Do not award ***A1*** if additional values are given.

[2 marks]

(b) $\int_0^{10} |v(t)| \mathrm{d}t$ **OR** $-\int_0^{6.74416...} v(t)\,\mathrm{d}t + \int_{6.74416...}^{9.08837...} v(t)\,\mathrm{d}t - \int_{9.08837...}^{10} v(t)\,\mathrm{d}t$ ***(A1)***

$= 37.0968...$

$= 37.1$ (m) ***A1***

[2 marks]

(c) recognising acceleration at $t = 7$ is given by $v'(7)$ ***(M1)***

acceleration $= 5.93430...$

$= 5.93$ (ms^{-2}) ***A1***

[2 marks]

Total [6 marks]

9. (a) $\tan\theta = \frac{50}{y-x}$ OR $\cot\theta = \frac{y-x}{50}$ ***A1***

$y = x + 50\cot\theta$ ***AG***

Note: $y - x$ may be identified as a length on a diagram, and not written explicitly.

[1 mark]

(b) attempt to differentiate with respect to t ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50\left(\operatorname{cosec}\theta\right)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

attempt to set speed of B equal to double the speed of A (***M1)***

$2\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - 50\left(\operatorname{cosec}\theta\right)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\operatorname{cosec}\theta\right)^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***A1***

$\theta = \arctan 5 \left(= 1.373\ldots = 78.69\ldots°\right)$ OR $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \left(\frac{1}{5}\right)^2 = \frac{26}{25}$ ***(A1)***

Note: This ***A1*** can be awarded independently of previous marks.

$\frac{\mathrm{d}x}{\mathrm{d}t} = -50\left(\frac{26}{25}\right) \times -0.1$

So the speed of boat A is 5.2 (ms^{-1}) ***A1***

Note: Accept 5.20 from the use of inexact values.

[6 marks]
Total [7 marks]

Section B.

10. (a) attempt to find the point of intersection of the graph of f and the line $y = x$ ***(M1)***

$x = 5.56619...$

$= 5.57$ ***A1***

[2 marks]

(b) $f'(x) = -45e^{-0.5x}$ ***A1***

attempt to set the gradient of f equal to -1 ***(M1)***

$-45e^{-0.5x} = -1$

Q has coordinates $\left(2\ln 45, 2\right)$ (accept $(-2\ln\frac{1}{45}, 2)$ ***A1A1***

Note: Award ***A1*** for each value, even if the answer is not given as a coordinate pair.

Do not accept $\frac{\ln\frac{1}{45}}{-0.5}$ or $\frac{\ln 45}{0.5}$ as a final value for x. Do not accept 2.0 or 2.00 as a final value for y.

[4 marks]

(c) attempt to substitute coordinates of Q (in any order) into an appropriate equation ***(M1)***

$y - 2 = -\left(x - 2\ln 45\right)$ OR $2 = -2\ln 45 + c$ ***A1***

equation of L is $y = -x + 2\ln 45 + 2$ ***AG***

[2 marks]

continued…

Question 10 continued

(d) (i) $x = \ln 45 + 1 \left(= 4.81\right)$ ***A1***

(ii) appropriate method to find the sum of two areas using integrals of the difference of two functions ***(M1)***

Note: Allow absent or incorrect limits.

$$\int_{4.806...}^{5.566...} \left(x - \left(-x + 2\ln 45 + 2\right)\right) \mathrm{d}x + \int_{5.566...}^{7.613...} \left(90\mathrm{e}^{-0.5x} - \left(-x + 2\ln 45 + 2\right)\right) \mathrm{d}x \quad \textbf{(A1)(A1)}$$

Note: Award ***A1*** for one correct integral expression including correct limits and integrand.
Award ***A1*** for a second correct integral expression including correct limits and integrand.

$= 1.51965...$
$= 1.52$ ***A1***

[5 marks]

(e) by symmetry 2×1.52 ***(M1)***
$= 3.03930...$
$= 3.04$ ***A1***

Note: Accept any answer that rounds to 3.0 (but do not accept 3).

[2 marks]
Total [15 marks]

12. (a) $\frac{1}{x(k-x)} \equiv \frac{a}{x} + \frac{b}{k-x}$

$a(k-x)+bx=1$ ***(A1)***

attempt to compare coefficients OR substitute $x=k$ and $x=0$ and solve ***(M1)***

$a=\frac{1}{k}$ and $b=\frac{1}{k}$ ***A1***

$f'(x)=\frac{1}{kx}+\frac{1}{k(k-x)}$

[3 marks]

(b) attempt to integrate their $\frac{a}{x}+\frac{b}{k-x}$ ***(M1)***

$f(x)=\frac{1}{k}\int\left(\frac{1}{x}+\frac{1}{k-x}\right)\mathrm{d}x$

$=\frac{1}{k}\left(\ln|x|-\ln|k-x|\right)(+c)\left(=\frac{1}{k}\ln\left|\frac{x}{k-x}\right|(+c)\right)$ ***A1A1***

[3 marks]

Note: Award ***A1*** for each correct term. Award ***A1A0*** for a correct answer without modulus signs. Condone the absence of $+c$.

continued…

Question 12 continued

(c) attempt to separate variables and integrate both sides ***M1***

$$5k\int \frac{1}{P(k-P)}\,\mathrm{d}P=\int 1\,\mathrm{d}t$$

$$5\left(\ln P-\ln\left(k-P\right)\right)=t+c$$ ***A1***

Note: There are variations on this which should be accepted, such as $\frac{1}{k}\left(\ln P-\ln\left(k-P\right)\right)=\frac{1}{5k}t+c$. Subsequent marks for these variations should be awarded as appropriate.

EITHER

attempt to substitute $t=0$, $P=1200$ into an equation involving c ***M1***

$$c=5\left(\ln 1200-\ln\left(k-1200\right)\right)\left(=5\ln\left(\frac{1200}{k-1200}\right)\right)$$ ***A1***

$$5\left(\ln P-\ln\left(k-P\right)\right)=t+5\left(\ln 1200-\ln\left(k-1200\right)\right)$$ ***A1***

$$\ln\left(\frac{P\left(k-1200\right)}{1200\left(k-P\right)}\right)=\frac{t}{5}$$

$$\frac{P\left(k-1200\right)}{1200\left(k-P\right)}=\mathrm{e}^{\frac{t}{5}}$$ ***A1***

OR

$$\ln\left(\frac{P}{k-P}\right)=\frac{t+c}{5}$$

$$\frac{P}{k-P}=A\mathrm{e}^{\frac{t}{5}}$$ ***A1***

attempt to substitute $t=0$, $P=1200$ ***M1***

$$\frac{1200}{k-1200}=A$$ ***A1***

$$\frac{P}{k-P}=\frac{1200\mathrm{e}^{\frac{t}{5}}}{k-1200}$$ ***A1***

continued…

Question 12 continued

THEN

attempt to rearrange and isolate P ***M1***

$Pk - 1200P = 1200k\mathrm{e}^{\frac{t}{5}} - 1200P\mathrm{e}^{\frac{t}{5}}$ OR $Pk\mathrm{e}^{-\frac{t}{5}} - 1200P\mathrm{e}^{-\frac{t}{5}} = 1200k - 1200P$

OR $\frac{k}{P} - 1 = \frac{k - 1200}{1200\mathrm{e}^{\frac{t}{5}}}$

$P\left(k - 1200 + 1200\mathrm{e}^{\frac{t}{5}}\right) = 1200k\mathrm{e}^{\frac{t}{5}}$ OR $P\left(k\mathrm{e}^{-\frac{t}{5}} - 1200\mathrm{e}^{-\frac{t}{5}} + 1200\right) = 1200k$ ***A1***

$P = \frac{1200k}{(k - 1200)\mathrm{e}^{-\frac{t}{5}} + 1200}$ ***AG***

[8 marks]

(d) attempt to substitute $t = 10$, $P = 2400$ ***(M1)***

$2400 = \frac{1200k}{(k - 1200)\mathrm{e}^{-2} + 1200}$ ***(A1)***

$k = 2845.34...$

$k = 2845$ ***A1***

Note: Award ***(M1)(A1)A0*** for any other value of k which rounds to 2850

[3 marks]

(e) attempt to find the maximum of the first derivative graph OR zero of the second derivative graph OR that $P = \frac{k}{2}(= 1422.67...)$ ***(M1)***

$t = 1.57814...$

$= 1.58$ (days) ***A2***

Note: Accept any value which rounds to 1.6.

[3 marks]

Total [20 marks]

9. (a) $1-t+t^2$ ***A1***

Note: Accept $1, -t$ and t^2.

[1 mark]

(b) $\sec x = \dfrac{1}{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)} \left(= \left(1-\dfrac{x^2}{2!}+\left(\dfrac{x^4}{4!}(-...)\right)\right)^{-1} \right)$ ***(M1)***

$t = \cos x - 1$ or $\sec x = 1-(\cos x-1)+(\cos x-1)^2$ ***(M1)***

$= 1-\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)\right)+\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}(-...)\right)^2$ ***A1***

$= 1+\dfrac{x^2}{2}-\dfrac{x^4}{24}+\dfrac{x^4}{4}$ ***A1***

so the Maclaurin series for $\sec x$ up to and including the term in x^4 is $1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}$ ***AG***

[4 marks]

Note: Condone the absence of '…'

(c) $\arctan 2x = 2x-\dfrac{(2x)^3}{3}+...$

$\lim\limits_{x\to 0}\left(\dfrac{x\arctan 2x}{\sec x-1}\right) = \lim\limits_{x\to 0}\left(\dfrac{x\left(2x-\dfrac{(2x)^3}{3}+...\right)}{\left(1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}\right)-1}\right)$ ***M1***

$= \lim\limits_{x\to 0}\left(\dfrac{2x^2-\dfrac{8x^4}{3}+...}{\dfrac{x^2}{2}+\dfrac{5x^4}{24}}\right)$ ***A1***

$= \lim\limits_{x\to 0}\left(\dfrac{2x^2\left(1-\dfrac{4x^2}{3}\right)}{\dfrac{x^2}{2}\left(1+\dfrac{5x^2}{12}\right)}\right)$

$= 4$ ***A1***

Note: Condone missing 'lim' and errors in higher derivatives.
Do not award ***M1*** unless x is replaced by $2x$ in arctan.

[3 marks]
Total [8 marks]

Section B

11. (a) attempt to use $V = \pi\int_a^b \left(f(x)\right)^2 \mathrm{d}x$ ***(M1)***

$V = \pi\int_0^{\ln 16}\left(\frac{k\mathrm{e}^{\frac{x}{2}}}{1+\mathrm{e}^x}\right)^2 \mathrm{d}x \quad \left(V = k^2\pi\int_0^{\ln 16}\frac{\mathrm{e}^x}{\left(1+\mathrm{e}^x\right)^2}\mathrm{d}x\right)$

EITHER

applying integration by recognition ***(M1)***

$= k^2\pi\left[-\frac{1}{1+\mathrm{e}^x}\right]_0^{\ln 16}$ ***A3***

OR

$u = 1+\mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 2$ and when $x = \ln 16$, $u = 17$

$V = k^2\pi\int_2^{17}\frac{1}{u^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{u}\right]_2^{17}$ ***A1***

OR

$u = \mathrm{e}^x \Rightarrow \mathrm{d}u = \mathrm{e}^x\mathrm{d}x$ ***(A1)***

attempt to express the integral in terms of u ***(M1)***

when $x = 0$, $u = 1$ and when $x = \ln 16$, $u = 16$

$V = k^2\pi\int_1^{16}\frac{1}{\left(1+u\right)^2}\mathrm{d}u$ ***(A1)***

$= k^2\pi\left[-\frac{1}{1+u}\right]_1^{16}$ ***A1***

Note: Accept equivalent working with indefinite integrals and original limits for x.

THEN

$= k^2\pi\left(\frac{1}{2}-\frac{1}{17}\right)$ ***A1***

so the volume of the solid formed is $\frac{15k^2\pi}{34}$ cubic units ***AG***

Note: Award ***(M1)(A0)(M0)(A0)(A0)(A1)*** when $\frac{15}{34}$ is obtained from GDC

[6 marks]

continued…

Question 11 continued

(b) a valid algebraic or graphical attempt to find k ***(M1)***

$k^2 = \frac{300 \times 34}{15\pi}$

$k = 14.7 \left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ (as $k \in \mathbb{R}^+$) ***A1***

Note: Candidates may use their GDC numerical solve feature.

[2 marks]

(c) (i) attempting to find $\text{OA} = f(0) = \frac{k}{2}$

with $k = 14.712...$ $\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{OA} = 7.36 \left(= \sqrt{\frac{170}{\pi}} \right)$ ***A1***

(ii) attempting to find $\text{BC} = f(\ln 16) = \frac{4k}{17}$

with $k = 14.712...$ $\left(= 2\sqrt{\frac{170}{\pi}} = \sqrt{\frac{680}{\pi}} \right)$ ***(M1)***

$\text{BC} = 3.46 \left(= \frac{8}{17}\sqrt{\frac{170}{\pi}} = \frac{8\sqrt{10}}{\sqrt{17\pi}} \right)$ ***A1***

[4 marks]

continued…

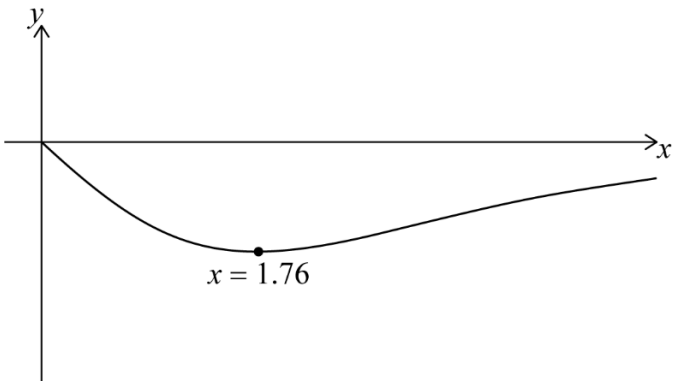
Question 11 continued

(d) (i) **EITHER**

recognising to graph $y = f'(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f'(x) = \frac{k\mathrm{e}^{\frac{x}{2}}\left(1-\mathrm{e}^{x}\right)}{2\left(1+\mathrm{e}^{x}\right)^{2}}$$



for $x > 0$ graph decreasing to the local minimum ***A1***

before increasing towards the x-axis ***A1***

continued…

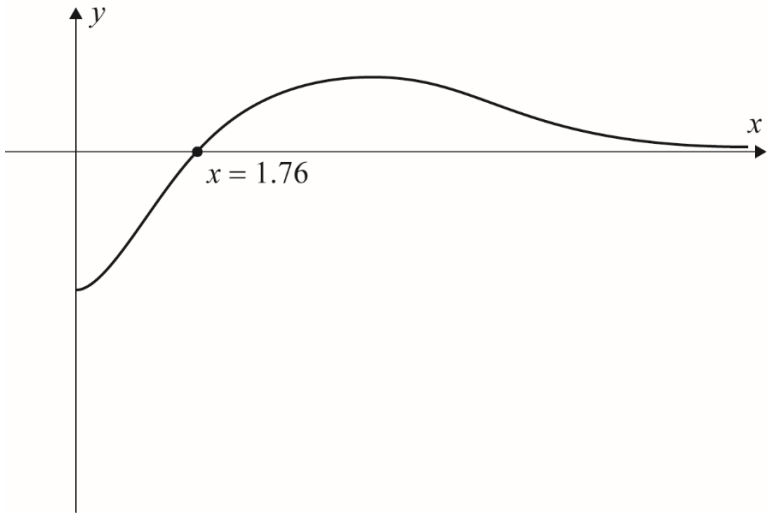
Question 11 continued

OR

recognising to graph $y = f''(x)$ ***(M1)***

Note: Award ***M1*** for attempting to use quotient rule or product rule differentiation.

$$f''(x) = \frac{k\mathrm{e}^{\frac{x}{2}}\left(\mathrm{e}^{2x} - 6\mathrm{e}^{x} + 1\right)}{4\left(1+\mathrm{e}^{x}\right)^{3}}$$



for $x > 0$, graph increasing towards and beyond the x-intercept ***A1***

recognising $f''(x) = 0$ for maximum rate ***(A1)***

THEN

$x = 1.76 \left(= \ln\left(2\sqrt{2}+3\right)\right)$ ***A1***

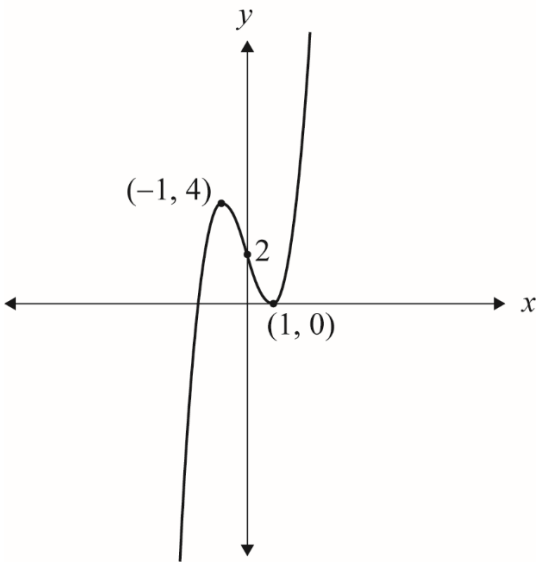
Note: Only award ***A*** marks if either graph is seen.

(ii) attempting to find $f(1.76...)$ ***(M1)***

the cross-sectional radius at this point is $5.20 \left(\sqrt{\frac{85}{\pi}}\right)$ (cm) ***A1***

[6 marks]
Total [18 marks]

1. (a) (i)

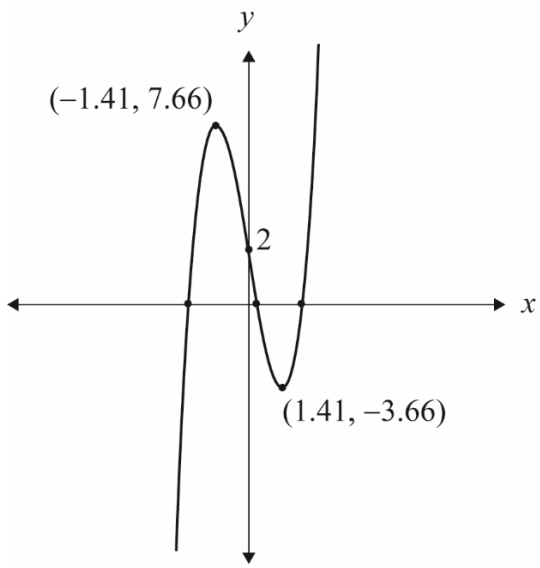


$c = 1$: positive cubic with correct *y*-intercept labelled	***A1***
local maximum point correctly labelled	***A1***
local minimum point correctly labelled	***A1***
	[3 marks]

continued…

Question 1 continued

(ii)



$c = 2$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

local minimum point correctly labelled ***A1***

Note: Accept the following exact answers:

Local maximum point coordinates $\left(-\sqrt{2}, 2+4\sqrt{2}\right)$.

Local minimum point coordinates $\left(\sqrt{2}, 2-4\sqrt{2}\right)$.

[3 marks]

continued…

Question 1 continued

(b) $f'(x) = 3x^2 - 3c$ ***A1***

Note: Accept $3x^2 - 3c$ (an expression).

[1 mark]

(c) (i) $c = 0$ ***A1***

[1 mark]

(ii) considers the number of solutions to their $f'(x) = 0$ ***(M1)***

$3x^2 - 3c = 0$

$c > 0$ ***A1***

[2 marks]

(iii) $c < 0$ ***A1***

Note: The ***(M1)*** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

[1 mark]

(d) attempts to solve their $f'(x)=0$ for x ***(M1)***

$x=\pm\sqrt{c}$ ***(A1)***

Note: Award ***(A1)*** if either $x=-\sqrt{c}$ or $x=\sqrt{c}$ is subsequently considered.
Award the above ***(M1)(A1)*** if this work is seen in part (c).

(i) correctly evaluates $f\left(-\sqrt{c}\right)$ ***A1***

$f\left(-\sqrt{c}\right)=-c^{\frac{3}{2}}+3c^{\frac{3}{2}}+2\ \left(=-c\sqrt{c}+3c\sqrt{c}+2\right)$

the y- coordinate of the local maximum point is $2c^{\frac{3}{2}}+2$ ***AG***

[3 marks]

(ii) correctly evaluates $f\left(\sqrt{c}\right)$ ***A1***

$f\left(\sqrt{c}\right)=c^{\frac{3}{2}}-3c^{\frac{3}{2}}+2\ \left(=c\sqrt{c}-3c\sqrt{c}+2\right)$

the y- coordinate of the local minimum point is $-2c^{\frac{3}{2}}+2$ ***AG***

[1 mark]

continued…

Question 1 continued

(e) (i) the graph of $y = f(x)$ will have one $x-$ axis intercept if

EITHER

$-2c^{\frac{3}{2}} + 2 > 0$ (or equivalent reasoning) ***R1***

OR

the minimum point is above the $x-$ axis ***R1***

Note: Award ***R1*** for a rigorous approach that does not (only) refer to sketched graphs.

THEN

$0 < c < 1$ ***A1***

Note: Condone $c < 1$. The ***A1*** is independent of the ***R1***.

[2 marks]

(ii) the graph of $y = f(x)$ will have two $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 = 0$ (or equivalent reasoning) ***(M1)***

OR

evidence from the graph in part(a)(i) ***(M1)***

THEN

$c = 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(iii) the graph of $y = f(x)$ will have three $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 < 0$ (or equivalent reasoning) ***(M1)***

OR

reasoning from the results in both parts (e)(i) and (e)(ii) ***(M1)***

THEN

$c > 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(f) case 1:

$c \leq 0$ (independent of the value of d) ***A1***

EITHER

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution) ***R1***

OR

$\Rightarrow g'(x) \geq 0$ for $x \in \tilde{}$ ***R1***

OR

the graph of $y = f(x)$ has no local maximum or local minimum points,

hence any vertical translation of this graph ($y = g(x)$) will also have no local maximum or local minimum points ***R1***

THEN

therefore there is only one x- axis intercept ***AG***

Note: Award at most ***A0R1*** if only $c < 0$ is considered.

continued…

Question 1 continued

case 2

$c > 0$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a local minimum point ***(A1)***

Note: Award ***(A1)*** for a correct y- coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point ***(M1)***

EITHER

considers both points above the x- axis or both points below the x- axis

OR

considers either the local minimum point only above the x- axis OR the local maximum point only below the x- axis

THEN

$d > 2c^{\frac{3}{2}}$ (both points above the x- axis) ***A1***

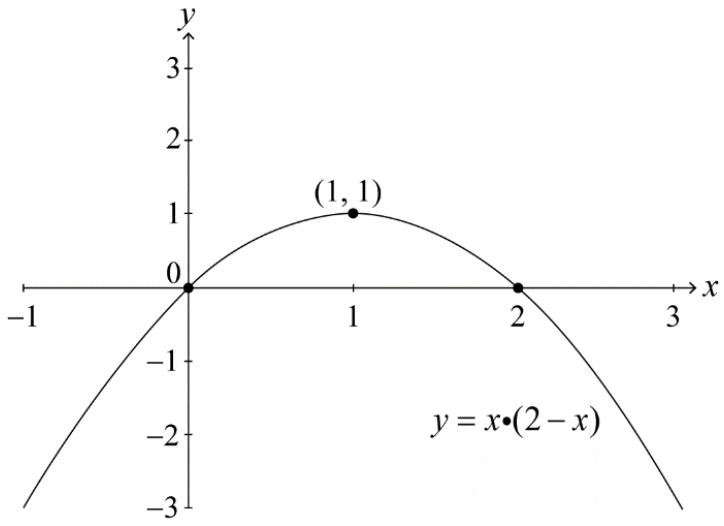
$d < -2c^{\frac{3}{2}}$ (both points below the x- axis) ***A1***

Note: Award at most ***(A1)(M1)A0A0*** for case 2 if $c > 0$ is not clearly stated.

[6 marks]

Total [27 marks]

1. (a)



inverted parabola extended below the x- axis ***A1***

x- axis intercept values $x = 0, 2$ ***A1***

Note: Accept a graph passing through the origin as an indication of $x = 0$.

local maximum at $(1,1)$ ***A1***

Note: Coordinates must be stated to gain the final ***A1***.
Do not accept decimal approximations.

[3 marks]

(b)

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$	1	0	2
$n = 2$ and $n = 4$	1	2	0

A1A1A1A1A1A1

Note: Award ***A1*** for each correct value.

For a table not sufficiently or clearly labelled, assume that their values are in the same order as the table in the question paper and award marks accordingly.

[6 marks]

(c) **METHOD 1**

attempts to use the product rule ***(M1)***

$f_n'(x) = -nx^n(a-x)^{n-1} + nx^{n-1}(a-x)^n$ ***A1A1***

Note: Award ***A1*** for a correct $u\dfrac{\mathrm{d}v}{\mathrm{d}x}$ and ***A1*** for a correct $v\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

EITHER

attempts to factorise $f_n'(x)$ (involving at least one of nx^{n-1} or $(a-x)^{n-1}$) ***(M1)***

$= nx^{n-1}(a-x)^{n-1}\big((a-x)-x\big)$ ***A1***

OR

attempts to express $f_n'(x)$ as the difference of two products with each product containing at least one of nx^{n-1} or $(a-x)^{n-1}$ ***(M1)***

$= (-x)\big(nx^{n-1}\big)(a-x)^{n-1} + (a-x)\big(nx^{n-1}\big)(a-x)^{n-1}$ ***A1***

THEN

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

Note: Award the final ***(M1)A1*** for obtaining any of the following forms:

$f_n'(x) = nx^n(a-x)^n\left(\dfrac{a-x-x}{x(a-x)}\right)$; $f_n'(x) = \dfrac{nx^n(a-x)^n}{x(a-x)}(a-x-x)$;

$f_n'(x) = nx^{n-1}\big((a-x)^n - x(a-x)^{n-1}\big)$;

$f_n'(x) = (a-x)^{n-1}\big(nx^{n-1}(a-x)^n - nx^n\big)$

METHOD 2

$f_n(x) = \big(x(a-x)\big)^n$ ***(M1)***

$= \big(ax-x^2\big)^n$ ***A1***

attempts to use the chain rule ***(M1)***

$f_n'(x) = n(a-2x)\big(ax-x^2\big)^{n-1}$ ***A1A1***

Note: Award ***A1*** for $n(a-2x)$ and ***A1*** for $\left(ax-x^2\right)^{n-1}$.

$f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$ ***AG***

[5 marks]

(d) $x=0,\ x=\frac{a}{2},\ x=a$ ***A2***

Note: Award ***A1*** for either two correct solutions or for obtaining
$x=0,\ x=-a,\ x=-\frac{a}{2}$.
Award ***A0*** otherwise.

[2 marks]

(e) attempts to find an expression for $f_n\left(\frac{a}{2}\right)$ ***(M1)***

$f_n\left(\frac{a}{2}\right)=\left(\frac{a}{2}\right)^n\left(a-\frac{a}{2}\right)^n$

$=\left(\frac{a}{2}\right)^n\left(\frac{a}{2}\right)^n\ \left(=\left(\frac{a}{2}\right)^{2n}\right),\left(=\left(\left(\frac{a}{2}\right)^n\right)^2\right)$ ***A1***

EITHER

since $a\in\mathbb{R}^+$, $\left(\frac{a}{2}\right)^{2n}>0$ (for $n\in\mathbb{Z}^+, n>1$ and so $f_n\left(\frac{a}{2}\right)>0$) ***R1***

Note: Accept any logically equivalent conditions/statements on a and n.
Award ***R0*** if any conditions/statements specified involving a, n or both are incorrect.

OR

(since $a\in\mathbb{R}^+$), $\frac{a}{2}$ raised to an even power ($2n$) (or equivalent reasoning) is always positive (and so $f_n\left(\frac{a}{2}\right)>0$) ***R1***

Note: The condition $a\in\mathbb{R}^+$ is given in the question. Hence some candidates will assume $a\in\mathbb{R}^+$ and not state it. In these instances, award ***R1*** for a convincing argument.
Accept any logically equivalent conditions/statements on a and n.
Award ***R0*** if any conditions/statements specified involving a, n or both are incorrect.

THEN

so $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ is always above the horizontal axis ***AG***

Note: Do not award ***(M1)A0R1***.

[3 marks]

(f) **METHOD 1**

$f_n'\left(\frac{a}{4}\right) = n\left(\frac{a}{4}\right)^{n-1}\left(a-\frac{a}{2}\right)\left(a-\frac{a}{4}\right)^{n-1} \left(= n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1}\right)$ ***A1***

EITHER

$n\left(\frac{a}{4}\right)^{n-1}\left(\frac{a}{2}\right)\left(\frac{3a}{4}\right)^{n-1} > 0$ as $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ ***R1***

OR

$n\left(\frac{a}{4}\right)^{n-1}$, $\left(a-\frac{a}{2}\right)$ and $\left(a-\frac{a}{4}\right)^{n-1}$ are all > 0 ***R1***

Note: Do not award ***A0R1***.

Accept equivalent reasoning on correct alternative expressions for $f_n'\left(\frac{a}{4}\right)$ and accept any logically equivalent conditions/statements on a and n.

Exceptions to the above are condone $n > 1$ and condone $n > 0$.

An alternative form for $f_n'\left(\frac{a}{4}\right)$ is $(2n)(3)^{n-1}\left(\frac{a}{4}\right)^{2n-1}$.

THEN

hence $f_n'\left(\frac{a}{4}\right) > 0$ ***AG***

[2 marks]

METHOD 2

$f_n(0) = 0$ and $f_n\left(\frac{a}{2}\right) > 0$ ***A1***

(since f_n is continuous and there are no stationary points between $x=0$ and $x=\frac{a}{2}$)

the gradient (of the curve) must be positive between $x=0$ and $x=\frac{a}{2}$ ***R1***

Note: Do not award ***A0R1***.

hence $f_n'\left(\frac{a}{4}\right)>0$ ***AG***

[2 marks]

(g) (i) $f_n'(-1)=n(-1)^{n-1}(a+2)(a+1)^{n-1}$

for n even:

$n(-1)^{n-1}(=-n)<0$ (and $(a+2),(a+1)^{n-1}$ are both >0) ***R1***

$f_n'(-1)<0$ ***A1***

$f_n'(0)=0$ and $f_n'\left(\frac{a}{4}\right)>0$ (seen anywhere) ***A1***

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If n is even, then $n-1$ is odd and hence $(-1)^{n-1}<0\ (=-1)$.

Do not award ***R0A1***.

The second ***A1*** is independent of the other two marks.

The ***A*** marks can be awarded for correct descriptions expressed in words.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0)=0$. The last ***A*** mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last ***A1*** can be awarded for use of a specific case (*e.g.* $n=2$).

hence $(0,0)$ is a local minimum point ***AG***

[3 marks]

(ii) for n odd:

$n(-1)^{n-1}\,(=n)>0$, (and $(a+2),(a+1)^{n-1}$ are both >0) so $f_n'(-1)>0$ ***R1***

Note: Candidates can give arguments based on the sign of $(-1)^{n-1}$ to obtain the ***R*** mark.

For example, award ***R1*** for the following:

If n is odd, then $n-1$ is even and hence $(-1)^{n-1}>0\ (=1)$.

$f_n'(0)=0$ and $f_n'\left(\frac{a}{4}\right)>0$ (seen anywhere) ***A1***

Note: The ***A1*** is independent of the ***R1***.

Candidates can state $(0,0)$ as a point of zero gradient from part (d) or show, state or explain (words or diagram) that $f_n'(0)=0$. The last *A* mark can be awarded for a clearly labelled diagram showing changes in the sign of the gradient.

The last *A1* can be awarded for use of a specific case (*e.g.* $n=3$).

hence $(0,0)$ is a point of inflexion with zero gradient ***AG***

[2 marks]

(h) considers the parity of n ***(M1)***

Note: Award ***M1*** for stating at least one specific even value of n.

n must be even (for four solutions) ***A1***

Note: The above 2 marks are independent of the 3 marks below.

$0 < k < \left(\frac{a}{2}\right)^{2n}$ ***A1A1A1***

Note: Award ***A1*** for the correct lower endpoint, ***A1*** for the correct upper endpoint and ***A1*** for strict inequality signs.

The third ***A1*** (strict inequality signs) can only be awarded if ***A1A1*** has been awarded.

For example, award ***A1A1A0*** for $0 \le k \le \left(\frac{a}{2}\right)^{2n}$. Award ***A1A0A0*** for $k > 0$.

Award ***A1A0A0*** for $0 < k < f_n\left(\frac{a}{2}\right)$.

[5 marks]

Total [31 marks]

1. METHOD 1

recognition that $y = \int \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)***

$y = \sin\left(x - \frac{\pi}{4}\right)(+c)$ ***(A1)***

substitute both x and y values into their integrated expression including c (***M1)***

$2 = \sin\frac{\pi}{2} + c$

$c = 1$

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

METHOD 2

$\int_{2}^{y} \mathrm{d}y = \int_{\frac{3\pi}{4}}^{x} \cos\left(x - \frac{\pi}{4}\right)\mathrm{d}x$ ***(M1)(A1)***

$y - 2 = \sin\left(x - \frac{\pi}{4}\right) - \sin\frac{\pi}{2}$ ***A1***

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

[4 marks]

5. (a) $f'(4) = 6$ ***A1***

[1 mark]

(b) $f(4) = 6 \times 4 - 1 = 23$ ***A1***

[1 mark]

(c) $h(4) = f\left(g(4)\right)$ ***(M1)***

$h(4) = f(4^2 - 3 \times 4) = f(4)$

$h(4) = 23$ ***A1***

[2 marks]

(d) attempt to use chain rule to find h' ***(M1)***

$f'\left(g\left(x\right)\right) \times g'\left(x\right)$ **OR** $\left(x^2 - 3x\right)' \times f'\left(x^2 - 3x\right)$

$h'(4) = \left(2 \times 4 - 3\right) f'\left(4^2 - 3 \times 4\right)$ ***A1***

$= 30$

$y - 23 = 30\left(x - 4\right)$ OR $y = 30x - 97$ ***A1***

[3 marks]

Total [7 marks]

8. $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x}=\frac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$\left(\mathrm{e}^{\int\frac{2}{x}\mathrm{d}x}=\mathrm{e}^{2\ln x}\right)=x^2$ ***(A1)***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=\ln 2x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right)=\ln 2x$

$x^2y=\int\ln 2x\ \mathrm{d}x$

attempt to use integration by parts ***(M1)***

$x^2y=x\ln 2x-x\left(+c\right)$ ***A1***

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{c}{x^2}$

substituting $x=\frac{1}{2}, y=4$ into an integrated equation involving c ***M1***

$4=0-2+4c$

$\Rightarrow c=\frac{3}{2}$

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{3}{2x^2}$ ***A1***

[7 marks]

10. (a) (i) valid approach to find turning point ($v' = 0$, $-\frac{b}{2a}$, average of roots) ***(M1)***

$4 - 6t = 0$ OR $-\frac{4}{2(-3)}$ OR $\frac{-\frac{2}{3} + 2}{2}$

$t = \frac{2}{3}$ (s) ***A1***

(ii) attempt to integrate v ***(M1)***

$\int v \, \mathrm{d}t = \int \left(4 + 4t - 3t^2\right) \mathrm{d}t = 4t + 2t^2 - t^3 (+c)$ ***A1A1***

Note: Award ***A1*** for $4t + 2t^2$, ***A1*** for $-t^3$.

attempt to substitute their t into their solution for the integral ***(M1)***

distance $= 4\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$

$= \frac{8}{3} + \frac{8}{9} - \frac{8}{27}$ (or equivalent) ***A1***

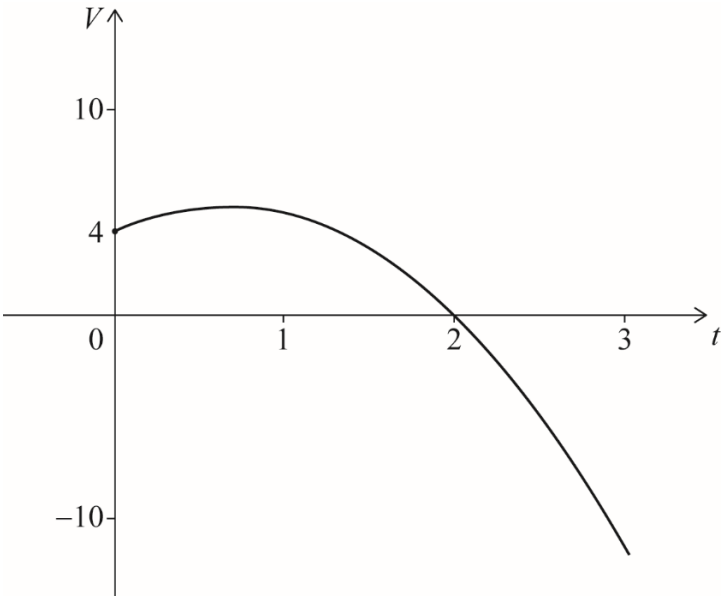
$= \frac{88}{27}$ (m) ***AG***

[7 marks]

continue…

Question 10 continued.

(b)



valid approach to solve $4+4t-3t^2=0$ (may be seen in part (a)) ***(M1)***

$(2-t)(2+3t)$ OR $\frac{-4\pm\sqrt{16+48}}{-6}$

correct x- intercept on the graph at $t=2$ ***A1***

Note: The following two ***A*** marks may only be awarded if the shape is a concave down parabola. These two marks are independent of each other and the ***(M1)***.

correct domain from 0 to 3 starting at $(0,4)$ ***A1***

Note: The 3 must be clearly indicated.

vertex in approximately correct place for $t=\frac{2}{3}$ and $v>4$ ***A1***

[4 marks]

continue…

Question 10 continued.

(c) recognising to integrate between 0 and 2, or 2 and 3 OR $\int_0^3 \left|4+4t-3t^2\right| \, \mathrm{d}t$ ***(M1)***

$\int_0^2 \left(4+4t-3t^2\right) \, \mathrm{d}t$

$=8$ ***A1***

$\int_2^3 \left(4+4t-3t^2\right) \, \mathrm{d}t$

$=-5$ ***A1***

valid approach to sum the two areas (seen anywhere) ***(M1)***

$\int_0^2 v \, \mathrm{d}t - \int_2^3 v \, \mathrm{d}t$ OR $\int_0^2 v \, \mathrm{d}t + \left|\int_2^3 v \, \mathrm{d}t\right|$

total distance travelled $=13$ (m) ***A1***

[5 marks]

Total [16 marks]

11. (a) For $n=1$

LHS: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

RHS: $\left(x^2+2(1)x+1(1-1)\right)\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

so true for $n=1$

now assume true for $n=k$; i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left[x^2+2kx+k\left(k-1\right)\right]\mathrm{e}^x$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$". Subsequent marks can still be awarded.

attempt to differentiate the RHS ***M1***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x^2+2kx+k\left(k-1\right)\right]\mathrm{e}^x\right)$

$=\left(2x+2k\right)\mathrm{e}^x+\left(x^2+2kx+k\left(k-1\right)\right)\mathrm{e}^x$ ***A1***

$=\left[x^2+2\left(k+1\right)x+k\left(k+1\right)\right]\mathrm{e}^x$ ***A1***

so true for $n=k$ implies true for $n=k+1$

therefore $n=1$ true and $n=k$ true $\Rightarrow$ $n=k+1$ true

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Award **R1** only if three of the previous four marks have been awarded

[7 marks]

continue…

Question 11 continued.

(b) **METHOD 1**

attempt to use $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n\left(n-1\right)\right]\mathrm{e}^x$ ***(M1)***

Note: For $x=0$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)_{|x=0}=n\left(n-1\right)$ may be seen.

$f\left(0\right)=0,\ f'\left(0\right)=0,\ f''\left(0\right)=2,\ f'''\left(0\right)=6,\ f^{(4)}\left(0\right)=12$

use of $f\left(x\right)=f\left(0\right)+xf'\left(0\right)+\frac{x^2}{2!}f''\left(0\right)+\frac{x^3}{3!}f'''\left(0\right)+\frac{x^4}{4!}f^{(4)}\left(0\right)+\ldots$ ***(M1)***

$\Rightarrow f\left(x\right)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

METHOD 2

'$x^2\times$ Maclaurin series of e^x' ***(M1)***

$x^2\left(1+x+\frac{x^2}{2!}+\ldots\right)$ ***(A1)***

$\Rightarrow f\left(x\right)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

[3 marks]

continue…

Question 11 continued.

(c) **METHOD 1**

attempt to substitute $x^2e^x \approx x^2 + x^3 + \frac{1}{2}x^4$ into $\frac{\left(x^2e^x - x^2\right)^3}{x^9}$ ***M1***

$$\frac{\left(x^2e^x - x^2\right)^3}{x^9} \approx \frac{\left(x^2 + x^3 + \frac{1}{2}x^4(+\ldots) - x^2\right)^3}{x^9}$$ ***(A1)***

EITHER

$$= \frac{\left(x^3 + \frac{1}{2}x^4 + \ldots\right)^3}{x^9}$$ ***A1***

$$= \frac{x^9(+\text{higher order terms})}{x^9}$$

OR

$$\left(\frac{x^3 + \frac{1}{2}x^4(+\ldots)}{x^3}\right)^3$$ ***A1***

$$\left(1 + \frac{1}{2}x(+\ldots)\right)^3$$

THEN

$= 1$ (+ higher order terms)

So $\lim_{x \to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = 1$ ***A1***

[4 marks]

continue…

Question 11 continued.

METHOD 2

$$\lim_{x\to 0}\left[\frac{\left(x^2e^x-x^2\right)^3}{x^9}\right]=\lim_{x\to 0}\left(\frac{x^2e^x-x^2}{x^3}\right)^3$$ ***M1***

$$=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^3$$ ***(A1)***

attempt to use L'Hôpital's rule ***M1***

$$=\lim_{x\to 0}\left(\frac{e^x-0}{1}\right)^3$$

$$=\left[\lim_{x\to 0}e^x\right]^3$$

$$=1$$ ***A1***

[4 marks]

Total [14 marks]

$\left(\quad\right)^3$

8. (a) **METHOD 1**

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$u = xy$, $v = \ln(xy)$, $\frac{du}{dx} = x\frac{dy}{dx} + y$, $\frac{dv}{dx} = \left(x\frac{dy}{dx} + y\right)\frac{1}{xy}$

$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$ ***A1***

Note: Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining $\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + x\frac{dy}{dx}\ln(xy) + y\ln(xy)\right]$.

$\frac{dy}{dx} = 1 - \left[\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$

$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))$ ***A1***

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 2

$y = x - xy\ln x - xy\ln y$

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$\frac{dy}{dx} = 1 - \left(\frac{xy}{x} + \left(x\frac{dy}{dx} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\ln y\right)$ ***A1***

or equivalent to the above, for example

$\frac{dy}{dx} = 1 - \left(x\ln x\frac{dy}{dx} + (1 + \ln x)y\right) - \left(y\ln y + x\left(\ln y\frac{dy}{dx} + \frac{dy}{dx}\right)\right)$

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$ ***A1***

or equivalent to the above, for example

$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$

$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 3

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$u = x\ln(xy),\ v = y,\ \dfrac{du}{dx} = \ln(xy) + \left(x\dfrac{dy}{dx} + y\right)\dfrac{x}{xy},\ \dfrac{dv}{dx} = \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = 1 - \left(x\dfrac{dy}{dx}\ln(xy) + y\ln(xy) + \dfrac{xy}{xy}\left(x\dfrac{dy}{dx} + y\right)\right)$ ***A1***

$\dfrac{dy}{dx} = 1 - x\dfrac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$ ***A1***

$\dfrac{dy}{dx} + \left(x\dfrac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

METHOD 4

lets $w = xy$ and attempts to find $\dfrac{dy}{dx}$ where $y = x - w\ln w$ ***M1***

$\dfrac{dy}{dx} = 1 - \left(\dfrac{dw}{dx} + \dfrac{dw}{dx}\ln w\right)\ \left(= 1 - \dfrac{dw}{dx}(1 + \ln w)\right)$ ***A1***

$\dfrac{dw}{dx} = x\dfrac{dy}{dx} + y$ ***A1***

$\dfrac{dy}{dx} = 1 - \left(x\dfrac{dy}{dx} + y + \left(x\dfrac{dy}{dx} + y\right)\ln(xy)\right)\ \left(= 1 - \left(x\dfrac{dy}{dx} + y\right)(1 + \ln(xy))\right)$

$\dfrac{dy}{dx} + \left(x\dfrac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$ ***AG***

[3 marks]

(b) **METHOD 1**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$ ***A1***

substitutes $x=1$ and their non-zero value of y into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(xy))=1$ ***(M1)***

$2\frac{\mathrm{d}y}{\mathrm{d}x}=0\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}=0\right)$ ***A1***

equation of the tangent is $y=1$ ***A1***

METHOD 2

substitutes $x=1$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(xy))=1$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$

EITHER

correctly substitutes $\ln y=\frac{1-y}{y}$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\frac{1}{y}\right)=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=0\ (y=1)$ ***A1***

OR

correctly substitutes $y+y\ln y=1$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}(2+\ln y)=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=0\ (y=1)$ ***A1***

THEN

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$

equation of the tangent is $y=1$ ***A1***

[5 marks]

Total [8 marks]

Section B

2. (a) (i) **METHOD 1**

$\frac{dy}{dt} = y$

$\int \frac{dy}{y} = \int dt$ ***(M1)***

$\ln y = t + c$ **OR** $\ln|y| = t + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ and ***A1*** for t and c.

$y = Ae^t$ ***AG***

METHOD 2

rearranging to $\frac{dy}{dt} - y = 0$ AND multiplying by integrating factor e^{-t} ***M1***

$ye^{-t} = A$ ***A1A1***

$y = Ae^t$ ***AG***

[3 marks]

(ii) substituting $y = Ae^t$ into differential equation in x ***M1***

$\frac{dx}{dt} = x - Ae^t$

$\frac{dx}{dt} - x = -Ae^t$ ***AG***

[1 mark]

(iii) integrating factor (IF) is $e^{\int -1 dt}$ ***(M1)***

$= e^{-t}$ ***(A1)***

$e^{-t}\frac{dx}{dt} - xe^{-t} = -A$

$xe^{-t} = -At + D$ (***A1)***

$x = (-At + D)e^t$ ***A1***

Note: The first constant must be A**,** and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[4 marks]

(b) (i) $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

EITHER

$= -x + y + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$= \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

OR

$= -x + y + (-x + y)$ ***(M1)***

$= 2(-x + y)$ ***A1***

THEN

$= 2\frac{\mathrm{d}y}{\mathrm{d}t}$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}Y}{\mathrm{d}t} = 2Y$ ***A1***

$\int \frac{\mathrm{d}Y}{Y} = \int 2\mathrm{d}t$ ***M1***

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ ***A1***

$Y = B\mathrm{e}^{2t}$ ***AG***

[3 marks]

(iii) $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$

$y = \int B\mathrm{e}^{2t}\,\mathrm{d}t$ ***M1***

$y = \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

Note: The first constant must be B, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[2 marks]

(iv) **METHOD 1**

substituting $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$ and their (iii) into $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ ***M1(M1)***

$B\mathrm{e}^{2t} = -x + \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

Note: Follow through from incorrect part (iii) cannot be awarded if it does not lead to the ***AG*.**

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -\frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}(x\mathrm{e}^{-t})}{\mathrm{d}t} = -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}$ ***M1***

$x\mathrm{e}^{-t} = \int -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}\ \mathrm{d}t$

$x\mathrm{e}^{-t} = -\frac{B}{2}\mathrm{e}^{t} + C\mathrm{e}^{-t} + D$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C + D\mathrm{e}^{t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y \Rightarrow B\mathrm{e}^{2t} = \frac{B}{2}\mathrm{e}^{2t} - C - D\mathrm{e}^{t} + \frac{B}{2}\mathrm{e}^{2t} + C \Rightarrow D = 0$ ***M1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

[3 marks]

(c) (i) $\frac{\mathrm{d}y}{\mathrm{d}t} = -4x + y$

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ seen anywhere ***M1***

METHOD 1

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4(x - y) + \frac{\mathrm{d}y}{\mathrm{d}t}$

attempt to eliminate x ***M1***

$= -4\left(\frac{1}{4}\left(y - \frac{\mathrm{d}y}{\mathrm{d}t}\right) - y\right) + \frac{\mathrm{d}y}{\mathrm{d}t}$

$= 2\frac{\mathrm{d}y}{\mathrm{d}t} + 3y$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$ ***AG***

METHOD 2

rewriting LHS in terms of x and y ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ ***A1***

$= 0$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda\mathrm{e}^{\lambda t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F\lambda^2\mathrm{e}^{\lambda t}$ ***(A1)***

$F\lambda^2\mathrm{e}^{\lambda t} - 2F\lambda\mathrm{e}^{\lambda t} - 3F\mathrm{e}^{\lambda t} = 0$ ***(M1)***

$\lambda^2 - 2\lambda - 3 = 0$ (since $\mathrm{e}^{\lambda t} \neq 0$) ***A1***

λ_1 and λ_2 are 3 and -1 (either order) ***A1***

[4 marks]

(iii) **METHOD 1**

$y = F\mathrm{e}^{3t} + G\mathrm{e}^{-t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = 3F\mathrm{e}^{3t} - G\mathrm{e}^{-t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t}$ ***(A1)(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t} - 2\left(3F\mathrm{e}^{3t} - G\mathrm{e}^{-t}\right) - 3\left(F\mathrm{e}^{3t} + G\mathrm{e}^{-t}\right)$ ***M1***

$= 9F\mathrm{e}^{3t} + G\mathrm{e}^{-t} - 6F\mathrm{e}^{3t} + 2G\mathrm{e}^{-t} - 3F\mathrm{e}^{3t} - 3G\mathrm{e}^{-t}$ ***A1***

$= 0$ ***AG***

METHOD 2

$y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda_1\mathrm{e}^{\lambda_1 t} + G\lambda_2\mathrm{e}^{\lambda_2 t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F\lambda_1^2\mathrm{e}^{\lambda_1 t} + G\lambda_2^2\mathrm{e}^{\lambda_2 t}$ ***(A1)(A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = F\lambda_1^2\mathrm{e}^{\lambda_1 t} + G\lambda_2^2\mathrm{e}^{\lambda_2 t} - 2\left(F\lambda_1\mathrm{e}^{\lambda_1 t} + G\lambda_2\mathrm{e}^{\lambda_2 t}\right) - 3\left(F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}\right)$ ***M1***

$= F\mathrm{e}^{\lambda_1 t}\left(\lambda^2 - 2\lambda - 3\right) + G\mathrm{e}^{\lambda_2 t}\left(\lambda^2 - 2\lambda - 3\right)$ ***A1***

$= 0$ ***AG***

[4 marks]

Total [30 marks]

3. attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}}\mathrm{d}x = \int \frac{2}{\sqrt{u}}\mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

Total [5 marks]

10. (a) attempt to use quotient rule ***(M1)***

correct substitution into quotient rule

$f'(x) = \dfrac{5kx\left(\dfrac{1}{5x}\right) - k\ln 5x}{(kx)^2}$ (or equivalent) ***A1***

$= \dfrac{k - k\ln 5x}{k^2x^2}, (k \in \mathbb{R}^+)$ ***A1***

$= \dfrac{1 - \ln 5x}{kx^2}$ ***AG***

[3 marks]

(b) $f'(x) = 0$ ***M1***

$\dfrac{1 - \ln 5x}{kx^2} = 0$

$\ln 5x = 1$ ***(A1)***

$x = \dfrac{e}{5}$ ***A1***

[3 marks]

(c) $f''(x) = 0$ ***M1***

$\dfrac{2\ln 5x - 3}{kx^3} = 0$

$\ln 5x = \dfrac{3}{2}$ ***A1***

$5x = e^{\frac{3}{2}}$ ***A1***

so the point of inflexion occurs at $x = \dfrac{1}{5}e^{\frac{3}{2}}$ ***AG***

[3 marks]

continued…

Question 10 continued

(d) attempt to integrate ***(M1)***

$u = \ln 5x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{\ln 5x}{kx}\mathrm{d}x = \frac{1}{k}\int u\ \mathrm{d}u$ ***(A1)***

EITHER

$= \frac{u^2}{2k}$ ***A1***

so $\frac{1}{k}\int_1^{\frac{3}{2}} u\ \mathrm{d}u = \left[\frac{u^2}{2k}\right]_1^{\frac{3}{2}}$ ***A1***

OR

$= \frac{(\ln 5x)^2}{2k}$ ***A1***

so $\int_{\frac{1}{5}}^{\frac{1}{5}\mathrm{e}^{\frac{3}{2}}} \frac{\ln 5x}{kx}\mathrm{d}x = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{1}{5}}^{\frac{1}{5}\mathrm{e}^{\frac{3}{2}}}$ ***A1***

THEN

$= \frac{1}{2k}\left(\frac{9}{4} - 1\right)$

$= \frac{5}{8k}$ ***A1***

setting **their** expression for area equal to 3 ***M1***

$\frac{5}{8k} = 3$

$k = \frac{5}{24}$ ***A1***

[7 marks]

Total [16 marks]

12. (a) attempting to use the chain rule to find the first derivative ***M1***

$f'(x) = (\cos x)\mathrm{e}^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x) = \mathrm{e}^{\sin x}(\cos^2 x - \sin x)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ ***M1***

$f(0) = 1;\ f'(0) = (\cos 0)\mathrm{e}^{\sin 0} = 1;\ f''(0) = \mathrm{e}^{\sin 0}(\cos^2 0 - \sin 0) = 1$ ***A1***

substitution into the Maclaurin formula $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$ ***M1***

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1 + x + \frac{x^2}{2}$ ***A1***

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ ***M1***

$f'''(x) = (\cos x)\mathrm{e}^{\sin x}(\cos^2 x - \sin x) - (\cos x)\mathrm{e}^{\sin x}(2\sin x + 1)$ (or equivalent) ***A2***

substituting $x = 0$ into **their** $f'''(x)$ ***M1***

$f'''(0) = 1(1 - 0) - 1(0 + 1) = 0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$\mathrm{e}^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \ldots$

substituting Maclaurin series for $\sin x$ ***M1***

$\mathrm{e}^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^3}{3!} + \ldots$ ***A1***

coefficient of x^3 is $-\frac{1}{3!} + \frac{1}{3!} = 0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

continued…

Question 12 continued

(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots$ ***A1***

substituting $\left(e^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(e^{3x} - 1\right) = \left(e^{3x} - 1\right) - \frac{\left(e^{3x} - 1\right)^3}{3} + \frac{\left(e^{3x} - 1\right)^5}{5} - \ldots$$

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots\right)^3}{3} + \ldots$$ ***A1***

selecting correct terms from above ***M1***

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$$

$$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + \ldots}{3x + \frac{9x^2}{2} + \ldots}$$ ***A1***

$$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + \ldots}{3 + \frac{9x}{2} + \ldots}$$

$$= \frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x \to 0} \frac{(\cos x)e^{\sin x}}{\frac{3e^{3x}}{1 + \left(e^{3x} - 1\right)^2}}$$ (or equivalent) ***A1***

$$= \frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]

5. (a) use of a graph to find the coordinates of the local minimum ***(M1)***

$s = -16.513...$ ***(A1)***

maximum distance is $16.5\,\text{cm}$ (to the left of O) ***A1***

[3 marks]

(b) attempt to find time when particle changes direction *eg* considering the first maximum on the graph of s or the first t – intercept on the graph of s'. ***(M1)***

$t = 1.51986...$ ***(A1)***

attempt to find the gradient of s' for **their** value of t, $s''(1.51986...)$ ***(M1)***

$= -8.92\ \ (\text{cm/s}^2)$ ***A1***

[4 marks]

Total [7 marks]

11. (a) ***METHOD 1***

using $I(t) = e^{\int P(t)dt}$ ***M1***

$e^{\int \frac{1}{t+1}dt}$

$= e^{\ln(t+1)}$ ***A1***

$= t+1$ ***AG***

METHOD 2

attempting product rule differentiation on $\frac{d}{dt}(x(t+1))$ ***M1***

$\frac{d}{dt}(x(t+1)) = \frac{dx}{dt}(t+1) + x$

$= (t+1)\left(\frac{dx}{dt} + \frac{x}{t+1}\right)$ ***A1***

so $t+1$ is an integrating factor for this differential equation ***AG***

[2 marks]

continued…

Question 11 continued

(b) attempting to multiply through by $(t+1)$ and rearrange to give ***(M1)***

$(t+1)\dfrac{\mathrm{d}x}{\mathrm{d}t}+x=10(t+1)\mathrm{e}^{-\frac{t}{4}}$ ***A1***

$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right)=10(t+1)\mathrm{e}^{-\frac{t}{4}}$

$x(t+1)=\int 10(t+1)\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t$ ***A1***

attempting to integrate the RHS by parts ***M1***

$=-40(t+1)\mathrm{e}^{-\frac{t}{4}}+40\int \mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t$

$=-40(t+1)\mathrm{e}^{-\frac{t}{4}}-160\mathrm{e}^{-\frac{t}{4}}+C$ ***A1***

Note: Condone the absence of C .

EITHER

substituting $t=0, x=0 \Rightarrow C=200$ ***M1***

$x=\dfrac{-40(t+1)\mathrm{e}^{-\frac{t}{4}}-160\mathrm{e}^{-\frac{t}{4}}+200}{t+1}$ ***A1***

using $-40\mathrm{e}^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)\mathrm{e}^{-\frac{t}{4}}$ and $-160\mathrm{e}^{-\frac{t}{4}}$ ***M1***

OR

using $-40\mathrm{e}^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)\mathrm{e}^{-\frac{t}{4}}$ and $-160\mathrm{e}^{-\frac{t}{4}}$ giving

$x(t+1)=-40\mathrm{e}^{-\frac{t}{4}}(t+5)+C$ (or equivalent) ***M1A1***

substituting $t=0,\ x=0 \Rightarrow C=200$ ***M1***

THEN

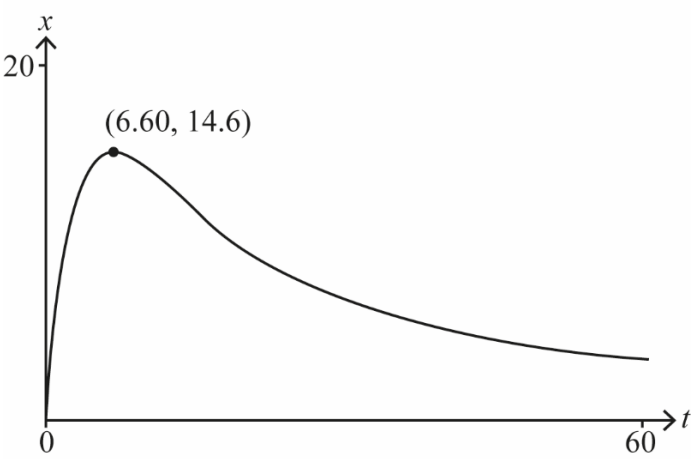
$x(t)=\dfrac{200-40\mathrm{e}^{-\frac{t}{4}}(t+5)}{t+1}$ ***AG***

[8 marks]

continued…

Question 11 continued

(c)



graph starts at the origin and has a local maximum (coordinates not required) ***A1***
sketched for $0 \leq t \leq 60$ ***A1***
correct concavity for $0 \leq t \leq 60$ ***A1***
maximum amount of salt is 14.6 (grams) at $t = 6.60$ (minutes) ***A1A1***

[5 marks]

(d) using an appropriate graph or equation (first or second derivative) ***M1***
amount of salt is decreasing most rapidly at $t = 12.9$ (minutes) ***A1***

[2 marks]

(e) **EITHER**

attempting to form an integral representing the amount of salt that left the tank ***M1***

$$\int_0^{60} \frac{x(t)}{t+1}\,\mathrm{d}t$$

$$\int_0^{60} \frac{200 - 40\mathrm{e}^{-\frac{t}{4}}(t+5)}{(t+1)^2}\,\mathrm{d}t$$ ***A1***

OR

attempting to form an integral representing the amount of salt that entered the tank minus the amount of salt in the tank at $t = 60$ (minutes) ***M1***

amount of salt that left the tank is $\int_0^{60} 10\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t - x(60)$ ***A1***

THEN

$= 36.7$ (grams) ***A2***

[4 marks]

Total [21 marks]

1. $\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = \int \left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}x$ ***(A1)***

$\int \frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}x = 3x-10x^{\frac{1}{2}}\ (+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR $27-10\times 3-(3-10)$

$=4$ ***A1***

[5 marks]

4. evidence of using product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x-1)\times\left(k\mathrm{e}^{kx}\right)+2\times\mathrm{e}^{kx} \quad \left(=\mathrm{e}^{kx}(2kx-k+2)\right)$ ***A1***

correct working for one of (seen anywhere) ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1 \Rightarrow k\mathrm{e}^{k}+2\mathrm{e}^{k}$

OR

slope of tangent is $5\mathrm{e}^{k}$

their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1$ equals the slope of $y=5\mathrm{e}^{k}x \left(=5\mathrm{e}^{k}\right)$ (seen anywhere) ***(M1)***

$k\mathrm{e}^{k}+2\mathrm{e}^{k}=5\mathrm{e}^{k}$

$k=3$ ***A1***

[5 marks]

12. (a) **METHOD 1**

recognition of both known series ***(M1)***

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

attempt to multiply the two series up to and including x^3 term ***(M1)***

$e^x \sin x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \ldots$ ***(A1)***

$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$ ***A1***

[4 marks]

METHOD 2

$f(x) = e^x \sin x$

$f'(x) = e^x \cos x + e^x \sin x$ ***A1***

$f''(x) = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \left(= 2e^x \cos x\right)$

$f'''(x) = 2e^x \cos x - 2e^x \sin x$

$f''(x) = 2e^x \cos x$ and $f'''(x) = 2e^x(\cos x - \sin x)$ ***A1***

substitute $x = 0$ into f or its derivatives to obtain Maclaurin series ***(M1)***

$e^x \sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 2 + \ldots$

$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$ ***A1***

[4 marks]

continued…

Question 12 continued

(b) $e^{x^2}\sin\left(x^2\right) = x^2 + x^4 + \frac{1}{3}x^6 + ...$ ***(A1)***

substituting their expression and attempt to integrate ***M1***

$$\int_0^1 e^{x^2}\sin(x^2)\,dx \approx \int_0^1 \left(x^2 + x^4 + \frac{1}{3}x^6 \right) dx$$

Note: Condone absence of limits up to this stage.

$$= \left[\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{21} \right]_0^1$$ ***A1***

$$= \frac{61}{105}$$ ***A1***

[4 marks]

continued…

Question 12 continued

(c) (i) attempt to use product rule at least once ***M1***

$g'(x) = e^x \cos x - e^x \sin x$ ***A1***

$g''(x) = e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x \left(= -2e^x \sin x\right)$ ***A1***

EITHER

$2\left(g'(x) - g(x)\right) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right) = -2e^x \sin x$ ***A1***

OR

$g''(x) = 2\left(e^x \cos x - e^x \sin x - e^x \cos x\right)$ ***A1***

THEN

$g''(x) = 2\left(g'(x) - g(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-2e^x \sin x$.

(ii) $g'''(x) = 2\left(g''(x) - g'(x)\right)$ ***A1***

$g^{(4)}(x) = 2\left(g'''(x) - g''(x)\right)$ ***AG***

Note: Accept working with each side separately to obtain $-4e^x \cos x$.

[5 marks]

continued…

Question 12 continued

(d) attempt to substitute $x = 0$ into a derivative ***(M1)***

$g(0) = 1,\ g'(0) = 1,\ g''(0) = 0$ ***A1***

$g'''(0) = -2,\ g^{(4)}(0) = -4$ ***(A1)***

attempt to substitute into Maclaurin formula ***(M1)***

$g(x) = 1 + x - \frac{2}{3!}x^3 - \frac{4}{4!}x^4 + ... \left(= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + ... \right)$ ***A1***

Note: Do not award any marks for approaches that do not use the part (c) result.

[5 marks]

(e) **METHOD 1**

$\lim_{x \to 0} \frac{e^x \cos x - 1 - x}{x^3} = \lim_{x \to 0} \frac{\left(1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 + ...\right) - 1 - x}{x^3}$ ***M1***

$= \lim_{x \to 0} \left(-\frac{1}{3} - \frac{1}{6}x + ... \right)$ ***(A1)***

$= -\frac{1}{3}$ ***A1***

Note: Condone the omission of $+...$ in their working.

continued…

Question 12 continued

METHOD 2

$\lim_{x\to 0}\frac{e^x\cos x-1-x}{x^3}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***M1***

$=\lim_{x\to 0}\frac{e^x\cos x-e^x\sin x-1}{3x^2}\left(=\lim_{x\to 0}\frac{g'(x)-1}{3x^2}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x}{6x}\left(=\lim_{x\to 0}\frac{g''(x)}{6x}\right)$

$=\frac{0}{0}$, using l'Hôpital's rule again

$=\lim_{x\to 0}\frac{-2e^x\sin x-2e^x\cos x}{6}\left(=\lim_{x\to 0}\frac{g'''(x)}{6}\right)$ ***A1***

$=-\frac{1}{3}$ ***A1***

[3 marks]

Total [21 marks]

6. (a) attempts to replace x with $-x$ ***M1***

$f(-x) = -x\sqrt{1-(-x)^2}$

$= -x\sqrt{1-(-x)^2}\ (= -f(x))$ ***A1***

Note: Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal.

Note: Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by $180°$ about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) attempts both product rule and chain rule differentiation to find $f'(x)$ ***M1***

$f'(x) = x \times \frac{1}{2} \times (-2x) \times (1-x^2)^{-\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} \times 1 \left(= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right)$ ***A1***

$= \frac{1-2x^2}{\sqrt{1-x^2}}$

sets their $f'(x) = 0$ ***M1***

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to evaluate $f(x)$ at least at one of their $f'(x) = 0$ roots.

$a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ***A1***

Note: Award ***A1*** for $-\frac{1}{2} \le y \le \frac{1}{2}$.

[6 marks]

Total [8 marks]

7. METHOD 1

$u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \ \mathrm{d}x$ ***(A1)***

attempts to express the integral in terms of u ***M1***

$\int_1^2 u^{n-1} \mathrm{d}u$ ***A1***

$= \frac{1}{n}\left[u^n\right]_1^2 \quad \left(= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}\right)$ ***A1***

Note: Condone the absence of or incorrect limits up to this point.

$= \frac{2^n - 1^n}{n}$ ***M1***

$= \frac{2^n - 1}{n}$ ***A1***

Note: Award ***M1*** for correct substitution of **their** limits for u into their antiderivative for u (or given limits for x into their antiderivative for x).

METHOD 2

$\int \sec^n x \tan x \ \mathrm{d}x = \int \sec^{n-1} x \sec x \tan x \ \mathrm{d}x$ ***(A1)***

applies integration by inspection ***(M1)***

$= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}$ ***A2***

Note: Award ***A2*** if the limits are not stated.

$= \frac{1}{n}\left(\sec^n \frac{\pi}{3} - \sec^n 0\right)$ ***M1***

Note: Award ***M1*** for correct substitution into their antiderivative.

$= \frac{2^n - 1}{n}$ ***A1***

[6 marks]

4. (a) recognizing at rest $v = 0$ ***(M1)***

$t = 3.34692...$

$t = 3.35$ (seconds) ***A1***

Note: Award ***(M1)A0*** for any other solution to $v = 0$ eg $t = -0.205$ or $t = 6.08$.

[2 marks]

(b) recognizing particle changes direction when $v = 0$ OR when $t = 3.34692...$ ***(M1)***

$a = -4.71439...$

$a = -4.71\,(\text{ms}^{-2})$ ***A2***

[3 marks]

(c) distance travelled = $\int_0^6 |v|\,\mathrm{d}t$ OR

$$\int_0^{3.34...} \left(\mathrm{e}^{\sin(t)} + 4\sin(t)\right)\mathrm{d}t - \int_{3.34...}^6 \left(\mathrm{e}^{\sin(t)} + 4\sin(t)\right)\mathrm{d}t \;\; (= 14.3104... + 6.44300...)$$ ***(A1)***

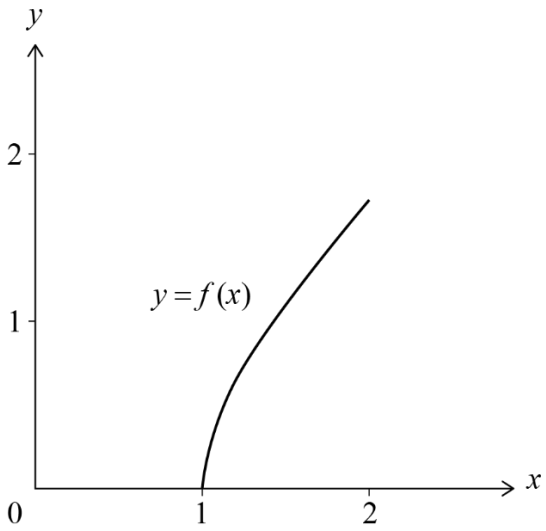
$= 20.7534...$

$= 20.8$ (metres) ***A1***

[2 marks]

Total [7 marks]

10. (a)



correct shape (concave down) within the given domain $1 \leq x \leq 2$ ***A1***

$(1,0)$ and $\left(2,\sqrt{3}\right)\left(=(2,1.73)\right)$ ***A1***

Note: The coordinates of endpoints may be seen on the graph or marked on the axes.

continued…

Question 10 continued

(b) (i) interchanging x and y (seen anywhere) ***M1***

$x = \sqrt{y^2 - 1}$

$x^2 = y^2 - 1$ ***A1***

$y = \sqrt{x^2 + 1}$ ***A1***

$f^{-1}(x) = \sqrt{x^2 + 1}$ ***AG***

(ii) $0 \le x \le \sqrt{3}$ OR domain $\left[0, \sqrt{3}\right] \left(= [0, 1.73]\right)$ ***A1***

$1 \le y \le 2$ OR $1 \le f^{-1}(x) \le 2$ OR range $[1, 2]$ ***A1***

[5 marks]

continued…

Question 10 continued

(c) (i) attempt to substitute $x = \sqrt{y^2 + 1}$ into the correct volume formula ***(M1)***

$V = \pi \int_0^h \left(\sqrt{y^2 + 1}\right)^2 \, \mathrm{d}y \left(= \pi \int_0^h \left(y^2 + 1\right) \, \mathrm{d}y \right)$ ***A1***

$= \pi \left[\frac{1}{3} y^3 + y \right]_0^h$ ***A1***

$= \pi \left(\frac{1}{3} h^3 + h \right)$ ***AG***

Note: Award marks as appropriate for correct work using a different variable e.g. $\pi \int_0^h \left(\sqrt{x^2 + 1}\right)^2 \mathrm{d}x$

(ii) attempt to substitute $h = \sqrt{3} \ (= 1.732...)$ into V ***(M1)***

$V = 10.8828...$

$V = 10.9$ (m³) $\left(= 2\sqrt{3}\pi\right)$ (m³) ***A1***

[5 marks]

continued…

Question 10 continued

(d) **METHOD 1**

$\text{time} = \frac{10.8828...}{0.4}\left(= \frac{2\sqrt{3}\pi}{0.4}\right)$ ***(M1)***

$= 27.207...$

$= 27.2\left(= 5\sqrt{3}\pi\right)(s)$ ***A1***

[2 marks]

continued…

Question 10 continued

(e) attempt to find the height of the tank when $V = 5.4414... \left(= \sqrt{3}\pi\right)$ ***(M1)***

$$\pi\left(\frac{1}{3}h^3 + h\right) = 5.4414... \left(= \sqrt{3}\pi\right)$$

$h = 1.1818...$ ***(A1)***

attempt to use the chain rule or differentiate $V = \pi\left(\frac{1}{3}h^3 + h\right)$ with respect to t ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi(h^2+1)} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ OR $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi\left(h^2+1\right)\frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

attempt to substitute **their** h and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.4$ ***(M1)***

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.4}{\pi\left(1.1818...^2+1\right)} = 0.053124...$$

$= 0.0531$ (m s^{-1}) ***A1***

[6 marks]

Total [20 marks]

12. (a) attempt to use Euler's method ***(M1)***

$x_{n+1} = x_n + 0.1;\ \ y_{n+1} = y_n + 0.1 \times \frac{\mathrm{d}y}{\mathrm{d}x}$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{x^2}$

correct intermediate y-values ***(A1)(A1)***

$3.7, 4.63140..., 5.92098..., 7.79542...$

Note: ***A1*** for any two correct y-values seen

$y = 10.6958...$

$y = 10.7$ ***A1***

Note: For the final ***A1,*** the value 10.7 must be the last value in a table or a list, or be given as a final answer, not just embedded in a table which has further lines.

[4 marks]

(b) $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

replacing y with vx and $\frac{\mathrm{d}y}{\mathrm{d}x}$ with $v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x^2 \Rightarrow x^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = v^2x^2 - 2x^2$ ***A1***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - 2$ (since $x > 0$)

$x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 - v - 2$ ***AG***

[3 marks]

continued…

Question 12 continued

(c) (i) attempt to separate variables v and x ***(M1)***

$$\int \frac{\mathrm{d}v}{v^2 - v - 2} = \int \frac{\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}v}{(v-2)(v+1)} = \int \frac{\mathrm{d}x}{x}$$ ***(A1)***

attempt to express in partial fraction form ***M1***

$$\frac{1}{(v-2)(v+1)} \equiv \frac{A}{v-2} + \frac{B}{v+1}$$

$$\frac{1}{(v-2)(v+1)} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)$$ ***A1***

$$\frac{1}{3}\int\left(\frac{1}{v-2} - \frac{1}{v+1}\right)\mathrm{d}v = \int \frac{\mathrm{d}x}{x}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| (+c)$$ ***A1***

Note: Condone absence of modulus signs throughout.

EITHER

attempt to find c using $x = 1, y = 3, v = 3$ ***M1***

$$c = \frac{1}{3}\ln\frac{1}{4}$$

$$\frac{1}{3}\left(\ln|v-2| - \ln|v+1|\right) = \ln|x| + \frac{1}{3}\ln\frac{1}{4}$$

expressing both sides as a single logarithm ***(M1)***

$$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(\frac{|x|^3}{4}\right)$$

continued…

Question 12 continued

OR

expressing both sides as a single logarithm ***(M1)***

$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(A|x|^3\right)$

attempt to find A using $x=1, y=3, v=3$ ***M1***

$A = \frac{1}{4}$

THEN

$\left|\frac{v-2}{v+1}\right| = \frac{1}{4}x^3$ (since $x>0$)

substitute $v = \frac{y}{x}$ (seen anywhere) ***M1***

$\frac{\frac{y}{x}-2}{\frac{y}{x}+1} = \frac{1}{4}x^3$ (since $y>2x$)

$\left(\Rightarrow \frac{y-2x}{y+x} = \frac{1}{4}x^3\right)$

attempt to make y the subject ***M1***

$y - \frac{x^3y}{4} = 2x + \frac{x^4}{4}$ ***A1***

$y = \frac{8x+x^4}{4-x^3}$ ***AG***

continued…

Question 12 continued

(ii) actual value at $y(1.5) = 27.3$ ***A1***

(iii) gradient changes rapidly (during the interval considered) OR

the curve has a vertical asymptote at $x = \sqrt[3]{4} \left(= 1.5874...\right)$ ***R1***

[12 marks]

Total [19 marks]

2. METHOD 1

recognises that $g(x) = \int \left(3x^2 + 5e^x\right) dx$ ***(M1)***

$g(x) = x^3 + 5e^x (+C)$ ***(A1)(A1)***

Note: Award ***A1*** for each integrated term.

substitutes $x = 0$ and $y = 4$ into their integrated function (must involve $+C$) ***(M1)***

$4 = 0 + 5 + C \Rightarrow C = -1$

$g(x) = x^3 + 5e^x - 1$ ***A1***

METHOD 2

attempts to write both sides in the form of a definite integral ***(M1)***

$\int_0^x g'(t)\, dt = \int_0^x \left(3t^2 + 5e^t\right) dt$ ***(A1)***

$g(x) - 4 = x^3 + 5e^x - 5e^0$ ***(A1)(A1)***

Note: Award ***A1*** for $g(x) - 4$ and ***A1*** for $x^3 + 5e^x - 5e^0$.

$g(x) = x^3 + 5e^x - 1$ ***A1***

[5 marks]

5. (a) recognises the need to find the value of t when $v = 0$ ***(M1)***

$t = 1.5707... \left(= \frac{\pi}{2} \right)$

$t = 1.57 \left(= \frac{\pi}{2} \right)$ (s) ***A1***

[2 marks]

(b) recognises that $a(t) = v'(t)$ ***(M1)***

$t_1 = 2.2627...$, $t_2 = 2.9573...$

$t_1 = 2.26$, $t_2 = 2.96$ (s) ***A1A1***

Note: Award ***M1A1A0*** if the two correct answers are given with additional values outside $0 \le t \le 3$.

[3 marks]

(c) speed is greatest at $t = 3$ ***(A1)***

$a = -1.8377...$

$a = -1.84$ (m s^{-2}) ***A1***

[2 marks]

Total [7 marks]

6. attempts to express x^2 in terms of y ***(M1)***

$V = \pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y$ ***A1***

Note: Correct limits are required.

Attempts to solve $\pi\int_h^4 36\left(1-\frac{(y-4)^2}{16}\right)\mathrm{d}y = 285$ for h ***(M1)***

Note: Award ***M1*** for attempting to solve $36\pi\left(\frac{h^3}{48}-\frac{h^2}{4}+\frac{8}{3}\right)=285$ or equivalent for h.

$h = 0.7926...$

$h = 0.793$ (cm) ***A2***

[5 marks]

7. (a) (as $\lim_{x\to 0} x^2 = 0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$\Rightarrow \lim_{x\to 0}\left(\arctan\left(\cos x\right) - k\right) = 0$ ***M1***

$\arctan 1 - k = 0 \ \left(k = \arctan 1\right)$ ***A1***

so $k = \frac{\pi}{4}$ ***AG***

Note: Award ***M1A0*** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

continued…

Question 7 continued

(b) $\lim_{x\to 0}\dfrac{\arctan(\cos x)-\frac{\pi}{4}}{x^2}\left(=\dfrac{0}{0}\right)$

$=\lim_{x\to 0}\dfrac{\frac{-\sin x}{1+\cos^2 x}}{2x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

recognises to apply l'Hôpital's rule again ***(M1)***

$=\lim_{x\to 0}\dfrac{\frac{-\sin x}{1+\cos^2 x}}{2x}\left(=\dfrac{0}{0}\right)$

Note: Award ***M0*** if their limit is not the indeterminate form $\frac{0}{0}$.

EITHER

$=\lim_{x\to 0}\dfrac{\frac{-\cos x\left(1+\cos^2 x\right)-2\sin^2 x\cos x}{\left(1+\cos^2 x\right)^2}}{2}$ ***A1A1***

Note: Award ***A1*** for a correct first term in the numerator and ***A1*** for a correct second term in the numerator.

OR

$\lim_{x\to 0}\dfrac{-\cos x}{2\left(1+\cos^2 x\right)-4x\sin x\cos x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

THEN

substitutes $x=0$ into the correct expression to evaluate the limit ***A1***

Note: The final ***A1*** is dependent on all previous marks.

$=-\dfrac{1}{4}$ ***AG***

[6 marks]

Total [8 marks]

10. (a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t+6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t+6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t+6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

continued…

Question 10 continued

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required ***(M1)***

attempts to solve $h_A'(t) = h_B'(t)$ for t ***(M1)***

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ ***(A1)***

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ ***(A1)***

attempts to calculate the total amount of time ***(M1)***

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) ***A1***

[6 marks]

Total [15 marks]

12. (a) rate of growth (change) of the (marsupial) population (with respect to time) ***A1***

[1 mark]

Note: Do not accept growth (change) in the (marsupials) population per year.

(b) **METHOD 1**

attempts implicit differentiation on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - \frac{kP^2}{N}$ by expanding $kP\left(1-\frac{P}{N}\right)$ ***(M1)***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t} - 2\frac{kP}{N}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***A1A1***

$= k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1-\frac{2P}{N}\right)$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ and so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ ***AG***

METHOD 2

attempts implicit differentiation (product rule) on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1-\frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)\frac{\mathrm{d}P}{\mathrm{d}t}\right)$ ***A1***

substitutes $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ into their $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\left(kP\left(1-\frac{P}{N}\right)\right)\left(1-\frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)kP\left(1-\frac{P}{N}\right)\right)$

$= k^2P\left(1-\frac{P}{N}\right)^2 - k^2P\left(1-\frac{P}{N}\right)\left(\frac{P}{N}\right)$

$= k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{P}{N}-\frac{P}{N}\right)$ ***A1***

so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = 0 \Rightarrow k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right) = 0$ ***(M1)***

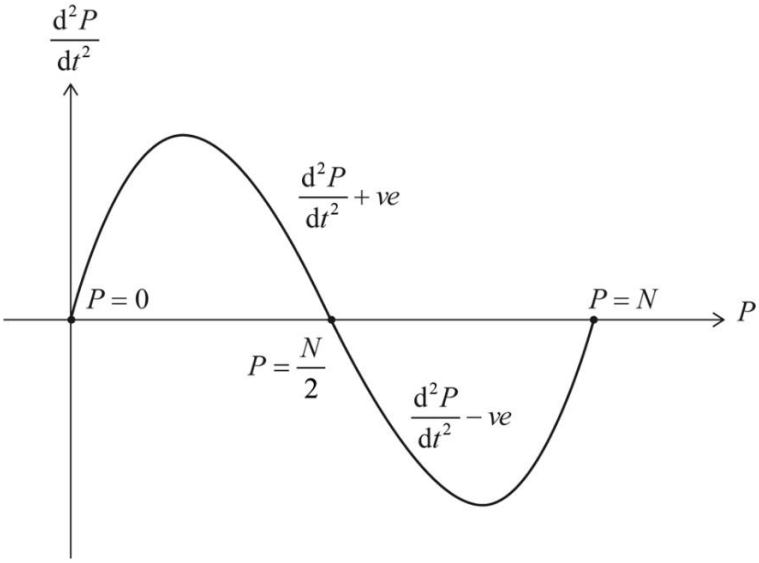
$P = 0, \frac{N}{2}, N$ ***A2***

Note: Award ***A1*** for $P = \frac{N}{2}$ only.

uses the second derivative to show that concavity changes at $P = \frac{N}{2}$ or the first derivative to show a local maximum at $P = \frac{N}{2}$ ***M1***

EITHER

a clearly labelled correct sketch of $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ versus P showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***



OR

a correct and clearly labelled sign diagram (table) showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

continued…

Question 12 continued

OR

for example, $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = \frac{3k^2N}{32}(>0)$ with $P = \frac{N}{4}$ and $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = -\frac{3k^2N}{32}(<0)$ with $P = \frac{3N}{4}$

showing $P = \frac{N}{2}$ corresponds to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

so the population is increasing at its maximum rate when $P = \frac{N}{2}$ ***AG***

[5 marks]

(d) substitutes $P = \frac{N}{2}$ into $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***(M1)***

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{N}{2}\right)\left(1 - \frac{\frac{N}{2}}{N}\right)$$

the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ is $\frac{kN}{4}$ ***A1***

[2 marks]

continued…

Question 12 continued

(e) **METHOD 1**

attempts to separate variables ***M1***

$$\int \frac{N}{P(N-P)}\mathrm{d}P = \int k\ \mathrm{d}t$$

attempts to write $\frac{N}{P(N-P)}$ in partial fractions form ***M1***

$$\frac{N}{P(N-P)} \equiv \frac{A}{P} + \frac{B}{(N-P)} \Rightarrow N \equiv A(N-P) + BP$$

$A=1,\ B=1$ ***A1***

$$\frac{N}{P(N-P)} \equiv \frac{1}{P} + \frac{1}{(N-P)}$$

$$\int \left(\frac{1}{P} + \frac{1}{(N-P)}\right)\mathrm{d}P = \int k\ \mathrm{d}t$$

$$\Rightarrow \ln P - \ln(N-P) = kt\,(+C)$$ ***A1A1***

Note: Award ***A1*** for $-\ln(N-P)$ and ***A1*** for $\ln P$ and $kt\,(+C)$. Absolute value signs are not required.

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$ and so $C = \ln P_0 - \ln(N-P_0)$

$$kt = \ln\left(\frac{P}{N-P}\right) - \ln\left(\frac{P_0}{N-P_0}\right)\left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

so $kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 2

attempts to separate variables ***M1***

$$\int \frac{1}{P\left(1-\frac{P}{N}\right)} \mathrm{d}P = \int k \ \mathrm{d}t$$

attempts to write $\frac{1}{P\left(1-\frac{P}{N}\right)}$ in partial fractions form ***M1***

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{A}{P} + \frac{B}{1-\frac{P}{N}} \Rightarrow 1 \equiv A\left(1-\frac{P}{N}\right) + BP$$

$A=1,\ B=\frac{1}{N}$ ***A1***

$$\frac{1}{P\left(1-\frac{P}{N}\right)} \equiv \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)}$$

$$\int \frac{1}{P} + \frac{1}{N\left(1-\frac{P}{N}\right)} \mathrm{d}P = \int k \ \mathrm{d}t$$

$$\Rightarrow \ln P - \ln\left(1-\frac{P}{N}\right) = kt\left(+C\right)$$ ***A1A1***

Note: Award ***A1*** for $-\ln\left(1-\frac{P}{N}\right)$ and ***A1*** for $\ln P$ and $kt\left(+C\right)$. Absolute value signs are not required.

continued…

Question 12 continued

$$\ln\left(\frac{P}{1-\frac{P}{N}}\right) = kt + C \Rightarrow \ln\left(\frac{NP}{N-P}\right) = kt + C$$

attempts to find C in terms of N and P_0 ***M1***

when $t = 0$, $P = P_0$ and so $C = \ln\left(\frac{NP_0}{N-P_0}\right)$

$$kt = \ln\left(\frac{NP}{N-P}\right) - \ln\left(\frac{NP_0}{N-P_0}\right)\left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

$$kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 3

lets $u = \frac{1}{P}$ and forms $\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***M1***

multiplies both sides of the differential equation by $-\frac{1}{P^2}$ and makes the above substitutions ***M1***

$$-\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{1}{N} - \frac{1}{P}\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = k\left(\frac{1}{N} - u\right)$$

$\frac{\mathrm{d}u}{\mathrm{d}t} + ku = \frac{k}{N}$ (linear first-order DE) ***A1***

$$\mathrm{IF} = \mathrm{e}^{\int k\,\mathrm{d}t} = \mathrm{e}^{kt} \Rightarrow \mathrm{e}^{kt}\frac{\mathrm{d}u}{\mathrm{d}t} + k\mathrm{e}^{kt}u = \frac{k}{N}\mathrm{e}^{kt}$$ ***(M1)***

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\mathrm{e}^{kt}\right) = \frac{k}{N}\mathrm{e}^{kt}$$

$$u\mathrm{e}^{kt} = \frac{1}{N}\mathrm{e}^{kt}\left(+C\right) \quad \left(\frac{1}{P}\mathrm{e}^{kt} = \frac{1}{N}\mathrm{e}^{kt}\left(+C\right)\right)$$ ***A1***

attempts to find C in terms of N and P_0 ***M1***

when $t = 0$, $P = P_0$, $u = \frac{1}{P_0}$ and so $C = \frac{1}{P_0} - \frac{1}{N}\left(= \frac{N - P_0}{NP_0}\right)$

$$\mathrm{e}^{kt}\left(\frac{N - P}{NP}\right) = \frac{N - P_0}{NP_0}$$

$$\mathrm{e}^{kt} = \left(\frac{P}{N - P}\right)\left(\frac{N - P_0}{P_0}\right)$$ ***A1***

$$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right)$$ ***AG***

[7 marks]

continued…

Question 12 continued

(f) substitutes $t = 10$, $P = 3P_0$ and $N = 4P_0$ into $kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right)$ ***M1***

$10k = \ln 3\left(\frac{4P_0 - P_0}{4P_0 - 3P_0}\right)\ \left(= \ln 9\right)$

$k = 0.220\ \left(= \frac{1}{10}\ln 9, = \frac{1}{5}\ln 3\right)$ ***A1***

[2 marks]

Total [21 marks]

2. (a) (i) $4-\mathrm{i}$ ***A1***

[1 mark]

(ii) $\text{mean} = \frac{1}{2}(4+\mathrm{i}+4-\mathrm{i})$ ***A1***

$= 4$ ***AG***

[1 mark]

(b) **METHOD 1**

attempts product rule differentiation ***(M1)***

Note: Award ***(M1)*** for attempting to express $f(x)$ as $f(x) = x^3 - 9x^2 + 25x - 17$.

$f'(x) = (x-1)(2x-8) + x^2 - 8x + 17 \ \left(f'(x) = 3x^2 - 18x + 25\right)$ ***A1***

$f'(4) = 1$ ***A1***

Note: Where $f'(x)$ is correct, award ***A1*** for solving $f'(x) = 1$ and obtaining $x = 4$.

EITHER

$y - 3 = 1(x-4)$ ***A1***

OR

$y = x + c$

$3 = 4 + c \Rightarrow c = -1$ ***A1***

OR

states the gradient of $y = x - 1$ is also 1 and verifies that $(4, 3)$ lies on the line $y = x - 1$ ***A1***

THEN

so $y = x - 1$ is the tangent to the curve at $\mathrm{A}(4, 3)$ ***AG***

Note: Award a maximum of ***(M0)A0A1A1*** to a candidate who does not attempt to find $f'(x)$.

continued…

Question 2 continued

METHOD 2

sets $f(x) = x - 1$ to form $x - 1 = (x - 1)(x^2 - 8x + 17)$ ***(M1)***

EITHER

$(x - 1)(x^2 - 8x + 16) = 0$ $(x^3 - 9x^2 + 24x - 16 = 0)$ ***A1***

attempts to solve a correct cubic equation ***(M1)***

$(x - 1)(x - 4)^2 = 0 \Rightarrow x = 1, 4$

OR

recognises that $x \neq 1$ and forms $x^2 - 8x + 17 = 1$ $(x^2 - 8x + 16 = 0)$ ***A1***

attempts to solve a correct quadratic equation ***(M1)***

$(x - 4)^2 = 0 \Rightarrow x = 4$

THEN

$x = 4$ is a double root ***R1***

so $y = x - 1$ is the tangent to the curve at $\mathrm{A}(4, 3)$ ***AG***

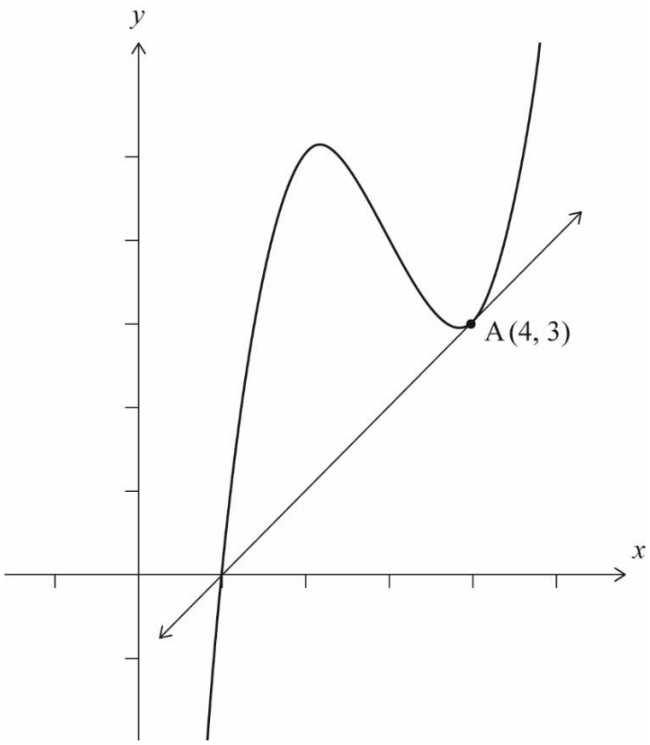
Note: Candidates using this method are not required to verify that $y = 3$.

[4 marks]

continued…

Question 2 continued

(c)



a positive cubic with an x- intercept $(x=1)$, and a local maximum and local minimum in the first quadrant both positioned to the left of A **A1**

Note: As the local minimum and point A are very close to each other, condone graphs that seem to show these points coinciding.
For the point of tangency, accept labels such as A, $(4,3)$ or the point labelled from both axes. Coordinates are not required.

a correct sketch of the tangent passing through A and crossing the x-axis at the same point ($x=1$) as the curve **A1**

Note: Award ***A1A0*** if both graphs cross the x- axis at distinctly different points.

[2 marks]

continued…

Question 2 continued

(d) (i) **EITHER**

$g'(x) = (x-r)(2x-2a) + x^2 - 2ax + a^2 + b^2$ ***(M1)A1***

OR

$g(x) = x^3 - (2a+r)x^2 + (a^2+b^2+2ar)x - (a^2+b^2)r$

attempts to find $g'(x)$ ***M1***

$g'(x) = 3x^2 - 2(2a+r)x + a^2 + b^2 + 2ar$

$= 2x^2 - 2(a+r)x + 2ar + x^2 - 2ax + a^2 + b^2$ ***A1***

$\left(= 2(x^2 - ax - rx + ar) + x^2 - 2ax + a^2 + b^2\right)$

THEN

$g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) **METHOD 1**

$g(a) = b^2(a-r)$ ***(A1)***

$g'(a) = b^2$ ***(A1)***

attempts to substitute their $g(a)$ and $g'(a)$ into $y - g(a) = g'(a)(x-a)$ ***M1***

$y - b^2(a-r) = b^2(x-a)$

EITHER

$y = b^2(x-r)$ $\left(y = b^2x - b^2r\right)$ ***A1***

sets $y = 0$ so $b^2(x-r) = 0$ ***M1***

$b > 0 \Rightarrow x = r$ OR $b \neq 0 \Rightarrow x = r$ ***R1***

OR

sets $y = 0$ so $-b^2(a-r) = b^2(x-a)$ ***M1***

$b > 0$ OR $b \neq 0$ $\Rightarrow -(a-r) = x-a$ ***R1***

$x = r$ ***A1***

THEN

so the tangent intersects the x- axis at the point $\mathrm{R}(r, 0)$ ***AG***

continued…

Question 2 continued

METHOD 2

$g'(a) = b^2$ ***(A1)***

$g(a) = b^2(a - r)$ ***(A1)***

attempts to substitute their $g(a)$ and $g'(a)$ into $y = g'(a)x + c$ and attempts to find c ***M1***

$c = -b^2 r$

EITHER

$y = b^2(x - r)$ $\left(y = b^2 x - b^2 r\right)$ ***A1***

sets $y = 0$ so $b^2(x - r) = 0$ ***M1***

$b > 0 \Rightarrow x = r$ OR $b \neq 0 \Rightarrow x = r$ ***R1***

OR

sets $y = 0$ so $b^2(x - r) = 0$ ***M1***

$b > 0$ OR $b \neq 0$ $\Rightarrow x - r = 0$ ***R1***

$x = r$ ***A1***

METHOD 3

$g'(a) = b^2$ ***(A1)***

the line through $\mathrm{R}(r, 0)$ parallel to the tangent at A has equation

$y = b^2(x - r)$ ***A1***

sets $g(x) = b^2(x - r)$ to form $b^2(x - r) = (x - r)\left(x^2 - 2ax + a^2 + b^2\right)$ ***M1***

$b^2 = x^2 - 2ax + a^2 + b^2$, $(x \neq r)$ ***A1***

$(x - a)^2 = 0$ ***A1***

since there is a double root ($x = a$), this parallel line through

$\mathrm{R}(r, 0)$ is the required tangent at A ***R1***

[6 marks]

continued…

Question 2 continued

(e) **EITHER**

$g'(a) = b^2 \Rightarrow b = \sqrt{g'(a)}$ (since $b > 0$) ***R1***

Note: Accept $b = \pm\sqrt{g'(a)}$.

OR

$(a \pm b\text{i} =) a \pm \text{i}\sqrt{b^2}$ and $g'(a) = b^2$ ***R1***

THEN

hence the complex roots can be expressed as $a \pm \text{i}\sqrt{g'(a)}$ ***AG***

[1 mark]

(f) (i) $b = 4$ (seen anywhere) ***A1***

EITHER

attempts to find the gradient of the tangent in terms of a and equates to 16 ***(M1)***

OR

substitutes $r = -2, x = a$ and $y = 80$ to form $80 = (a - (-2))(a^2 - 2a^2 + a^2 + 16)$ ***(M1)***

OR

substitutes $r = -2, x = a$ and $y = 80$ into $y = 16(x - r)$ ***(M1)***

THEN

$\frac{80}{a+2} = 16 \Rightarrow a = 3$

roots are -2 (seen anywhere) and $3 \pm 4\text{i}$ ***A1A1***

Note: Award ***A1*** for -2 and ***A1*** for $3 \pm 4\text{i}$. Do not accept coordinates.

[4 marks]

(ii) $(3, -4)$ ***A1***

Note: Accept "$x = 3$ and $y = -4$".
Do not award ***A1FT*** for $(a, -4)$.

[1 mark]

continued…

Question 2 continued

(g) (i) $g'(x) = 2(x-r)(x-a) + x^2 - 2ax + a^2 + b^2$

attempts to find $g''(x)$ ***M1***

$g''(x) = 2(x-a) + 2(x-r) + 2x - 2a \ (= 6x - 2r - 4a)$

sets $g''(x) = 0$ and correctly solves for x ***A1***

for example, obtaining $x - r + 2(x-a) = 0$ leading to $3x = 2a + r$

so $x = \frac{1}{3}(2a + r)$ ***AG***

Note: Do not award ***A1*** if the answer does not lead to the ***AG***.

[2 marks]

(ii) point P is $\frac{2}{3}$ of the horizontal distance (way) from point R to point A ***A1***

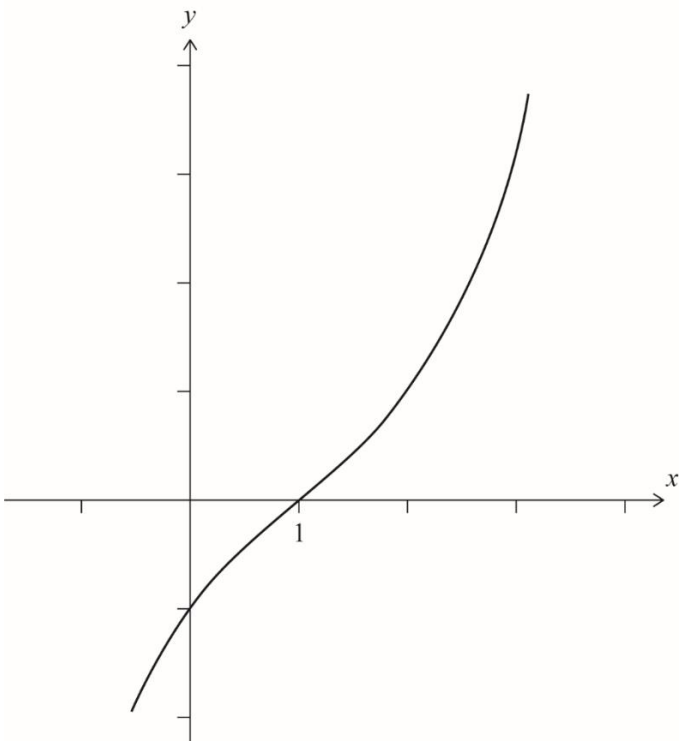
Note: Accept equivalent numerical statements or a clearly labelled diagram displaying the numerical relationship.
Award ***A0*** for non-numerical statements such as "P is between R and A, closer to A".

[1 mark]

continued…

Question 2 continued

(h) (i) $y = (x-1)(x^2-2x+5)$ ***(A1)***



a positive cubic with no stationary points and a non-stationary point of inflexion at $x = 1$ ***A1***

Note: Graphs may appear approximately linear. Award this ***A1*** if a change of concavity either side of $x = 1$ is apparent. Coordinates are not required and the y- intercept need not be indicated.

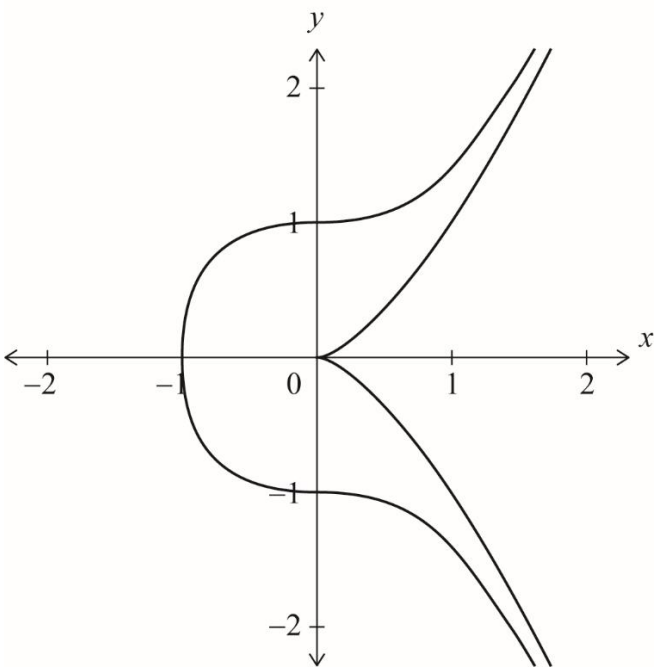
[2 marks]

(ii) $(r, 0)$ ***A1***

[1 mark]

Total [28 marks]

1. (a) (i)



approximately symmetric about the x-axis graph of $y^2 = x^3$ ***A1***

including cusp/sharp point at (0, 0) ***A1***

[2 marks]

(ii) approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts ***A1***

some indication of position of intersections at $x = -1,\ y = \pm 1$ ***A1***

[2 marks]

Note: Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct. Award ***A1A0A0A0*** if only one correct branch of both curves are seen.
Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0.***

continued…

Question 1 continued

(b) (i) $(0, 1)$ and $(0, -1)$ ***A1***

[1 mark]

(ii) Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different x-axis intercepts (one goes through the origin, and the other does not)

graphs have different y-axis intercepts ***A1***

Note: Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent.

[1 mark]

continued…

Question 1 continued

(c) Any **two** from:

as $x \to \infty$, $y \to \pm\infty$

as $x \to \infty$, $y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have x intercepts at $x = -\sqrt[3]{b}$

they have y intercepts at $y = (\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or x-axis) is a line of symmetry

they all have the same line of symmetry $(y = 0)$

they have one x-axis intercept

they have two y-axis intercepts

they have two points of inflexion

at x-axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at x-axis intercepts ***A1A1***

Note: The last example is the only valid answer for things "not" present. Do not credit an answer of "they are all symmetrical" without some reference to the line of symmetry.
Note: Do not allow same/ similar shape or equivalent.
Note: In accordance with marking rules, mark their first two responses and ignore any subsequent.

[2 marks]

continued…

Question 1 continued

(d) (i) **METHOD 1**

attempt to differentiate implicitly ***M1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+1}{2y}$ OR $(\pm)2\sqrt{x^3+x}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2+1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

METHOD 2

attempt to use chain rule $y = (\pm)\sqrt{x^3+x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+1)$ ***A1A1***

Note: Award ***A1*** for $(\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}$, ***A1*** for $(3x^2+1)$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

[3 marks]

(ii) **EITHER**

local minima/maxima occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$1+3x^2 = 0$ has no (real) solutions (or equivalent) ***R1***

OR

$\left(x^2 \geq 0 \Rightarrow\right)\ 3x^2+1 > 0$, so $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ ***R1***

THEN

so, no local minima/maxima exist ***AG***

[1 mark]

continued…

Question 1 continued

(e) **EITHER**

attempt to use quotient rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\pm)\frac{12x\sqrt{x+x^3} - (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)}{4(x+x^3)}$ ***A1A1***

Note: Award ***A1*** for correct $12x\sqrt{x+x^3}$ and correct denominator, ***A1*** for correct $-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$.

Note: Future ***A*** marks may be awarded if the denominator is missing or incorrect.

stating or using $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ (may be seen anywhere) ***(M1)***

$12x\sqrt{x+x^3} = (1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$

OR

attempt to use product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2}\left(3x^2+1\right)\left(-\frac{1}{2}\right)\left(3x^2+1\right)\left(x^3+x\right)^{-\frac{3}{2}} + 3x\left(x^3+x\right)^{-\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct first term, ***A1*** for correct second term.

setting $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ ***(M1)***

continued…

Question 1 continued

OR

attempts implicit differentiation on $2y\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+1$ ***M1***

$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x$ ***A1***

recognizes that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{3x}$

$(\pm)\frac{3x^2+1}{2\sqrt{x^3+x}}=(\pm)\sqrt{3x}$ ***(A1)***

THEN

$12x(x+x^3)=(1+3x^2)^2$

$12x^2+12x^4=9x^4+6x^2+1$

$3x^4+6x^2-1=0$ ***A1***

attempt to use quadratic formula or equivalent ***(M1)***

$x^2=\frac{-6\pm\sqrt{48}}{6}$

$(x>0\Rightarrow)\,x=\sqrt{\frac{2\sqrt{3}-3}{3}}\ \ (p=2,\ q=-3,\ r=3)$ ***A1***

Note: Accept any integer multiple of p, q and r (e.g. $4, -6$ and 6).

[7 marks]

continued…

Question 1 continued

(f) (i) attempt to find tangent line through $(-1,-1)$ ***(M1)***

$y+1=-\frac{3}{2}(x+1)$ OR $y=-1.5x-2.5$ ***A1***

[2 marks]

(ii) attempt to solve simultaneously with $y^2=x^3+2$ ***(M1)***

Note: The ***M1*** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. (4.25, -8.875)).

obtain $\left(\frac{17}{4},-\frac{71}{8}\right)$ ***A1***

[2 marks]

continued…

Question 1 continued

(g) attempt to find equation of $[\mathrm{QS}]$ ***(M1)***

$\frac{y-1}{x+1} = -\frac{79}{42} \left(= -1.88095\ldots\right)$ ***(A1)***

solve simultaneously with $y^2 = x^3 + 2$ ***(M1)***

$x = 0.28798\ldots \left(= \frac{127}{441}\right)$ ***A1***

$y = -1.4226\ldots \left(= \frac{13175}{9261}\right)$ ***A1***

$(0.288, -1.42)$

OR

attempt to find vector equation of $[\mathrm{QS}]$ ***(M1)***

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{21}{4} \\ -\frac{79}{8} \end{pmatrix}$ ***(A1)***

$x = -1 + \frac{21}{4}\lambda$

$y = 1 - \frac{79}{8}\lambda$

attempt to solve $\left(1 - \frac{79}{8}\lambda\right)^2 = \left(-1 + \frac{21}{4}\lambda\right)^3 + 2$ ***(M1)***

$\lambda = 0.2453\ldots$

$x = 0.28798\ldots \left(= \frac{127}{441}\right)$ ***A1***

$y = -1.4226\ldots \left(= \frac{13175}{9261}\right)$ ***A1***

$(0.288, -1.42)$

[5 marks]

[Total 28 marks]

1. $g'(x) = 2x\mathrm{e}^{x^2+1}$ ***(A2)***

substitute $x = -1$ into **their** derivative ***(M1)***

$g'(-1) = -2\mathrm{e}^2$ ***A1***

Note: Award ***A0M0A0*** in cases where candidate's incorrect derivative is $g'(x) = \mathrm{e}^{x^2+1}$.

[4 marks]

4. recognizing need to integrate ***(M1)***

$\int \frac{6x}{x^2+1}\mathrm{d}x$ OR $u = x^2+1$ OR $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\int \frac{3}{u}\mathrm{d}u$ OR $3\int \frac{2x}{x^2+1}\mathrm{d}x$ ***(A1)***

$= 3\ln\left(x^2+1\right)(+c)$ or $3\ln u\,(+c)$ ***A1***

correct substitution of $x = 1$ and $f(x) = 5$ or $x = 1$ and $u = 2$ into equation

using **their** integrated expression (must involve *c*) ***(M1)***

$5 = 3\ln 2 + c$

$f(x) = 3\ln\left(x^2+1\right) + 5 - 3\ln 2$ $\left(= 3\ln\left(x^2+1\right) + 5 - \ln 8 = 3\ln\left(\frac{x^2+1}{2}\right) + 5\right)$ (or equivalent) ***A1***

Note: Accept the use of the modulus sign in working and the final answer.

[5 marks]

7. attempt at implicit differentiation, including use of the product rule ***(M1)***

EITHER

$\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2+\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}=8x$ ***A1A1A1***

Note: Award ***A1*** for each of $\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2$, $\left(x^2+y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

OR

$x^2y^2+y^4=4x^2$

$2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}+4y^3\frac{\mathrm{d}y}{\mathrm{d}x}=8x$ ***A1A1A1***

Note: Award ***A1*** for each of $2xy^2+2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

THEN

at a local maximum or minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***(M1)***

$2xy^2=8x$

$x=0$ or $y^2=4\left(\Rightarrow y=\pm2\right)$ ***A1***

Note: Award ***A0*** for $x=0$ or $y=2$

since $x>0$ and $-2<y<2$ there are no solutions ***R1***

hence there are no local maximum or minimum points ***AG***

[7 marks]

9. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{xy}$

let $y = vx$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2 - 2x^2}{vx^2}$ ***(M1)***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-2}{v}$ (***A1)***

$\Rightarrow \int v\mathrm{d}v = -\int \frac{2}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \frac{v^2}{2} = -2\ln|x|(+c)$ ***A1***

Note: Condone the absence of the modulus sign up to this point.

$\Rightarrow \frac{y^2}{2x^2} = -2\ln|x| + c$ ***A1***

attempt to substitute $x = 1, y = 2$ into their integrated expression to find c ***M1***

$\Rightarrow 2 = -2\ln|1| + c \Rightarrow c = 2$

$\Rightarrow \frac{y^2}{2x^2} = -2\ln|x| + 2$

$\Rightarrow y^2 = 2x^2\left(-2\ln|x| + 2\right)\left(= 4x^2\left(1 - \ln|x|\right)\right)$ ***A1***

[8 marks]

continued…

Question 9 continued

(b) attempt to set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ in the differential equation ***(M1)***

$y = \sqrt{2}x$ and $y = -\sqrt{2}x$ or $m = \pm\sqrt{2}$ ***A1***

[2 marks]

Total [10 marks]

Section B

2. (a) $(0.708519..., 0.639580...)$

$(0.709, 0.640)$ $(x = 0.709,\ y = 0.640)$ ***A1A1***

[2 marks]

(b) $1.09885...$

$x = 1.10$ (accept $(1.10, 0)$) ***A1***

[1 mark]

(c) **METHOD 1**

$\int_0^2 |f(x)|\,dx$ ***(A1)***

$4.61117...$

area = 4.61 ***A2***

METHOD 2

$-\int_{1.09885...}^{2} f(x)\,dx$ OR $\int_{1.09885...}^{2} |f(x)|\,dx$ OR $4.17527\ldots$ ***(A1)***

$\int_0^{1.09885...} f(x)\,dx - \int_{1.09885...}^{2} f(x)\,dx$ OR $0.435901\ldots + 4.17527\ldots$ ***(A1)***

$4.61117...$

area = 4.61 ***A1***

[3 marks]

Total [6 marks]

8. EITHER

$\left(\frac{\mathrm{d}V}{\mathrm{d}h} =\right) 10\pi h - \pi h^2$ ***(A1)***

Note: This ***A1*** may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h} = 76.5616...$.

attempt to use chain rule to find a relationship between $\frac{\mathrm{d}h}{\mathrm{d}t}$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ ***(M1)***

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} \left(= \frac{1}{\left(\frac{\mathrm{d}V}{\mathrm{d}h}\right)} \times \frac{\mathrm{d}V}{\mathrm{d}t} \right)$

OR

attempt to differentiate $V = 5\pi h^2 - \frac{1}{3}\pi h^3$ throughout with respect to t ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t} = 10\pi h \frac{\mathrm{d}h}{\mathrm{d}t} - \pi h^2 \frac{\mathrm{d}h}{\mathrm{d}t}$ ***(A1)***

continued…

Question 8 continued

THEN

$\left(10\pi h - \pi h^2\right)\frac{\mathrm{d}h}{\mathrm{d}t} = 2$ OR $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{2}{10\pi h - \pi h^2}$ ***(A1)***

Note: Award this ***A1*** if the correct expression is seen with their h already substituted.

attempt to solve $200 = 5\pi h^2 - \frac{1}{3}\pi h^3$ ***(M1)***

$h = 4.20648...$ ***(A1)***

Note: This ***(M1)(A1)*** can be awarded independently of all previous marks, and may be implied by the value $\frac{\mathrm{d}V}{\mathrm{d}h} = 76.5616...$
Ignore extra values of h -3.24 and 14.0.

$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.0261227...$

$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.0261\left(\mathrm{cms}^{-1}\right)$ ***A1***

[6 marks]

11. (a) attempt to use product rule ***(M1)***

$f'(x) = 3e^{2x} + 2e^{2x}(3x-4)\left(= e^{2x}(6x-5)\right)$ ***A2***

Note: Award ***A1*** for 2 out of 3 of $3e^{2x}$, $6xe^{2x}$ and $-8e^{2x}$ seen or implied.

[3 marks]

(b) $f'(x) = 1$ ***(M1)***

$x = 0.86299...$

$x = 0.863$ ***A1***

$y = -7.92719...$

$y = -7.93$ ***A1***

$(0.863, -7.93)$

[3 marks]

(c) x-intercept is at $\frac{4}{3}(1.33)$ ***(A1)***

attempt to use formula for volume of revolution ***(M1)***

Note: Award ***(M1)*** for an integral involving π and $(f(x))^2$. Condone use of 2π and incorrect or absent limits.

$\pi\int_0^{\frac{4}{3}} \left(e^{2x}(3x-4)\right)^2 dx$ ***(A1)***

Note: This ***(A1)*** can be awarded if the dx is omitted.

$= 164.849...$

$= 165$ ***A1***

[4 marks]

continued…

Question 11 continued

(d) (i) attempt to compose functions in the correct order ***(M1)***

$(f \circ g)(0) = f(g(0)) = f(1)$

$= -7.38905...$

$= -7.39 (= -e^2)$ ***A1***

(ii) attempt to use the chain rule ***(M1)***

$(f \circ g)'(0) = f'(g(0))g'(0)$

Note: For this ***(M1)*** to be awarded, multiplication of two derivatives should be seen or implied.

$= 2f'(1) (= 2 \times 7.38905...)$ ***(A1)***

$= 14.7781...$

$= 14.8 (= 2e^2)$ ***A1***

[5 marks]

Total [15 marks]

1. (a) **EITHER**

$S_n = \frac{n}{2}\left(2\times 1+\left(n-1\right)\times 1\right)$ ***A1***

OR

$u_1 = 1$ and either $u_n = n$ or $d = 1$ stated explicitly ***A1***

OR

$1+2+...+n$ (or equivalent) stated explicitly ***A1***

THEN

$S_n = \frac{n}{2}\left(1+n\right)$ ***AG***

Note: Award ***A0*** for a numerical verification.

[1 mark]

(b) (i) 14 ***A1***

[1 mark]

(ii) $a_1 + a_2 + a_3 = 1$ ***A1***

$2a_1 + 4a_2 + 8a_3 = 5$ ***A1***

$3a_1 + 9a_2 + 27a_3 = 14$ ***A1***

Note: For the third ***A*** mark, award ***A1FT*** for $3a_1 + 9a_2 + 27a_3 = p$ where p is their answer to part (b) (i).

[3 marks]

continued...

Question 1 continued

(iii) $a_1 = \frac{1}{6}\ (= 0.166666... \approx 0.167)$, $a_2 = \frac{1}{2}\ (= 0.5)$,

$a_3 = \frac{1}{3}\ (= 0.333333... \approx 0.333)$ ***A2***

Note: Award ***A1*** if only two of a_1, a_2, a_3 are correct.
Only award ***FT*** if three linear equations, each in a_1, a_2 and a_3 are stated in part (b) (ii) or (iii).
Award ***A2FT*** for their a_1, a_2 and a_3.
Award ***A1FT*** for their a_1, a_2 and $a_3 = 0$.

[2 marks]

(c) $f'(x) = 1 + 2x + 3x^2 + ... + nx^{n-1}$ ***A1***

Note: Award ***A1*** for $f'(x) = \sum_{i=1}^{n} ix^{i-1}$.

$\Rightarrow x f'(x) = x + 2x^2 + 3x^3 + ... + nx^n$ ***AG***

[1 mark]

continued...

Question 1 continued

(d) (i) **METHOD 1**

$f_2(x) = xf_1'(x)$

$f_1'(x) = 1^2 + 2^2 x + \left(3^2 x^2\right) + \ldots + n^2 x^{n-1} \ \left(= 1 + 4x + \left(9x^2\right) + \ldots + n^2 x^{n-1}\right)$ ***A1***

Note: Award ***A1*** for

$xf_1'(x) = x\left(1^2 + 2^2 x + \left(3^2 x^2\right) + \ldots + n^2 x^{n-1}\right) \ \left(= x\left(1 + 4x + \left(9x^2\right) + \ldots + n^2 x^{n-1}\right)\right)$

.

$xf_1'(x) = 1^2 x + 2^2 x^2 + \left(3^2 x^3\right) + \ldots + n^2 x^n \ \left(= x + 4x^2 + \left(9x^3\right) + \ldots + n^2 x^n\right)$ ***A1***

Note: Award ***A1*** for $f_1'(x) = \sum_{i=1}^{n} i^2 x^{i-1}$ and ***A1*** for $xf_1'(x) = x\sum_{i=1}^{n} i^2 x^{i-1}$.

The second ***A1*** is dependent on the first ***A1***.

Award a maximum of ***A0A1*** if a general term is not considered.

$= \sum_{i=1}^{n} i^2 x^i$ ***AG***

METHOD 2

$f_2(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(xf'(x)\right)$

$= x\left(f'(x) + xf''(x)\right) \ \left(= xf'(x) + x^2 f''(x)\right)$ ***A1***

$= x\sum_{i=1}^{n} i x^{i-1} + x^2 \sum_{i=1}^{n} i(i-1)x^{i-2}$

$= \sum_{i=1}^{n} i x^i + \sum_{i=1}^{n} i(i-1)x^i \ \left(= \sum_{i=1}^{n}\left(i + i^2 - i\right)x^i\right)$ ***A1***

$= \sum_{i=1}^{n} i^2 x^i$ ***AG***

[2 marks]

continued...

Question 1 continued

(ii) consider $q = 1$

$f_1(x) = x + 2x^2 + ... + nx^n$ (reference to part (c)) and $f_1(x) = \sum_{i=1}^{n} ix^i$ ***R1***

assume true for $q = k$, ($f_k(x) = \sum_{i=1}^{n} i^k x^i$) ***M1***

Note: Do not award ***M1*** for statements such as "let $q = k$" or "$q = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $q = k + 1$

$f_{k+1}(x) = xf_k'(x)$ ***M1***

$= x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ **OR** $x(1 + 2^{k+1}x + 3^{k+1}x^2 + ... + n^{k+1}x^{n-1})$ ***A1***

Note: Award the above ***M1*** if $f_{k+1}(x) = x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ or $xf_k'(x) = x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ (or equivalent) is stated.

$= \sum_{i=1}^{n} i^{k+1}x^i$ **OR** $x + 2^{k+1}x^2 + 3^{k+1}x^3 + ... + n^{k+1}x^n$ ***A1***

since true for $q = 1$ and true for $q = k + 1$ if true for $q = k$, hence true for all q $(\in \mathbb{Z}^+)$ ***R1***

Note: To obtain the final ***R1***, three of the previous five marks must have been awarded.

[6 marks]

(iii) $f_q(1) = 1^q + 2^q + 3^q + ... + n^q$

$= \sum_{i=1}^{n} i^q \left(= \sum_{i=1}^{n} 1^i i^q \right)$ ***A1***

[1 mark]

continued...

Question 1 continued

(e) uses $S_n = \frac{u_1\left(r^n - 1\right)}{r-1}$ with $r = x$ and $u_1 = 1$ ***M1***

clear indication there are $(n+1)$ terms ***R1***

$f(x) = \frac{x^{n+1} - 1}{x - 1}$ ***AG***

[2 marks]

(f) **METHOD 1**

$f_1(x) = xf'(x)$

$= x\frac{(x-1)(n+1)x^n - 1 \times \left(x^{n+1} - 1\right)}{(x-1)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to use the quotient or the product rule to find $f'(x)$.

$= x\frac{(nx + x - n - 1)x^n - \left(x^{n+1} - 1\right)}{(x-1)^2}\left(= x\frac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2}\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$= \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$ ***AG***

METHOD 2

attempts to form $(x-1)f_1(x)$ ***M1***

$(x-1)f_1(x) = nx^{n+1} - \left(x + x^2 + x^3 + \ldots + x^n\right)$

$f_1(x) = \frac{1}{x-1}\left(nx^{n+1} - \left(\frac{x^{n+1} - 1}{x-1} - 1\right)\right)$ ***A1***

$f_1(x) = \frac{1}{x-1}\left(\frac{nx^{n+1}(x-1) - x^{n+1} + x}{x-1}\right)\left(= \frac{1}{x-1}\left(\frac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{x-1}\right)\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$$f_1(x) = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$ ***AG***

[3 marks]

(g) (i) $\lim_{x \to 1} f_1(x) = \frac{n - (n+1) + 1}{0} \left(= \frac{0}{0} \right)$ ***R1***

Note: Only award ***R1*** for sufficient simplification of the numerator, for example, as shown above.
Do not award ***R1*** if $\lim_{x \to 1}$ is not referred to or stated.

[1 mark]

continued...

Question 1 continued

(ii) attempts to differentiate both the numerator and the denominator ***M1***

$$\lim_{x\to 1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}$$ ***A1***

Note: Award ***A1*** for $\left(\left(\lim_{x\to 1}\right)\frac{n(n+2)x^{n+1}-n(n+1)x^n-(n+1)x^n+1}{2(x-1)}\right)$. This form can be used in subsequent work.

(l'Hôpital's rule applies again since)

$$\lim_{x\to 1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}=\frac{0}{0}$$ ***R1***

Note: Do not award ***R1*** if $\lim_{x\to 1}$ is not referred to or stated.
Subsequent marks are independent of this ***R*** mark.

attempts to differentiate both the numerator and the denominator ***M1***

$$\lim_{x\to 1}\frac{n(n+2)(n+1)x^n-n(n+1)^2x^{n-1}}{2}$$

$$=\frac{n(n+2)(n+1)-n(n+1)^2}{2}\quad\left(=\frac{n^3+3n^2+2n-\left(n^3+2n^2+n\right)}{2}\right)$$

$$=\frac{n(n+1)\left((n+2)-(n+1)\right)}{2}\quad\left(=\frac{n^2+n}{2}\right)$$ ***A1***

$$=\frac{1}{2}n(n+1)$$ ***AG***

[5 marks]

Total [28 marks]

2. (a) **EITHER**

$A = 2\pi\int_0^3 2x\sqrt{1+2^2}\ \mathrm{d}x \left(= 4\sqrt{5}\pi\int_0^3 x\,\mathrm{d}x\right)$ ***(A1)***

$= 2\pi\sqrt{5}\ \left[x^2\right]_0^3\ \left(= 2\pi\sqrt{5}\left(3^2 - 0^2\right)\right)$ ***A1***

OR

$A = 2\pi m\sqrt{1+m^2}\ \left[\frac{x^2}{2}\right]_0^h\ \left(= 2\pi m\sqrt{1+m^2}\left(\frac{h^2}{2}\right)\right)$ ***(A1)***

$= 2\pi(2)\sqrt{5}\ \left[\frac{x^2}{2}\right]_0^3\ \left(= 2\pi(2)\sqrt{5}\left(\frac{3^2}{2}\right)\right)$ ***A1***

THEN

$= 18\sqrt{5}\,\pi$ ***AG***

[2 marks]

(b) (i) $r = mh$ ***A1***

[1 mark]

(ii) $l = \sqrt{h^2 + r^2}$ ***(M1)***

$l = \sqrt{h^2 + h^2m^2}\ (= h\sqrt{1+m^2})$ ***A1***

[2 marks]

continued...

Question 2 continued

(iii) $A = 2\pi\int_0^h mx\sqrt{1+m^2}\ \mathrm{d}x$ ***(A1)***

$= 2\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$ ***(M1)***

Note: Award ***(M1)*** for $(c)\,\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$.

At least one of the above two lines needs to be seen.

$= \pi h^2 m\sqrt{1+m^2}\ \left(= \pi hm \times \sqrt{\left(h^2+h^2m^2\right)}\right)$ ***A1***

$= \pi rl$ ***AG***

Note: Award as above if $\frac{l}{h} = \sqrt{1+m^2}$ is used, for example.

[3 marks]

(c) **METHOD 1**

attempts to use the chain rule ***(M1)***

Note: Award ***(M1)*** for $\frac{\mathrm{d}y}{\mathrm{d}x} = (c)\,x\left(r^2-x^2\right)^{-\frac{1}{2}}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2-x^2\right)^{-\frac{1}{2}}\ (-2x)\ \left(= -x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***A1***

[2 marks]

continued...

Question 2 continued

(d) **EITHER**

attempts to substitute $y = \sqrt{r^2 - x^2}$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into A ***(M1)***

$$A = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \left(-x\left(r^2 - x^2\right)^{-\frac{1}{2}} \right)^2} \, \mathrm{d}x$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, \mathrm{d}x$$ ***A1***

OR

attempts to substitute y and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y into A ***(M1)***

$$A = 2\pi \int_{-r}^{r} y \sqrt{1 + \left(-\frac{x}{y} \right)^2} \, \mathrm{d}x$$

$$= 2\pi \int_{-r}^{r} y \sqrt{1 + \frac{x^2}{y^2}} \, \mathrm{d}x \left(= 2\pi \int_{-r}^{r} \sqrt{x^2 + y^2} \, \mathrm{d}x \right)$$ ***A1***

THEN

attempts to perform valid algebraic simplification to form a definite integral in terms of r only ***M1***

$$= 2\pi \int_{-r}^{r} r \, \mathrm{d}x$$

$$= 2\pi r \left[x \right]_{-r}^{r} \left(= 2\pi r \left(r - \left(-r \right) \right) \right)$$ ***A1***

$$= 4\pi r^2$$ ***AG***

Note: Award marks as above for $A = 4\pi \int_{0}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \left(-x\left(r^2 - x^2\right)^{-\frac{1}{2}} \right)^2} \, \mathrm{d}x$.

[4 marks]

continued...

Question 2 continued

(e) (i) **EITHER**

horizontal stretch ***A1***

factor $\frac{1}{k}$ ***A1***

OR

horizontal compression ***A1***

factor k (invariant line y- axis) ***A1***

Note: Award ***A1A1*** as above for correct alternative descriptions.

For example, dilation by a factor of $\frac{1}{k}$ from the y- axis.

[2 marks]

(ii) $\pm\frac{r}{k}$ ***A1***

Note: Award ***A0*** for $\frac{r}{k}$ only and ***A0*** for $-\frac{r}{k}$ only.

[1 mark]

continued...

Question 2 continued

(iii) **METHOD 1**

attempts to use the chain rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2 - (kx)^2\right)^{-\frac{1}{2}} \times \left(-k^2 2x\right)$ $\left(= -k^2 x\left(r^2 - (kx)^2\right)^{-\frac{1}{2}} \left(= \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}\right)\right)$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - k^2 x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{k^2 x}{y}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-k^2 x}{\sqrt{r^2 - k^2 x^2}}$ ***A1***

[2 marks]

continued...

Question 2 continued

(iv) **EITHER**

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2}\sqrt{1 + \frac{k^4x^2}{r^2 - k^2x^2}}\ \mathrm{d}x \qquad \textbf{(A1)}$$

Note: Award ***(A1)*** for the correct substitution of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of $r^2 - k^2x^2$ and then cancelling $r^2 - k^2x^2$ ***(M1)***

OR

$$A = 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \left(-\frac{k^2x}{y}\right)^2}\ \mathrm{d}x \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} y\sqrt{1 + \frac{k^4x^2}{y^2}}\ \mathrm{d}x\right) \qquad \textbf{(A1)}$$

Note: Award ***(A1)*** for the correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x}$.

attempts to simplify to find $p(x)$, eg. forming a common denominator of y^2 and cancelling y^2 ***(M1)***

THEN

$$= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 - k^2x^2 + k^4x^2}\ \mathrm{d}x \left(= 2\pi \int_{-\frac{r}{k}}^{\frac{r}{k}} \sqrt{r^2 + \left(k^4 - k^2\right)x^2}\mathrm{d}x\right) \qquad \textbf{A1A1}$$

Note: Award ***A1*** for correct limits (seen anywhere) and ***A1*** for $p\left(x\right)$ correct.
The above ***A1*** for correct limits is independent of the ***(M1)***.

[4 marks]

continued...

Question 2 continued

(v) $r = 6378\ (\text{km})$ ***(A1)***

$k = 1.00330...\left(= \dfrac{6378}{6357} = \dfrac{2126}{2119}\right)$ ***(A1)***

attempts to form a definite integral for surface area ***(M1)***

$$= 2\pi \int_{-6357}^{6357} \sqrt{6378^2 - \left(\frac{6378}{6357}\right)^2 x^2 + \left(\frac{6378}{6357}\right)^4 x^2}\ \mathrm{d}x$$

$$= 510064226.3...$$

$= 5.101 \times 10^8\ (\text{km}^2)$ ***A1***

Note: Award ***A0A1FT(M1)A0*** for using $r = 6357\ (\text{km})$ and $k = 0.996707...$ leading to an answer of $5.089 \times 10^8\ (\text{km}^2)$.

[4 marks]

Total [27 marks]

6. $A = \frac{1}{2}x^2 \sin\frac{\pi}{3}$ OR $A = \frac{1}{2}x^2 \sin 60^\circ$ OR triangle height $h = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}$ $\left(= \frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$= \frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(= \frac{\sqrt{3}}{4}x^2\right)$ ***A1***

Note: Award ***A1*** for $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\sqrt{3}}{4} \times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} \times \sin\frac{\pi}{3} \times 2x\frac{\mathrm{d}x}{\mathrm{d}t}$ ***(A1)***

$= \frac{2\sqrt{3}}{4} \times 5\sqrt{3} \times 4$

$\frac{\mathrm{d}A}{\mathrm{d}t} = 30\,(\mathrm{cm^2 s^{-1}})$ ***A1***

Note: Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

9. METHOD 1 (subtracting volumes)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2h}{2}-\frac{h^3}{24}$

recognition that the volume of the ring is $\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi R^2h$ where $R\neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi\left(r^2-\frac{h^2}{4}\right)h$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{r^2h}{2}-\frac{h^3}{24}\right)-\pi r^2h+\frac{\pi h^3}{4}=\pi$ OR $\frac{h^3}{4}-\frac{h^3}{12}=1$ (or equivalent)

OR

recognition that the volume of the ring is $\pi\int\left(\left(r^2-y^2\right)-\left(r^2-\frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$\frac{h^2}{4}y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{h^3}{8}-\frac{h^3}{24}$

correct equation ***(A1)***

$2\pi\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=\pi$ OR $2\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=1$ (or equivalent)

THEN

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $=\pi\int\left(r^2-y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi R^2h$ where $R\neq r$ ***(M1)***

$\pi\int\left(r^2-y^2\right)\mathrm{d}y+\pi\left(r^2-\frac{h^2}{4}\right)h\ \left(=\frac{4}{3}\pi r^3-\pi\right)$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{2r^3}{3}-\frac{r^2h}{2}+\frac{h^3}{24}\right)+\pi r^2h-\frac{\pi h^3}{4}=\frac{4}{3}\pi r^3-\pi$ OR $\frac{h^3}{12}-\frac{h^3}{4}=-1$ (or equivalent)

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

continued…

Question 9 continued

METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$2\pi\int x\sqrt{r^2-x^2}\,\mathrm{d}x$

correct limits *r* and $\sqrt{r^2-\frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$-\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression ***(M1)***

$-\frac{1}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$

correct equation ***(A1)***

$2\times\frac{-2\pi}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)=\pi$ OR $2\left(\frac{2\pi}{3}\times\frac{h^3}{8}\right)=\pi$

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

Section B

11. (a) $L = \text{AC} + \text{CB}$

$\frac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\alpha \left(\Rightarrow \text{AC} = \frac{\frac{3}{4}}{\cos\alpha} \Rightarrow \text{AC} = \frac{3}{4}\sec\alpha \right)$ ***A1***

$\frac{6}{\text{CB}} = \sin\alpha \left(\Rightarrow \text{CB} = \frac{6}{\sin\alpha} \Rightarrow \text{CB} = 6\text{cosec}\,\alpha \right)$ ***A1***

so $L = \frac{3}{4}\sec\alpha + 6\text{cosec}\,\alpha$ ***AG***

[2 marks]

(b) (i) $\frac{\text{d}L}{\text{d}\alpha} = \frac{3}{4}\sec\alpha\tan\alpha - 6\text{cosec}\,\alpha\cot\alpha$ ***A1***

(ii) attempt to write $\frac{\text{d}L}{\text{d}\alpha}$ in terms of $\sin\alpha, \cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\frac{\text{d}L}{\text{d}\alpha} = \frac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \frac{6\cos\alpha}{\sin^2\alpha}$ OR $\frac{\text{d}L}{\text{d}\alpha} = \frac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \frac{6}{\sin\alpha\tan\alpha} \left(= \frac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha} \right)$

$\frac{\text{d}L}{\text{d}\alpha} = 0 \Rightarrow \frac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\frac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

[5 marks]

continued…

Question 11 continued

(c) (i) attempt to use product rule (at least once) ***(M1)***

$$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$. Allow unsimplified correct answer.

$$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\alpha\tan^2\alpha + \frac{3}{4}\sec^3\alpha + 6\operatorname{cosec}\alpha\cot^2\alpha + 6\operatorname{cosec}^3\alpha\right)$$

(ii) attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha = \arctan 2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$$\sec\alpha = \sqrt{5} \text{ OR } \operatorname{cosec}\alpha = \frac{\sqrt{5}}{2} \text{ OR } \cot\alpha = \frac{1}{2} \text{ OR } \cos\alpha = \frac{1}{\sqrt{5}} \text{ OR } \sin\alpha = \frac{2}{\sqrt{5}}$$

Note: ***M1A1*** may be seen in part (d).

$$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3 \text{ (or equivalent)} \quad \textbf{A2}$$

$$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$$

Note: Award ***A1*** for only two or three correct terms.
Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5} \quad \textbf{AG}$$

[7 marks]

continued…

Question 11 continued

(d) (i) $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

(ii) $\left(L_{\min} =\right)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

(e) $\left(11.25 =\right)\frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1***.

[2 marks]

Total [19 marks]

4. $A = \int_0^c \frac{x}{x^2+2}\,\mathrm{d}x$

EITHER

attempts to integrate by inspection or substitution using $u = x^2 + 2$ or $u = x^2$ ***(M1)***

Note: If candidate simply states $u = x^2 + 2$ or $u = x^2$, but does not attempt to integrate, do not award the ***(M1)*** .

Note: If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k \ln u$ or $k \ln(u+2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln(u+2)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

Note: Award the ***(M1)*** only for expressions of the form $k \ln\left(x^2+2\right)$.

$\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln\left(c^2+2\right) - \ln 2\right)\left(= \ln 3\right)$ OR $\frac{1}{2}\ln\left(c^2+2\right) - \frac{1}{2}\ln 2\left(= \ln 3\right)$

continued…

Question 4 continued

correctly applies at least one log law to their expression ***(M1)***

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)(=\ln 3)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}(=\ln 3)$ OR $\ln\left(\frac{c^2+2}{2}\right)=\ln 9$

OR $\ln(c^2+2)-\ln 2=\ln 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}(=\ln 3)$ OR $\ln\frac{\sqrt{c^2+2}}{\sqrt{2}}(=\ln 3)$

Note: Condone the absence of $\ln 3$ up to this stage.

$\frac{c^2+2}{2}=9$ OR $\sqrt{\frac{c^2+2}{2}}=3$ ***A1***

$c^2=16$

$c=4$ ***A1***

Note: Award ***A0*** for $c=\pm 4$ as a final answer.

Total [6 marks]

10. (a) $y^2 = 9 - x^2$ OR $y = \pm\sqrt{9-x^2}$ ***A1***

(since $y > 0$) $\Rightarrow y = \sqrt{9-x^2}$ ***AG***

[1 mark]

(b) $b = 2y \left(= 2\sqrt{9-x^2}\right)$ or $h = x + 3$ ***(A1)***

attempts to substitute their base expression and height expression into $A = \frac{1}{2}bh$ ***(M1)***

$A = \sqrt{9-x^2}(x+3)$ (or equivalent) $\left(= \frac{2(x+3)\sqrt{9-x^2}}{2} = x\sqrt{9-x^2} + 3\sqrt{9-x^2}\right)$ ***A1***

[3 marks]

(c) **METHOD 1**

attempts to use the product rule to find $\frac{\mathrm{d}A}{\mathrm{d}x}$ ***(M1)***

attempts to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^2}$ ***(M1)***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) \sqrt{9-x^2} + (3+x)\left(\frac{1}{2}\right)\left(9-x^2\right)^{-\frac{1}{2}}(-2x) \left(= \sqrt{9-x^2} - \frac{x^2+3x}{\sqrt{9-x^2}}\right)$ ***A1***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right) \frac{9-x^2}{\sqrt{9-x^2}} - \frac{x^2+3x}{\sqrt{9-x^2}} \left(= \frac{9-x^2-(x^2+3x)}{\sqrt{9-x^2}}\right)$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

continued…

Question 10 continued

METHOD 2

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{\mathrm{d}A}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}x}$

attempts to find $\frac{\mathrm{d}A}{\mathrm{d}y}$ where $A=y(x+3)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y^2=9-x^2$ ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}y}=y\frac{\mathrm{d}x}{\mathrm{d}y}+x+3$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$ (or equivalent) ***A1***

substitutes their $\frac{\mathrm{d}A}{\mathrm{d}y}$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into $\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{\mathrm{d}A}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}x}=\left(y\left(-\frac{y}{x}\right)+x+3\right)\left(-\frac{x}{y}\right)$ (or equivalent)

$=\frac{9-x^2-x^2-3x}{\sqrt{9-x^2}}$ (or equivalent) ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

[4 marks]

continued…

Question 10 continued

(d) $\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \left(\frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}} = 0 \right)$ ***(M1)***

attempts to solve $9 - 3x - 2x^2 = 0$ (or equivalent) ***(M1)***

$-(2x - 3)(x + 3)(= 0)$ OR $x = \frac{3 \pm \sqrt{(-3)^2 - 4(-2)(9)}}{2(-2)}$ (or equivalent) ***(A1)***

$x = \frac{3}{2}$ ***A1***

Note: Award the above ***A1*** if $x = -3$ is also given.

substitutes their value of x into either $y = \sqrt{9 - x^2}$ or $y = -\sqrt{9 - x^2}$ ***(M1)***

Note: Do not award the above ***(M1)*** if $x \leq 0$.

$y = -\sqrt{9 - \left(\frac{3}{2}\right)^2}$

$= -\frac{\sqrt{27}}{2} \left(= -\frac{3\sqrt{3}}{2}, = -\sqrt{\frac{27}{4}}, = -\sqrt{6.75} \right)$ ***A1***

[6 marks]

Total [14 marks]

12. (a) let $t = \sqrt{x}$ ***M1***

$t^2 = x \Rightarrow 2t\ \mathrm{d}t = \mathrm{d}x$ (or equivalent) ***A1***

so $\int \cos\sqrt{x}\ \mathrm{d}x = 2\int t\cos t\ \mathrm{d}t$ ***A1***

attempts integration by parts ***(M1)***

$u = 2t,\ \mathrm{d}v = \cos t\ \mathrm{d}t,\ \mathrm{d}u = 2\ \mathrm{d}t,\ v = \sin t$

$2\int t\cos t\ \mathrm{d}t = 2t\sin t - 2\int \sin t\ \mathrm{d}t$ ***(A1)***

$= 2t\sin t + 2\cos t + C$ ***A1***

substitution of $t = \sqrt{x}\ \Rightarrow \int \cos\sqrt{x}\ \mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$ ***AG***

[6 marks]

(b) $x_{n+1} = \dfrac{\left(2\left(n+1\right)-1\right)^2\pi^2}{4}\ \left(= \dfrac{\left(2n+1\right)^2\pi^2}{4}\right)$ ***A1***

[1 mark]

continued…

Question 12 continued

(c) area of R_n is $\left|\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x\right|$ ***(M1)***

Note: Modulus may be seen at a later stage.

$$= \left|\left[2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}\right]_{\frac{(2n-1)^2\pi^2}{4}}^{\frac{(2n+1)^2\pi^2}{4}}\right| \quad \textbf{A1}$$

Note: Condone $+C$ at this stage.

attempts to substitute their limits into their integrated expression ***(M1)***

$$= 2\left|\frac{(2n+1)\pi}{2}\times\sin\frac{(2n+1)\pi}{2} + \cos\frac{(2n+1)\pi}{2} - \left(\frac{(2n-1)\pi}{2}\times\sin\frac{(2n-1)\pi}{2} + \cos\frac{(2n-1)\pi}{2}\right)\right|$$

A1

$$= 2\left|(-1)^n\frac{(2n+1)\pi}{2} - \left((-1)^{n+1}\frac{(2n-1)\pi}{2}\right)\right| \text{ (or equivalent)} \quad \textbf{A1}$$

$$= 2\left|(-1)^n\frac{(2n+1)\pi}{2} + (-1)^n\frac{(2n-1)\pi}{2}\right| \quad \textbf{A1}$$

$$= 2\left|(-1)^n\frac{4n\pi}{2}\right|$$

$$= 4n\pi \quad \textbf{A1}$$

Note: Award a maximum of (***M1)A1M1A1A1A0A0*** for only attempting to calculate $\int_{x_n}^{x_{n+1}} \cos\sqrt{x}\ \mathrm{d}x$, and not applying the modulus.

[7 marks]

continued…

Question 12 continued

(d) **EITHER**

attempts to find $(d =) R_{n+1} - R_n$ ***M1***

$(d =) 4(n+1)\pi - 4n\pi$

$= 4\pi$ ***A1***

Note: Award ***M0*** for consideration of special cases for example R_3 and R_2.
Accept $d = k\pi$.

which is a constant (common difference is 4π) ***R1***

OR

an arithmetic sequence is of the form $u_n = dn + c$ $(u_n = dn + u_1 - d)$ ***M1***

attempts to compare $u_n = dn + c$ $(u_n = dn + u_1 - d)$ and $R_n = 4n\pi$ ***M1***

$d = 4\pi$ and $c = 0$ $(u_1 - d = 0)$ ***A1***

Note: Accept $d = k\pi$.

THEN

so the areas of the regions form an arithmetic sequence ***AG***

[3 marks]

Total [17 marks]

12.

Note: Penalise only once for an answer not given to six significant figures in parts (a), c(ii) and (d)(ii).

(a) attempt to use Euler's method ***(M1)***

$$y_{n+1} = y_n + 0.03\left(\frac{x_n{}^2 y_n - y_n}{x_n{}^2 + 1}\right), y_1 = 3 + 0.03\left(\frac{0-3}{0+1}\right)$$

$y_1 = 2.91$ ***(A1)***

at least one **correct** further intermediate value given to at least 3 significant figures ***(A1)***

y_0	3
y_1	2.91
y_2	2.82285...
y_3	2.73877...
y_4	2.65793...
y_5	2.58046...

$y(0.15) \approx y_5 = 2.58046160...$

$= 2.58046$ ***A1***

Note: Award final ***A1*** for the correct answer seen as the last line in a table.
If the table goes beyond this value and the correct answer is not explicitly identified award maximum ***(M1)(A1)(A1)A0***

[4 marks]

continued…

Question 12 continued

(b) (i) $\frac{dy}{dx} = -3$ ***A1***

(ii) **METHOD 1**

attempt to use quotient (or product) rule on

$\frac{dy}{dx} = \frac{x^2y - y}{x^2 + 1}\left(= \left(x^2 + 1\right)^{-1}\left(x^2y - y\right)\right)$ ***(M1)***

attempt to use product rule and implicit differentiation on x^2y ***(M1)***

$\frac{d}{dx}\left(x^2y - y\right) = x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}$ (seen anywhere) ***A1***

$= 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$)

$$\frac{d^2y}{dx^2} = \frac{\left(x^2 + 1\right)\left(x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}\right) - \left(x^2y - y\right)(2x)}{\left(x^2 + 1\right)^2}$$

OR $\frac{d^2y}{dx^2} = \left(x^2 + 1\right)^{-1}\left(x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}\right) - 2x\left(x^2 + 1\right)^{-2}\left(x^2y - y\right)$ ***A1***

$\frac{d^2y}{dx^2} = 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$) ***AG***

METHOD 2

$\left(x^2 + 1\right)\frac{dy}{dx} = x^2y - y$ ***A1***

attempt to use product rule and implicit differentiation ***(M1)***

$2x\frac{dy}{dx} + \left(x^2 + 1\right)\frac{d^2y}{dx^2} = x^2\frac{dy}{dx} + (2x)y - \frac{dy}{dx}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$\frac{d^2y}{dx^2} = 3$ (when $x = 0, y = 3, \frac{dy}{dx} = -3$) ***AG***

[5 marks]

continued…

Question 12 continued

(c) (i) $3-3x+\frac{x^2}{2!}(3)+\frac{x^3}{3!}(9)+...\left(=3-3x+\frac{3}{2}x^2+\frac{3}{2}x^3+...\right)$ ***A1A1***

Note: Award ***A1*** for first three terms, ***A1*** for fourth term

(ii) $y(0.15)=2.58881125...$

$=2.58881$ ***A1***

[3 marks]

continued…

Question 12 continued

(d) (i) **EITHER**

attempt to separate variables ***M1***

$\int \frac{1}{y}\mathrm{d}y = \int \frac{x^2-1}{x^2+1}\mathrm{d}x$ OR $\int \frac{1}{y}\mathrm{d}y = \int \left(1-\frac{2}{x^2+1}\right)\mathrm{d}x$

$\ln y = x - 2\arctan x + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ or $\ln|y|$, ***A1*** for $x - 2\arctan x$.
Condone missing $+c$ at this stage.

OR

attempt to use integrating factor ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} - y\left(1-\frac{2}{x^2+1}\right) = 0$

IF $= \mathrm{e}^{\int -\left(1-\frac{2}{x^2+1}\right)\mathrm{d}x} = \mathrm{e}^{-x+2\arctan x}$ ***A1***

$\mathrm{e}^{-x+2\arctan x}\frac{\mathrm{d}y}{\mathrm{d}x} - y\mathrm{e}^{-x+2\arctan x}\left(1-\frac{2}{x^2+1}\right) = 0$

$\mathrm{e}^{-x+2\arctan x}y = A$ ***A1***

THEN

$\ln y = x - 2\arctan x + c$ OR $y = A\mathrm{e}^{x-2\arctan x}$

attempt to find c or A using $x = 0, y = 3$ ***(M1)***

$\ln 3 = 0 - 2\arctan 0 + c$ OR $3 = A\mathrm{e}^{0-2\arctan 0}$

$c = \ln 3$ OR $A = 3$ ***(A1)***

Note: This ***A1*** should not be awarded if a correct value of c or A is preceded by incorrect working.

$y = \mathrm{e}^{x-2\arctan x+\ln 3}\left(=3\mathrm{e}^{x-2\arctan x}\right)$ ***A1***

continued…

Question 12 continued

(ii) $y(0.15) = 2.58786288...$

$= 2.58786$ ***A1***

[7 marks]

(e) the graph of $y = f(x)$ is concave up OR $\frac{d^2y}{dx^2} > 0$ (for $0 \leq x < 1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

hence tangents used (in Euler's method) give an underestimate,

so the approximate value for y when $x = 1.5$ is less than the actual value. ***R1***

[2 marks]

Note: ***R1*** is dependent on ***A1***, as well as reference to tangents, in words or on a diagram.

Total [21 marks]

5. (a) recognize that acceleration is zero when $v'(t) = 0$ OR at a local maximum on the graph of v ***(M1)***

$t_1 = 0.394791...$

$t_1 = 0.395 \left(= \arctan\left(\frac{5}{12}\right)\right)$ (seconds) ***A1***

[2 marks]

(b) recognition that $v = 0$ ***(M1)***

sketch OR $t = 4.71238...$ OR $t = 10.9955...$

$t_2 = 10.9955...$

$t_2 = 11.0 \left(= \frac{7\pi}{2}\right)$ ***A1***

[2 marks]

(c) $\int_{t_1}^{t_2} |v| \, \mathrm{d}t$ OR $\int_{0.394791...}^{10.9955...} |v| \, \mathrm{d}t$ OR $\int_{0.394791...}^{4.71238...} v \, \mathrm{d}t + \int_{4.71238...}^{10.9955...} |v| \, \mathrm{d}t \left(= 6.53806... + 1.29313...\right)$

OR $\int_{0.394791...}^{4.71238...} v \, \mathrm{d}t - \int_{4.71238...}^{10.9955...} v \, \mathrm{d}t \left(= 6.53806... - \left(-1.29313...\right)\right)$ ***(A1)***

distance = $7.83118\ldots$

= 7.83 (m) ***A1***

[2 marks]

Total [6 marks]

7. attempts to express x in terms of $\tan y$ ***(M1)***

$x = \tan y + 2$ ***(A1)***

let V be the volume of the solid

correctly uses $V = \pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

Note: Award ***M0*** for $V = \pi\int_a^b \left(\arctan(x-2)\right)^2 dy$

$V = \pi\int_0^{\frac{\pi}{3}} \left(\tan y + 2\right)^2 \mathrm{d}y \left(= \pi \int_0^{1.0472...} \left(\tan y + 2\right)^2 \mathrm{d}y \right)$ ***(A1)***

$= 24.0213...$

$= 24.0 \ \left(= \pi\left(4\ln 2 + \pi + \sqrt{3}\right)\right)$ ***A1***

Note: GDC in degrees gives13.3

[5 marks]

12. (a) attempts to use $y_1 = y_0 + h \times f(x_0, y_0)$ ***(M1)***

$$y_1 = 2 + 0.1 \times \frac{1^2 + 3(2)^2}{2}$$

$$= 2.65$$ ***A1***

Note: Award ***(M1)A0*** for 2.35.

[2 marks]

(b) let $y = vx$ ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$ ***(A1)***

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x^2 + 3v^2x^2}{vx^2}$$ ***(M1)***

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1 + 2v^2}{v}$$ ***A1***

attempt to separate variables x and v ***M1***

$$\int \frac{v}{2v^2 + 1}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$$

$$\frac{1}{4}\ln(2v^2 + 1) = \ln x + C$$ ***A1***

Note: Condone the absence of C to this stage.

continued…

Question 12 continued

EITHER

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+C$$

when $x=1,\ y=2\Rightarrow C=\frac{1}{4}\ln 9$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+\frac{1}{4}\ln 9$$

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=\sqrt{3}x$$

OR

$$\ln\left(\frac{2y^2}{x^2+1}\right)=\ln\left(x^4\right)+\ln C$$

$$\frac{2y^2}{x^2}+1=Cx^4$$

when $x=1,\ y=2\Rightarrow C=9$ ***M1***

THEN

$$\frac{2y^2}{x^2}+1=9x^4$$ ***A1***

$$y=x\sqrt{\frac{9x^4-1}{2}}$$ ***AG***

[8 marks]

continued…

Question 12 continued

(c) $y = 2.71422...$

$y = 2.71$ ***A1***

[1 mark]

(d) **EITHER**

the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ is concave up (for $1 \leq x \leq 1.1$) ***A1***

OR

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$ (for $1 \leq x \leq 1.1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

THEN

hence the tangent drawn using Euler's method gives an underestimate of the true value, so the value of y when $x = 1.1$ is greater than the approximate value found in part (a) ***R1***

Note: Only award ***R1*** if there is reference to tangent (in words or in a diagram).

[2 marks]

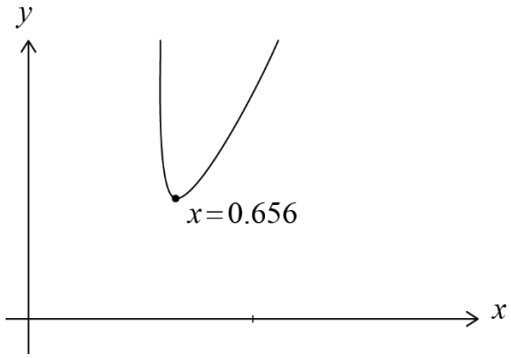
continued…

Question 12 continued

(e) **EITHER**

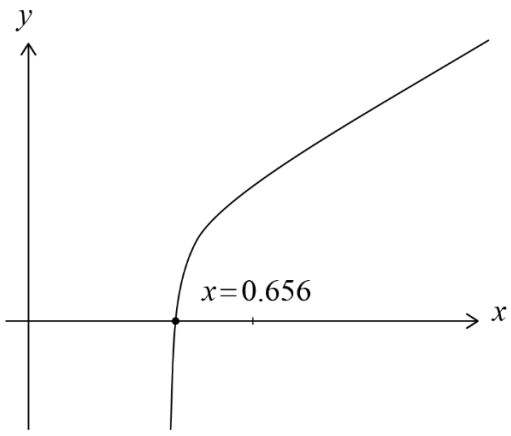
Note: Award the first ***A*** mark for a correct graph seen in part (d).

correct graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***



OR

correct graph of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***A1***



THEN

$x = 0.655996...$

$x = 0.656$ ***A1***

[2 marks]

continued…

Question 12 continued

(f) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ (seen anywhere) ***(A1)***

Note: Award ***(A1)*** for equivalent answers (seen anywhere) such as $\frac{-x^4+x^2y^2+6y^4}{x^2y^3}=0$ or $-x^4+x^2y^2+6y^4=0$.

EITHER

divides $-x^4+x^2y^2+6y^4(=0)$ through by y^4 ***(M1)***

$-\frac{x^4}{y^4}+\frac{x^2}{y^2}+6(=0)$

$m=\frac{y}{x}\Rightarrow-\frac{1}{m^4}+\frac{1}{m^2}+6(=0)$ ***(M1)***

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$ ***A1***

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

OR

divides $-x^4+x^2y^2+6y^4(=0)$ through by x^2y^2 ***(M1)***

$-\frac{x^2}{y^2}+1+6\frac{y^2}{x^2}(=0)$

$m=\frac{y}{x}\Rightarrow-\frac{1}{m^2}+1+6m^2(=0)$ ***(M1)***

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$ ***A1***

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

continued…

Question 12 continued

OR

attempts to factorize $-x^4 + x^2y^2 + 6y^4 (= 0)$ ***(M1)***

$-\left(x^2 - 3y^2\right)\left(x^2 + 2y^2\right)(= 0)$ ***A1***

attempts to solve their factorized equation ***(M1)***

$\Rightarrow y = \pm\frac{1}{\sqrt{3}}x \left(y^2 = -\frac{1}{2}x^2\right)$

THEN

$y = \frac{1}{\sqrt{3}}x$ (and so $m = \frac{1}{\sqrt{3}}$) ***A1***

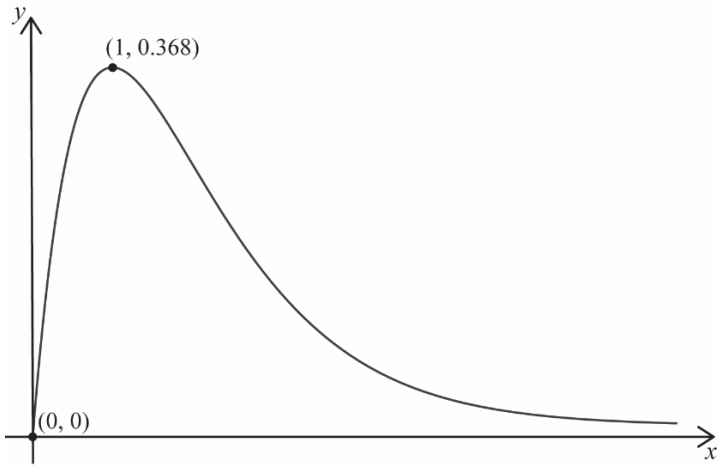
as $x > 0, y > 0$ (or equivalent reasoning/justification) ***R1***

Note: Award ***R1*** for $y = \frac{1}{\sqrt{3}}x$ (and so $m = \frac{1}{\sqrt{3}}$) as $y \neq -\frac{1}{\sqrt{3}}x$ and $x^2 + 2y^2 = 0$ for $x = 0$ and $y = 0$ only.

[6 marks]

Total [21 marks]

1. (a)



A1 for $(1, 0.368)$ or $\left(1, \frac{1}{e}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through,the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

continued…

Question 1 continued

(b) $\int_0^b x\mathrm{e}^{-x}\,\mathrm{d}x$ ***(A1)***

Note: Award (***A1)*** for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$=\left[-x\mathrm{e}^{-x}\right]_0^b+\int_0^b \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for each part (including the correct sign with each)

$=\left[-x\mathrm{e}^{-x}\right]_0^b-\left[\mathrm{e}^{-x}\right]_0^b$ ***A1***

Note: Award ***A1*** for correct second term.
Condone absence of limits to this point

attempt to substitute limits ***M1***

$=-b\mathrm{e}^{-b}-\mathrm{e}^{-b}+1$ ***A1***

$=\dfrac{\mathrm{e}^b-b-1}{\mathrm{e}^b}$ ***AG***

[6 marks]

(c) (i) $\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^b-b-1}{\mathrm{e}^b}=\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^b-1}{\mathrm{e}^b}$ ***A1***

Note: Award ***A1*** for correct quotient. Condone absence of limit.

$\left(=\lim\limits_{b\to\infty}\dfrac{\mathrm{e}^b}{\mathrm{e}^b}\right)=1$ ***A1***

[2 marks]

(ii) $\left(\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x=\right)\ 1$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) (i) correct integral ***(M1)***

Note: Award ***M1*** for correct integrand with limits from 0 to a larger number.

24 ***A1***

[2 marks]

(ii) 120 ***A1***

Note: The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) $A_n = n!$ ***A1***

[1 mark]

continued…

Question 1 continued

(f)

Note: Accept starting at $n = 0$, throughout this proof.

$n = 1$

$A_1 = 1$ and $1! = 1$ ***M1A1***

Note: Award ***M1*** for considering the case where $n = 1$, and ***A1*** if it is clear that both $A_1 = 1$ and $1! = 1$ have been considered.

so true for $n = 1$

assume true for $n = k$, ($A_k = \int_0^\infty x^k \mathrm{e}^{-x}\,\mathrm{d}x = k!$) ***M1***

Note: Award ***M0*** for statements such as "let $n = k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

when $n = k + 1$

attempt to integrate by parts ***M1***

Note: To obtain the ***M1***, a minimum of an expression +/- an integral must be seen.

$\int_0^\infty x^{k+1}\mathrm{e}^{-x}\,\mathrm{d}x = \left[-x^{k+1}\mathrm{e}^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1***

$(k+1)\int_0^\infty x^k \mathrm{e}^{-x}\,\mathrm{d}x$ simplified to $(k+1)\,k!$ seen ***A1***

$= 0 + (k+1)\,k!$

Note: Condone omission of the zero.

$= (k+1)!$ ***A1***

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$. ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[8 marks]

Total [25 marks]

2. (a) $x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b) (i) $(x_1 =)\ 6$ ***A1***

Note: Award the ***A1*** if 6 seen in part (ii).

[1 mark]

(ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at $(6, 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

(c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

continued…

Question 2 continued

(d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(ii) $x_1 + x_2 + ... + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times ... \times x_n)^{\frac{1}{n}} \leq \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times ... \times x_n \leq \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n(S) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) (i) $M_3(12) = 4^3 = 64$ ***A1***

[1 mark]

(ii) $M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(iii) $M_5(12) = 2.4^5 = 79.6 \ \left(79.6262...\right)$ ***A1***

[1 mark]

continued…

Question 2 continued

(f) considering $M_n(12)$ for higher values of n

$P(12) = 81$ ***A1***

$n = 4$
A1

Note: Award ***A0A0*** for $P(12) = 82.6$ and $n = 4.41$.

[2 marks]

(g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n = 7$ ***A1***

$P(20) = \left(\frac{20}{7}\right)^7 = 1550 \ (1554.260...)$ ***A1***

Note: Award ***(M1)A0A1*** for $n = 7.36$ and $P(20) = 1570$.

[3 marks]

continued…

Question 2 continued

(h) **EITHER**

$\ln(g(x)) = x\left(\ln(S) - \ln x\right)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g'(x)}{g(x)} = \ln S - \ln x - x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g'(x)}{g(x)} = \ln\frac{S}{x} + \left(x\frac{x}{S}\times\frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g\left(x\right) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g'\left(x\right) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x\times\frac{x}{S}\times\left(-Sx^{-2}\right) + \ln\left(\frac{S}{x}\right)\right]$

$= \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right) - 1\right]$ ***A1***

THEN

$g'(x) = \left(\ln\frac{S}{x} - 1\right)g(x)$

$g(x) \neq 0$

$g'(x) = 0 \Rightarrow \ln\frac{S}{x} - 1 = 0$ ***M1***

$x = \frac{S}{\mathrm{e}}\ (0.368S,\ 0.36789...S)$ ***A1***

[6 marks]

continued…

Question 2 continued

(i) $\ln\left(g(x)\right) = x\ln\left(\frac{S}{x}\right) \Rightarrow \ln\left(g(x)\right) = \ln\left(\frac{S}{x}\right)^x$ ***M1***

$g(x) = \left(\frac{S}{x}\right)^x$ ***A1***

$= M_x(S)$ for $x \in \mathbb{Z}^+$ ***AG***

[2 marks]

(j) $\frac{100}{\mathrm{e}} = 36.8$ ***M1***

$\left(\frac{100}{36}\right)^{36} = 9.3996...\times10^{15}$ **AND** $\left(\frac{100}{37}\right)^{37} = 9.47406...\times10^{15}$ ***R1***

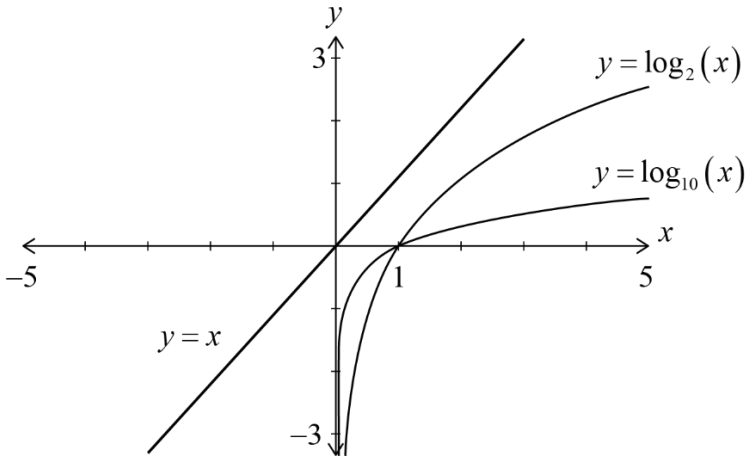
largest possible product is 9.47×10^{15} $(9.47406...\times10^{15})$ ***A1***

Note: Award ***A1*** independently of the ***R1*** (but not independently of the ***M1***).

[3 marks]

Total [30 marks]

1. (a)



clearly labelled graphs of $y = \log_2 x$ and $y = \log_{10} x$ with correct domain, asymptotic behaviour and concavity evident ***A1***

correct relative positions of the two log graphs both above and below the x-axis ***A1***

$(1, 0)$ indicated (coordinates not required) ***A1***

correct graph of $y = x$ ***A1***

[4 marks]

continued…

Question 1 continued

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x-\ln x\right)$

$=1-\frac{1}{x}$ ***A1***

attempts to solve their $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ for x ***(M1)***

$1-\frac{1}{x}=0 \Rightarrow x=1$

(when $x=1$,) $x-\ln x=1$ ***A1***

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-\frac{1}{x}\right)$

$=\frac{1}{x^2}$ ***A1***

$\frac{1}{x^2}>0$ (when $x=1$) ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1*** for either '$1>0$' or a graph of $y=\frac{1}{x^2}>0$ or 'the graph of $y=x-\ln x$ is concave-up'. Do not award ***R1*** if the second derivative is incorrect.

continued…

Question 1 continued

OR

for $(0<)x<1$, $1-\frac{1}{x}<0$ ***R1***

for $x>1$, $1-\frac{1}{x}>0$ ***R1***

hence $x-\ln x$ has a minimum value of 1

Note: Award ***R1R1*** for either a clearly labelled sign diagram/table (accept correct numerical values) or the graph of $y=1-\frac{1}{x}$ with sign change in gradient indicated.

Note: Award a maximum of ***A0(M1)A1A0R1*** or ***A0(M1)A1R0R1*** if no symbolic derivatives are seen.

[5 marks]

(c)

EITHER

$x-\ln x\geq 1$ $\left(x\in\mathbb{R}^+\right)$ ***R1***

OR

$x-\ln x>0$ $\left(x\in\mathbb{R}^+\right)$ ***R1***

THEN

so $x>\ln x$ ***AG***

[1 mark]

continued…

Question 1 continued

(d)

Interval	Number of intersection points
$0 < a < 1$	$p = 1$
$1 < a < 1.4$	$q = 2$
$1.5 < a < 2$	$r = 0$

A1A2A1

Note: Award ***A1*** for $p = 1$, ***A2*** for $q = 2$ and ***A1*** for $r = 0$.

[4 marks]

(e) **METHOD 1**

EITHER

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

continued…

Question 1 continued

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\frac{\ln x}{\ln a} = \frac{1}{\ln a}$ OR $\ln x = 1$ OR $\left(e^{\frac{1}{x}}\right)^x = x$ for x ***(M1)***

$x = e$

coordinates of P are (e, e) (accept $x = e$, $y = e$) ***A1A1***

attempts to solve $\frac{1}{\ln a} = e$ OR $\log_a e = e$ for a analytically ***(M1)***

$\ln a = \frac{1}{e}$ OR $a^e = e$

$a = e^{\frac{1}{e}}$ ***A1***

continued…

Question 1 continued

METHOD 2

EITHER

$y = \log_a x$

$\frac{dy}{dx} = \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $\frac{1}{x \ln a} = 1$ for x ***(M1)***

OR

$y = x - \log_a x$

$\frac{dy}{dx} = 1 - \frac{1}{x \ln a}$ ***(A1)***

attempts to solve $1 - \frac{1}{x \ln a} = 0$ for x ***(M1)***

THEN

$x = \frac{1}{\ln a}$ OR $x \ln a = 1$ OR $\ln a = \frac{1}{x}$ OR $\ln a^x = 1$ OR $\frac{1}{a^x \ln a} = 1$ ***A1***

at $x = \frac{1}{\ln a}$, $\log_a x = x$

attempts to solve $\log_a\left(\frac{1}{\ln a}\right) = \frac{1}{\ln a}$ for a ***(M1)***

EITHER

$\frac{\ln\left(\frac{1}{\ln a}\right)}{\ln a} = \frac{1}{\ln a} \Rightarrow \ln\left(\frac{1}{\ln a}\right) = 1$

OR

for example, writes $a^{\log_a\left(\frac{1}{\ln a}\right)} = a^{\frac{1}{\ln a}}$ and then attempts to apply appropriate index/log laws to both sides: $\ln a = \frac{\log_a a}{\log_a \mathrm{e}}$ and so $\frac{1}{\ln a} = \log_a \mathrm{e}$

$a^{\frac{1}{\ln a}} = a^{\log_a \mathrm{e}} = \mathrm{e}$

continued…

Question 1 continued

THEN

attempts to solve $\frac{1}{\ln a} = \mathrm{e}$ OR $\log_a \mathrm{e} = \mathrm{e}$ for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ OR $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

$x = \frac{1}{\ln \mathrm{e}^{\frac{1}{\mathrm{e}}}} = \frac{1}{\frac{1}{\mathrm{e}}}$

coordinates of P are $\left(\mathrm{e}, \mathrm{e}\right)$ (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

continued…

Question 1 continued

METHOD 3

$y = \log_a x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$ ***(A1)***

(equation of the tangent at (x_1, y_1) is) $y = \frac{1}{x_1 \ln a}(x - x_1) + \frac{\ln x_1}{\ln a}$ (or equivalent) ***A1***

compares this equation with $y = x$ and attempts to form at least one of the following ***M1***

$\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$

attempts to solve $\frac{1}{x_1 \ln a} = 1$ OR $\frac{\ln x_1 - 1}{\ln a} = 0$ for x_1 ***(M1)***

$x_1 = \mathrm{e}$

coordinates of P are (e, e) (accept $x = \mathrm{e}$, $y = \mathrm{e}$) ***A1A1***

attempts to solve $\frac{1}{\mathrm{e} \ln a} = 1$ (or equivalent) for a analytically ***(M1)***

$\ln a = \frac{1}{\mathrm{e}}$ **OR** $a^{\mathrm{e}} = \mathrm{e}$

$a = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***A1***

[8 marks]

continued…

Question 1 continued

(f) (i) $1 < a < e^{\frac{1}{e}}$ ***A1***

Note: Award ***A0*** for $a < e^{\frac{1}{e}}$.

[1 mark]

(ii) $a > e^{\frac{1}{e}}$ ***A1***

Note: Only award ***FT*** for $1.4 < a < 1.5$. If the value of a is not exact, e.g. 1.44, award at most ***A0A1*** in part (f) for a consistent approximate endpoint value.
If a value of a is not found in part (e), award at most ***A0A1*** in part (f) for a consistent approximate endpoint value provided that $1.4 < a < 1.5$.

[1 mark]

Total [24 marks]

8. (a) **METHOD 1**

attempt to integrate by parts ***(M1)***

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ \mathrm{d}x$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ ***(M1)***

$\int x \ln x \ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$ ***A1***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ \mathrm{d}x = x \ln x - x$ **)**

attempt to integrate by parts ***(M1)***

$u = x \ln x$, $\mathrm{d}v = \ln x \mathrm{d}x$ ***(M1)***

$\int x \ln x (\ln x) \ \mathrm{d}x = x \ln x (x \ln x - x) - \int (\ln x + 1)(x \ln x - x) \mathrm{d}x$ ***A1***

$= x \ln x (x \ln x - x) - \int x(\ln x)^2 \ \mathrm{d}x + \int x \mathrm{d}x$ ***A1***

$2I = x \ln x (x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x\ \mathrm{d}x$ **)**

$$\int x\ln x\ \mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts ***(M1)***

$u = \ln x\ ,\ \mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$$\int (x\ln x)\ln x\ \mathrm{d}x = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x) = \dfrac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$= \dfrac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$= \dfrac{\sin^2(kx)}{x^2} (= f(x))$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ (***M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim_{x\to 0}\left(f\left(x\right)\right)=\lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$=\lim_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+...\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+...\right)$ ***A1***

$\lim_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{k^2x^2\left(+...\right)}{x^2}\right)$ ***M1***

$=\lim_{x\to 0}\left(k^2+\text{terms in }x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits (***M1)***

$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = \lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{k\cos kx}{1}\right) = k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx) = kx(+...)$

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{kx+...}{x}\right) = \lim_{x\to 0}(k + \text{terms in } x) = k$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = k\times k = k^2$ ***A1***

$k^2 = 16 \Rightarrow k = 4(k>0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)\mathrm{e}^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}), \left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right), (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***

For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x} - (4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4\mathrm{e} < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4\mathrm{e} < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{\mathrm{e}} > 0$ so minimum ***R1***

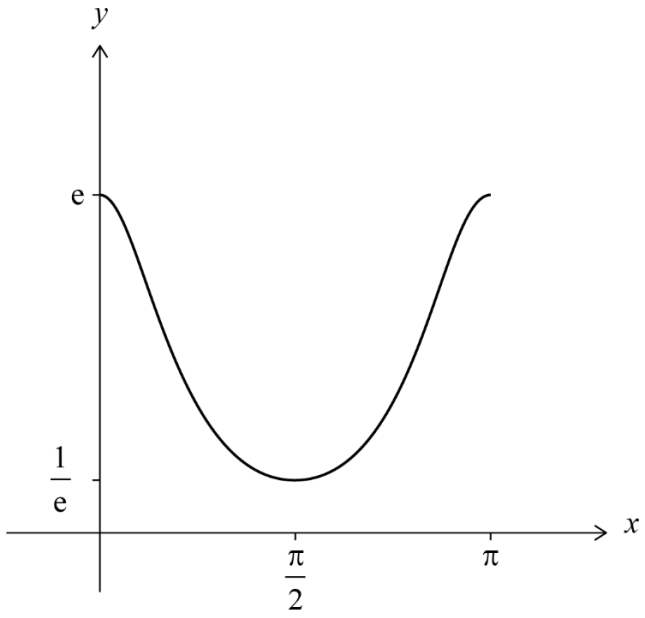
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0,e),(\pi,e)$ and minimum point $\left(\frac{\pi}{2},\frac{1}{e}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots$; $e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + \ldots$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x) \approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x \approx \mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

8. (a) **METHOD 1**

attempt to integrate by parts ***(M1)***

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ \mathrm{d}x$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ ***(M1)***

$\int x \ln x \ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$ ***A1***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x \right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ \mathrm{d}x = x \ln x - x$ **)**

attempt to integrate by parts ***(M1)***

$u = x \ln x$, $\mathrm{d}v = \ln x \mathrm{d}x$ ***(M1)***

$\int x \ln x(\ln x) \ \mathrm{d}x = x \ln x(x \ln x - x) - \int (\ln x + 1)(x \ln x - x)\mathrm{d}x$ ***A1***

$= x \ln x(x \ln x - x) - \int x(\ln x)^2 \ \mathrm{d}x + \int x \mathrm{d}x$ ***A1***

$2I = x \ln x(x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x\ \mathrm{d}x$ **)**

$\int x\ln x\ \mathrm{d}x = \dfrac{x^2(\ln x)}{2} - \dfrac{x^2}{4}$

attempt to integrate by parts ***(M1)***

$u = \ln x\ ,\ \mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$\int (x\ln x)\ln x\ \mathrm{d}x = \ln x\left(\dfrac{x^2(\ln x)}{2} - \dfrac{x^2}{4}\right) - \int \dfrac{1}{x}\left(\dfrac{x^2(\ln x)}{2} - \dfrac{x^2}{4}\right)\mathrm{d}x$ ***A1***

$= \ln x\left(\dfrac{x^2(\ln x)}{2} - \dfrac{x^2}{4}\right) - \int\left(\dfrac{x(\ln x)}{2} - \dfrac{x}{4}\right)\mathrm{d}x$ ***A1***

$= \ln x\left(\dfrac{x^2(\ln x)}{2} - \dfrac{x^2}{4}\right) - \dfrac{x^2\ln x}{4} + \dfrac{x^2}{8} + \dfrac{x^2}{8}$ ***A1***

$= \dfrac{x^2(\ln x)^2}{2} - \dfrac{x^2\ln x}{2} + \dfrac{x^2}{4}(+c)$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$\left[\dfrac{x^2(\ln x)^2}{2} - \dfrac{x^2\ln x}{2} + \dfrac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\dfrac{1}{4}\right)$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \dfrac{1}{4}$ ***A1***

$= 32(\ln 2)^2 - 16\ln 2 + \dfrac{15}{4}$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x)=\dfrac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$=\dfrac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$=\dfrac{\sin^2(kx)}{x^2}\left(=f(x)\right)$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x\to 0}\left(f\left(x\right)\right)=\frac{0}{0}$ (***M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim_{x\to 0}\left(f\left(x\right)\right)=\lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(=\lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$=\lim_{x\to 0}\left(\frac{2k^2\left(\cos^2 kx-\sin^2 kx\right)}{2}\right)\left(=\lim_{x\to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right)=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right)=kx\left(+...\right)$

$\sin^2\left(kx\right)=k^2x^2\left(+...\right)$ ***A1***

$\lim_{x\to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right)=\lim_{x\to 0}\left(\frac{k^2x^2\left(+...\right)}{x^2}\right)$ ***M1***

$=\lim_{x\to 0}\left(k^2+\text{terms in }x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$=k^2$ ***A1***

$\left(k^2=16\Rightarrow\right)k=4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits (***M1***)

$\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = \lim_{x\to 0}\left(\frac{\sin kx}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{k\cos kx}{1}\right) = k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx) = kx(+...)$

$\lim_{x\to 0}\left(\frac{\sin kx}{x}\right) = \lim_{x\to 0}\left(\frac{kx+...}{x}\right) = \lim_{x\to 0}(k + \text{terms in } x) = k$ ***(A1)***

THEN

hence $\lim_{x\to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = k\times k = k^2$ ***A1***

$k^2 = 16 \Rightarrow k = 4(k>0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x\to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)\mathrm{e}^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, \mathrm{e}), \left(\frac{\pi}{2}, \frac{1}{\mathrm{e}}\right), (\pi, \mathrm{e})$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***

For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)\mathrm{e}^{\cos 2x} - (4\cos 2x)\mathrm{e}^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4\mathrm{e} < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4\mathrm{e} < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{\mathrm{e}} > 0$ so minimum ***R1***

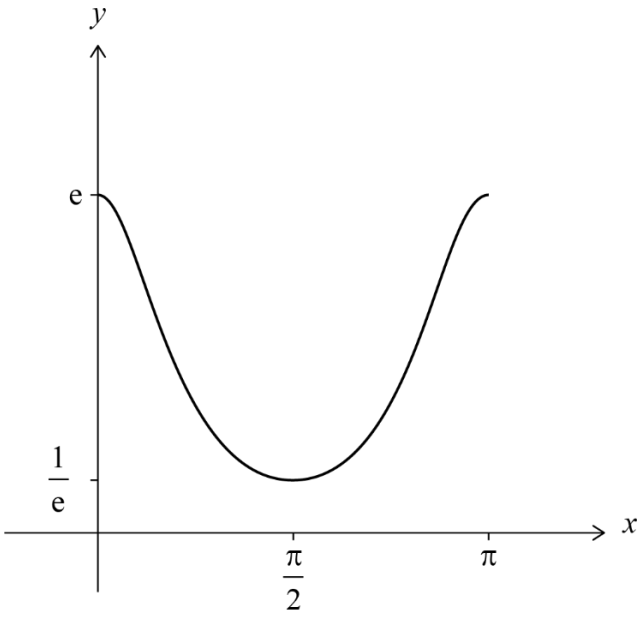
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0,e),(\pi,e)$ and minimum point $\left(\frac{\pi}{2},\frac{1}{e}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + ...$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + ...;\ e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + ...$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + ...\right)\left(1 + \frac{2}{3}x^4 + ...\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f(x) \approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x \approx \mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note: If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

4. (a) recognizing at rest when $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ OR s is a minimum ***(M1)***

$q=5.73553...$

$=5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q=5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...}|v(t)|\,\mathrm{d}t$ OR $\int_0^{5.73...}\left|\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right|\mathrm{d}t$ OR $\left|\int_0^{5.73...}v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...}v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of dt.

Only accept $\left|\int_0^q v(t)\,\mathrm{d}t\right|$ if their value of q does not result in the particle changing direction in the first q seconds.

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: Special Cases:

Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q=1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ $(3.37644...$ from 3sf).

Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...}v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302...-(-4.3)$

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x = 0$, $y = 2$ to find the value of A ***M1***

$2 = A\mathrm{e}^{0} \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $|y-4| = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{dy}{dx} + \frac{1}{10}y = \frac{4}{10}$ so IF $e^{\int \frac{1}{10}dx} = e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}\frac{dy}{dx} + \frac{1}{10}e^{\frac{1}{10}x}y = \frac{4}{10}e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x}(+c)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(2e^0 = 4e^0 + c \Rightarrow) c = -2$ ***A1***

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x} - 2$

$y = 4 - 2e^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $= 2.0980199... - \left(4 - 2e^{-0.05}\right) = 0.000478749...$

$= 0.000479 \left(= 4.79 \times 10^{-4}\right)$ ***A1***

Note: Accept $0.000459 \left(= 4.59 \times 10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

4. (a) recognizing at rest when $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ OR s is a minimum ***(M1)***

$q=5.73553...$

$=5.74$ ***A1***

Note: If no working shown, award ***(M1)A0*** for $q=5.7$ (2sf).

[2 marks]

(b) **METHOD 1**

recognizing that integral of $v(t)$ is required ***(M1)***

$\int_0^{5.73...}|v(t)|\,\mathrm{d}t$ OR $\int_0^{5.73...}\left|\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right|\mathrm{d}t$ OR $\left|\int_0^{5.73...}v(t)\,\mathrm{d}t\right|$ OR $-\int_0^{5.73...}v(t)\,\mathrm{d}t$ ***(A1)***

Note: Condone absence of dt.

Only accept $\left|\int_0^q v(t)\,\mathrm{d}t\right|$ if their value of q does not result in the particle changing direction in the first q seconds.

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: Special Cases:

Award a maximum of ***(M1)(A1FT)A0FT*** if the candidate obtains $q=1.62320...$ in part (a), and uses that value to find the total distance to be $3.38302...$ $(3.37644...$ from 3sf).

Award ***(M1)(A0)A1*** if the candidate writes $\int_0^{5.73...}v(t)\,\mathrm{d}t$ followed by the correct answer.

METHOD 2

recognition that total distance travelled is the difference between the initial displacement and the displacement at minimum ***(M1)***

initial displacement is $3.38302...$ AND at minimum is -4.3 ***(A1)***

total distance travelled = $3.38302...-(-4.3)$

$=7.68302...$

$=7.68\ (\mathrm{m})$ ***A1***

Note: If no working shown, award ***(M1)(A0)A0*** for 7.7 (2sf).

[3 marks]

Total [5 marks]

9. (a) attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4 - y_{n-1}}{10}\right)$ ***(M1)***

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Note: Condone absence of absolute value signs throughout

$\int \frac{\mathrm{d}y}{4-y} = \int \frac{\mathrm{d}x}{10}$ ***M1***

$-\ln|4-y| = \frac{x}{10}(+c)$ ***A1***

EITHER

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(-\ln 2 = 0 + c \Rightarrow) c = -\ln 2$ ***A1***

$-\ln|4-y| = \frac{x}{10} - \ln 2 \Rightarrow \ln\frac{|4-y|}{2} = -\frac{x}{10}$

$|4-y| = 2\mathrm{e}^{-\frac{x}{10}}$ ***A1***

OR

$|4-y| = \mathrm{e}^{-\frac{x}{10}-c}$ (so $4-y = \pm\mathrm{e}^{-c}\mathrm{e}^{-\frac{x}{10}}$)

$4-y = A\mathrm{e}^{-\frac{x}{10}}$ ***A1***

substituting initial conditions $x = 0$, $y = 2$ to find the value of A ***M1***

$2 = A\mathrm{e}^{0} \Rightarrow A = 2$ ***A1***

THEN

$y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$ ***AG***

Note: Candidates may use $-\int \frac{\mathrm{d}y}{y-4} = \int \frac{\mathrm{d}x}{10}$ and correctly obtain $|y-4| = 2\mathrm{e}^{-\frac{x}{10}}$ leading to $4-y = 2\mathrm{e}^{-\frac{x}{10}}$ after consideration of the boundary conditions. In this case, the absence of absolute value signs should not be condoned until the sign has been resolved.

continued…

Question 9 continued

METHOD 2

attempt to rearrange and find an integrating factor ***M1***

$\frac{dy}{dx} + \frac{1}{10}y = \frac{4}{10}$ so IF $e^{\int \frac{1}{10}dx} = e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}\frac{dy}{dx} + \frac{1}{10}e^{\frac{1}{10}x}y = \frac{4}{10}e^{\frac{1}{10}x}$

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x}(+c)$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

substituting initial conditions $x = 0, y = 2$ to find the value of c ***M1***

$(2e^0 = 4e^0 + c \Rightarrow) c = -2$ ***A1***

$e^{\frac{1}{10}x}y = 4e^{\frac{1}{10}x} - 2$

$y = 4 - 2e^{-\frac{x}{10}}$ ***AG***

[5 marks]

(c) absolute error $= 2.0980199... - \left(4 - 2e^{-0.05}\right) = 0.000478749...$

$= 0.000479 \left(= 4.79 \times 10^{-4}\right)$ ***A1***

Note: Accept $0.000459 \left(= 4.59 \times 10^{-4}\right)$ from use of 4sf value.

[1 mark]

Total [9 marks]

Section B

2. (a) (i) $\frac{y}{x}$ ***A1***

[1 mark]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(\frac{y}{x}\right)} \left(= -\frac{1}{m}\right)$ ***A1***

Note: Award ***A1*** for responses such as 'the gradient is the negative (opposite) reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).

Award ***A1*** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.

Do not award ***FT*** from part (a) (i).

so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***AG***

[1 mark]

Question 2 continued

(iii) attempts to separate variables x and y ***(M1)***

$$\int y \, \mathrm{d}y = -\int x \, \mathrm{d}x$$

Note: Award ***(M1)*** for $y \, \mathrm{d}y = -x \, \mathrm{d}x$.

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$ ***A1***

Note: Award ***A1*** for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.

Award ***A0*** for $\frac{y^2}{2} = -\frac{x^2}{2}$.

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

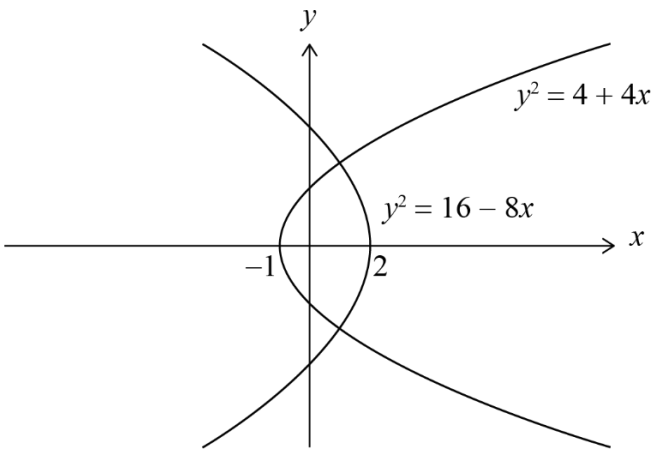
$\Rightarrow x^2 + y^2 = k$ (where $k = 2c$) ***AG***

[2 marks]

continued…

Question 2 continued

(b)



two parabolic shaped curves with approximately correct shape/position
(e.g. two intersection points, in first and fourth quadrant) ***A1A1***

x-intercepts -1 and 2 ***A1***

[3 marks]

continued…

Question 2 continued

(c) at intersection, $4a^2 - 4ax = 4b^2 + 4bx$

$4a^2 - 4b^2 = 4ax + 4bx$ ($a^2 - b^2 = ax + bx$, $4a^2 - 4b^2 - 4ax - 4bx = 0$) ***A1***

attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of $a^2 - b^2 - ax - bx = 0$ (or equivalent) ***(M1)***

$(a+b)(a-b) = (a+b)x$

Note: Accept alternative forms such as $4(a+b)(a-b) = 4(a+b)x$ or $(a+b)((a-b)-x) = 0$.

recognition that $a + b > 0$ (or equivalent, eg. $a > 0, b > 0$) (allows division by $a + b$) ***R1***

Note: Subsequent marks are not dependent on this ***R1***.

$x = a - b$ ***A1***

Note: As $x = a - b$ is an ***AG***, only award the above ***A1*** if $a^2 - b^2 = (a+b)(a-b)$ has been used.

substitutes $x = a - b$ into either $y^2 = 4a^2 - 4ax$ or $y^2 = 4b^2 + 4bx$ and attempts to simplify ***(M1)***

$y^2 = 4a^2 - 4a(a-b)$ OR $y^2 = 4b^2 + 4b(a-b)$

$y^2 = 4a^2 - 4a^2 + 4ab \Rightarrow y = \pm 2\sqrt{ab}$ ***A1***

so $\mathrm{M}\left(a-b, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b, -2\sqrt{ab}\right)$ ***AG***

[6 marks]

continued…

Question 2 continued

(d) **METHOD 1**

attempts implicit differentiation on either curve ***(M1)***

$\frac{dy}{dx} = -\frac{4a}{2y}$ (or equivalent) and $\frac{dy}{dx} = \frac{4b}{2y}$ (or equivalent) ***A1***

substitutes $y = 2\sqrt{ab}$ into either $\frac{dy}{dx} = -\frac{4a}{2y}$ or $\frac{dy}{dx} = \frac{4b}{2y}$ ***(M1)***

$\frac{dy}{dx} = -\sqrt{\frac{a}{b}}$ $\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{dy}{dx} = \sqrt{\frac{b}{a}}$ $\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

OR

eg. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

continued…

Question 2 continued

METHOD 2

attempts chain rule differentiation on either $y = \sqrt{4a^2 - 4ax}$ or $y = \sqrt{4b^2 + 4bx}$ ***(M1)***

$\frac{dy}{dx} = -\frac{2a}{\sqrt{4a^2 - 4ax}}$ (or equivalent) and $\frac{dy}{dx} = \frac{2b}{\sqrt{4b^2 + 4bx}}$ (or equivalent) ***A1***

substitutes $x = a - b$ into either $\frac{dy}{dx} = -\frac{2a}{\sqrt{4a^2 - 4ax}}$ or $\frac{dy}{dx} = \frac{2b}{\sqrt{4b^2 + 4bx}}$ ***(M1)***

$\frac{dy}{dx} = -\sqrt{\frac{a}{b}}$ $\left(= -\frac{a}{\sqrt{ab}}\right)$ and $\frac{dy}{dx} = \sqrt{\frac{b}{a}}$ $\left(= \frac{b}{\sqrt{ab}}\right)$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}} \times \sqrt{\frac{b}{a}} = -1$ OR $-\frac{a}{\sqrt{ab}} \times \frac{b}{\sqrt{ab}} = -1$ (or equivalent) ***A1***

OR

eg. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

[5 marks]

continued…

Question 2 continued

(e) (i) $g(x, y) = \dfrac{-\dfrac{x}{y} + \tan\dfrac{\pi}{4}}{1 - \left(-\dfrac{x}{y}\right)\tan\dfrac{\pi}{4}}$ ***(A1)***

$g(x, y) = \dfrac{-\dfrac{x}{y} + 1}{1 + \dfrac{x}{y}} \left(= \dfrac{\dfrac{-x+y}{y}}{\dfrac{y+x}{y}} \right)$ ***A1***

so $g(x, y) = \dfrac{y-x}{y+x}$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) let $y = vx$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x} =\right) \frac{vx - x}{vx + x} \left(= \frac{v-1}{v+1}\right)$ ***(A1)***

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single fraction in v ***(M1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{v+1}$ (or equivalent) ***(A1)***

attempts to separate variables x and v ***(M1)***

$\int \frac{v+1}{v^2+1}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$ (or equivalent)

$\frac{1}{2}\ln\left(v^2+1\right) + \arctan v = -\ln|x|\left(+d\right)$ (or equivalent) ***A1A1***

$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\frac{y}{x} + \ln|x| = d$ (or equivalent) ***A1***

[9 marks]

continued…

Question 2 continued

(f) **METHOD 1**

$$g(x,\ y)=\frac{\frac{1}{\tan\alpha}f\left(x,y\right)+1}{\frac{1}{\tan\alpha}-f\left(x,y\right)}$$ ***M1***

EITHER

as $\alpha\to\frac{\pi}{2}$, $\frac{1}{\tan\alpha}\to 0$, (hence $g(x,\ y)\to-\frac{1}{f(x,\ y)}$) ***R1***

OR

as $\alpha\to\frac{\pi}{2}$, $\tan\alpha\to\infty$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

THEN

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

METHOD 2

uses either $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$ or $\frac{1}{\tan\alpha}=\frac{\cos\alpha}{\sin\alpha}$ to form $g(x,\ y)=\frac{\cos\alpha f\left(x,y\right)+\sin\alpha}{\cos\alpha-\sin\alpha f\left(x,y\right)}$ ***M1***

as $\alpha\to\frac{\pi}{2}$, $\cos\alpha\to 0$ and $\sin\alpha\to 1$ and so $g(x,\ y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\ y)=-\frac{1}{f(x,\ y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

[2 marks]

Total [31 marks]

4. (a) recognising $\cos x = 2\sin x \cos x$ ***(M1)***

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) ***(A1)***

x - coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ ***A1***

Note: Award ***(M1)(A1)A0*** for solutions of 30° and 150°.

[3 marks]

(b) **METHOD 1**

attempt to integrate $\pm(\cos x - \sin 2x)$ ***(M1)***

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - \sin 2x)\,dx$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - 2\sin x \cos x)\,dx$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ OR $= \left[\sin x - \sin^2 x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for $\pm$ correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract ***M1***

$= \left(\sin\left(\frac{5\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi)\right)$ OR

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(1 - \frac{1}{2}\right)$ OR $= \left(\frac{1}{2} - \frac{1}{4}\right) - (1 - 1)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B.
Accept work done in degrees.

continued…

Question 4 continued

METHOD 2

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x = \left[\sin x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ and $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x = \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) ***M1***

$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)$ and $-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)$

attempt to subtract the two integrals in either order (seen anywhere) ***(M1)***

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)\right)$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x$

$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \quad \left(= -\frac{1}{4}\right)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B.

Accept work done in degrees.

[4 marks]

Total [7 marks]

6. METHOD 1

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1.***

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}\left(f(x)\right)^2\mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

EITHER

attempt to use integration by substitution ***M1***

$\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin(u)\mathrm{d}u$

Note: Award ***M1*** for $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$=\left[-\frac{\pi}{2}\cos(u)\right]_0^{\frac{\pi}{4}}$ ***A1***

OR

attempt to integrate by inspection ***(M1)***

$\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}2x\sin\left(x^2\right)\mathrm{d}x$ OR $\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}\sin\left(x^2\right)\mathrm{d}\left(x^2\right)$

$=\left[-\frac{\pi}{2}\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ ***A1***

Note: Condone incorrect or absent limits for ***M1***.
The correct limits may be seen or implied by later work for the ***A1***.

THEN

$=\left(-\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\right)-\left(-\frac{\pi}{2}\cos(0)\right)$ (or equivalent) ***(A1)***

$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{2}$ OR $-\frac{\pi\sqrt{2}}{4}+\frac{\pi}{2}$ OR $\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}}+1\right)$ OR $\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$ ***A1***

$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

continued…

Question 6 continued

METHOD 2

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1***.

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} \left(f(x)\right)^2 \mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

attempt to use integration by substitution ***M1***

$u = \cos\left(x^2\right) \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = -2x\sin\left(x^2\right)$

Note: Award ***M1*** for $u = \cos\left(x^2\right)$

$= -\dfrac{\pi}{2}\int_{-\frac{1}{\sqrt{2}}}^{-1} \mathrm{d}u$

$= \left[-\dfrac{\pi}{2}u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$ (or equivalent) ***A1A1***

Note: Condone incorrect or absent limits for ***M1***.

A1 for $-\dfrac{\pi}{2}u$ and ***A1*** for both correct limits.

$= \dfrac{\pi}{2} - \dfrac{\pi}{2\sqrt{2}}$ OR $\dfrac{\pi}{2} - \dfrac{\pi\sqrt{2}}{4}$ OR $\dfrac{\pi}{2}\left(1 - \dfrac{1}{\sqrt{2}}\right)$ OR $\dfrac{\pi}{2}\left(1 - \dfrac{\sqrt{2}}{2}\right)$ ***A1***

$= \dfrac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

Total [6 marks]

8. (a) (i) $\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \ldots \left(= x^2 - \frac{x^6}{6} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

(ii) **METHOD 1**

attempt to square their series for $\sin\left(x^2\right)$ ***(M1)***

$\left(\sin\left(x^2\right)\right)^2 = \left(x^2 - \frac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\frac{x^6}{3!}\right)^2 + \ldots.$

$= x^4 - \frac{2x^8}{3!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\left(x^2\right) = \frac{1 - \cos\left(2x^2\right)}{2}$ ***(M1)***

$\sin^2\left(x^2\right) = \frac{1}{2}\left(1 - \left(1 - \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^4}{4!}\right)\right)$

$= x^4 - \frac{8x^8}{4!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[5 marks]

continued…

Question 8 continued

(b) **METHOD 1**

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=\frac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

METHOD 2

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=2x\sin\left(2x^2\right)$ ***(M1)***

$=2x\left(2x^2-\frac{\left(2x^2\right)^3}{3!}+\ldots\right)$

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

METHOD 3

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$=4x\left(x^2-\frac{x^6}{3!}+\ldots\right)\left(1-\frac{x^4}{2!}+\ldots\right)$ ***(A1)***

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

continued…

Question 8 continued

METHOD 4

recognition that $2x\cos\left(x^2\right)=\frac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$=2\left(x^2-\frac{x^6}{3!}+\ldots\right)\left(2x-\frac{6x^5}{2!}+\ldots\right)$

$=4x^3-\frac{8x^7}{3}+\ldots$ ***A1***

[2 marks]

Total [7 marks]

11. (a) attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division ***M1***

$3(-1)^3+5(-1)^2+(-1)-1=0$ OR

	3	5	1	-1
-1		-3	-2	1
	3	2	-1	0

OR $x+1\overline{)3x^3+5x^2+x-1}$ ***A1***

$$\begin{array}{r} 3x^2+2x-1 \\ \hline 3x^3+3x^2 \\ \hline 2x^2+x \\ 2x^2+2x \\ \hline -x-1 \\ -x-1 \\ \hline 0 \end{array}$$

[2 marks]

(b) attempt to divide $P(x)$ by $(x+1)$ e.g. using long division or synthetic division ***(M1)***

$P(x)=(x+1)(3x^2+2x-1)$ ***(A1)***

$=(x+1)(x+1)(3x-1)\left(=(x+1)^2(3x-1)\right)$ ***A1***

[3 marks]

(c) $\dfrac{1}{(x+1)(2x+1)}\equiv\dfrac{A}{x+1}+\dfrac{B}{2x+1}\Rightarrow 1\equiv A(2x+1)+B(x+1)$

attempt to equate both coefficients OR substitute two values eg -1 and $-\dfrac{1}{2}$ ***(M1)***

$2A+B=0$ and $A+B=1$ OR $1=-A$ and $1=\dfrac{1}{2}B$

$A=-1$ and $B=2$ ***A1A1***

Note: Award ***A1*** for each value.

$$\frac{1}{(x+1)(2x+1)}=-\frac{1}{x+1}+\frac{2}{2x+1}$$

[3 marks]

continued…

Question 11 continued

(d) $\dfrac{1}{(x+1)(x+1)(2x+1)}$

$= \dfrac{1}{(x+1)}\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)$ ***(A1)***

$= -\dfrac{1}{(x+1)^2}+\dfrac{2}{(2x+1)(x+1)}\left(= -\dfrac{1}{(x+1)^2}+2\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)\right)$ ***A1***

$= \dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}$ ***AG***

Note: Award ***A1A0*** for follow through from incorrect values in part (c).

[2 marks]

(e) attempt to integrate at least one term in $\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)$ ***(M1)***

$\int\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)\mathrm{d}x$

$= 2\ln|2x+1|-2\ln|x+1|+\dfrac{1}{x+1}(+c)$ ***A1A1A1***

Note: Award ***A1*** for each correct term.
Award a maximum of ***M1A1A0A1*** if modulus signs are omitted.
Condone the absence of $+c$.

[4 marks]

(f) (i) **METHOD 1**

attempt to cancel factors and substitute $x=-1$ ***(M1)***

$\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1}\left(\dfrac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim\limits_{x\to -1}\left(\dfrac{3x-1}{2x+1}\right) = \dfrac{3(-1)-1}{2(-1)+1}$

$= 4$ ***A1***

continued…

Question 11 continued

METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x = -1$ ***(M1)***

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x\to -1}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x\to -1}\left(\frac{18x+10}{12x+10}\right) = \frac{18(-1)+10}{12(-1)+10}$$

$= 4$ ***A1***

(ii) **METHOD 1**

attempt to consider coefficients of x^3 or divide all terms by x^3

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(\frac{3x^3+\ldots}{2x^3+\ldots}\right) \text{ or } \lim_{x\to \infty}\left(\frac{3+\text{terms which tend to } 0}{2+\text{terms which tend to } 0}\right)$$

$= \frac{3}{2}$ ***A1***

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim_{x\to \infty}\left(\frac{3x-1}{2x+1}\right) \text{ or } \lim_{x\to \infty}\left(\frac{3-\frac{1}{x}}{2+\frac{1}{x}}\right)$$

$= \frac{3}{2}$ ***A1***

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x\to \infty}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x\to \infty}\left(\frac{18x+10}{12x+10}\right) = \lim_{x\to \infty}\left(\frac{18}{12}\right)$$

$= \frac{3}{2}$ ***A1***

Note: If the ***M1*** has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

Total [17 marks]

8.

Note: To award full marks limit notation $\lim_{x\to 0}$ must be seen at least once in their working. If no limit notation is seen but otherwise all correct, do not award the final ***A1.***

$$\lim_{x\to 0}\frac{4\sec^4 x\tan x+2\sin x\cos x}{4x^3-2x}$$ ***A1A1***

Note: Award ***A1*** for numerator and ***A1*** for denominator.

$$=\lim_{x\to 0}\frac{16\sec^4 x\tan^2 x+4\sec^6 x-2\sin^2 x+2\cos^2 x}{12x^2-2}$$ ***M1A1A1***

Note: Award ***M1 for second use of*** l'Hôpital's rule providing their expression is in indeterminate form as $x\to 0$ and providing there is **no third attempt** at using l'Hôpital's Rule.

$$=-3$$ ***A1***

[6 marks]

12. (a) $n=1: LHS = f^{(1)}(x) = -\frac{1}{2}\times -a(1-ax)^{-\frac{3}{2}}\left(=\frac{a}{2}(1-ax)^{-\frac{3}{2}}\right)$ ***A1***

$RHS = \frac{a(1)!(1-ax)^{-\frac{3}{2}}}{2^1(0)!}$ therefore true for $n=1$ ***R1***

assume (that the result is) true for $n=k$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$" or "assume that $n=k$ is true". The assumption of truth must be clear.

$$f^{(k)}(x) = \frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}}}{2^{2k-1}(k-1)!}$$

attempt to differentiate the right-hand side with respect to x: ***M1***

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$$

$$= \frac{-(2k+1)\times -a}{2}\times\frac{a^k(2k-1)!(1-ax)^{-\frac{(2k+1)}{2}-1}}{2^{2k-1}(k-1)!}$$ (or equivalent) ***A1***

attempt to multiply top and bottom by $2k$ ***M1***

$$= \frac{(2k+1)}{2}\times\frac{a^{k+1}(2k-1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k-1}(k-1)!}\times\frac{2k}{2k}$$

$$= \frac{a^{k+1}(2k+1)!(1-ax)^{-\frac{2k+3}{2}}}{2^{2k+1}(k)!}$$ ***A1***

hence if the result is true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=1$ it is true for all $n\in\mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, at least five of the previous marks must have been awarded.

[8 marks]

continued…

Question 12 continued.

(b) $f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \ldots$

$f''(x) = \frac{a^2(3)!(1-ax)^{-\frac{5}{2}}}{2^3(1)!}$ OR $\frac{3}{4}a^2(1-ax)^{-\frac{5}{2}}$ OR $f''(0) = \frac{a^2(3)!}{2^3}$ ***A1***

$f(0) = 1,\ f'(0) = \frac{a}{2},\ f''(0) = \frac{6a^2}{8}$ ***A1***

$f(x) = 1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + \ldots.$ ***AG***

[2 marks]

continued…

Question 12 continued.

(c) attempt to use $a=2$ or $a=4$ in the expansion ***M1***

$$(1-2x)^{-\frac{1}{2}} = 1+\frac{2x}{2}+\frac{3\times 4x^2}{8}\left(=1+x+\frac{3}{2}x^2+\ldots\right)$$

$$(1-4x)^{-\frac{1}{2}} = 1+\frac{4x}{2}+\frac{3\times 16x^2}{8}\left(=1+2x+6x^2+\ldots\right)$$ ***A1***

Note: Award ***A1*** for at least one correct.

attempt to multiply their two expansions together ***M1***

$$\left(1+x+\frac{3}{2}x^2+\ldots\right)\left(1+2x+6x^2+\ldots\right)=1+2x+6x^2+x+2x^2+\frac{3}{2}x^2+\ldots$$

$$=1+3x+\frac{19}{2}x^2+\ldots \text{ OR } \frac{2+4x+12x^2+2x+4x^2+3x^2+\ldots}{2}$$ ***A1***

$$(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}} \approx \frac{2+6x+19x^2}{2}$$ ***AG***

[4 marks]

(d) $|x|<\frac{1}{4}$ ***A1***

[1 mark]

continued…

Question 12 continued.

(e)

$\left(\frac{2+6x+19x^2}{2}\right)=\frac{2+0.6+0.19}{2}\left(=\frac{279}{200}\right)$ (or equivalent) ***A1***

attempt to substitute $x=\frac{1}{10}$ into $(1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}$ ***(M1)***

$\left((1-2x)^{-\frac{1}{2}}(1-4x)^{-\frac{1}{2}}=\right)\frac{10}{\sqrt{48}}$ (or equivalent) ***A1***

$\frac{10}{4\sqrt{3}}\approx\frac{279}{200}$ (or equivalent in terms of $\sqrt{3}$) ***A1***

$\frac{1}{\sqrt{3}}\approx\frac{279}{500}$

$\sqrt{3}\approx\frac{500}{279}$ ***A1***

[5 marks]

Total [20 marks]

4. (a) recognition that velocity is zero ***(M1)***

$v = 2\sin(0.5t) + 0.3t - 2 = 0$

$t = 1.68694\ldots$

$t = 1.69$ ***A1***

[2 marks]

(b) recognition that $v > 0$ ***(M1)***

$1.68694... < t < 6.11857...$

$1.69 < t < 6.12$ ***A1***

[2 marks]

(c) attempt to substitute into the total displacement formula (condone missing or incorrect limits, and absence of $\mathrm{d}t$) ***(M1)***

$\int_0^{10}(2\sin(0.5t) + 0.3t - 2)\,\mathrm{d}t$ OR $\int_0^{10} v(t)\,\mathrm{d}t$

$= -2.13464...$

$= -2.13\,(\mathrm{m})$ ***A1***

Note: Award ***(M1)A0*** if -2.13 is followed by 2.13.

[2 marks]

Total [6 marks]

7. $\frac{\mathrm{d}y}{\mathrm{d}x} - y = x$

recognition that an integrating factor is required ***(M1)***

$\mathrm{e}^{\int P(x)dx} \left(= \mathrm{e}^{\int -1dx}\right)$

$= \mathrm{e}^{-x}$ ***(A1)***

$\mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{-x}y = x\mathrm{e}^{-x}$

$\mathrm{e}^{-x}y\left(= \int x\mathrm{e}^{-x}\mathrm{d}x\right)$ ***A1***

attempt to integrate right hand side using integration by parts ***(M1)***

$\left(\mathrm{e}^{-x}y =\right) - x\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\mathrm{d}x$

$= -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$ ***A1***

$y = -x - 1 + c\mathrm{e}^{x}$

substitute initial values $y = 2$ and $x = 0$ into an integrated expression involving c ***(M1)***

$2 = -1 + c \Rightarrow c = 3$

$y = 3\mathrm{e}^{x} - x - 1$ ***A1***

[7 marks]

11. (a) attempt to use implicit differentiation ***M1***

$\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{e}^{x+y}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

attempt to expand brackets and collect $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms on the same side ***M1***

$\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}=2x-\mathrm{e}^{x+y}$ OR $\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^{x+y}=2x$ OR $\left(\mathrm{e}^{x+y}-2y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2x-\mathrm{e}^{x+y}$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-\mathrm{e}^{x+y}}{\mathrm{e}^{x+y}-2y}$ ***AG***

[5 marks]

(b) (i) recognition that $2x-\mathrm{e}^{x+y}=0$ at P and Q ***(M1)***

$\Rightarrow x+y=\ln(2x)$ ***A1***

recognition that $\mathrm{e}^{x+y}=x^2+y^2$ at P and Q ***(M1)***

$\Rightarrow 2x=x^2+\left(\ln(2x)-x\right)^2$ or equivalent ***A1***

$2x^2+\left(\ln(2x)\right)^2-2x\ln(2x)-2x=0$ ***AG***

continued…

Question 11 continued

(ii) $x = 0.331077\ldots$ or $1.84273\ldots$

$x = 0.331$ or 1.84 ***A1A1***

attempt to use $x + y = \ln(2x)$ OR $e^{x+y} = x^2 + y^2$ to find y-coordinates ***(M1)***

$y = -0.743332\ldots$, $y = -0.538335...$

coordinates $(0.331, -0.743)$ and $(1.84, -0.538)$ ***A1A1***

Note: If no working shown, award ***A1A1(M1)A0A0*** for $(0.33, -0.74)$ and $(1.8, -0.54)$ (2 sf)

[9 marks]

(c) coordinates $(-0.743, 0.331)$ and $(-0.538, 1.84)$ ***A1***

Note: Do not award ***FT*** from (b) if only one coordinate pair is given.

[1 mark]

(d) setting $\frac{2x - e^{x+y}}{e^{x+y} - 2y} = -1$ OR recognition that the point lies on line of symmetry ***(M1)***

$2x - e^{x+y} = 2y - e^{x+y}$

$y = x$ ***(A1)***

attempt to substitute $y = x$ into $e^{x+y} = x^2 + y^2$ ***(M1)***

$e^{2x} = 2x^2$ OR $e^{2y} = 2y^2$

$x = -0.451$, $y = -0.451$ ***A1***

coordinates $(-0.451, -0.451)$

Note: If no working shown, award ***M1A1(M1)A0*** for $x = -0.45$, $y = -0.45$ (2 sf)

[4 marks]

Total [19 marks]

1. (a) attempts to find an intersection point ***(M1)***

$a = -0.916562...$ or $b = 0$

$a = -0.917$, $b = 0$ ***A1A1***

[3 marks]

(b) let A be the area of the region

EITHER

attempts to form the required integral involving subtraction (in any order). Accept absence of limits or incorrect limits. Accept absence of dx. ***(M1)***

OR

shows a graph with the required area shaded ***(M1)***

THEN

$A = \left(\int_{a}^{b} \left(f(x) - g(x) \right) \mathrm{d}x \right) = \int_{-0.916562...}^{0} \left(1 - x^2 - \mathrm{e}^{2x} \right) \mathrm{d}x$ (or equivalent) ***(A1)***

$A = 0.239855...$

$A = 0.240$ ***A1***

[3 marks]

Total [6 marks]

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4. (a) $v = -0.996114...$

$v = -0.996$ (ms^{-1}) ***A1***

[1 mark]

(b)

considers $v'(t) = 0$ ***(M1)***

$t = 0.405833...$

$v_{\max} = 1.18230...$

$v_{\max} = 1.18$ (ms^{-1}) ***A1***

[2 marks]

(c) recognizes that the particle changes direction when $v = 0$ ***(M1)***

Note: Award ***(M1)*** for $t = 1.65840...$ seen.

finds acceleration for their value of *t* for which $v(t) = 0$ ***(M1)***

$v'(1.65840...)$

$a = -2.53487...$

$a = -2.53$ (ms^{-2}) ***A1***

[3 marks]

Total [6 marks]

6. attempts to solve $(V=)\frac{4}{3}\pi r^3 = 20$ for r ***(M1)***

$r = 1.68389...\left(=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}\right)$ (seen anywhere) ***A1***

attempts to use the chain rule ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ OR $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$ (or equivalent)

EITHER

$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}\left(=4\pi(1.68389...)^2\frac{\mathrm{d}r}{\mathrm{d}t}\right)\left(=4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}\frac{\mathrm{d}r}{\mathrm{d}t}\right)$ ***A1***

OR

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4\pi r^2}\frac{\mathrm{d}V}{\mathrm{d}t}\left(=\frac{1}{4\pi(1.68389...)^2}\frac{\mathrm{d}V}{\mathrm{d}t}\right)\left(=\frac{1}{4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}}\frac{\mathrm{d}V}{\mathrm{d}t}\right)$ ***A1***

THEN

attempts to find $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $\frac{\mathrm{d}V}{\mathrm{d}t} = 5$ ***(M1)***

$\frac{\mathrm{d}r}{\mathrm{d}t} = 0.140324...$

$\frac{\mathrm{d}r}{\mathrm{d}t} = 0.140\left(=\frac{5}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}\right)$ (cm s^{-1}) (accept 0.14) ***A1***

[6 marks]

7. attempts to express x (or x^2) in terms of y ***(M1)***

Note: Only award ***(M1)*** if base e is applied to both sides.

$x = \mathrm{e}^{\frac{y}{4}} + 2 \quad \left(x^2 = \left(\mathrm{e}^{\frac{y}{4}} + 2 \right)^2 = \mathrm{e}^{\frac{y}{2}} + 4\mathrm{e}^{\frac{y}{4}} + 4 \right)$ ***(A1)***

let V be the volume of the solid formed

forms a definite integral of the form $\pi \int_c^d x^2 \mathrm{d}y$ with their expression for x^2 in terms of y ***(M1)***

$V = \pi \int_0^4 \left(\mathrm{e}^{\frac{y}{4}} + 2 \right)^2 \mathrm{d}y \quad \left(= \pi \int_0^4 \left(\mathrm{e}^{\frac{y}{2}} + 4\mathrm{e}^{\frac{y}{4}} + 4 \right) \mathrm{d}y \right)$ ***(A1)***

$= 176.779...$

$= 177 \ \left(= 2\pi\left(\mathrm{e}^2 + 8\mathrm{e} - 1 \right) \right)$ (cubic units) ***A1***

[5 marks]

9. recognizes that $x=1 \Rightarrow ax^2+bx+c=0$ ***(M1)***

$a+b+c=0$ (seen anywhere) ***A1***

passes through $(2,1)$ so:

$1=\dfrac{2-4}{4a+2b+c}$ $(4a+2b+c=-2)$ (seen anywhere) ***A1***

local minimum point at $(2,1)$ so:

attempts to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ using quotient or product rule ***M1***

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(ax^2+bx+c\right)-\left(x-4\right)\left(2ax+b\right)}{\left(ax^2+bx+c\right)^2}$$

substitutes $x=2$ into the numerator of their $\dfrac{\mathrm{d}y}{\mathrm{d}x}(=0)$ ***(M1)***

$(4a+2b+c)-(2-4)(4a+b)=0$ $\left(\dfrac{(4a+2b+c)-(2-4)(4a+b)}{(4a+2b+c)^2}=0\right)$ ***A1***

$(12a+4b+c=0)$

Note: An incorrect numerator may lead to a correct equation.
In this instance, award ***A0*** here and do not award the final ***A*** mark.

attempts to solve their 3 linear equations in a,b and c ***(M1)***

$a=3, b=-11$ and $c=8$ ***A1***

Note: Three linear equations and a value for each of a,b and c need to be seen to gain the last ***M*** mark.

Note*:* The last ***M*** mark is dependent on an equation formed from the numerator of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

[8 marks]

SECTION B

12. (a) attempts to use Euler's method ***(M1)***

$x_{n+1} = x_n + \frac{\pi}{12}$; $y_{n+1} = y_n + \frac{\pi}{12} \times \frac{\mathrm{d}y}{\mathrm{d}x}$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = y\operatorname{cosec} 2x + \sqrt{\tan x}$

$y_1 = 1.25281... \left(= \frac{\pi}{4} + \frac{\pi}{12}\left(\frac{\pi}{4} + 1\right)\right)$ ***(A1)***

$y_2 = 1.97608...$

$y = 1.98$ ***A1***

[3 marks]

(b) attempts chain rule differentiation with multiplication of two derivatives ***(M1)***

$u = \cot x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\operatorname{cosec}^2 x$ and $y = \frac{1}{2}\ln u \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2u}$ OR

$u = \frac{\cos x}{\sin x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sin^2 x}$ and $y = \frac{1}{2}\ln u \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2u}$ OR

$u = \sqrt{\cot(x)} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{\cot(x)}} \times -\operatorname{cosec}^2 x$ and $\frac{dy}{du} = \frac{1}{\sqrt{\cot(x)}}$ OR

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln f(x)\right) = \frac{f'(x)}{2f(x)}$

THEN

$= -\frac{\operatorname{cosec}^2 x}{2\cot x}$ OR $-\frac{1}{\sin^2 x} \times \frac{\sin x}{2\cos x}$ ***A1***

$= -\frac{1}{2\sin x \cos x}$ ***(A1)***

$= -\frac{1}{\sin 2x}$ ***A1***

$= -\operatorname{cosec} 2x$ ***AG***

[4 marks]

continued…

Question 12 continued.

(c) **METHOD 1**

attempts to use $I(x) = e^{\int P(x)\,dx}$ ***(M1)***

$e^{\int -\operatorname{cosec} 2x\,dx}$ ***A1***

$= e^{\frac{1}{2}\ln(\cot x)}$ (***A1)***

$= e^{\ln\left(\sqrt{\cot x}\right)}$ ***A1***

$= \sqrt{\cot x}$ ***AG***

METHOD 2

attempts product rule differentiation on $\frac{d}{dx}\left(y\sqrt{\cot x}\right)$ ***M1***

$= \frac{dy}{dx}\sqrt{\cot x} - \frac{y\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$ ***A1***

$= \sqrt{\cot x}\left(\frac{dy}{dx} - y\frac{\operatorname{cosec}^2 x}{2\cot x}\right)$ ***A1***

$= \sqrt{\cot x}\left(\frac{dy}{dx} - y\operatorname{cosec} 2x\right)$ ***A1***

so $\sqrt{\cot x}$ is an integrating factor ***AG***

[4 marks]

continued…

Question 12 continued.

(d) $\sqrt{\cot x}\dfrac{\mathrm{d}y}{\mathrm{d}x} - y\operatorname{cosec} 2x\sqrt{\cot x} = \sqrt{\tan x}\sqrt{\cot x}$ (or equivalent) ***(M1)***

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{\cot x}\right) = 1$ ***(A1)***

$y\sqrt{\cot x} = \int 1\,\mathrm{d}x$ ***A1***

$y\sqrt{\cot x} = x(+C)$ or equivalent ***A1***

substitutes $x = \dfrac{\pi}{4}, y = \dfrac{\pi}{4} \Rightarrow C = 0$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = x\sqrt{\tan x}$ ***AG***

[5 marks]

(e) (i) $y = 2.52878...$

$y = 2.53$ ***A1***

(ii) the gradient changes substantially (in the neighbourhood of $x = \dfrac{5\pi}{12}$) ***R1***

Note: Award ***R0*** for saying the gradient is very large at $x = \dfrac{5\pi}{12}$

(iii) **EITHER**

the curve is concave up (over the interval) ***A1***

OR

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ (over the interval) ***A1***

[3 marks]

continued…

Question 12 continued.

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = y\operatorname{cosec} 2x + \sqrt{\tan x}\ \left(= x\sqrt{\tan x}\operatorname{cosec} 2x + \sqrt{\tan x}\right)$

$\operatorname{cosec} 2x, \sqrt{\tan x} > 0$ (for $0 < x < \frac{\pi}{2}$) ***R1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} > 0$ ***A1***

so the curve has a positive gradient for $0 < x < \frac{\pi}{2}$ ***AG***

Note: Do not award ***R0A1***.

[2 marks]

Total [21 marks]

2. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 + 3$ ***(M1)(A1)***

Note: Award ***M1*** for at least two correct terms.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 18x$ ***A1***

[3 marks]

(b) valid attempt to find x-coordinates (e.g. solving $12x^2 - 18x = 0$ or graphing $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 9x^2 + 3$ ***(M1)***

$x = 0, 1.5 \left(\frac{3}{2}\right)$ ***(A1)***

point B is $(0, 0)$ ***A1***

point C is $(1.5, -0.5625)$ $\left((1.5, -0.563), \left(\frac{3}{2}, -\frac{9}{16}\right)\right)$ ***A1***

Note: Award ***M0A0A1A0*** for an unsupported answer of point B is $(0, 0)$.

[4 marks]

(c) y-intercept $= 0$ (as line passes through $(0, 0)$) ***R1***

Note: Award ***R1*** for correctly substituting point B or point C to show that $y = 0$.

Award ***R1*** for $y = -0.375x + 0$.

gradient $= \frac{-0.5625}{1.5}$ $\left(= \frac{-\frac{9}{16}}{\frac{3}{2}}\right)$ ***A1***

Note: Award ***A0FT*** if their answer to (b) doesn't lead to the given answer, but condone $\frac{-0.563}{1.5}$.

$= -0.375$

so equation is $y = -0.375x$ ***AG***

Note: Award at most ***A1R0*** for working backwards to verify points B and C lie on the given line.

[2 marks]

continued…

Question 2 continued

(d) $x^4 - 3x^3 + 3x = -0.375x$ ***(M1)***

2.427 ***A1***

Note: Correct answer must be given to three decimal places for the ***A1*** to be awarded.

Award ***(M1)A1*** for $(2.427, -0.910)$.

Award ***(M1)A0*** for an unsupported answer of 2.43.

[2 marks]

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 3mx^2 + n$ (***A1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 6mx$ (***A1)***

$6x(2x - m) = 0$

$x = 0, \frac{m}{2}$ ***A1***

Note: Accept $0m$ in place of 0.

[3 marks]

(f) (i) point B is $(0, 0)$ ***A1***

[1 mark]

(ii) $y_c = \left(\frac{m}{2}\right)^4 - m\left(\frac{m}{2}\right)^3 + n\left(\frac{m}{2}\right) \quad \left(= \frac{m^4}{16} - \frac{m^4}{8} + \frac{nm}{2}\right)$ ***(A1)***

$= -\frac{m^4}{16} + \frac{nm}{2}$ OR $\frac{8nm - m^4}{16}$ OR $\frac{m}{2}\left(n - \frac{m^3}{8}\right)$ OR equivalent simplification ***A1***

Point C is $\left(\frac{m}{2}, -\frac{m^4}{16} + \frac{nm}{2}\right)$

Note: Award the second ***A1*** if the simplified y-coordinate is seen here or in part (g).

[2 marks]

continued…

Question 2 continued

(g) attempt to divide their y_C by their x_C OR substitute into $\frac{y_2 - y_1}{x_2 - x_1}$ ***(M1)***

e.g. $\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \div \frac{m}{2}$

$\left(-\frac{m^4}{16} + \frac{nm}{2}\right) \times \frac{2}{m}$ **OR** equivalent manipulation leading to given answer ***A1***

Note: Award ***A0 FT*** if their answer does not lead to the ***AG***.

$= -\frac{m^3}{8} + n$ and y-intercept $= 0$,

so equation is $y = \left(-\frac{m^3}{8} + n\right)x$ ***AG***

Note: Award at most ***M1A0*** for working backwards to verify points B and C lie on the given line.

[2 marks]

(h) **METHOD 1**

$x^4 - mx^3 + nx = -\frac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \frac{m^3}{8}x = 0$

recognizing their $(x - x_C)$ is a factor of this expression ***(M1)***

$x\left(x - \frac{m}{2}\right)\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)\ (= 0)$ OR equivalent ***A1M1A1***

Note: Award ***A1*** for $x\left(x - \frac{m}{2}\right)$ **,** ***M1A1*** for $\left(x^2 - \frac{m}{2}x - \frac{m^2}{4}\right)$.

If a candidate divides by x without justification, do not award the first ***A1***, but all subsequent marks can still be awarded.

continued…

Question 2 continued

use of quadratic formula to find roots, x_A (and x_D), of the quadratic ***M1***

$x_A = \dfrac{\dfrac{m}{2} - \sqrt{\dfrac{m^2}{4} + m^2}}{2}$ OR equivalent ***A1***

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$x_A = \dfrac{m}{4} - \dfrac{m}{4}\sqrt{5}$ ***AG***

Note: Award ***(M1)(M0)A0M0A0M0A0*** for attempting to verify the given answer satisfies $x^4 - mx^3 + nx = -\dfrac{m^3}{8}x + nx$.

METHOD 2

$x^4 - mx^3 + nx = -\dfrac{m^3}{8}x + nx$

attempt to rearrange this equation to equal zero ***(M1)***

$x^4 - mx^3 + \dfrac{m^3}{8}x = 0$

$x\left(x^3 - mx^2 + \dfrac{m^3}{8}\right) = 0$

Recognise that their x_C is a root of this equation ***(M1)***

Attempt to find sum and product of roots ***(M1)***

$x_A + x_D + \dfrac{m}{2} = m$ AND $x_A \times x_D \times \dfrac{m}{2} = -\dfrac{m^3}{8}$ ***A1***

$x_A \times \left(\dfrac{m}{2} - x_A\right) \times \dfrac{m}{2} = -\dfrac{m^3}{8}$

$x_A{}^2 - \dfrac{m}{2}x_A - \dfrac{m^2}{4} = 0$ ***A1***

use of quadratic formula to find roots, x_A (and x_D), of the quadratic ***M1***

continued…

Question 2 continued

$$x_A = \frac{\frac{m}{2} - \sqrt{\frac{m^2}{4} + m^2}}{2}$$ OR equivalent ***A1***

Note: Condone a $\pm$ in place of the minus sign, provided given answer is restated.

$$x_A = \frac{m}{4} - \frac{m}{4}\sqrt{5}$$ ***AG***

[7 marks]

(i) $$\frac{0 - \left(\frac{m}{4} - \frac{m}{4}\sqrt{5}\right)}{\frac{m}{2} - 0}$$ ***(A1)***

$$= \frac{\sqrt{5} - 1}{2}$$ ***A1***

Note: Answer must be exact.

[2 marks]

Total [28 marks]

1. (a) (i) attempts chain rule differentiation to find $f'(x)$ ***(M1)***

$f'(x) = \frac{2}{(2-x)^3}$ $\left(= (-1)(-2)(2-x)^{-3}\right)$ ***A1***

Note: Award ***(M1)*** for attempting chain rule differentiation on $(4-4x+x^2)^{-1}$ or attempting quotient rule differentiation on $\frac{1}{(4-4x+x^2)}$ $\left(= \frac{1}{(2-x)^2}\right)$.

Award ***A1*** for $f'(x) = \frac{2}{(2-x)^3}$ $\left(= (-1)(-2)(2-x)^{-3}\right)$.

[2 marks]

(ii) $g'(x) = 2x$ ***(A1)***

$f'(x)g'(x) = \left(2(2-x)^{-3}\right)(2x)$ $\left(= \frac{2(2x)}{(2-x)^3}\right)$ (or equivalent) ***A1***

$= \frac{4x}{(2-x)^3}$ ***AG***

[2 marks]

continued…

Question 1 continued.

(iii)

Note: Award a maximum of ***(M1)A1(M1)A0FT*** from parts (a) (i) and (ii).

substitutes $f(x), g(x)$ and their $g'(x), f'(x)$ into the given expression ***(M1)***

EITHER

$f(x)g'(x)+g(x)f'(x)=2x(2-x)^{-2}+2x^2(2-x)^{-3}$ ***A1***

Note: Award ***A1*** if $f(x)g'(x)=2x(2-x)^{-2}$ and $g(x)f'(x)=2x^2(2-x)^{-3}$ are stated separately.

attempts to factorise their expression ***(M1)***

$=2x(2-x)^{-3}\left((2-x)+x\right)$ ***A1***

OR

$f(x)g'(x)+g(x)f'(x)=\frac{2x}{(2-x)^2}+\frac{2x^2}{(2-x)^3}$ ***A1***

Note: Award ***A1*** if $f(x)g'(x)=\frac{2x}{(2-x)^2}$ and $g(x)f'(x)=\frac{2x^2}{(2-x)^3}$ are stated separately.

attempts to form an expression with a common denominator ***(M1)***

Note: Award ***(M1)*** for $(2-x)^2(2-x)^3$ as a common denominator.

$=\frac{2x(2-x)}{(2-x)^3}+\frac{2x^2}{(2-x)^3}\left(=\frac{4x-2x^2+2x^2}{(2-x)^3}\right)$ ***A1***

THEN

$=\frac{4x}{(2-x)^3}$

AG

Note: Award marks as appropriate for attempting to find the derivative of $f(x)g(x)=\frac{x^2}{(2-x)^2}$ (or equivalent).

[4 marks]

continued…

Question 1 continued.

(b) **METHOD 1**

$f'(x)g'(x)-g(x)f'(x)=f(x)g'(x)$ ***(A1)***

$\left(f'(x)g'(x)-g(x)f'(x)-f(x)g'(x)=0\right)$

$f'(x)\left(g'(x)-g(x)\right)=f(x)g'(x)\ \ \left(f'(x)\left(g(x)-g'(x)\right)=-f(x)g'(x)\right)$ ***A1***

$\dfrac{f'(x)}{f(x)}=\dfrac{g'(x)}{g'(x)-g(x)}$ ***AG***

Note: Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 2

$g'(x)=\dfrac{f'(x)g'(x)}{f(x)}-\dfrac{g(x)f'(x)}{f(x)}$ ***(A1)***

$g'(x)=\dfrac{f'(x)}{f(x)}\left(g'(x)-g(x)\right)$ ***A1***

$\dfrac{f'(x)}{f(x)}=\dfrac{g'(x)}{g'(x)-g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

continued…

Question 1 continued.

METHOD 3

$g'(x) = \frac{f(x)g'(x)}{f'(x)} + g(x)$ ***(A1)***

$\frac{f(x)}{f'(x)} = \frac{g'(x) - g(x)}{g'(x)}$ ***A1***

$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

METHOD 4

$\frac{f(x)}{f'(x)} + \frac{g(x)}{g'(x)} = 1$ ***(A1)***

$\frac{f(x)}{f'(x)} = \frac{g'(x) - g(x)}{g'(x)}$ ***A1***

$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g'(x) - g(x)}$ ***AG***

Note: Candidates may not show the steps exactly as shown above.
Award ***(A0)A0*** for use of $f(x)$ and $g(x)$ from part (a).

[2 marks]

continued…

Question 1 continued.

(c) **METHOD 1**

Note: Condone the absence of '$\mathrm{d}x$' and the modulus sign throughout.
Only award the second ***A*** mark if the constant of integration has been dealt with correctly.

EITHER

$$\ln f(x) = \int \frac{g'(x)}{g'(x) - g(x)} \mathrm{d}x \ (+C)$$ ***A1***

Note: Condone the absence of '$+C$' when awarding the first ***A*** mark.

$$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} \mathrm{e}^{C} \quad \left(f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x + C\right)} \right)$$ ***A1***

Note: Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$, where $A = \mathrm{e}^{C}$.

OR

$$\ln f(x)(+C) = \int \frac{g'(x)}{g'(x) - g(x)} \mathrm{d}x$$ ***A1***

Note: Condone the absence of '$+C$' when awarding the first ***A*** mark.

$$f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)} \mathrm{e}^{-C} \quad \left(f(x)\mathrm{e}^{C} = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}, f(x) = \mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x - C\right)} \right)$$ ***A1***

Note: Award ***A1*** for $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$, where $A = \mathrm{e}^{-C}$.

THEN

$$f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)} \mathrm{d}x\right)}$$ ***AG***

continued…

Question 1 continued.

METHOD 2

Note: Condone the absence of '$\mathrm{d}x$' throughout.

$$f'(x)-\frac{g'(x)}{g'(x)-g(x)}f(x)=0$$

Integrating factor: $\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$ ***A1***

Note: Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=C$.

$$\frac{d}{dx}\left[f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}\right]=0\Rightarrow f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=A$$ ***A1***

Note: Award ***A1*** for $f(x)\mathrm{e}^{\left(-\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}=C$.

$$f(x)=A\mathrm{e}^{\left(\int\frac{g'(x)}{g'(x)-g(x)}\mathrm{d}x\right)}$$ ***AG***

[2 marks]

continued…

Question 1 continued.

(d) $g'(x) = xe^x + e^x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\frac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$= \frac{xe^x + e^x}{e^x} \left(= \frac{e^x(x+1)}{e^x} \right)$

$= x + 1$ (as $e^x \neq 0$) ***(A1)***

attempts to integrate their $\frac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$\int (x+1)\,dx = \frac{1}{2}x^2 + x (+C)$

$f(x) = e^{\left(\frac{1}{2}x^2 + x\right)}$ ***A1***

Note: Award ***A0*** for $f(x) = e^{\left(\frac{1}{2}x^2 + x + C\right)}$ (or equivalent expressed with an arbitrary constant).

[5 marks]

continued…

Question 1 continued.

(e) $g'(x) = \cos x - \sin x$ (seen anywhere) ***(A1)***

attempts to find an expression for $\dfrac{g'(x)}{g'(x) - g(x)}$ ***(M1)***

$= \dfrac{\cos x - \sin x}{\cos x - \sin x - \sin x - \cos x} \left(= \dfrac{\sin x - \cos x}{2\sin x} \right)$

$= \dfrac{1}{2} - \dfrac{1}{2}\cot x$ (as $\sin x \neq 0$) OR $= \dfrac{1}{2} - \dfrac{1}{2}\dfrac{\cos x}{\sin x}$ (as $\sin x \neq 0$) ***A1***

$f(x) = \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{2}\cot x\right) \mathrm{d}x}$

attempts to find the indefinite integral of $(\pm k)\cot x$ OR $(\pm k)\dfrac{\cos x}{\sin x}$ ***(M1)***

Note: As $|\sin x| = \sin x$ for $0 < x < \pi$, condone the presence or omission of the modulus sign throughout a candidate's solution.
Condone the presence of an arbitrary constant except when awarding the final ***A*** mark.

$\int \left(\dfrac{1}{2} - \dfrac{1}{2}\cot x \right) \mathrm{d}x = \dfrac{x}{2} - \dfrac{1}{2}\ln|\sin x| (+C) \left(= \dfrac{1}{2}\left(x - \ln|\sin x| (+C)\right) \right)$ ***A1***

$f(x) = \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{-\frac{1}{2}\ln|\sin x|} \left(\mathrm{e}^{C}\right)$

$= \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{\ln\sqrt{\frac{1}{\sin x}}} \left(\mathrm{e}^{C}\right) \left(= \mathrm{e}^{\frac{x}{2}} \mathrm{e}^{\frac{1}{2}\ln\left(\frac{1}{\sin x}\right)} \left(\mathrm{e}^{C}\right), = \sqrt{\mathrm{e}^{x - \ln(\sin x)}} \left(\mathrm{e}^{C}\right) \right)$ ***A1***

$= \mathrm{e}^{\frac{x}{2}} \sqrt{\dfrac{1}{\sin x}}$

$= \sqrt{\mathrm{e}^{x} \operatorname{cosec} x} \left(= \sqrt{\dfrac{\mathrm{e}^{x}}{\sin x}} \right)$ (where $h(x) = \dfrac{1}{\sin x}$) ***A1***

[7 marks]

Total [24 marks]

7. (a) attempt to find $f(0)=\sqrt{2}, f\left(\frac{\pi}{4}\right)=1$ or $f\left(\frac{\pi}{2}\right)=\sqrt{2}$ or sketch of graph ***(M1)***

$1 \leq f(x) \leq \sqrt{2}$ ***A1A1***

Note: Award ***A1A0*** for strong inequality seen. Allow equivalent, and/or interval notation.

[3 marks]

(b) consider $\pi\int_0^{\frac{\pi}{2}} \sec^2\left(x-\frac{\pi}{4}\right) \mathrm{d}x$ ***M1***

Note: For the ***M1,*** condone incorrect or missing limits and omission of π.

$= \pi\left[\tan\left(x-\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \pi\left[\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)\right]$ ***(A1)***

$= \pi\left(1-\left(-1\right)\right)$

$= 2\pi$ ***A1***

[4 marks]

Total [7 marks]

10. (a) outer curved surface area is $2\pi(4r)h$ AND inner curved surface area is $2\pi rh$ ***(A1)***

area of each base (top and bottom) is $\pi(4r)^2 - \pi r^2$ ***(A1)***

$S = 2\left[\pi(4r)^2 - \pi r^2\right] + 2\pi(4r)h + 2\pi rh$ ***A1***

$= 30\pi r^2 + 10\pi rh$ ***AG***

[3 marks]

(b) $30\pi r^2 + 10\pi rh = 240\pi$

attempt to solve their equation for h or rh in terms of r (must isolate h or rh) ***(M1)***

$h = \dfrac{240 - 30r^2}{10r}\left(= \dfrac{24 - 3r^2}{r}\right)$ OR $rh = \dfrac{240 - 30r^2}{10}\left(= 24 - 3r^2\right)$ (or equivalent)

A1

uses volume = large cylinder – small cylinder ***(M1)***

$V = \pi(4r)^2 h - \pi r^2 h \quad \left(= 16\pi r^2 h - \pi r^2 h = 15\pi r^2 h\right)$ ***A1***

attempt to substitute in for h or rh ***(M1)***

$V = 15\pi r^2\left(\dfrac{24 - 3r^2}{r}\right)$ OR $V = 15\pi r\left(\dfrac{240 - 30r^2}{10}\right)\left(= 15\pi r\left(24 - 3r^2\right)\right)$ OR

$384\pi r - 48\pi r^3 - 24\pi r + 3\pi r^3$ ***A1***

$= 360\pi r - 45\pi r^3$ ***AG***

[6 marks]

(c) $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 360\pi - 135\pi r^2$ ***A1A1***

[2 marks]

continued…

Question 10 continued.

(d) **METHOD 1** (working with r)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi r^2=0$

$r^2=\frac{360}{135}\left(=\frac{8}{3}\right)$

$r=\sqrt{\frac{360}{135}}\left(=\sqrt{\frac{8}{3}}\right)$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

METHOD 2 (working with $p\sqrt{\frac{2}{3}}$)

recognition that (for a maximum) $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ ***M1***

$360\pi-135\pi\left(p\sqrt{\frac{2}{3}}\right)^2=0$

$360-90p^2=0$

$p^2=4$ ***A1***

$p=2$ OR $r=2\sqrt{\frac{2}{3}}$ ***A1***

[3 marks]

continued…

Question 10 continued.

(e) attempt to substitute their value of r into $V = 360\pi r - 45\pi r^3$ ***M1***

$$V = 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times \left(2\sqrt{\frac{2}{3}}\right)^3$$

$$= 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times 8 \times \frac{2}{3} \times \sqrt{\frac{2}{3}}$$

$$\left(= 720\pi\sqrt{\frac{2}{3}} - 240\pi\sqrt{\frac{2}{3}}\right) \quad \textbf{(A1)}$$

$$= 480\pi\sqrt{\frac{2}{3}} \quad \textbf{A1}$$

[3 marks]

Total [17 marks]

11. (a) **EITHER**

attempt to use l'Hôpital's ***(M1)***

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\frac{2e^{2x}}{2e^{2x}}$$ ***A1***

OR

attempt to divide each term by e^{2x} ***(M1)***

$$\lim_{x\to\infty}\left(\frac{e^{2x}-1}{e^{2x}+1}\right)=\lim_{x\to\infty}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)$$ ***A1***

THEN

$=1$ ***AG***

[2 marks]

(b) (i) attempt to use quotient rule or product rule ***M1***

$$\frac{dy}{dx}=\frac{\left(e^{2x}+1\right)2e^{2x}-\left(e^{2x}-1\right)2e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for second term in numerator. If denominator is incorrect award ***A1A0.***

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***AG***

(ii) attempt to substitute for y and express as a single fraction ***M1***

$$1-y^2=1-\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}$$

$$=\frac{\left(e^{2x}+1\right)^2-\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}\text{ or equivalent}$$ ***(A1)***

$$=\frac{e^{4x}+2e^{2x}+1-\left(e^{4x}-2e^{2x}+1\right)}{\left(e^{2x}+1\right)^2}\quad\text{OR}\quad\frac{2e^{2x}\times 2}{\left(e^{2x}+1\right)^2}$$ ***A1***

$$=\frac{4e^{2x}}{\left(e^{2x}+1\right)^2}$$ ***AG***

[6 marks]

continued…

Question 11 continued.

(c) (i) attempt to use implicit differentiation ***M1***

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} =\right) -2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***

$= -2y\left(1-y^2\right)$ ***A1***

$= 2y^3 - 2y$ ***AG***

(ii) $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3} =\right)\left(6y^2-2\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***

$= \left(6y^2-2\right)\left(1-y^2\right)\left(= 8y^2 - 6y^4 - 2\right)$ ***A1***

[5 marks]

(d) attempt to evaluate at least three of y, y', y'', y''' at $x = 0$ ***(M1)***

$y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = -2$ ***A1***

attempt to use Maclaurin series $y \approx y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$ ***M1***

$y \approx 0 + 1x + 0x^2 + \frac{(-2)}{3!}x^3 + \ldots$

$y \approx x - \frac{x^3}{3} + \ldots$ ***A1***

[4 marks]

Total [17 marks]

10. (a) (i) $a = 0.358$ (exact); $b = 30.5$ (exact answer is 30.52) ***A1A1***

Note: Award ***A1A0*** if the values of a and b are interchanged or not labeled.

(ii) a represents the (average) rate of increase (change) in population (0.358 millions of people per year). (or equivalent) ***R1***

[3 marks]

(b) It is unreliable because 2030 is outside the range of data (extrapolation). ***A1***

[1 mark]

(c) (i) attempt to find $B(100)$ ***(M1)***

$55.1633\ldots$

55.2 million OR $55{,}200{,}000$ ***A1***

(ii) The annual growth rate of the population is 0.5%. ***A1***

Note: Description must include some reference to annual rate.

[3 marks]

(d) $54.6094\ldots$

54.6 million OR $54{,}600{,}000$ ***A1***

[1 mark]

(e) consideration of the difference function $C(t) - B(t)$ or $B(t) - C(t)$ or $|C(t) - B(t)|$ ***(M1)***

evidence of finding the maximum (or minimum) of this function. ***(M1)***

$t = 58.6283\ldots$

2058 (accept 2059) ***A1***

[3 marks]

continued…

Question 10 continued.

(f) (i) $0.242876\ldots$
$B'(75) = 0.243$ ***A1***

(ii) $0.184941...$
$C'(75) = 0.185$ ***A1***

[2 marks]

(g) $B'(75) > C'(75)$ (or equivalent in words) ***A1***

the population in Benoit's model is increasing at a faster rate than in Cecilia's model (in 2075) (or equivalent) ***R1***

Note: Do not award ***A0R1***.

[2 marks]

Total [15 marks]

1. $f'(x) = 4x^2 - 16$ ***A1***

sets their derivative equal to zero ***(M1)***

$4x^2 - 16 = 0$, $(x = \pm 2)$

$p = 2$ (accept $x = 2$) ***A1***

substitutes their **positive** *p* into $f(x)$ ***(M1)***

$y = \frac{4\left(2^3\right)}{3} - 16(2)\left(= \frac{32}{3} - 32 = -\frac{64}{3}\right)$

$q = -\frac{64}{3}$ (accept $y = \frac{-64}{3}$) ***A1***

Total [5 marks]

3. METHOD 1

attempt to set up integral $e^x - \left(-e^x\right) = 2e^x$ or e^x and then double ***(M1)***

$\int \left(e^x - \left(-e^x\right)\right)dx$ OR $2\int e^x\,dx$

$= 2\int_{-1}^{1} e^x\,dx$

$= 2\left[e^x\right]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into their integrated function and subtract ***(M1)***

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

METHOD 2

$\int_{-1}^{1} e^x\,dx = \left[e^x\right]_{-1}^{1}$ and $\int_{-1}^{1} -e^x\,dx = \left[-e^x\right]_{-1}^{1}$ ***(A1)***

attempt to substitute correct limits into both their integrated functions and subtract ***(M1)***

$e^1 - e^{-1}$ and $-e^1 - (-e^{-1})$

subtracts their two integrals in correct order ***(M1)***

$e^1 - e^{-1} - \left(-e^1 + e^{-1}\right)$

$= 2\left(e - \frac{1}{e}\right)$, $2e - \frac{2}{e}$, $2e - 2e^{-1}$ ***A1***

Total [4 marks]

9. $y = vx,\ \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{x - vx}{x + vx}\left(= \frac{1 - v}{1 + v}\right)$

$x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{1 - v}{1 + v}$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1 - v}{1 + v} - v\left(= \frac{1 - 2v - v^2}{1 + v}\right)$

attempt to separate variables and form two integrals ***(M1)***

$\int \frac{1 + v}{1 - 2v - v^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$

$\int \frac{1}{x}\mathrm{d}x = \ln x + c$ ***A1***

use of substitution or inspection to integrate $\frac{1 + v}{1 - 2v - v^2}$ or equivalent ***(M1)***

$u = 1 - 2v - v^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}v} = -2 - 2v = -2(1 + v)$

$\int \frac{1 + v}{1 - 2v - v^2}\mathrm{d}v = -\frac{1}{2}\ln\left|1 - 2v - v^2\right|$ OR $-\frac{1}{2}\ln\left|v^2 + 2v - 1\right| = \ln|x| + c$ ***A1***

$-\frac{1}{2}\ln\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| - \ln|x| = c$ OR $-\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| - \ln|x| = c$

EITHER

attempt to substitute $x = 2$ and either $y = 0$ or $v = 0$ to find a constant c ***(M1)***

$c = -\ln 2$

$-\frac{1}{2}\ln\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| - \ln|x| = -\ln 2$ OR $-\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| - \ln|x| = -\ln 2$ ***A1***

OR

$x^2\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| = A$ OR $x^2\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = A$ ***A1***

attempt to substitute $x = 2$ either $y = 0$ or $v = 0$ to find a constant A ***(M1)***

THEN

$x^2\left|1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right| = 4$ OR $x^2\left(1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right) = 4$ ***A1***

checking boundary values confirms $x^2 - 2xy - y^2 = 4$ ***AG***

continued…

Question 9 continued.

Note: Condone absence of absolute value signs even if removed incorrectly until the final ***A1*** mark where they must be seen or have been removed to form a correct equation.

Total [8 marks]

Section B

12. (a) attempt to use integration by parts on $f_n(x) = \cos^{n-1} x \cos x$ ***M1***

$u = \cos^{n-1} x,\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = -(n-1)\cos^{n-2} x \sin x,\ v = \sin x$

$\int f_n(x)\mathrm{d}x = \cos^{n-1} x \sin x + \int (n-1)\cos^{n-2} x \sin^2 x \mathrm{d}x$ ***A1A1***

Note: ***A1*** for each term with correct signs

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1 - \cos^2 x\right)\mathrm{d}x$ ***A1***

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \mathrm{d}x - (n-1)\int \cos^n x \mathrm{d}x$ ***AG***

[4 marks]

(b) attempt to rearrange the equation given in part (a) to collect terms in $\int f_n(x)\mathrm{d}x$ or $\int \cos^n x \mathrm{d}x$ ***M1***

$\int f_n(x)\mathrm{d}x + (n-1)\int f_n(x)\mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)\mathrm{d}x$ OR

$\int \cos^n x \mathrm{d}x + (n-1)\int \cos^n x \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \mathrm{d}x$

$n\int f_n(x)\mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int f_{n-2}(x)\mathrm{d}x$ OR

$n\int \cos^n x \mathrm{d}x = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \mathrm{d}x$ ***A1***

$\int f_n(x)\mathrm{d}x = \dfrac{1}{n}\cos^{n-1} x \sin x + \dfrac{n-1}{n}\int f_{n-2}(x)\mathrm{d}x$ ***AG***

[2 marks]

continued…

Question 12 continued.

(c) attempt to use equation from part (a) to reduce the power of $\cos x$ ***(M1)***

$$\int \cos^4 x\mathrm{d}x = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\mathrm{d}x$$

EITHER

attempt to use equation from part (a) again to reduce the power of $\cos x$ ***(M1)***

$$\int \cos^4 x\mathrm{d}x = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\mathrm{d}x\right)$$ ***(A1)***

OR

attempt to use double angle formula to rewrite $\cos^2 x$ in terms of $\cos 2x$ ***(M1)***

$$\int \cos^4 x\mathrm{d}x = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int\left(\frac{\cos 2x+1}{2}\right)\mathrm{d}x$$ ***(A1)***

THEN

$$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + c$$ ***A1***

[4 marks]

(d) attempt to use the formula for volume of revolution using π and $\left(\cos^2 x\right)^2$ ***(M1)***

$$\text{volume} = \int \pi\left(\cos^2 x\right)^2\mathrm{d}x$$

$$\text{volume} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2\mathrm{d}x \quad \text{OR} \quad 2\int_{0}^{\frac{\pi}{2}} \pi\left(\cos^2 x\right)^2\mathrm{d}x$$ ***(A1)***

Note: Condone omission $\mathrm{d}x$ for the ***A1***.

attempt to substitute correct limits into their (c) and subtract ***(M1)***

$$\pi\left(\frac{1}{4}\cos^3\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3}{8}\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - \pi\left(\frac{1}{4}\cos^3\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \frac{3}{8}\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{3\pi^2}{8}$$ ***A1***

Note: ***FT*** marks may be awarded for a final answer of $r\pi^2$ based on non-zero values of p, q and r.

[4 marks]

continued…

Question 12 continued.

(e) (i) **METHOD 1**

attempt to raise Maclaurin expansion for $\cos x$ to the power of n ***(M1)***

$$f_n(x) = (\cos x)^n = \left(1 - \frac{x^2}{2} + \ldots\right)^n$$

$$= 1 - \frac{nx^2}{2} + \ldots$$ ***A2***

METHOD 2

attempt to differentiate $f_n(x)$ twice ***(M1)***

$f_n'(x) = -n\cos^{n-1} x \sin x,\ f_n''(x) = -n\cos^n x + n(n-1)\cos^{n-2} x \sin^2 x$ ***A1***

$f_n(0) = 1,\ f_n'(0) = 0,\ f_n''(0) = -n$

$$f_n(x) = 1 - \frac{nx^2}{2} + \ldots$$ ***A1***

(ii) **METHOD 1**

attempt to use Maclaurin expansion for $f_n(x)$ ***(M1)***

$$\lim_{x\to 0} \frac{f_n(x) - 1}{x^2} = \lim_{x\to 0} \frac{-\frac{nx^2}{2} + \ldots}{x^2} \left(= \lim_{x\to 0}\left(-\frac{n}{2} + \text{ powers of } x\right)\right)$$

$$= -\frac{n}{2}$$ ***A1***

METHOD 2

attempt to use l'Hôpital's rule twice on $\frac{\cos^n(x) - 1}{x^2}$ ***(M1)***

$$\lim_{x\to 0} \frac{\cos^n(x) - 1}{x^2} = \lim_{x\to 0} \frac{-n\cos^{n-1} x \sin x}{2x} = \lim_{x\to 0} \frac{n(n-1)\cos^{n-2} x \sin^2 x - n\cos^n x}{2}$$

$$= -\frac{n}{2}$$ ***A1***

Note: Do not award ***FT*** marks for an expression that does not involve n.

[5 marks]

Total [19 marks]

6.

(a) correct substitution into distance formula ***A1***

$l^2 = x^2 \ln x + 4 - x^2 + x^2$ OR $\sqrt{(x-0)^2 + \left(\sqrt{x^2 \ln x + 4 - x^2} - 0\right)^2}$ OR

$\sqrt{x^2 + x^2 \ln x + 4 - x^2}$

$l = \sqrt{x^2 \ln x + 4}$ ***AG***

[1 mark]

(b) recognising $\frac{\mathrm{d}l}{\mathrm{d}x} = 0$ (seen anywhere) ***(M1)***

EITHER

attempt to use chain rule with l ***(M1)***

$\frac{1}{2}\left(x^2 \ln x + 4\right)^{-\frac{1}{2}} \times \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x + 4\right)$

attempt to use product rule with $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x + 4\right)$ ***(M1)***

$\frac{1}{2}\left(x^2 \ln x + 4\right)^{-\frac{1}{2}} \times \left[x^2 \times \frac{1}{x} + \ln x \times 2x\right]$ ***A1***

OR

recognising to minimise $x^2 \ln x + 4$ ***(M1)***

attempt to use product rule ***(M1)***

$x^2 \times \frac{1}{x} + \ln x \times 2x$ ***A1***

THEN

$x + 2x \ln x = 0$ (or equivalent) ***A1***

$\Rightarrow \ln x = -\frac{1}{2}$

$x = \mathrm{e}^{-\frac{1}{2}} \left(= \frac{1}{\sqrt{\mathrm{e}}}\right)$ ***A1***

Note: Award ***A0*** for including $x = 0$ in the final answer.

[6 marks]

Total [7 marks]

7. $x^2 = r^4 - y^4$ (seen anywhere) (***A1)***

substituting their expression for x^2 into $\int_a^b \pi x^2 \mathrm{d}y$ ***(M1)***

$\pi\int_{-r}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ **OR** $2\pi\int_{0}^{r}\left(\mathrm{r}^4 - y^4\right)\mathrm{d}y$ (***A1)***

$= \pi\left[r^4 y - \frac{y^5}{5}\right]_{-r}^{r}$ **OR** $2\pi\left[r^4 y - \frac{y^5}{5}\right]_{0}^{r}$ ***A1***

Note: Award ***A1*** for $r^4 y - \frac{y^5}{5}$ seen.

$= \pi\left(\left(r^5 - \frac{r^5}{5}\right) - \left(-r^5 + \frac{r^5}{5}\right)\right)$ **OR** $2\pi\left(r^5 - \frac{r^5}{5}\right)$ ***(A1)***

$= \frac{8}{5}\pi\mathrm{r}^5$ ***A1***

Total [6 marks]

9. METHOD 1

$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}=\frac{0}{0}$ indeterminate form, attempt to apply l'Hôpital's rule ***R1***

Note: Subsequent marks are independent of this ***R*** mark.

$=\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}\ \left(=\lim_{x\to 0}\left(1+\frac{x\cos x}{\sin x}\right)\right)$ ***A1***

EITHER

Applying l'Hôpital's rule again (since $\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=\frac{0}{0}$) ***(M1)***

Note: Award ***(M0)*** if their limit is not the indeterminate form $\frac{0}{0}$.

$\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{\cos x}$ ***A1***

OR

$=\lim_{x\to 0}\left(1+\frac{\cos x}{\left(\frac{\sin x}{x}\right)}\right)$ ***A1***

recognizing fundamental trig limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ ***R1***

THEN

substituting $x=0$ ***(M1)***

$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$ ***A1***

continued…

Question 9 continued

METHOD 2

multiplying numerator and denominator by $1+\cos x$ ***M1***

$$\lim_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}\right)$$

$$\lim_{x\to 0}\left(\frac{x\sin x\left(1+\cos x\right)}{1-\cos^2 x}\right)$$

$$\lim_{x\to 0}\left(\frac{x\left(1+\cos x\right)}{\sin x}\right)$$ ***A1***

$$=\lim_{x\to 0}\left(\frac{1+\cos x}{\left(\frac{\sin x}{x}\right)}\right)$$ ***(A1)***

recognizing fundamental trig limit $\lim_{x\to 0}\frac{\sin x}{x}=1$ OR $\lim_{x\to 0}\frac{x}{\sin x}=1$ ***(R1)***

substituting $x=0$ ***(M1)***

$$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$$ ***A1***

continued…

Question 9 continued

METHOD 3

expressing either $x\sin x$ or $1-\cos x$ as a Maclaurin series with at least two terms ***(M1)***

$$\lim_{x\to 0}\left(\frac{x\left(x-\frac{x^3}{3!}\left(+\frac{x^5}{5!}\ldots\right)\right)}{\frac{x^2}{2!}-\frac{x^4}{4!}\left(+\frac{x^6}{6!}+\ldots\right)}\right)$$ ***A1***

recognizing numerator and denominator are multiples of x^2 ***(M1)***

$$\lim_{x\to 0}\left(\frac{1-\frac{x^2}{3!}\left(+\frac{x^4}{5!}-\ldots\right)}{\frac{1}{2!}-\frac{x^2}{4!}\left(+\frac{x^4}{6!}+\ldots\right)}\right)$$ ***A1***

substituting $x=0$ ***(M1)***

$$\left(\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\right)=2$$ ***A1***

METHOD 4

Expressing in terms of half angles ***M1***

$$\lim_{x\to 0}\frac{2x\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$$ ***A1A1***

$$=\lim_{x\to 0}\frac{x\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=2\lim_{x\to 0}\frac{\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}$$ ***(M1)***

Using $\lim_{x\to 0}\frac{x}{\tan x}=1$ ***M1***

=2 ***A1***

[6 marks]

Section B

3. METHOD 1

attempt to find change in population using a definite integral (***M1)***

$t = 4$ at the start of 2026 (seen anywhere) ***(A1)***

$\int_0^4 -104000\mathrm{e}^{-0.0145t}\mathrm{d}t$ (***A1)***

$= -404165.8...$ (***A1)***

attempt to add initial population to their change in population from a definite integral (***M1)***

population at the start of 2026 $= 6.78\times10^6 - 404165.8...$

$= 6375834.1...$

$= 6380000\left(= 6.38\times10^6\right)$ ***A1***

METHOD 2

attempt to find population using an indefinite integral (***M1)***

$P = \int -104000\mathrm{e}^{-0.0145t}\mathrm{d}t$

$\frac{-104000\mathrm{e}^{-0.0145t}}{-0.0145} + c\left(= 7172413.7...\mathrm{e}^{-0.0145t} + c\right)$ ***(A1)***

attempt to substitute $t = 0, P = 6.78\times10^6$ into equation with c. ***(M1)***

$6.78\times10^6 = 7172413.7... + c \Rightarrow c = -392413.7...$

$P = 7172413.7...\mathrm{e}^{-0.0145t} - 392413.7...$ ***(A1)***

$t = 4$ at the start of 2026 (seen anywhere) ***(A1)***

population at the start of 2026 $7172413.7...\mathrm{e}^{-0.0145(4)} - 392413.7...$

$= 6375834.1...$

$= 6380000\left(= 6.38\times10^6\right)$ ***A1***

Total [6 marks]

5. (a) recognition of the need to differentiate (***M1***)

$h'(x) = \frac{\pi}{50}\left(-15\sin\left(\frac{\pi x}{50}\right)\right)\left(= -\frac{15\pi}{50}\sin\left(\frac{\pi x}{50}\right) = -\frac{3\pi}{10}\sin\left(\frac{\pi x}{50}\right)\right)$ ***A1A1***

$h'(k) = -\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right)\left(= -\frac{3\pi}{10}\sin\left(\frac{\pi k}{50}\right)\right)$

Note: Award ***A1*** for $-15\sin\left(\frac{\pi k}{50}\right)$ and ***A1*** for factor of $\frac{\pi}{50}$.

Award ***A1A0*** for a correct expression with additional terms or additional factors.

[3 marks]

(b) recognition that gradient of tangent = $-\tan\left(\frac{\pi}{8}\right)$ OR $\tan\left(\frac{7\pi}{8}\right)$ (***M1***)

Note: Accept $\tan\left(\frac{\pi}{8}\right)$ OR $-\tan\left(\frac{7\pi}{8}\right)$ for the (***M1***)

setting their $h'(k)$ equal to $-\tan\left(\frac{\pi}{8}\right)$ OR $\tan\left(\frac{7\pi}{8}\right)(= -0.414213...)$ (***A1***)

$-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right) = -\tan\left(\frac{\pi}{8}\right)$ OR $-\frac{15\pi}{50}\sin\left(\frac{\pi k}{50}\right) = \tan\left(\frac{7\pi}{8}\right)$

$k = 7.24211...$, $k = 42.7578...$

$k = 7.24$, $k = 42.8$ ***A1***

[3 marks]
Total [6 marks]

9. (a) attempt to use Euler's method with a step of 0.25 ***(M1)***

$\left(x_{n+1} = x_n + 0.25\right), y_{n+1} = y_n + 0.25\left(\frac{2x_n}{{x_n}^2 + y_n}\right)$

$\left(y_0 = 0\right)$
$y_1 = 0.5$
$y_2 = 0.803030...$ ***(A1)***
$y_3 = 1.04868...$

Note: Award ***(A1)*** for at least two correct intermediate values given to 3sf.

If $x = 2, y = 1.26152...$

$y = 1.26$ ***A1***

[3 marks]

(b) (i) The estimate will be an overestimate ***A1***

because the tangents to the curve lie above the curve (the curve is concave down). ***R1***

(ii) The gradient of the curve (at $(1, 0)$) is positive (so the method works in the direction of the upper part of the curve). ***R1***

[3 marks]

Total [6 marks]

Section B

3.

(a) one critical value (seen anywhere) ***(A1)***

recognizing that for f to be decreasing $f' < 0$ ***(M1)***

$x = -1.73554...$ or $x = 0.517999...$

$x \leq -1.74$ and $x \geq 0.518$ OR $x < -1.74$ and $x > 0.518$ ***A1***

[3 marks]

(b) correct critical value $x = -0.405465... \left(= \ln \frac{2}{3} \right)$ (seen anywhere) ***(A1)***

attempt to find maximum of f' OR recognising that $f''(x) > 0$ OR sketching graph of f'' OR $f''(x) = 2 - 3e^x$ ***(M1)***

concave-up for $x < \ln \frac{2}{3}$ OR $x < -0.405$ ***A1***

[3 marks]

Total [6 marks]

5.

(a) recognition that maximum speed occurs when $|v|$ is greatest ***(M1)***

one correct coordinate for minimum $(4.71238..., -2.71828...)$

$2.72\ (=e)\ \ (\text{ms}^{-1})$ ***A1***

Note: Award ***(M1)A0*** for the answer 1, from working with degrees.

[2 marks]

(b) substitution of limits into correct formula ***(M1)***

$\int_0^5 |v|\,dt$

$3.84591...$

$3.85\ (\text{metres})$ ***A1***

[2 marks]

(c) $v = 0$ (seen anywhere) ***(M1)***

$t = 2.35619...\left(=\frac{3\pi}{4}\right)$ ***(A1)***

$a = 0.986137...$

$a = 0.986\left(=2e^{-\frac{1}{\sqrt{2}}}\right)\ (\text{ms}^{-2})$ ***A1***

[3 marks]

Total [7 marks]

11. (a) **METHOD 1**

$\frac{{}^{15}C_5 \times {}^{10}C_5 \times {}^{5}C_5}{3!}\left(=\frac{3003\times 252\times 1}{6}=\frac{756756}{6}\right)$ OR $\frac{15!}{5!\times 5!\times 5!\times 3!}$ ***(A1)(A1)***

Note: Award ***A1*** for ${}^{15}C_5 \times {}^{10}C_5 \left(=756756\right)$ OR $\frac{15!}{5!\times 5!\times 5!}$ and ***A1*** for dividing by $3!\left(=6\right)$.

$=126126$ ***A1***

METHOD 2

${}^{14}C_4 \times {}^{9}C_4 \times {}^{4}C_4$ ***(A1)(A1)***

Note: Award ***A1*** for ${}^{14}C_4$ and ***A1*** for ${}^{14}C_4 \times {}^{9}C_4$.

If a random student comes out and picks a team of 4 on their turn.

$=126126$ ***A1***

[3 marks]

(b) attempt to use integration by parts ***(M1)***

$u=t$ and $\frac{\mathrm{d}v}{\mathrm{d}t}=\mathrm{e}^{-3t}$

$\int \mathrm{e}^{-3t}dt=-\frac{1}{3}\mathrm{e}^{-3t}$ (seen anywhere) ***A1***

$\int_0^a f(t)\,\mathrm{d}t = k\left\{\left[t\left(-\frac{1}{3}\mathrm{e}^{-3t}\right)\right]_0^a-\int_0^a -\frac{1}{3}\mathrm{e}^{-3t}dt\right\}$

$=k\left\{\left[t\left(-\frac{1}{3}\mathrm{e}^{-3t}\right)\right]_0^a+\frac{1}{3}\left[-\frac{1}{3}\mathrm{e}^{-3t}\right]_0^a\right\}$ ***A1***

Note: Condone the absence of k and limits up to this stage.

continued…

Question 11 continued.

$$= k\left\{\left(-\frac{1}{3}a\mathrm{e}^{-3a} - 0\right) + \frac{1}{3}\left(-\frac{1}{3}\mathrm{e}^{-3a} + \frac{1}{3}\right)\right\}$$

$$= k\left(-\frac{1}{3}a\mathrm{e}^{-3a} - \frac{1}{9}\mathrm{e}^{-3a} + \frac{1}{9}\right) \quad \textbf{A1}$$

$$= \frac{k}{9}\left[1 - (3a+1)\mathrm{e}^{-3a}\right] \quad \textbf{AG}$$

[4 marks]

(c) (i) $\lim\limits_{x\to\infty}\dfrac{3x+1}{\mathrm{e}^{3x}} = \lim\limits_{x\to\infty}\dfrac{3}{3\mathrm{e}^{3x}}\left(= \lim\limits_{x\to\infty}\dfrac{1}{\mathrm{e}^{3x}}\right)$ ***A1***

Note: This first ***A1*** must be seen.

$= 0$ ***A1***

(ii) $\lim\limits_{a\to\infty}\dfrac{k}{9}\left[1-(3a+1)\mathrm{e}^{-3a}\right] = \dfrac{k}{9}\left(1 - \lim\limits_{a\to\infty}\dfrac{3a+1}{\mathrm{e}^{3a}}\right)$

$= \dfrac{k}{9}(1-0)$ ***(A1)***

recognising area of probability density function $= 1$ (seen anywhere) ***(M1)***

$\dfrac{k}{9} = 1$

$k = 9$ ***A1***

[5 marks]

(d) recognition that the integral or their expression equals 0.5 for $a = m$ ***(M1)***

$\int_0^m f(t)\,dt = 0.5$ OR $1-(3m+1)\mathrm{e}^{-3m} = 0.5$

$0.559448\ldots$ OR $33.566939\ldots$

median = 0.559 (min) OR 33.6 (sec) ***A1***

Note: ***FT*** can only be awarded for a positive value of k.

Award ***(M1)(A1)*** for a final answer of 0.56 with no working shown.

[2 marks]

Total [14 marks]

12. (a) **METHOD 1**

attempt to differentiate implicitly ***(M1)***

$8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 24 + 4\frac{\mathrm{d}y}{\mathrm{d}x}(= 0)$ ***A1A1***

Note: Award ***A1*** for $8x - 24$ and ***A1*** for $2y\frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}y}{\mathrm{d}x}$.

Award ***A1A0*** for $8x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 24 + 4\frac{\mathrm{d}y}{\mathrm{d}x} = -20$ (or equivalent).

The implicit differentiation must be seen to award the above ***A*** marks.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{24 - 8x}{2y + 4}\ \left(= \frac{12 - 4x}{y + 2}\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(3 - x)}{y + 2}$ ***AG***

METHOD 2

$\frac{d}{dx}\left(4x^2 + y^2 - 24x + 4y + 20\right) = 8x - 24$ ***A1***

$\frac{d}{dy}\left(4x^2 + y^2 - 24x + 4y + 20\right) = 2y + 4$ ***A1***

$\frac{dy}{dx} = -\frac{\frac{dC}{dx}}{\frac{dC}{dy}}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{24 - 8x}{2y + 4}\ \left(= \frac{12 - 4x}{y + 2}\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(3 - x)}{y + 2}$ ***AG***

[4 marks]

continued…

Question 12 continued.

(b) recognising boundaries of domain when $\frac{\mathrm{d}y}{\mathrm{d}x}$ undefined ***(M1)***

$y+2=0$

$y=-2$ ***(A1)***

substituting their y value into curve equation and attempt to solve for x ***(M1)***

$4x^2+(-2)^2-24x+4(-2)+20=0 \quad (x^2-6x+4=0)$

$x=\frac{6\pm\sqrt{20}}{2}$ OR $(x-3)^2-5=0$

$3-\sqrt{5}\le x\le 3+\sqrt{5}$ (accept $a=5$) ***A1***

Note: Award a maximum of ***M1A1M1A0*** for a final answer of $0.764\le x\le 5.24$.

[4 marks]

(c) $x_{\mathrm{A}}=3$ ***A1***

substituting their x value into curve equation ***(M1)***

$4(3)^2+y^2-24(3)+4y+20=0$

$2.472135\ldots$OR $-2+2\sqrt{5}$

$y_{\mathrm{A}}=2.47$ OR $y_{\mathrm{A}}=-2+2\sqrt{5}$ ***A1***

$A(3,2.47)$ OR $A\left(3,-2+2\sqrt{5}\right)$

Note: Award ***A1(M1)A1*** for a final answer of $A(3,2.5)$ with no working shown.

[3 marks]

(d) $4x^2+(mx)^2-24x+4(mx)+20=0 \quad \left(\left(4+m^2\right)x^2+(4m-24)x+20=0\right)$ ***(A1)***

$(4m-24)^2-4\left(4+m^2\right)(20)$ ***A1***

recognising discriminant = 0 (seen anywhere) ***M1***

both $m=-4$ and $m=1$ ***A1***

[4 marks]

continued…

Question 12 continued.

(e) **METHOD 1**

attempt to form an equation in y by substituting $x = -\frac{1}{4}y$ ***(M1)***

$4\left(-\frac{1}{4}y\right)^2 + y^2 - 24\left(-\frac{1}{4}y\right) + 4y + 20 = 0$ ***(A1)***

$y_B = -4$ ***A1***

Note: Award ***(M1)A1A0*** if an extra value is given for y_B.

METHOD 2

attempt to form an equation in x by substituting $y = -4x$ ***(M1)***

$4x^2 + (-4x)^2 - 24x + 4(-4x) + 20 (= 0)$

$x = 1$ ***(A1)***

$y_B = -4$ ***A1***

Note: Award ***(M1)A1A0*** if an extra value is given for y_B.

[3 marks]

(f) attempt to solve for x by quadratic formula or completing the square ***(M1)***

$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4)(y^2 + 4y + 20)}}{2 \times 4}$ OR $4(x-3)^2 - 36 = -y^2 - 4y - 20$

$x = \frac{24 - \sqrt{256 - 64y - 16y^2}}{8}$ $\left(x = 3 - \sqrt{\frac{16 - 4y - y^2}{4}}\right)$ ***A1***

Note: Condone $\pm$ at this stage.

$V = \pi \int_{-4}^{-2+2\sqrt{5}} \left(\frac{24 - \sqrt{256 - 64y - 16y^2}}{8}\right)^2 \mathrm{d}y \left(= \pi \int_{-4}^{2.47\ldots} \left(3 - \sqrt{\frac{16 - 4y - y^2}{4}}\right)^2 \mathrm{d}y\right)$ ***A1***

$29.708027\ldots$

volume = 29.7 ***A1***

Note: Award ***(M1)A1A1*** for a final answer of 30 with no working shown.

[4 marks]

Total [22 marks]

5. (a) use of quotient rule or product rule ***(M1)***

EITHER

$$f'(x) = \frac{3\times 2\times (2x+a)^2(x+5)^2 - 2(2x+a)^3(x+5)}{\left((x+5)^2\right)^2}$$ ***(A1)***

$$f'(x) = \frac{6(2x+a)^2(x+5)^2 - 2(2x+a)^3(x+5)}{(x+5)^4} \left(= \frac{2(x-a+15)(2x+a)^2}{(x+5)^3}\right)$$ ***A1***

OR

$$f'(x) = 3\times 2\times (2x+a)^2(x+5)^{-2} - 2(2x+a)^3(x+5)^{-3}$$ ***(A1)***

$$f'(x) = 6(2x+a)^2(x+5)^{-2} - 2(2x+a)^3(x+5)^{-3} \left(= 2(x-a+15)(2x+a)^2(x+5)^{-3}\right)$$ ***A1***

[3 marks]

(b) recognizing $f'(1)$ is equal to $\tan 70^\circ$ (=2.74747) ***(M1)***

$\frac{6(2+a)^2 6^2 - 2(2+a)^3\times 6}{6^4} = \tan 70^\circ$ **OR** $6(2+a)^2 6^{-2} - 2(2+a)^3\times 6^{-3} = \tan 70^\circ$ ***(A1)***

$2.72844\ldots,\ 14.96968\ldots$
$a = 2.73, 15.0$ ***A1A1***

[4 marks]

Total [7 marks]

9. (a) Let $\hat{\text{APG}} = \alpha$

$\theta = \hat{\text{BPG}} - \hat{\text{APG}}$ (may be seen in diagram) ***(M1)***

$\tan(\theta + \alpha) = \frac{x+2}{6}\left(\Rightarrow \theta + \alpha = \arctan\left(\frac{x+2}{6}\right)\right)$ AND $\tan\alpha = \frac{x}{6}\left(\Rightarrow \alpha = \arctan\left(\frac{x}{6}\right)\right)$ ***A1***

$\theta = \arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right)$

AG

[2 marks]

(b) **METHOD 1**

setting $\theta = 0.178$ ***(M1)***

$\arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right) = 0.178$

$x = 4.63047...$ ***(A1)***

Note: The following three marks are independent of the first two marks.

EITHER

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\left(\frac{x+2}{6}\right)\right) = \frac{6}{(x+2)^2+36}$ OR $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\left(\frac{x}{6}\right)\right) = \frac{6}{x^2+36}$ (seen anywhere) ***(A1)***

attempt to use the chain rule to differentiate $\arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right)$ with respect to t ***(M1)***

$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} =\right) \frac{6}{(x+2)^2+36} \times \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{6}{x^2+36} \times \frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} =\right) -0.0294199... \times \frac{\mathrm{d}x}{\mathrm{d}t}$ (or equivalent) ***(A1)***

substituting their x and $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 12.5$ ***(M1)***

$12.5 = -0.0294199... \times \frac{\mathrm{d}x}{\mathrm{d}t}$

continued…

Question 9 continued

OR

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{12}{x^2+2x+36}\right)=-\frac{12(2x+2)}{\left(x^2+2x+36\right)^2}$ (seen anywhere) ***(A1)***

attempt to use the chain rule to differentiate $\tan\theta=\frac{12}{x^2+2x+36}$ with respect to t ***(M1)***

$\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{12(2x+2)}{\left(x^2+2x+36\right)^2}\frac{\mathrm{d}x}{\mathrm{d}t}$ OR $\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-0.0303721...\times\frac{\mathrm{d}x}{\mathrm{d}t}$ (or equivalent) ***(A1)***

substituting $\theta=0.178$, their x and $\frac{\mathrm{d}\theta}{\mathrm{d}t}=12.5$ ***(M1)***

$\sec^2(0.178)\times12.5=-0.0303721...\times\frac{\mathrm{d}x}{\mathrm{d}t}$

THEN

$\frac{\mathrm{d}x}{\mathrm{d}t}=-424.881...$

$(\text{speed}=)\ 425\ (\text{km hr}^{-1})$ ***A1***

continued…

Question 9 continued

METHOD 2

setting $\theta = 0.178$ ***(M1)***

$$\arctan\left(\frac{x+2}{6}\right) - \arctan\left(\frac{x}{6}\right) = 0.178$$

$x = 4.63047...$ ***(A1)***

substitutes their value of x to find $\frac{\mathrm{d}\theta}{\mathrm{d}x}$ when $\theta = 0.178$ ***(M1)***

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = -0.0294199...$$

recognition that chain rule required (seen anywhere) ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$ (or equivalent, must include x, t and θ)

substituting their value for $\frac{\mathrm{d}\theta}{\mathrm{d}x}\left(= -0.0294199...\right)$ and $\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} =\right) 12.5$ ***(M1)***

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t} =\right) \frac{1}{-0.0294199...} \times 12.5$$

$= -424.881...$

$(\text{speed} =)\ 425\ (\text{km hr}^{-1})$ ***A2***

[7 marks]

Total [9 marks]

Section B

10. (a) (i) $7.43076\ldots$

$v(1) = 7.43 \ (\text{m s}^{-1})$ ***A1***

(ii) equating $v(t)$ and 5 ***(M1)***

$0.348114\ldots$

0.348 (seconds) ***A1***

[3 marks]

(b) recognizing $a = v'(t)$ ***(M1)***

$0.590930...$

0.591 (seconds) ***A1***

[2 marks]

(c) (i) considering large values of t ***(M1)***

$\left(\lim_{t \to \infty}(v(t)) =\right) 8.14 \ (\text{ms}^{-1})$ ***A1***

(ii) **EITHER**

the race lasts a finite time (e.g. it ends after (Fiona) crosses the line) ***R1***

OR

the graph approaches the limiting value but Fiona will never attain that speed ***R1***

OR

Fiona cannot maintain that speed for a long period of time ***R1***

Note: Award ***R1*** for a valid reason that does not contradict their answer to part (c)(i).

[3 marks]

continued…

Question 10 continued

(d) Fiona takes t_f seconds to travel $200\ \text{m}$

EITHER

recognizing distance travelled by either athlete in the first t seconds is

$\int_0^t v(t)\,\mathrm{d}t$ **OR** $\int_0^t w(t)\,\mathrm{d}t$ (seen anywhere) ***(M1)***

equating distance travelled by Fiona to 200 (m) ***(M1)***

$$\int_0^{t_f} v(t)\,\mathrm{d}t = 200$$

OR

attempt to integrate $v(t)$ ***(M1)***

$$s(t) = 8.14\sqrt{t^2 + 0.2} - 3.64031...$$

equating 200 to *their* $s(t)$ (must include a constant of integration) ***(M1)***

$$8.14\sqrt{t^2 + 0.2} - 3.64031... = 200$$

THEN

$t_f = 25.0132...$ (accept 25) ***(A1)***

recognition that a definite integral of $w(t)$ with 0 and *their* t_f is required ***(M1)***

(Lucy's distance =) $\int_0^{25.0132...} w(t)\,\mathrm{d}t\ \ (=195.772...)$

distance from finishing line = 200 – *their* Lucy's distance ***(M1)***

$4.22788\ldots$

$4.23\ (\text{m})$ ***A1***

[6 marks]

Total [14 marks]

12. (a) attempt to use integration by parts ***(M1)***

$\left(x^2-5\right)\mathrm{e}^x-2\int x\mathrm{e}^x\,\mathrm{d}x$ **OR** $-5\mathrm{e}^x+x^2\mathrm{e}^x-2\int x\mathrm{e}^x\,\mathrm{d}x$ ***A1***

attempt to use integration by parts a second time ***(M1)***

$\left(x^2-5\right)\mathrm{e}^x-2\left(x\mathrm{e}^x-\int \mathrm{e}^x\,\mathrm{d}x\right)$ **OR** $-5\mathrm{e}^x+x^2\mathrm{e}^x-2\left(x\mathrm{e}^x-\int \mathrm{e}^x\,\mathrm{d}x\right)$ ***A1***

$\left(x^2-5\right)\mathrm{e}^x-2\left(x\mathrm{e}^x-\mathrm{e}^x\right)$ ***(A1)***

$\left(x^2-2x-3\right)\mathrm{e}^x+c$ OR $\left(x^2-5\right)\mathrm{e}^x-2\left(x\mathrm{e}^x-\mathrm{e}^x\right)+c$ (or equivalent) ***A1***

[6 marks]

(b) rearranging to $\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2-5$ **AND** attempt to find integrating factor ***(M1)***

$\mathrm{e}^{\int \mathrm{d}x}=\mathrm{e}^x$ (seen anywhere) ***A1***

$\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^x y=\left(x^2-5\right)\mathrm{e}^x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x y\right)=\left(x^2-5\right)\mathrm{e}^x$ **OR** $\mathrm{e}^x y=\int\left(x^2-5\right)\mathrm{e}^x\,\mathrm{d}x$ ***M1***

$\mathrm{e}^x y=\left(x^2-2x-3\right)\mathrm{e}^x+C$ **OR** $\mathrm{e}^x y=x^2\mathrm{e}^x-2x\mathrm{e}^x-3\mathrm{e}^x+C$ ***A1***

$y=x^2-2x-3+C\mathrm{e}^{-x}$ ***AG***

[4 marks]

continued…

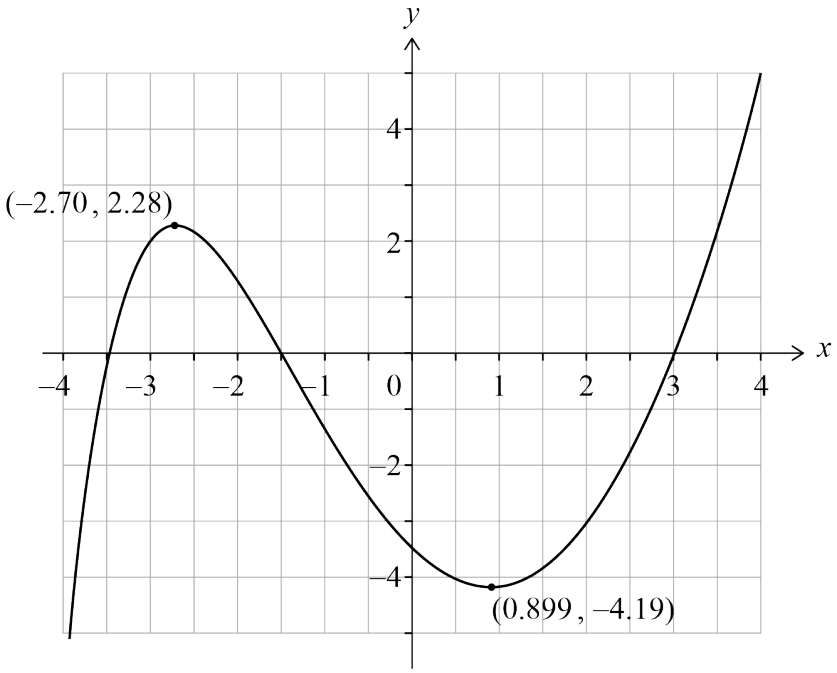
Question 12 continued

(c) substituting $x = -3, y = 2$ into $y = x^2 - 2x - 3 + Ce^{-x}$ ***(M1)***

$2 = 9 + 6 - 3 + Ce^3$

$\Rightarrow C = -0.497870... \left(= -10e^{-3}\right)$ ***A1***

$y = x^2 - 2x - 3 - 0.498e^{-x}$ (accept $y = x^2 - 2x - 3 - 10e^{-x-3}$)



A1A1A1

Note: Award marks as follows:

A1 for approximately correct shape AND x-intercepts in approximately correct place (exactly two negative and one positive x-intercepts)

A1 for maximum in approximately correct position and labelled $(-2.70, 2.28)$ AND minimum in approximately correct position and labelled $(0.899, -4.19)$

A1 for domain.

[5 marks]

continued…

Question 12 continued

(d) substituting -3 and p **OR** -3 and q into $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

gradient of tangent at $(-3,\ p)$ is $4-p$ and at $(-3,\ q)$ is $4-q$

finding equation of tangents ***(M1)***

$y-p=(4-p)(x+3)$ **AND** $y-q=(4-q)(x+3)$ ***A1***

solving their equations simultaneously ***(M1)***

$q-p=(x+3)\big(4-p-(4-q)\big)\ \big(=(x+3)(q-p)\big)$

$x+3=1$ (since $p\neq q$)

$x=-2$ ***A1***

$y-p=4-p$ **OR** $y-q=4-q$

$y=4$ ***A1***

hence all tangents intersect at $(-2,\ 4)$ ***AG***

[6 marks]

Total [21 marks]

2. (a) $g'(x) = m$ OR $g''(x) = 0$ (seen anywhere) ***(A1)***

Note: Award ***A0*** for $g''(x) = 0$ obtained from an incorrect $g'(x)$, for example, $g'(x) = 1$ OR $g'(x) = 0$.

EITHER

$k(x) = \dfrac{|0|}{\left(1+m^2\right)^{\frac{3}{2}}}$ OR $k(x) = \dfrac{0}{\left(1+m^2\right)^{\frac{3}{2}}}$ OR $k(x) = \dfrac{0}{\sqrt{1+3m^2+3m^4+m^6}}$ ***A1***

OR

the numerator of $k(x)$ is zero and the denominator is non-zero ***A1***

THEN

$\Rightarrow k(x) = 0$ ***AG***

[2 marks]

(b) (i) $x = -\dfrac{b}{2a}$ ***A1***

Note: Do not award ***A1*** if $a = 0$ is included as a solution.

[1 mark]

(ii) substitutes their value of x into $k(x)$ ***(M1)***

$$k\left(-\frac{b}{2a}\right) = \frac{2|a|}{\left(1+\left(2a\left(-\frac{b}{2a}\right)+b\right)^2\right)^{\frac{3}{2}}} \quad \left(= \frac{2|a|}{\left(1+(-b+b)^2\right)^{\frac{3}{2}}}\right)$$

$= 2|a|$ ***A1***

Note: Award ***(M1)A0*** if the modulus sign is not considered.

[2 marks]

continued…

Question 2 continued.

(iii) **EITHER**

$\lim_{x \to \infty} k(x) = 0$ ***A1***

OR

the curvature (of a quadratic) approaches zero as $x \to \infty$ ***A1***

THEN

for large positive values of x, quadratic functions are close to being straight ***R1***

Note: Award ***R1*** for similar statements such as 'for large positive values of x, a quadratic function behaves like a linear function.

[2 marks]

(iv) **C** ($k_{\max}$ of $p = k_{\max}$ of q) ***A1***

$k_{\max}$ of p is $(2 \times |-2|) = 4$ and $k_{\max}$ of q is $(2 \times |2|) = 4$ ***R1***

Note: Accept arguments such as '$k_{\max} = 2|a|$ and both quadratics have $|a| = 2$.
Do not award ***A1R0***.
If modulus signs were omitted in part (b)(ii), award ***FT*** in part (b)(iv) for an answer of **B** with appropriate values shown.

[2 marks]

(c) (i) sets $k'(x) = 0$ and attempts to solve for x ***(M1)***

$x^2 = \frac{1}{2}$

$x = \frac{1}{\sqrt{2}}$ ***A1***

Note: Award ***A0*** for $x = \pm \frac{1}{\sqrt{2}} (= \pm 0.707)$. Award ***A0*** for $x = 0.707$.

[2 marks]

continued…

Question 2 continued.

(ii) substitutes their value of x into $k(x)$ ***(M1)***

Note: Award a maximum of ***(M1)(A0)A0A0*** if their value of x is a decimal.
Award ***(M1)(A0)A0A0*** for $(0.707, 0.385)$.

$$k\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above denominator.

$\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^{\frac{3}{2}}$ ***(A1)***

$= \frac{3\sqrt{3}}{2\sqrt{2}}$ (or equivalent unsimplified fractional surd e.g. $\sqrt{\frac{27}{8}}$) ***A1***

$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}} \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}\right)$

$= \frac{2}{3\sqrt{3}}$ (or equivalent surd e.g. $\frac{\sqrt{12}}{9}$) ***A1***

$k_{\max} = \frac{2\sqrt{3}}{9}$ ***AG***

[4 marks]

continued…

Question 2 continued.

(d) (i) **EITHER**

sets $k'(x) = 0$ and attempts to solve for e^{2x} ***(M1)***

$e^{2x} = \frac{1}{2}$ ***A1***

$e^{x} = \frac{1}{\sqrt{2}}$

substitutes their values for e^{2x} and e^{x} into $k(x)$ ***(M1)***

OR

attempts to solve $k'(x) = 0$ for x ***(M1)***

$x = \frac{1}{2}\ln\frac{1}{2}\ \left(= \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2\right)$ ***A1***

Note: Award a maximum of ***(M1)(A0)(M0)A0A0*** if their value of x is a decimal. Award ***(M1)(A0)(M0)A0A0*** for $(-0.347, 0.385)$.

substitutes their value of x into $k(x)$ ***(M1)***

THEN

for example, $\left(k\left(\ln\frac{1}{\sqrt{2}}\right) = \right)\frac{\frac{1}{\sqrt{2}}}{\left(1+\left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{3}{2}}}\left(= \frac{\frac{1}{\sqrt{2}}}{\left(\frac{3}{2}\right)^{\frac{3}{2}}}\right)$ ***A1***

$k_{\max} = \frac{\frac{1}{\sqrt{2}}}{\frac{3\sqrt{3}}{2\sqrt{2}}}\ \left(= \frac{1}{\sqrt{2}} \times \frac{2\sqrt{2}}{3\sqrt{3}}\right)$ ***A1***

$= \frac{2}{3\sqrt{3}}$

$k_{\max} = \frac{2\sqrt{3}}{9}$ ***AG***

Note: After obtaining the penultimate ***A1*** above, candidates may state $k_{\max} = \frac{2\sqrt{3}}{9}$ as per correct work seen in part (c) (ii). In such cases, award the final ***A1***.

[5 marks]

continued…

Question 2 continued.

(ii) **EITHER**

w and v are inverse functions of each other (and as such are equally 'curved') (or equivalent) ***R1***

OR

the graphs of w and v are reflections of each other in the line $y = x$ ***R1***

OR

w and v have the same shape (or equivalent) ***R1***

[1 mark]

(e) (i) **METHOD 1**

attempts to use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(-2x)\left(r^2 - x^2\right)^{-\frac{1}{2}} \left(= -\frac{x}{\sqrt{r^2 - x^2}}, = -\frac{x}{y}\right)$ ***A1***

attempts to use the quotient or product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-y + x\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2} \left(= \frac{x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y}\right)$ ***A1***

correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ to form $= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} \left(= \frac{x}{y^2}\left(-\frac{x}{y}\right) - \frac{1}{y}\right)$ ***A1***

$= \frac{-y^2 - x^2}{y^3} \left(= -\frac{y^2 + x^2}{y^3}\right)$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

continued…

Question 2 continued.

METHOD 2

attempts to use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(-2x)\left(r^2 - x^2\right)^{-\frac{1}{2}} \left(= -\frac{x}{\sqrt{r^2 - x^2}}\right)$ ***A1***

attempts to use the quotient or product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***(M1)***

$= \frac{-\left(r^2 - x^2\right)^{\frac{1}{2}} - x^2\left(r^2 - x^2\right)^{-\frac{1}{2}}}{r^2 - x^2} \left(= -x^2\left(r^2 - x^2\right)^{-\frac{3}{2}} - \left(r^2 - x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

$= -\frac{r^2}{\left(r^2 - x^2\right)^{\frac{3}{2}}} \left(= \frac{-\left(r^2 - x^2\right) - x^2}{\left(r^2 - x^2\right)^{\frac{3}{2}}}, = -\frac{r^2}{\left(y^2\right)^{\frac{3}{2}}}\right)$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

Candidates may express their $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of r and y instead of r and x.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

continued…

Question 2 continued.

METHOD 3

attempts to use implicit differentiation on $y^2 = r^2 - x^2$ ***(M1)***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \left(y\frac{\mathrm{d}y}{\mathrm{d}x} = -x\right)$ ***A1***

attempts to use the product rule and implicit differentiation ***(M1)***

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$ (or equivalent) ***A1***

correct substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ to form $\left(-\frac{x}{y}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{y}\left(-1 - \frac{x^2}{y^2}\right)$

$= \frac{-y^2 - x^2}{y^3} \quad \left(= -\frac{y^2 + x^2}{y^3}\right)$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{r^2}{y^3}$ ***AG***

[6 marks]

continued…

Question 2 continued.

(ii) **METHOD 1**

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left|-\frac{r^2}{y^3}\right|}{\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left|-\frac{r^2}{y^3}\right| = \frac{r^2}{y^3}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{y}\right)^2\right)$ as $\frac{y^2+x^2}{y^2}$ ***M1***

denominator is $\left(\frac{r^2}{y^2}\right)^{\frac{3}{2}}\left(=\frac{r^3}{y^3}\right)$ ***A1***

Note: Award the following ***A*** mark for showing that $k = \frac{1}{r}$.

$k = \frac{\frac{r^2}{y^3}}{\frac{r^3}{y^3}}$ $\left(k = \frac{r^2}{y^3} \times \frac{y^3}{r^3}\right) \Rightarrow k = \frac{1}{r}$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left|-\frac{r^2}{y^3}\right| = \frac{r^2}{y^3}$ is not considered.

continued…

Question 2 continued.

METHOD 2

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left| -\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \right|}{\left(1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following three marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left| -\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \right| = \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)$ as $\frac{r^2}{r^2-x^2}$ ***M1***

denominator is $\left(\frac{r^2}{r^2-x^2}\right)^{\frac{3}{2}}$ ***A1***

Note: Award the following ***A*** mark for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}}{\frac{r^3}{\left(r^2-x^2\right)^{\frac{3}{2}}}} \quad \left(k = \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \times \frac{\left(r^2-x^2\right)^{\frac{3}{2}}}{r^3} \Rightarrow k = \frac{1}{r} \right)$$ ***A1***

which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left| -\frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}} \right| = \frac{r^2}{\left(r^2-x^2\right)^{\frac{3}{2}}}$ is not considered.

continued…

Question 2 continued.

METHOD 3

substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the given $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into $k(x)$ ***(M1)***

$$= \frac{\left|-\frac{x^2+y^2}{y^3}\right|}{\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}}$$

Note: Award the following two marks for evaluating the above numerator and denominator as appropriate.

numerator: recognizes that $\left|-\frac{x^2+y^2}{y^3}\right| = \frac{x^2+y^2}{y^3}$ (seen anywhere) ***A1***

denominator: attempts to express $\left(1+\left(-\frac{x}{y}\right)^2\right)^{\frac{3}{2}}$ as $\left(\frac{y^2+x^2}{y^2}\right)^{\frac{3}{2}}$ ***M1***

Note: Award the following ***A1A1*** for showing that $k = \frac{1}{r}$.

$$k = \frac{\frac{x^2+y^2}{y^3}}{\left(\frac{x^2+y^2}{y^2}\right)^{\frac{3}{2}}} \left(= \frac{\frac{x^2+y^2}{y^3}}{\frac{\left(x^2+y^2\right)^{\frac{3}{2}}}{y^3}} \right)$$

$$= \frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}$$ ***A1***

$$k = \frac{1}{\left(r^2\right)^{\frac{1}{2}}} \Rightarrow k = \frac{1}{r}$$ ***A1***

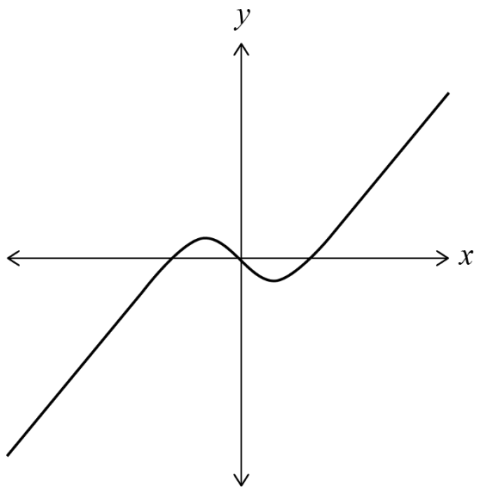
which is a constant ***AG***

Note: Award a maximum of ***(M1)A0M1A1A1*** if $\left|-\frac{x^2+y^2}{y^3}\right| = \frac{x^2+y^2}{y^3}$ is not considered.

[5 marks]

Total [32 marks]

1. (a) (i)



general shape of curve passing through $(0,0)$ ***A1***

[1 mark]

(ii) $(-4,0),(0,0),(4,0)$ ***A1***

Note: Award ***A0*** if additional points are given.

[1 mark]

(iii) Max $(-1.94,1.20)(-1.94347...,1.20113...)$ ***A1***

Min $(1.94,-1.20)$ $(1.94347...,-1.20113...)$ ***A1***

[2 marks]

continued

Question 1 continued.

(b) **EITHER**

odd ***A1***

since $f(-x)=\dfrac{(-x)\left((-x)^2-16\right)}{(-x)^2+16}=\dfrac{-x\left(x^2-16\right)}{x^2+16}=-f(x)$ (or equivalent) ***R1***

OR

odd ***A1***

since graph of f has rotational symmetry order 2 (of 180 degrees) about $(0,0)$ ***R1***

OR

odd ***A1***

since graph has reflection symmetry over the x-axis followed by reflection symmetry over the y-axis (or vice versa) ***R1***

Note: Do not award ***A1R0.***

[2 marks]

(c) attempt to use quotient rule or product rule ***(M1)***

$$f'(x)=\frac{\left(x^2+A\right)\left(3x^2-A\right)-2x\left(x^3-Ax\right)}{\left(x^2+A\right)^2} \quad \left(f'(x)=\frac{2x^2+\left(x^2-A\right)}{x^2+A}-\frac{2x^2\left(x^2-A\right)}{\left(x^2+A\right)^2}\right)$$

(or equivalent) ***A1A1***

Note: Award ***A1*** for numerator, ***A1*** for denominator.

Note: If product rule used, award ***A1*** for each correct term.

Note: Award ***(M1)*** for an attempt to use quotient rule or product rule with $A=16$ or any other constant. Award no further marks.

$$f'\left(\sqrt{A}\right)\left(=\frac{(2A)(2A)-0}{(2A)^2}\right)=1$$ ***A1***

hence independent of A ***AG***

[4 marks]

continued

Question 1 continued.

(d) (i) **METHOD 1**

$x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 + A\right) - 2Ax}{x^2 + A} \left(\equiv \frac{x^3 + Ax - 2Ax}{x^2 + A} \right)$ ***M1***

$\equiv \frac{x^3 - Ax}{x^2 + A} \left(\equiv \frac{x\left(x^2 + A - 2A\right)}{x^2 + A} \right)$ ***A1***

$x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$ ***AG***

Note: Award no marks for numerical approaches.

METHOD 2

EITHER

attempt to do long division on $\frac{x\left(x^2 - A\right)}{x^2 + A}$ or $\frac{x^3 - Ax}{x^2 + A}$ ***M1***

quotient of x and remainder of $-2Ax$ ***A1***

$x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$ ***AG***

OR

$\frac{x^3}{x^2 + A} - \frac{Ax}{x^2 + A} \equiv \frac{x^3 + Ax}{x^2 + A} - \frac{Ax + Ax}{x^2 + A}$ ***M1***

$\equiv \frac{x\left(x^2 + A\right)}{\left(x^2 + A\right)} - \frac{2Ax}{\left(x^2 + A\right)}$ ***A1***

$\equiv x - \frac{2Ax}{x^2 + A} \equiv \frac{x\left(x^2 - A\right)}{x^2 + A}$ ***AG***

Note: Award no marks for numerical approaches.

[2 marks]

continued

Question 1 continued.

(ii) hence oblique asymptote is $y = x$ (since $\frac{2Ax}{x^2 + A} \to 0$ as $x \to \pm\infty$) ***A1***

Note: Award ***A0*** if not given as an equation.

[1 mark]

(iii) $\left(\sqrt{A}, 0\right)$ OR $\left(-\sqrt{A}, 0\right)$ ***A1***

[1 mark]

continued

Question 1 continued.

(e)

Note: Award no marks for attempts at integration that do not involve partial fractions.

attempt to write $\frac{1}{1-v^2}$ in the form $\frac{B}{1+v}+\frac{C}{1-v}$ ***(M1)***

$B=\frac{1}{2}$ and $C=\frac{1}{2}$ ***A1***

$$\int \frac{1}{1-v^2}\,\mathrm{d}v = \int \frac{\frac{1}{2}}{1+v}+\frac{\frac{1}{2}}{1-v}\,\mathrm{d}v$$

attempt to integrate their partial fractions ***(M1)***

EITHER

$\frac{1}{2}\ln|1+v|-\frac{1}{2}\ln|1-v|+\frac{1}{2}\ln A$ ***A1A1***

Note: Award ***A1*** for first two terms. The moduli must be present for this mark. Award ***A1*** for $\frac{1}{2}\ln A$.

OR

$\frac{1}{2}\ln|1+v|-\frac{1}{2}\ln|1-v|(+c)$ ***A1***

$=\frac{1}{2}\ln\left|\frac{\mathrm{e}^{2c}(1+v)}{1-v}\right|$ where $\mathrm{e}^{2c}=A$ ***A1***

Note: The moduli must be present for each of these marks.

THEN

$=\frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right|$ ***AG***

[5 marks]

continued

Question 1 continued.

(f) attempt to separate variables and integrate ***(M1)***

$$\int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}v}{1-v^2}$$

$$\ln|x| = \frac{1}{2}\ln\left|\frac{A(1+v)}{1-v}\right| \quad \textbf{A1}$$

attempt to use power rule of logs ***(M1)***

$$\ln|x| = \ln\left|\frac{A(1+v)}{1-v}\right|^{\frac{1}{2}} \text{ OR } \ln|x|^2 = \ln\left|\frac{A(1+v)}{1-v}\right|$$

$$|x| = \left|\frac{A(1+v)}{1-v}\right|^{\frac{1}{2}} \text{ OR } x^2 = \left|\frac{A(1+v)}{1-v}\right| \quad \textbf{A1}$$

$$x^2 = \left|\frac{A\left(1+\frac{y}{x}\right)}{1-\frac{y}{x}}\right| \quad \textbf{A1}$$

$$x^2 = \left|\frac{A(x+y)}{x-y}\right| \quad \textbf{AG}$$

Note: Award at most ***(M1)A1(M1)A1A0*** for incorrect use or omission of moduli.

Note: The substitution of $\frac{y}{x}$ may be seen earlier.

[5 marks]

continued

Question 1 continued.

(g) $x^2 = \frac{A(x+y)}{x-y}$ ***(A1)***

attempt to clear denominator ***M1***

$x^2(x-y) = A(x+y)$

$x^3 - x^2y = Ax + Ay$

attempt to isolate y ***M1***

$x^3 - Ax = x^2y + Ay$

$x(x^2 - A) = y(x^2 + A)$ $\left(x^3 - Ax = y(x^2 + A)\right)$ ***A1***

$y = \frac{x(x^2 - A)}{x^2 + A}$ ***AG***

Note: The y must be factored out for the final ***A1***.

[4 marks]

Total [28 marks]

2. (a) (i) 0 ***A1***

[1 mark]

(ii) 0 ***A1***

[1 mark]

(b) attempts to expand a product of two of the side lengths ***(M1)***

EITHER

$V = x\left(4x^2 - 2(a+b)x + ab\right)$ $\left(V = x\left(4x^2 - 2ax - 2bx + ab\right)\right)$ ***A1***

OR

$V = (ax - 2x^2)(b - 2x)$

$= 4x^3 - 2ax^2 - 2bx^2 + abx$ ***A1***

OR

$V = (a - 2x)(bx - 2x^2)$

$= 4x^3 - 2ax^2 - 2bx^2 + abx$ ***A1***

THEN

$V = 4x^3 - 2(a+b)x^2 + abx$ ***AG***

[2 marks]

continued…

Question 2 continued

(c) attempts to find $\frac{\mathrm{d}V}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 4(a+b)x + ab$ ***A1***

Note: Award ***(M1)*** for obtaining the equivalent of two correct terms.

EITHER

attempts to solve $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ for x using the quadratic formula ***(M1)***

$x = \frac{-\left(-4(a+b)\right) \pm \sqrt{\left(-4(a+b)\right)^2 - 4(12)(ab)}}{24}$ ***(A1)***

Note: Award ***A0*** for incorrect expressions, for example:

$x = \frac{-\left(-4(a+b)\right) \pm \sqrt{-4(a+b)^2 - 4(12)(ab)}}{24}$

attempts to expand brackets within the square root sign and/or to take a common factor of 4 ***(M1)***

$x = \frac{-\left(-4(a+b)\right) \pm \sqrt{16a^2 + 16b^2 + 32ab - 48ab}}{24}$ ***A1***

or $x = \frac{4(a+b) \pm 4\sqrt{a^2 + 2ab + b^2 - 3ab}}{24}$

OR

attempt to substitute either of the given x values into their $\frac{\mathrm{d}V}{\mathrm{d}x}$ ***M1***

correct manipulation with each solution to reduce each expression to zero ***A1A1***

$\frac{\mathrm{d}V}{\mathrm{d}x} = 0$ is a quadratic equation and hence these are the only two solutions ***R1***

THEN

$x = \frac{(a+b) \pm \sqrt{a^2 - ab + b^2}}{6}$ ***AG***

[6 marks]

continued…

Question 2 continued

(d) attempts to find their $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24x - 4(a+b)$ ***A1***

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} < 0$ ***R1***

Note: Award ***R1*** for equivalent statement in words.

substitutes $x = \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$ into their $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}$ ***(M1)***

Note: Award ***(M1)*** for substituting $x = \frac{(a+b)\pm\sqrt{a^2-ab+b^2}}{6}$

$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24\left(\frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}\right) - 4(a+b)$

$= -4\sqrt{a^2-ab+b^2}\ (<0)$ ***A1***

so $x_m = \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}$ gives a local maximum of V ***AG***

[5 marks]

(e) the maximum volume will occur at either the endpoints or at the local maximum. ***R1***

Note: Award ***R1*** for an answer which refers to the volume being 0 at both endpoints ($x = 0$ and $x = \frac{a}{2}$) and the volume of the box at $x_m > 0$

hence the maximum volume of the box occurs at x_m / the local maximum is a global maximum. ***AG***

[1 mark]

continued…

Question 2 continued

(f) $2st - s^2 < t^2 - s^2$ ***A1***

$t^2 > 2st$ and since $t \in \mathbb{Z}^+$ (dividing both sides by t) ***R1***

Note: Award ***R1*** for equivalent reasoning, $t > 0$ or t is positive

$\Rightarrow t > 2s$ ***AG***

[2 marks]

(g) attempts to substitute for each of a, b and $\sqrt{a^2 - ab + b^2}$ in terms of s and t ***(M1)***

$(x_m =) \dfrac{\left(2st - s^2\right) + \left(t^2 - s^2\right) - \left(s^2 - st + t^2\right)}{6}$ ***A1***

$= \dfrac{-3s^2 + 3st}{6}$ ***A1***

$x_m = \dfrac{s(t-s)}{2}$ ***AG***

[3 marks]

(h) (i) **METHOD 1**

when s is even, x_m is an integer ***R1***

hence V_m is a product of integers (a, $b \in \mathbb{Z}^+$) ***R1***

hence V_m is an integer ***AG***

METHOD 2

$V_m = \dfrac{s(t-s)}{2}\left(2st - s^2 - 2\left(\dfrac{s(t-s)}{2}\right)\right)\left(t^2 - s^2 - 2\left(\dfrac{s(t-s)}{2}\right)\right)$ (or equivalent) ***A1***

Note: Award ***A1*** for $V_m = \dfrac{s(t-s)}{2}\left(a - 2\left(\dfrac{s(t-s)}{2}\right)\right)\left(b - 2\left(\dfrac{s(t-s)}{2}\right)\right)$

when s is even, $\dfrac{s(t-s)}{2}$ is an integer **AND**

hence V_m is a product of integers (a, $b \in \mathbb{Z}^+$) ***R1***

hence V_m is an integer ***AG***

[2 marks]

continued…

Question 2 continued

(ii) when s is odd **AND** t is odd ($s(t-s)$ is even) ***A1***

x_m and V_m are both integers ***AG***

[1 mark]

(iii) the following table shows some possible sets of values for s, t, a, b, x_m and V_m

s	t	a	b	x_m	V_m
1	3	5	8	1	18
1	5	9	24	2	200
2	5	16	21	3	450
2	6	20	32	4	1152
2	7	24	45	5	2450
2	8	28	60	6	4608
3	7	33	40	6	3528
4	10	64	84	12	28800
…	…	…	…	…	…

attempts to use values of s and t to find

EITHER x_m **OR** a and b **OR** V_m ***(M1)***

a corresponding correct x_m value ***A1***

corresponding correct a and b values ***(A1)***

a corresponding correct V_m value ***A1***

Note: Award at most ***(M1)A0A0A0*** if the values of s and t do not satisfy the conditions: s, $t \in \mathbb{Z}^+$, $t > 2s$, and s is even / s and t are odd.

The three ***A*** marks can be awarded in any order.

The ***A1*** for x_m and the ***A1*** for a and b can be awarded independently.

Some candidates may determine in part (h) that $V_m = \frac{s^2t^2(t-s)^2}{2}$.

[4 marks]

Total [28 marks]

3. (a) (i) $c = 9$ ***A1***

(ii) attempt to substitute (3,0) into $y = ax^2 + c$ with *their* c ***(M1)***

$9a + 9 = 0$

$a = -1$ ***A1***

[3 marks]

(b) **METHOD 1** (single integral)

attempt to subtract areas (seen anywhere) ***(M1)***

$\int_0^3 \left(9 - x^2\right) - \left(9 - 3x\right)\ \mathrm{d}x \quad \left(= \int_0^3 3x - x^2\ \mathrm{d}x\right)$ OR $\int_0^3 \left(9 - x^2\right)\mathrm{d}x - \text{triangle area}$

correct integration $\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$ **OR** $\left[9x - \frac{1}{3}x^3\right]_0^3$ ***(A1)***

attempt to substitute their limits into their integrated expression (condone absence of the 0 substitution) ***(M1)***

$\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3 - \left(\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3\right)$ OR $9(3) - \frac{1}{3}(3)^3$

$\frac{27}{2} - 9$ OR $18 - \frac{27}{2}$

Area $= \frac{9}{2}$ (square units) ***A1***

continued…

Question 3 continued.

METHOD 2 (evaluating two integrals separately)

$\int_0^3 \left(9-x^2\right)\,\mathrm{d}x = \left[9x-\frac{1}{3}x^3\right]_0^3$ **OR** $\int_0^3 (9-3x)\,\mathrm{d}x = \left[9x-\frac{3}{2}x^2\right]_0^3$ ***(A1)***

Note: Award ***A1*** for correct integration. Ignore absence of limits.

attempt to substitute their limits into both of their integrated expressions (condone absence of the 0 substitution) ***(M1)***

$9(3)-\frac{1}{3}(3)^3-\left(9(0)-\frac{1}{3}(0)^3\right)$ and $9(3)-\frac{3}{2}(3)^2$

attempt to subtract the two integrals in any order (seen anywhere) ***(M1)***

$9(3)-\frac{1}{3}(3)^3-\left(9(3)-\frac{3}{2}(3)^2\right)$ OR $\int\left(9-x^2\right)\mathrm{d}x-\int(9-3x)\mathrm{d}x$

$18-\frac{27}{2}$

Area $=\frac{9}{2}$ (square units) ***A1***

[4 marks]

Total [7 marks]

5. (a) (i) **METHOD 1**

$f'(x) = 4(1-x)^{-5}$ ***A1***

substitution of x=0 into derivative ***M1***

$4(1-0)^{-5}$ OR $f'(0) = 4$

$a = 4$ ***AG***

Note: Do not award ***M1*** for $1+4x$ seen without clear evidence of a substitution.

METHOD 2

attempt to create binomial expansion to at least two terms ***M1***

$1+\binom{-4}{1}x+\left(\binom{-4}{2}(x)^2+...\right)$

correct expansion

$1+(-4)(-x)+\left(\frac{(-4)(-5)}{2}(-x)^2+...\right)$ ***A1***

$a = 4$ ***AG***

Note: Do not award ***A1*** for $1+4x$ seen without clear evidence of correct working.

(ii) **METHOD 1**

$f''(x) = (-5)\times(4)(1-x)^6\times(-1),\ \ 20(1-x)^{-6}$ ***(A1)***

$(f''(0) =)\,20$ ***(A1)***

$\frac{20}{2!}$

$b = 10$ ***A1***

METHOD 2

$\frac{(-4)(-5)}{2!}(-x)^2$ (may be seen in their expansion in (i) ***(A1)***

$\frac{20}{2}$ ***(A1)***

$b = 10$ ***A1***

[5 marks]

continued…

Question 5 continued.

(b) attempt to substitute 0.1 into the expansion ***(M1)***

1.52

1520 (dollars) ***A1***

[2 marks]

Total [7 marks]

7. (a) attempt to separate variables ***(M1)***

$$\int \frac{\mathrm{d}y}{y^2} = \int \cos^2 x \ \mathrm{d}x$$

attempt to use an identity for $\cos^2 x$, may be seen above ***(M1)***

$$\frac{1}{2}(1+\cos 2x)$$

$$-y^{-1} = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$ ***A1A1***

Note: Award ***A1*** for LHS and award ***A1*** for RHS. Award ***A1A0*** for at least one correct side and missing constant of integration.

$$-y^{-1} = \frac{2x+\sin 2x}{4} + C_1 \quad \textbf{OR} \quad y = \frac{-4}{2x+\sin 2x + C_2}$$ ***(A1)***

Note: Condone solutions that do not deal correctly with the constant(s) of integration.

$a = -4$ ***A1***

[6 marks]

(b) attempt to substitute initial conditions with *their* a. ***(M1)***

$$1 = \frac{-4}{\pi + 0 + b}$$

$b = -4 - \pi$ ***A1***

[2 marks]

Total [8 marks]

9. (a) recognition that $f'(x) = 0$ (seen anywhere) ***(M1)***

attempt to solve quadratic equation (condone not seeing $=0$) ***(M1)***

$a = -5,\ b = 1$ ***A1***

[3 marks]

(b) (local) maximum ***A1***

($f'(x)$/gradient) changes from positive to negative (at $x = a$)

OR $f(x)$ changes from increasing to decreasing (at $x = a$)

OR $f''(a) = -18$ (accept $f''(a) < 0$) ***R1***

Note: Do not award ***A0R1***.

[2 marks]

(c) $f''(x) = 6x + 12$ ***(A1)***

$6c + 12 = 0$ ***(A1)***

$c = -2$ ***A1***

[3 marks]

(d) yes (there is a point of inflexion at $x = c$) ***A1***

($f''(x)$/second derivative) changes sign at $x = c$

OR concavity (of f) changes at $x = c$

OR $f'(x)$/gradient decreasing to the left of c, and increasing to the right of c ***R1***

[2 marks]

Note: Award ***R0*** if their only reason is that $f''(c) = 0$. Do not award ***A0R1***.

continued…

Question 9 continued.

(e) recognize to find $\int f'(x)\mathrm{d}x$ ***(M1)***

$x^3+6x^2-15x(+C)$ ***(A1)***

substitution of $(-2,36)$ into their integrated expression (must contain $+C$) ***(M1)***

$36=(-2)^3+6(-2)^2-15(-2)+C$

$C=-10$ ***A1***

$f(x)=x^3+6x^2-15x-10$

[4 marks]

Total [14 marks]

10. (a) (i) attempt to create binomial expansion to four terms with $n=-1$ ***(M1)***

$$1+\binom{-1}{1}x^2+\binom{-1}{2}\left(x^2\right)^2+\binom{-1}{3}\left(x^2\right)^2+\ldots$$

$$1-x^2+\frac{(-1)(-2)}{2}\left(x^2\right)^2+\frac{(-1)(-2)(-3)}{6}\left(x^2\right)^3$$ ***A1A1***

Note: Award ***A1*** for the third term and ***A1*** for the fourth term.

$$1-x^2+x^4-x^6$$ ***AG***

[3 marks]

(ii) $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7\left(+C\right)$ ***A2***

[2 marks]

Note: Award ***A1A0*** for any three correct terms.

(b) (i) attempt to create binomial expansion the term in x^4 with $n=-\frac{1}{2}$ ***(M1)***

$$1+\binom{-\frac{1}{2}}{1}x^2+\binom{-\frac{1}{2}}{2}\left(x^2\right)^2+\ldots$$

$$1-\frac{1}{2}\left(-x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2$$ ***(A1)(A1)***

Note: Award ***(A1)*** for the first two terms and ***(A1)*** for the third term.

$$1+\frac{1}{2}x^2+\frac{3}{8}x^4$$ ***A1***

(ii) $x+\frac{1}{6}x^3+\frac{3}{40}x^5\left(+C\right)$ ***A1***

[5 marks]

continued…

Question 10 continued.

(c) $\arcsin x = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ (seen anywhere) ***(A1)***

attempt to substitute $x = \frac{1}{2}$ into their antiderivative ***(M1)***

$\left(\arcsin\frac{1}{2} \approx\right)\frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5$, $\frac{1}{2} + \frac{1}{48} + \frac{3}{1280}$ ***A1***

$k = 3$ (accept $\frac{25}{48} + \frac{3}{1280}$) ***A1***

[4 marks]

(d) $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (seen anywhere) ***(A1)***

equating their value of $\arcsin\left(\frac{1}{2}\right)$ to their expression from (c) ***(M1)***

$\frac{\pi}{6} \approx \frac{25}{48} + \frac{3}{1280}$, $\pi \approx 6 \times \left(\frac{25}{48} + \frac{3}{1280}\right)$

$\pi \approx \frac{25}{8} + \frac{18}{1280}$, $\frac{4000}{1280} + \frac{18}{1280}$ ***(A1)***

$\pi \approx \frac{2009}{640}$ (accept $p = 2009$) ***A1***

Note: Condone use of "$=$" instead of "$\approx$".

[4 marks]

Total [18 marks]

6. $x\dfrac{\mathrm{d}y}{\mathrm{d}x}+3y=\dfrac{\operatorname{cosec}^2 x}{x}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{3}{x}y=\dfrac{\operatorname{cosec}^2 x}{x^2}$ ***(A1)***

IF $=\mathrm{e}^{\int\frac{3}{x}\mathrm{d}x}\left(=x^3\right)$ ***(A1)***

attempt to multiply through by their IF: ***(M1)***

$x^3\dfrac{\mathrm{d}y}{\mathrm{d}x}+3x^2y=x\operatorname{cosec}^2 x$

Note: Award first 3 marks if candidates directly multiply through the DE by x^2.

attempt to write LHS as an exact differential ***(M1)***

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^3y\right)=x\operatorname{cosec}^2 x$

$x^3y=-x\cot x+\ln\left|\sin x\right|+c$ ***A1***

$y=\dfrac{-x\cot x+\ln\left|\sin x\right|+c}{x^3}$

substituting $x=\dfrac{\pi}{2}$, $y=\dfrac{8}{\pi^2}$ into their integrated expression ***(M1)***

$\Rightarrow c=\pi$

$y=\dfrac{\pi-x\cot x+\ln\left|\sin x\right|}{x^3}$ ***A1***

Note: Do not penalise omission of absolute value signs as for $0<x<\pi$, $\sin x$ is always positive.

[7 marks]

8. (a) attempt to sum times for both sections using $T = \frac{D}{S}$ ***(M1)***

$T = \frac{400\sec\theta}{0.8} + \frac{1000 - 400\tan\theta}{1.2}$ ***A1***

$T = 500\sec\theta + \frac{2500 - 1000\tan\theta}{3}$ ***AG***

[2 marks]

(b) attempt to differentiate T $\left(\sec\theta\tan\theta \text{ or } \sec^2\theta \text{ seen}\right)$ ***M1***

$\left(\frac{\mathrm{d}T}{\mathrm{d}\theta} =\right) 500\sec\theta\tan\theta - \frac{1000}{3}\sec^2\theta$ ***A1A1***

[3 marks]

(c) set their $\frac{\mathrm{d}T}{\mathrm{d}\theta} = 0$ and attempt to solve ***(M1)***

$500\sin\theta - \frac{1000}{3} = 0$

$\sin\theta = \frac{2}{3}$ ***A1***

$\tan\theta = \frac{2}{\sqrt{5}}$ OR $\sec\theta = \frac{3}{\sqrt{5}}$ or equivalent ***A1***

$\mathrm{PX} = 400\tan\theta$ (seen anywhere) OR $\mathrm{PX} = \frac{800}{\sqrt{5}}$ ***A1***

$= 160\sqrt{5}$ ***AG***

[4 marks]

Total [9 marks]

Section B

9. (a)

Note: In this question, the marks are independent and may be awarded in either order.

finding correct discriminant $k^2 - 26$ ***A1***

recognition that discriminant $\left(b^2 - 4ac\right) < 0$ ***M1***

$b^2 - 4ac < 0$ or $k^2 - 26 < 0$ or $k^2 < 26$

(since $k \in \mathbb{Z}^+$) $k = 5$ ***AG***

[2 marks]

(b) (i) $x = -5$ (must be an equation) ***A1***

(ii) attempt to substitute their value of x (from axis of symmetry) into $\frac{1}{2}x^2 + 5x + 13$

OR attempt to use completing the square ***(M1)***

$(y =) \frac{1}{2}(-5)^2 + 5(-5) + 13 \left(= \frac{1}{2}\right)$ OR $(y =) \frac{1}{2}(x+5)^2 + \frac{1}{2}$

$\left(-5, \frac{1}{2}\right)$ ***A1***

[3 marks]

(c) at $x = -3,\ y = \frac{5}{2}$ (seen anywhere) ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = x + 5$ ***A1***

at $x = -3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ ***A1***

normal gradient is negative reciprocal of their $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -3$ ***(M1)***

gradient of normal $= -\frac{1}{2}$

$y - \frac{5}{2} = -\frac{1}{2}(x+3)$ OR $\frac{5}{2} = -\frac{1}{2}(-3) + c \Rightarrow c = 1$ ***A1***

$y = -\frac{1}{2}x + 1$ ***AG***

[5 marks]

continued…

Question 9 continued

(d) attempt to describe shaded area as the difference of two integrals or one integral and the area of a trapezium ***(M1)***

$(A=)\ \int_{-3}^{0}\left(\frac{1}{2}x^2+5x+13\right)\mathrm{d}x-\int_{-3}^{0}\left(-\frac{1}{2}x+1\right)\mathrm{d}x$ OR $\int_{-3}^{0}\left(\frac{1}{2}x^2+\frac{11}{2}x+12\right)\mathrm{d}x$

OR $\int_{-3}^{0}\left(\frac{1}{2}x^2+5x+13\right)\mathrm{d}x-\frac{1}{2}\left(\frac{5}{2}+1\right)3$

correct integration ***A2***

$\left[\frac{x^3}{6}+\frac{5x^2}{2}+13x\right]_{-3}^{0}-\left[-\frac{x^2}{4}+x\right]_{-3}^{0}$ OR $\left[\frac{x^3}{6}+\frac{11x^2}{4}+12x\right]_{-3}^{0}$ OR $\left[\frac{x^3}{6}+\frac{5x^2}{2}+13x\right]_{-3}^{0}-\frac{21}{4}$

Note: If not completely correct, award ***A1*** for at least two terms correctly integrated.

substituting 0 and -3 into their integrated expression and subtracting ***(M1)***

$0-\left(\frac{(-3)^3}{6}+\frac{11(-3)^2}{4}+12(-3)\right)$ OR $0-\left(\frac{(-3)^3}{6}+\frac{5(-3)^2}{2}+13(-3)\right)-\frac{21}{4}$ (or equivalent)

$=\frac{63}{4}$ ***A1***

[5 marks]
Total [15 marks]

$\begin{pmatrix} -1 \\ \end{pmatrix} \quad \begin{pmatrix} 7 \\ \end{pmatrix}$

8. attempt to write at least one function in terms of y ***(M1)***

$x^2 = y^2 + 16$ **OR** $x^2 = 25y - 50$ (or equivalent) ***(A1)***

one correct expression for volume

$\pi\int_0^3 y^2 + 16 \, \mathrm{d}y$ **OR** $\pi\int_2^3 25y - 50 \, \mathrm{d}y$ ***(A1)***

attempt to subtract two volumes with limits for y ***(M1)***

$\pi\int_0^3 \left(y^2 + 16\right) \mathrm{d}y - \pi\int_2^3 \left(25y - 50\right) \mathrm{d}y$ ***(A1)***

Note: Award at most ***M1A1A1M0A0A0*** for a candidate who writes $\pi\int_0^3 \left(y^2 + 16\right) - \left(25y - 50\right) \mathrm{d}y$.

$179.0707... - 39.269908$

$139.8008\ldots$

$V = 140$ (cubic units) $\left(= \frac{89}{2}\pi\right)$ ***A1***

Note: Ignore absence of π throughout if their final answer is correct. Otherwise award at most ***M1A1A1M1A0A0.***

Total [6 marks]

9. attempt to find a relationship between x and θ using cosine rule ***(M1)***

$x^2 = 15^2 + 20^2 - 2(15)(20)\cos\theta,\ \ x = \sqrt{625 - 600\cos\theta}$

attempt to relate the rates (seen anywhere) ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{1}{2}(625 - 600\cos\theta)^{-\frac{1}{2}} \times 600\sin\theta$ ***(A1)***

OR

Attempt to differentiate implicitly with respect to t ***(M1)***

$2x\frac{\mathrm{d}x}{\mathrm{d}t} = 600\sin\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$ **OR** $\frac{\mathrm{d}x}{\mathrm{dt}} = \frac{1}{2}(625 - 600\cos\theta)^{-\frac{1}{2}} \times 600\sin\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$ ***(A1)***

OR

Using GDC to find $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***(M1)***

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 13.835110...$ ***(A1)***

attempt to find $\cos\theta$ or $\sin\theta$ or θ when area is 140 ***(M1)***

$\sin\theta = \frac{14}{15}$ or $\cos\theta = \frac{\sqrt{29}}{15}$ or $\theta = 1.20358...$ (seen anywhere) ***(A1)***

(accept $68.9605...^\circ$)

$\frac{1}{2}\left(625 - 600\left(\frac{\sqrt{29}}{15}\right)\right)^{\left(-\frac{1}{2}\right)}\left(600 \times \frac{280}{300}\right) \times \left(-\frac{\pi}{60}\right)$ (***A1)***

$-0.724402...$

$\frac{\mathrm{d}x}{\mathrm{d}t} = -0.724\ \left(\mathrm{cm\,min^{-1}}\right)$ ***A1***

Note: Award **(M1)(M1)(A1)(M1)(A1)(A1)A0** for $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.724\ \left(\mathrm{cm\,min^{-1}}\right)$.

Total [7 marks]

Section B

10. (a) (i) speed of airplane $= 60\ (\text{ms}^{-1})$ ***A1***

(ii) speed of car $= 0\ (\text{ms}^{-1})$ ***A1***

[2 marks]

(b) (i) recognizing $v_{\text{air}} = v_{\text{car}}$ when speeds are the same ***(M1)***

$6.35564...$

$t = 6.36\ (\text{s})$ ***A1***

(ii) $31.7782\ldots$

speed $= 31.8\ (\text{ms}^{-1})$ ***A1***

[3 marks]

(c) let $s_{\text{air}}(t)$ be the displacement of the airplane and $s_{\text{car}}(t)$ be the displacement of the car at time t

recognizing to integrate either velocity function ***(M1)***

$\int 60\text{e}^{-0.1t}\ \text{d}t,\ \int 5t\ \text{d}t$

correct integration ***(A1)(A1)***

$-600\text{e}^{-0.1t},\ \frac{5}{2}t^2\ (+C)$ (award ***A1*** for each term)

recognize to subtract displacements (seen anywhere) ***(M1)***

EITHER

substituting initial condition into their $d(t)$ ***(M1)***

$100 = -600\text{e}^{-1(0)} - \frac{5}{2}\times 0^2 + C$ OR $-100 = 600\text{e}^{-1(0)} + \frac{5}{2}\times 0^2 + C$ (or equivalent)

OR

substituting initial condition into their $s_{\text{air}}(t)$ ***(M1)***

$\left(s_{\text{car}}(0) = 0 \Rightarrow\right) 100 = -600\text{e}^{-1(0)} + C$

continued…

Question 10 continued.

THEN

$C = 700$ (or -700) ***(A1)***

$d(t) = -600\mathrm{e}^{-0.1t} - \frac{5}{2}t^2 + 700$ (accept $d(t) = \left|600\mathrm{e}^{-0.1t} + \frac{5}{2}t^2 - 700\right|$) ***A1***

[7 marks]

(d) recognizing to solve their $d(t) = 0$ ***(M1)***

$15.0586...$

$t = 15.1$ (s) ***A1***

[2 marks]

(e) attempt to find the distance travelled by the car ***(M1)***

$\int_0^{15.0586} 5t \, \mathrm{d}t$, $\frac{5}{2}(15.0586)^2$ ***(A1)***

$566.904\ldots$

567 (m) (accept 570 (m) from 3 sf answer in (d)) ***A1***

Note: Award ***(M1)(A1FT)A0***, where appropriate, if the candidate has an incorrect $d(t)$ and/or incorrect value of t which leads to their distance travelled by the car to be less than 100 (m).

For example, where the candidate finds $d(t) = 600\mathrm{e}^{-0.1t} - \frac{5}{2}t^2 - 500$ in part (c), award ***(M1)(A1FT)A0*** to 7.08, or 7.06 from using 3 sf value.

[3 marks]

Total [17 marks]

5. attempt to rearrange $f(x) = e^{\frac{3x}{4}}$ to get an expression for x in terms of y OR find $f^{-1}(x)$ ***(M1)***

$x = \frac{4}{3}\ln y$ OR $f^{-1}(x) = \frac{4}{3}\ln x$ ***(A1)***

attempt to use $\pi\int x^2 dy$ OR $\pi\int \left(f^{-1}(x)\right)^2 dx$ for their rearrangement of $y = e^{\frac{3x}{4}}$ ***(M1)***

$\pi\int \left(\frac{4}{3}\ln y\right)^2 dy$ OR $\pi\int \left(\frac{4}{3}\ln x\right)^2 dx$

$\pi\int_1^8 \left(\frac{4}{3}\ln y\right)^2 dy$ OR $\pi\int_1^8 \left(\frac{4}{3}\ln x\right)^2 dx$ ***(A1)***

$= 85.6(85.5716...)\left(= \frac{16\pi}{9}\left(8(\ln 8)^2 - 16\ln 8 + 14\right)\right)$ ***A1***

[5 marks]

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10. (a) (i) attempt to find a maximum point ***(M1)***

$(x =) 0.718442...$ or $(y =) 1.80645\ldots$ or $(x =) 3.71844...$ or $(y =) 1.19181\ldots$ etc

$1.80645\ldots$

(maximum displacement) $= 1.81$ (cm) ***A1***

(ii) recognition that $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ ***(M1)***

recognition that the gradient at $t = 0$ represents the maximum velocity ***(M1)***

$v = \frac{\mathrm{d}s}{\mathrm{d}t}$ at $t = 0$

$v = 4.18879025...\left(\frac{4\pi}{3}\right)$

$v = 4.19\left(\frac{4\pi}{3}\right)$ (cms^{-1}) ***A1***

[5 marks]

(b) (i) attempt to find a minimum point ***(M1)***

$(x =) 2.21844...$ or $(y =) -1.46729...$ OR $(x =) 5.21844...$ OR $(y =) -0.968053\ldots$

$-1.46729...$

-1.47 ***A1***

(ii) $(t = 3.5)\, s = 1.06620...$

$s = 1.07$ ***A1***

[3 marks]

continued…

Question 10 continued

(c) **METHOD 1**

attempt to sum **distances** from O involving maximum, minimum and distance at $t = 3.5$ ***(M1)***

$1.80645\ldots \times 2 + 1.46729... \times 2 + 1.06620...$ ***(A1)***

$= 7.61369...$

$7.61\,(\text{cm})$ ***A1***

METHOD 2

attempt to integrate the absolute value of $\frac{\mathrm{d}s}{\mathrm{d}t}$ ***(M1)***

$\int_0^{3.5} \left| \frac{d}{dt}\left(2^{1-\frac{t}{5}} \sin\left(\frac{2\pi t}{3}\right)\right)\right| \mathrm{d}t$ ***(A1)***

$= 7.61369...$

$7.61\,(\text{cm})$ ***A1***

[3 marks]

(d) $(T =) 1.5$ ***A1***

[1 mark]

continued…

Question 10 continued

(e)

(i) attempt to divide two of $u_1, u_2, u_3 \ldots$ ***(M1)***

$\frac{u_2}{u_1}\left(=\frac{1.46729...}{1.80645...}\right)$ or $\frac{u_3}{u_2}\left(=\frac{1.19181...}{1.46729...}\right)$ or $r^2=\frac{u_3}{u_1}\left(=\frac{1.19181...}{1.80645...}\right)$

$0.812252...$

$(r=)0.812$ ***A1***

(ii) attempt to use the sum to infinity of the GP (***M1)***

$S_\infty=\frac{1.80645...}{1-0.812252...}(=9.62167...)$

recognition that the total distance is double their S_∞ (***M1)***

$2\times S_\infty=2\times\frac{1.80645...}{1-0.812252...}$ OR $2\times 9.62167..$

$19.2433...$

$19.2(\text{cm})$ ***A1***

Note: Award (***M1)***(***M1)A0*** for values of r that lead to an answer that is smaller than 2 x first maximum displacement (3.6129).

[5 marks]

Total [17 marks]

12. (a) (i) $\frac{dy}{dx} = -1.7$ ***A1***

(ii) attempt to use Euler's formula with step length 0.2 ***(M1)***

step	x	y
0	1	1.5
1	1.2	1.16
2	1.4	1.07491524

$1.5 + 0.2 \times -1.7$ or 1.16 ***(A1)***

1.07 ***A1***

Note: Accept 1.075.

(iii) change (decrease) in the y values is decreasing. OR the gradient is increasing as the gradient at 1.2 is $-0.425423\ldots$ ***R1***

concave up ***A1***

Note: Do not award ***R0A1.***
Do not award ***FT*** from either part (i) or part (ii).

[6 marks]

continued…

Question 12 continued

(b) (i) **METHOD 1**

$\left(\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\right)x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ (seen anywhere) ***A1***

attempt to differentiate either the LHS or the RHS implicitly using the product rule ***M1***

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y)\frac{\mathrm{d}y}{\mathrm{d}x}=2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y-6y\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***A1A1***

Note: Award **A1** for LHS and **A1** for RHS.

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2x+y-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-(x+7y)\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***AG***

METHOD 2

$\left(\frac{\mathrm{d}}{\mathrm{d}x}(xy)=\right)x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ (seen anywhere) ***A1***

attempt to rearrange into the form $\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{x^2+xy-3y^2}{x^2+xy}$ and attempt to differentiate the RHS implicitly using quotient rule ***M1***

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\frac{\left(x^2+xy\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y-6y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(x^2+xy-3y^2\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)}{\left(x^2+xy\right)^2}$$ ***A1***

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\frac{\left(x^2+xy\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y-6y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-\left(\left(x^2+xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(2x+x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)}{\left(x^2+xy\right)^2}$$ ***A1***

$$\left(x^2+xy\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2x+y-x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-(x+7y)\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***AG***

continued…

Question 12 continued

(ii) attempt to substitute their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $x = 1, y = \frac{3}{2}$ ***(M1)***

$$\frac{5}{2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{7}{2} - (-1.7)^2 - \frac{23}{2}(-1.7)$$

$8.064\left(\frac{1008}{125}\right)$ *OR* 8.06 ***A1***

Note: Award ***(M1)A0*** for 8.1 seen with no working.

[6 marks]

(c) attempt to make the numerators of common denominator equivalent ***(M1)***

$$1 + v = A(1 + 2v) + B(1 - 2v)$$

$A = \frac{3}{4}(0.75), B = \frac{1}{4}(0.25)$ ***A1***

[2 marks]

continued…

Question 12 continued

(d) attempt to replace y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ by substitution of $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

$\left(x^2 + x^2v\right)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = x^2 + vx^2 - 3x^2v^2$

$\left(1+v\right)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = 1 + v - 3v^2$ or $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - \frac{3v^2}{1+v}$ ***A1***

$v\left(1+v\right) + (1+v)x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v - 3v^2$

$(1+v)x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v - 3v^2 - v - v^2$

$(1+v)x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 - 4v^2$

$\int \frac{1+v}{1-4v^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***A1***

hence $\ln x = -\frac{3}{8}\ln\left|1-2v\right| + \frac{1}{8}\ln\left|1+2v\right|(+c)$ $\left(= -\frac{3}{8}\ln\left(2v-1\right) + \frac{1}{8}\ln\left(2v+1\right)(+c)\right)$ ***A1A1***

substitution of $x = 1,\ v = 1.5$ OR $x = 1, y = 1.5$ to find a value of c ***(M1)***

$c = \frac{1}{8}\ln 2\ \left(0.086643\ldots\right)$

$8\ln x + 3\ln\left(\frac{2y}{x} - 1\right) = \ln\left(1 + \frac{2y}{x}\right) + \ln 2$ or equivalent ***A1***

$x^8\left(\frac{2y}{x} - 1\right)^3 = 2\left(1 + \frac{2y}{x}\right)$ ***A1***

$x^6\left(2y - x\right)^3 = 2\left(x + 2y\right)$ ***AG***

[8 marks]

Total [22 marks]

1. (a) (i) **METHOD 1**

attempts to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}$ AND $m_N = \frac{-1}{-\frac{1}{t^2}}$ (or equivalent) ***A1***

Note: Accept $m_1 m_2 = -1$ (or equivalent) seen anywhere.
Award ***A1*** for $m_N = x^2$ and as $x = t$, then $m_N = t^2$.

so gradient of N is t^2 ***AG***

METHOD 2

uses $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{t^2}}{1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2}$ AND $m_N = \frac{-1}{-\frac{1}{t^2}}$ (or equivalent) ***A1***

Note: Accept $m_1 m_2 = -1$ (or equivalent) seen anywhere.
Award ***A1*** for $m_N = x^2$ and as $x = t$, then $m_N = t^2$.

so gradient of N is t^2 ***AG***

[2 marks]

continued…

Question 1 continued.

(ii) **EITHER**

$y-\frac{1}{t}=t^2\left(x-t\right)$ OR $y=t^2\left(x-t\right)+\frac{1}{t}$ ***A1***

OR

$\frac{1}{t}=t^2\left(t\right)+c\Rightarrow c=\frac{1}{t}-t^3$ ***A1***

OR

$t^2x-y-\left(t^2\left(t\right)-\frac{1}{t}\right)=0$ OR $t^2x-y=t^2\left(t\right)-\frac{1}{t}$ ***A1***

THEN

leading to $y=t^2x+\frac{1}{t}-t^3$ ***AG***

[1 mark]

(b) (i) $m=t^2\left(m^2=t^4\right)$ OR $c=\frac{1}{t}-t^3$ OR $c=\frac{1}{t}\left(1-t^4\right)$ ***(A1)***

$t=\pm\sqrt{m}$ (seen anywhere) ***A1***

substitutes one of their $t=\sqrt{m}$ OR $t=-\sqrt{m}$ into $c=\frac{1}{t}-t^3$ OR $c=\frac{1}{t}\left(1-t^4\right)$ ***M1***

$c=\pm\frac{1}{\sqrt{m}}\left(1-m^2\right)$ ***A1***

so either $c=\frac{1}{\sqrt{m}}\left(1-m^2\right)$ OR $c=\frac{1}{\sqrt{m}}\left(m^2-1\right)$ ***AG***

[4 marks]

(ii) $m\in\mathbb{R}^+$ (or equivalent) ***A1***

[1 mark]

continued…

Question 1 continued.

(c) (i) $m = 1$ ***A1***

[1 mark]

(ii) $m \in \mathbb{R}^+, m \neq 1$ (or equivalent) ***A1A1***

Note: Award ***A1*** for $0 < m < 1$ and ***A1*** for $m > 1$.

[2 marks]

(d) (i) a correct sketch of $y = \frac{1}{x}$ ***A1***

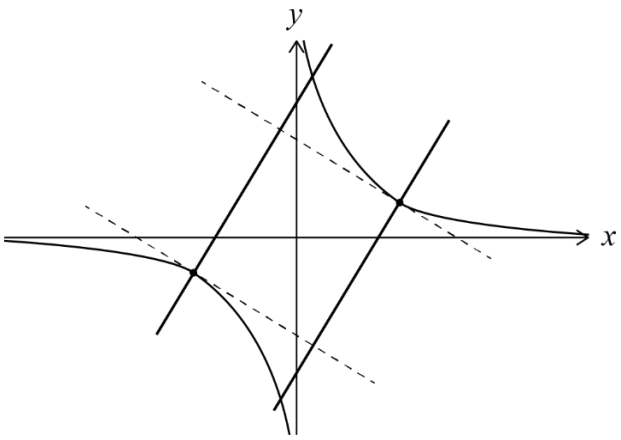
[1 mark]

(ii) a line normal to H such that $c \neq 0$ ***A1***

a second parallel line which is normal to H such that $c \neq 0$ ***A1***

Note: For normal lines that cross the y- axis, award ***A1A0*** if the y- axis intercepts are not approximately equidistant from O.

eg.



Note: Accept correct sketches that show the point(s) where the line(s) is/are normal. Candidates are not required to show dashed lines as shown above.

Award a maximum ***A1A0*** for a normal line sketched for $0 < m < 1$ and a normal line sketched for $m > 1$.

[2 marks]

continued…

Question 1 continued.

(e) (i) $\frac{k^2}{x} = t^2x - kt^3 + \frac{k}{t}$ ***(A1)***

attempts to form a quadratic equation in x^2 ***M1***

eg. $t^2x^2 - kt^3x + \frac{k}{t}x = k^2$ OR $x^2 - ktx + \frac{k}{t^3}x - \frac{k^2}{t^2} = 0$ ***A1***

OR $\frac{k^2}{t^2} = x^2 - \left(kt - \frac{k}{t^3}\right)x$ OR $t^2x^2 - k\left(t^3 - \frac{1}{t}\right)x - k^2 = 0$

$x^2 - k\left(t - \frac{1}{t^3}\right)x - \frac{k^2}{t^2} = 0$ ***AG***

Note: Award ***(A0)M1A0*** for attempting to verify that kt is a root of the ***AG*** (or its equivalent quadratic equation).
Award ***(A0)M0A0*** for showing that A lies on F.

[3 marks]

continued…

Question 1 continued.

(ii) **METHOD 1**

let α be the x- coordinate of A and let β be the x- coordinate of B

EITHER

recognizes that the sum of roots is $k\left(t-\frac{1}{t^3}\right)$ ***(M1)***

Note: Award ***(M1)*** for $-k\left(t-\frac{1}{t^3}\right)$.

$kt+\beta=k\left(t-\frac{1}{t^3}\right)$

OR

recognizes that the product of roots is $-\frac{k^2}{t^2}$ ***(M1)***

Note: Award ***(M1)*** for $\frac{k^2}{t^2}$.

$kt\beta=-\frac{k^2}{t^2}$

THEN

$\left(\beta=\right)-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},-kt^3\right)$

continued…

Question 1 continued.

METHOD 2

recognizes that $(x-kt)$ is a factor of $x^2-k\left(t-\frac{1}{t^3}\right)x-\frac{k^2}{t^2}(=0)$ ***(M1)***

$(x-kt)\left(x+\frac{k}{t^3}\right)(=0)$

$\Rightarrow x=kt,-\frac{k}{t^3}$

$-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},-kt^3\right)$

METHOD 3

attempts to use the quadratic formula ***(M1)***

$$x=\frac{k\left(t-\frac{1}{t^3}\right)\pm\sqrt{\left(k\left(t-\frac{1}{t^3}\right)\right)^2+4\left(\frac{k^2}{t^2}\right)}}{2}$$

$\Rightarrow x=kt,-\frac{k}{t^3}$

$-\frac{k}{t^3}$ is the other root ***A1***

$y=-kt^3$ ***A1***

coordinates of B are $\left(-\frac{k}{t^3},-kt^3\right)$

[3 marks]

continued…

Question 1 continued.

(f) **METHOD 1**

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ ***(A1)***

attempts to find their m_{AC} or m_{BC} ***(M1)***

$m_{\mathrm{AC}} = \frac{1}{t^2}$ AND $m_{\mathrm{BC}} = \frac{-\frac{k}{t} - \left(-kt^3\right)}{-kt - \left(-\frac{k}{t^3}\right)} = -t^2 \left(= \frac{kt^3 - \frac{k}{t}}{\frac{k}{t^3} - kt} = -t^2 \right) \left(t \neq \pm 1\right)$ (or equivalent) ***A1***

attempts to find their $m_{\mathrm{AC}} \times m_{\mathrm{BC}}$ OR recognizes that $m_{\mathrm{AC}} = -\frac{1}{m_{\mathrm{BC}}}$ (or equivalent) ***(M1)***

$= \frac{1}{t^2} \times \left(-t^2\right)$ OR eg. $\frac{1}{t^2} \times \frac{kt^4 - k}{t} \times \frac{t^3}{k - kt^4} = \frac{t^3}{t^3} \times \frac{kt^4 - k}{-\left(kt^4 - k\right)}$ OR $-\frac{1}{\frac{1}{t^2}} = -t^2$ (or equivalent)

A1

$= -1$ and so $\left[\mathrm{BC}\right] \perp \left[\mathrm{CA}\right]$ ***R1***

hence $\mathrm{B\hat{C}A}$ is a right angle ***AG***

METHOD 2

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ ***(A1)***

$m_{\mathrm{AC}} = \frac{1}{t^2}$ ***A1***

attempts to find their equation of the perpendicular to $\left[\mathrm{AC}\right]$ at C ***(M1)***

$y - \left(-\frac{k}{t}\right) = -t^2\left(x - \left(-kt\right)\right)$

substitutes $x = -\frac{k}{t^3}$ into their equation of the perpendicular to $\left[\mathrm{AC}\right]$ at C ***(M1)***

$y = -kt^3$ ***A1***

B lies on the perpendicular to $\left[\mathrm{AC}\right]$ at C $(\mathrm{B} \neq \mathrm{C})$ and so $\left[\mathrm{BC}\right] \perp \left[\mathrm{CA}\right]$ ***R1***

hence $\mathrm{B\hat{C}A}$ is a right angle ***AG***

continued…

Question 1 continued.

METHOD 3

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ OR $\vec{\mathrm{OC}} = \begin{pmatrix} -kt \\ -\frac{k}{t} \end{pmatrix}$ ***(A1)***

attempts to find their $\vec{\mathrm{CA}}$ OR their $\vec{\mathrm{CB}}$ ***(M1)***

$\vec{\mathrm{CA}} = \begin{pmatrix} 2kt \\ \frac{2k}{t} \end{pmatrix}$ AND $\vec{\mathrm{CB}} = \begin{pmatrix} -\frac{k}{t^3} + kt \\ -kt^3 + \frac{k}{t} \end{pmatrix}$ ***A1***

attempts to find their $\vec{\mathrm{CA}} \cdot \vec{\mathrm{CB}}$ ***(M1)***

$\vec{\mathrm{CA}} \cdot \vec{\mathrm{CB}} = 2kt\left(-\frac{k}{t^3} + kt\right) + \frac{2k}{t}\left(-kt^3 + \frac{k}{t}\right)$

$= -\frac{2k^2}{t^2} + 2k^2t^2 - 2k^2t^2 + \frac{2k^2}{t^2}$ ***A1***

$= 0$ (and since $t \neq \pm 1$, $\left|\vec{\mathrm{CB}}\right| \neq 0$) ***R1***

then $\mathrm{B\hat{C}A}$ is a right angle ***AG***

Note: Award as above if $\vec{\mathrm{OA}} = \begin{pmatrix} kt \\ \frac{k}{t} \end{pmatrix}$ is used instead of $\vec{\mathrm{CA}} = \begin{pmatrix} 2kt \\ \frac{2k}{t} \end{pmatrix}$.

continued …

Question 1 continued.

METHOD 4

$\mathrm{A}\left(kt, \frac{k}{t}\right)$ and $\mathrm{B}\left(-\frac{k}{t^3}, -kt^3\right)$

$\mathrm{C}\left(-kt, -\frac{k}{t}\right)$ ***(A1)***

attempts to find one of their AB^2, AC^2 or BC^2 ***(M1)***

EITHER

$\mathrm{AB}^2 = \left(kt - \left(-\frac{k}{t^3}\right)\right)^2 + \left(\frac{k}{t} - \left(-kt^3\right)\right)^2$ ***A1***

OR

$\mathrm{AC}^2 = \left(kt - (-kt)\right)^2 + \left(\frac{k}{t} - \left(-\frac{k}{t}\right)\right)^2$ ***A1***

OR

$\mathrm{BC}^2 = \left(-kt - \left(-\frac{k}{t^3}\right)\right)^2 + \left(-\frac{k}{t} - \left(-kt^3\right)\right)^2$ ***A1***

THEN

attempts to show that their $\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{BC}^2$ ***(M1)***

$\mathrm{AB}^2 = \frac{k^2}{t^6}\left(t^4+1\right)^3$ (or equivalent) AND $\mathrm{AC}^2 + \mathrm{BC}^2 = \frac{k^2}{t^6}\left(t^4+1\right)^3$ (or equivalent) ***A1***

$\mathrm{AB}^2 = k^2t^6 + 3k^2t^2 + \frac{3k^2}{t^2} + \frac{k^2}{t^6}$ AND $\mathrm{AC}^2 + \mathrm{BC}^2 = k^2t^6 + 3k^2t^2 + \frac{3k^2}{t^2} + \frac{k^2}{t^6}$

$\mathrm{AB}^2 = \mathrm{AC}^2 + \mathrm{BC}^2$ (and so, by the converse of Pythagoras' theorem) ***R1***

$\mathrm{B\hat{C}A}$ is a right angle ***AG***

[6 marks]

Total [26 marks]

2. (a) (i) that the stable/long term population will be the same as before the reduction. ***A1***

Note: Condone 'the maximum value / carrying capacity of the population will be 200'.

[1 mark]

(ii) attempt to substitute into logistic function ***(M1)***

$$40 = \frac{200}{1 + C\mathrm{e}^{0}}$$

$$40 + 40C = 200$$

$$C = 4$$ ***A1***

[2 marks]

(iii) correct substitution into their logistic function ***(A1)***

$$70 = \frac{200}{1 + 4\mathrm{e}^{-5k}}$$

$$k = 0.153\ (0.153451..., -\frac{1}{5}\ln\left(\frac{130}{280}\right))$$ ***A1***

[2 marks]

(iv) correct substitution into their logistic function ***(A1)***

$$x = \frac{200}{1 + 4\mathrm{e}^{-10\times 0.153451...}} \left(= 107.397....\right)$$

$$x = 107$$ ***A1***

[2 marks]

continued…

Question 2 continued

(b) (i) $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ ***A1***

[1 mark]

(ii) the population is steady/stable (at 200) **OR** this (200) is the equilibrium population ***A1***

Note: Accept phrases such as 'the rate of change of wolf population is zero'.
Do not accept 200 is the maximum. Do not follow through from an incorrect (b)(i).

[1 mark]

(iii) **EITHER**

$\frac{1}{\ln 200 - \ln x} \times$ an attempt to differentiate $\ln 200 - \ln x$ ***M1***

$f'(x) = \frac{1}{\ln 200 - \ln x} \times \left(-\frac{1}{x}\right)$ ***A1***

OR

$\frac{1}{\ln\left(\frac{200}{x}\right)} \times \frac{x}{200} \times -\frac{200}{x^2}$ ***M1A1***

Note: Award ***M1*** if there is a product of three expressions and two are correct.

$= \frac{-1}{x \ln\left(\frac{200}{x}\right)}$ ***AG***

[2 marks]

continued…

Question 2 continued

(iv) attempt to separate variables ***M1***

$$\int \frac{-1}{x\ln\left(\frac{200}{x}\right)}\mathrm{d}x = \int -a\mathrm{d}t$$ ***A1***

Note: Award ***A1*** also for $\int \frac{1}{x\left(\ln\left(\frac{200}{x}\right)\right)}\mathrm{d}x = \int a\mathrm{d}t$.

$$\ln\left(\ln 200 - \ln x\right) = -at\left(+c\right)$$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

$\ln 200 - \ln x = \mathrm{e}^{-at+c}$ **OR** $\ln 200 - \ln x = A\mathrm{e}^{-at}$, $\left(A = \mathrm{e}^{c}\right)$ ***A1***

$\ln x = \ln 200 - A\mathrm{e}^{-at}$ ***AG***

[5 marks]

(v) correct substitution $\ln 40 = \ln 200 - A\mathrm{e}^{-0}$ ***(A1)***

$A = \ln\left(5\right)$ ***A1***

$\left(p = 5\right)$

Note: Award ***A1A0*** for an unsupported 1.61 $(1.60943\ldots)$.

[2 marks]

continued…

Question 2 continued

(vi) correct substitution $\ln 70 = \ln 200 - \ln 5 \times e^{-5a}$ ***A1***

$a = 0.0854528...$ ***A1***

Note: Award ***A1*** if their unrounded answer has at least four significant figures and rounds to 0.0855 OR if an exact value equivalent to $0.2 \ln \left(\frac{\ln 5}{\ln \left(\frac{200}{70} \right)} \right)$ is seen.

$a = 0.0855$ to 3 significant figures ***AG***

[2 marks]

(vii) correct substitution $\ln x = \ln 200 - \ln 5 \times e^{-10 \times 0.0854528...}$ ***(A1)***

$100.839\ldots$ ***(A1)***

Note: Accept $100.871\ldots$ from use of 3sf 0.0855.

101 to the nearest whole number ***A1***

[3 marks]

(c) both are overestimates **OR** the Gompertz is closer to the true value ***A1***

[1 mark]

Note: Follow through on their answers for 2(a)(iv) and 2 (b)(vii) if both positive values and these values are compared with 85.

continued…

Question 2 continued

(d) (i) attempt to use Euler ***(M1)***

Note: Condone maximum of one error from missing $x_n +$ at start, or incorrect/missing subscripts, absence of 0.5.

$$x_{n+1} = x_n + 0.5\left(0.0855x_n \ln\left(\frac{200}{x_n}\right) - 0.03x_n\right)$$ ***(A1)***

evidence of correct use of Euler's method ***(A1)***

Note: Award ***(A1)*** for any correct value of x_{n+1} for $n = 2,3,...$ seen (including this intermediate value presented as a final answer).

after 10 years the population will be 89 wolves $(89.42803\ldots)$ ***A1***

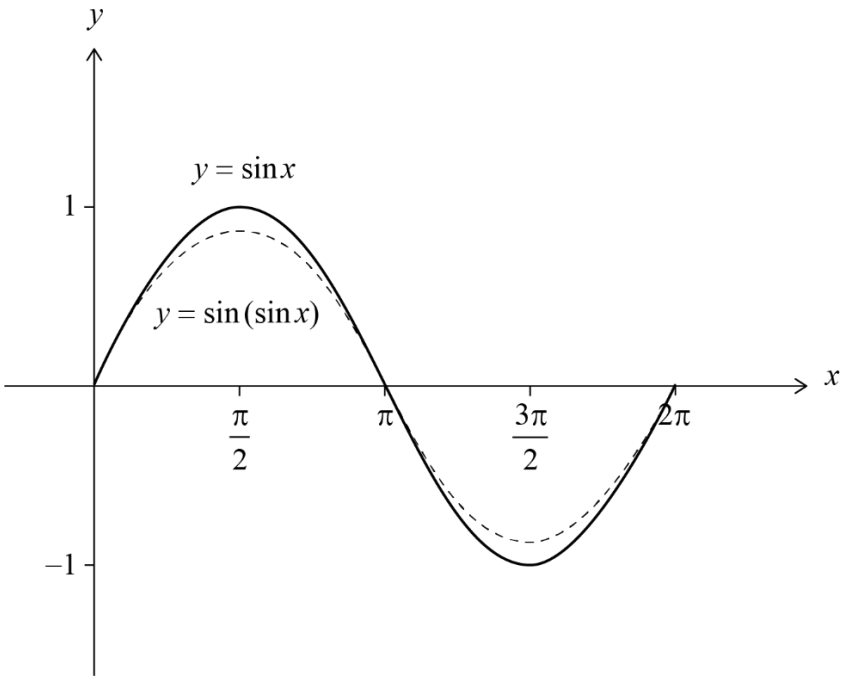
[4 marks]

(ii) correctly compare their result to the actual number of wolves or to the other models or both. Allow even if this is implicit, for example, it is more accurate / realistic than the other models ***A1***

[1 mark]

Total [29 marks]

1. (a)



correct shape $y = \sin x$ ***A1***

correct domain ***A1***

intercepts on x-axis at $0, \pi, 2\pi$ ***A1***

correct shape $y = \sin(\sin x)$ with a smaller maximum value and a larger minimum value than $y = \sin x$ and the same x-intercepts as $y = \sin x$ ***A1***

Note: For candidates who do not label their graphs with $\sin x$ and $\sin(\sin x)$ award at most ***A1A1A1A0.***

For candidates who sketch their graphs on separate axes award at most ***A1A1A1A0.***

These marks are independent of each other.

[4 marks]

continued…

Question 1 continued

(b) (i) 1 ***A1***

[1 mark]

(ii) $0.841(0.841470\ldots)$ ***A1***

[1 mark]

(iii) $0.746(0.745624\ldots)$ ***A1***

[1 mark]

(c) use of GDC eg tables (to vary the value of n) ***(M1)***

$\left(S_4\left(\frac{\pi}{2}\right)=0.678430...,S_5\left(\frac{\pi}{2}\right)=0.627571...,S_6\left(\frac{\pi}{2}\right)=0.587180...\right)$

$n=6$ ***A2***

[3 marks]

continued…

Question 1 continued

(d) attempt to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ using chain rule ***(M1)***

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} =\right) \cos(\sin x)\cos x$ ***A1***

$\cos(\sin x)\cos x = 0$

considers $\cos x = 0$ ***M1***

$x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ ***A1***

considers $\cos(\sin x) = 0$ ***M1***

$\sin x = \frac{\pi}{2} \Rightarrow$ no other solution for x ***R1***

Note: The ***M1R1*** above are independent of the previous ***M1A1***.

hence curve has exactly two points of zero gradient at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ ***AG***

[6 marks]

(e) $S_3'(x) = \cos(S_2(x))\cos(S_1(x))\cos x$ ***A1***

$S_3'(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$ ***AG***

[1 mark]

continued…

Question 1 continued

(f) consider case when $n=1$ $\text{LHS}=\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x=\text{RHS}$ ***A1***

Note: Subsequent marks after this ***A1*** are independent of this mark and can be awarded.

assume true for some k, ***M1***

i.e. $\frac{\mathrm{d}}{\mathrm{d}x}\left(S_k(x)\right)=\cos\left(S_{k-1}(x)\right)\cos\left(S_{k-2}(x)\right)\ldots\cos\left(S_1(x)\right)\cos x$

Note: Award ***M0*** for statements such as "let $n=k$" or "assume that $n=k$ is true". The assumption of truth of the statement must be clear. Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

EITHER

recognition that $S_{k+1}(x)=\sin\left(S_k(x)\right)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(S_{k+1}(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(S_k(x)\right)$

$=\cos\left(S_k(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(S_k(x)\right)$ ***A1***

$=\cos\left(S_k(x)\right)\cos\left(S_{k-1}(x)\right)\cos\left(S_{k-2}(x)\right)\ldots\cos x$ (by assumption) ***A1***

OR

recognition that $S_{k+1}(x)=S_k(\sin x)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(S_{k+1}(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(S_k(\sin x)\right)$

$=S_k'(\sin x)\cos x$ ***A1***

$=\cos\left(S_{k-1}(\sin x)\right)\cos\left(S_{k-2}(\sin x)\right)\ldots\cos x(\sin x)\cos x$ (by assumption)

$=\cos\left(S_k(x)\right)\cos\left(S_{k-1}(x)\right)\cos\left(S_{k-2}(x)\right)\ldots\cos x$ ***A1***

THEN

hence true for case $n=k+1$

true for the case $n=1$, when case $n=k$ assumed true, case $n=k+1$ proven to be true. Hence true for all positive integers n. ***R1***

Note: Only award **R1** if at least 3 of the previous 5 marks have been awarded.

[6 marks]

continued…

Question 1 continued

(g) attempt to differentiate numerator and denominator ***(M1)***

$$\lim_{x\to 0}\frac{S_n(x)}{x}=\lim_{x\to 0}\frac{\cos\left(S_{n-1}(x)\right)\cos\left(S_{n-2}(x)\right)\ldots\cos\left(S_1(x)\right)\cos x}{1}$$ ***A1***

$$=\frac{\cos(0)\cos(0)\ldots\cos(0)\cos(0)}{1}$$ ***A1***

$$=1$$ ***AG***

Note: For candidates who don't include the limit award a maximum of ***(M1)A1A0***

[3 marks]

Total [26 marks]

2. (a) attempt to use $S_\infty = \frac{u_1}{1-r}$ (***M1)***

$= \frac{1}{1+x^2}$ ***A1***

Note: Allow working with $u_1 = 1$ and $r = -x^2$ or $u_1 = 1 - x^2$ and $r = x^4$.

[2 marks]

(b) $\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x\ (+c)$ ***A1***

$\int (1 - x^2 + x^4 - x^6 + \ldots)\mathrm{d}x = (c+)\,x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots (= \arctan x)$ ***A1***

Note: The ***A1*** marks above are independent.

when $x = 0, \arctan 0 = c + 0 \Rightarrow c = 0$ ***R1***

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***AG***

[3 marks]

(c) $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}\left(\approx \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5\right)$ ***A1***

attempt to evaluate $\frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5$ ***M1***

$\pi \approx 3.156$ ***A1***

[3 marks]

continued…

Question 2 continued

(d)

METHOD 1

recognition that in the Maclaurin expansion of $\arctan x$, the error term is $\frac{x^{2m-1}}{2m-1}$ ***(M1)***

attempt to solve $\frac{1}{2m-1}\times\left(\frac{1}{\sqrt{3}}\right)^{2m-1} < 0.0001$ OR $m = 6.60583...(=7)$ ***M1***

6 (non-zero) terms are needed ***A1***

Note: Accept the error term in an absolute value.

Give marks as above for candidates who use $2m+1$ instead of $2m-1$. This gives $m = 5.60583...(=6)$ and leads to needing 6 terms.

Give marks as above for candidates who use n as the exponent and divisor. This gives $n = 12.211675...$ and leads to needing 6 terms.

METHOD 2

recognition that in the Maclaurin expansion of $\arctan x$, the error term is $\frac{x^{2m-1}}{2m-1}$ ***(M1)***

attempt to find the value of the error term for at least two consecutive values of m ***M1***

error terms are

$$\frac{\left(\frac{1}{\sqrt{3}}\right)^7}{7} = 0.003054..., \frac{\left(\frac{1}{\sqrt{3}}\right)^9}{9} = 0.0007919..., \frac{\left(\frac{1}{\sqrt{3}}\right)^{11}}{11} = 0.0002159..., \frac{\left(\frac{1}{\sqrt{3}}\right)^{13}}{13} = 0.00006092...$$

6 (non-zero) terms are needed ***A1***

Note: Accept the error term in an absolute value.

Accept $2m+1$ or n in place of $2m-1$.

continued…

Question 2 continued

METHOD 3

attempt to find the actual error using $\arctan x -$ (first m non-zero terms) ***(M1)***

attempt to find the error for at least two consecutive values of m ***M1***

$(m = 4 \Rightarrow 0.000623, m = 5 \Rightarrow 0.000169, m = 6 \Rightarrow 0.0000473)$

6 (non-zero) terms are needed ***A1***

[3 marks]

(e) $u = \arctan x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$ (or equivalent) (***A1)***

$$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, \mathrm{d}x = \left[x \arctan x\right]_0^{\frac{1}{\sqrt{3}}} - \int_0^{\frac{1}{\sqrt{3}}} \frac{x}{1+x^2} \, \mathrm{d}x \quad \textbf{A1A1}$$

Note: Award ***A1*** for each term.

$$= \left[x \arctan x\right]_0^{\frac{1}{\sqrt{3}}} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^{\frac{1}{\sqrt{3}}} \left(= \frac{1}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}\right) \quad \textbf{A1}$$

$$= \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3} \quad \textbf{AG}$$

[4 marks]

(f) $\left(\int_0^{\frac{1}{\sqrt{3}}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) \mathrm{d}x = \right) 0.158422$ ***A2***

[2 marks]

(g) attempt to solve (for π) $\frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3} \approx 0.158422$ ***M1***

$\pi \approx 3.1412$ ***A1***

[2 marks]

continued…

Question 2 continued

(h) theorem error is given by $\left|\int_0^{\frac{1}{\sqrt{3}}} \frac{x^9}{9}\mathrm{d}x\right| \left(= \left[\frac{x^{10}}{90}\right]_0^{\frac{1}{\sqrt{3}}}\right)$ ***(M1)***

$= 4.57\times10^{-5}$ ***A1***

attempt to evaluate their error ***(M1)***

$\left(\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\frac{4}{3}\right) - 0.158422$ OR $0.158458558\ldots - 0.158422$

$= 3.69\times10^{-5}$ ***A1***

Note: Accept $= 3.73\times10^{-5}$ from using the unrounded answer from part (f).
The (***M1)A1*** above are independent of the previous (***M1)A1***.

$3.69\times10^{-5} < 4.57\times10^{-5}$ ***R1***

hence theorem holds ***AG***

[5 marks]

continued…

Question 2 continued

(i) **METHOD 1**

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^n}{n}\right) \mathrm{d}x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\frac{\left(\frac{1}{\sqrt{3}}\right)^{n+1}}{n(n+1)} < 1\times10^{-6}$ ***A1***

attempt to solve using GDC ***(M1)***

$n = 14.3336\ldots$

$\Rightarrow n = 15$ ***A1***

(hence 8th term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 2

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \frac{x^{2m-1}}{2m-1}dx$ OR $\int_0^{\frac{1}{\sqrt{3}}} \frac{x^{2m+1}}{2m+1}dx$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\frac{\left(\frac{1}{\sqrt{3}}\right)^{2m}}{2m(2m-1)} < 1\times10^{-6}$ OR $\frac{\left(\frac{1}{\sqrt{3}}\right)^{2m+2}}{(2m+1)(2m+2)} < 1\times10^{-6}$ ***A1***

attempt to solve using GDC ***(M1)***

$\Rightarrow m = 7.66681\ldots$ OR $\Rightarrow m = 6.66681\ldots$ ***A1***

(hence 8th term is the error term so) we require 7 (non-zero) terms ***A1***

continued…

Question 2 continued

METHOD 3

recognition error is of the form $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^n}{n}\right) \mathrm{d}x$ ***(M1)***

Note: Condone absence of limits of integration for ***(M1)***

$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{13}}{13}\right) \mathrm{d}x = 2.51234... \times 10^{-6}$ OR $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{15}}{15}\right) \mathrm{d}x = 6.35065... \times 10^{-7}$ ***A1***

attempt to find the value of the integral for at least two consecutive odd values of n, $n > 9$ ***(M1)***

$\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{13}}{13}\right) \mathrm{d}x = 2.51234... \times 10^{-6}$, $\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{x^{15}}{15}\right) \mathrm{d}x = 6.35065... \times 10^{-7}$ ***A1***

Note: Award this ***A1*** for the second correct value.

(hence 8th term is the error term so) we require 7 (non-zero) terms ***A1***

METHOD 4

attempt to find the actual error using $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx -$ (first m non-zero terms) ***(M1)***

error with 6 terms is $2.00830\ldots \times 10^{-6}$ **OR** error with 7 terms is $5.04041\ldots \times 10^{-7}$ ***A1***

attempt to find the error for at least two consecutive values of m ***(M1)***

error with 6 terms is $2.00830\ldots \times 10^{-6}$ and error with 7 terms is $5.04041\ldots \times 10^{-7}$ ***A1***

Note: Award this ***A1*** for the second correct value.

we require 7 (non-zero) terms ***A1***

[5 marks]

Total [29 marks]