

APOLLO

Functions

HL

Mathematics: analysis and approaches

First assessment 2021

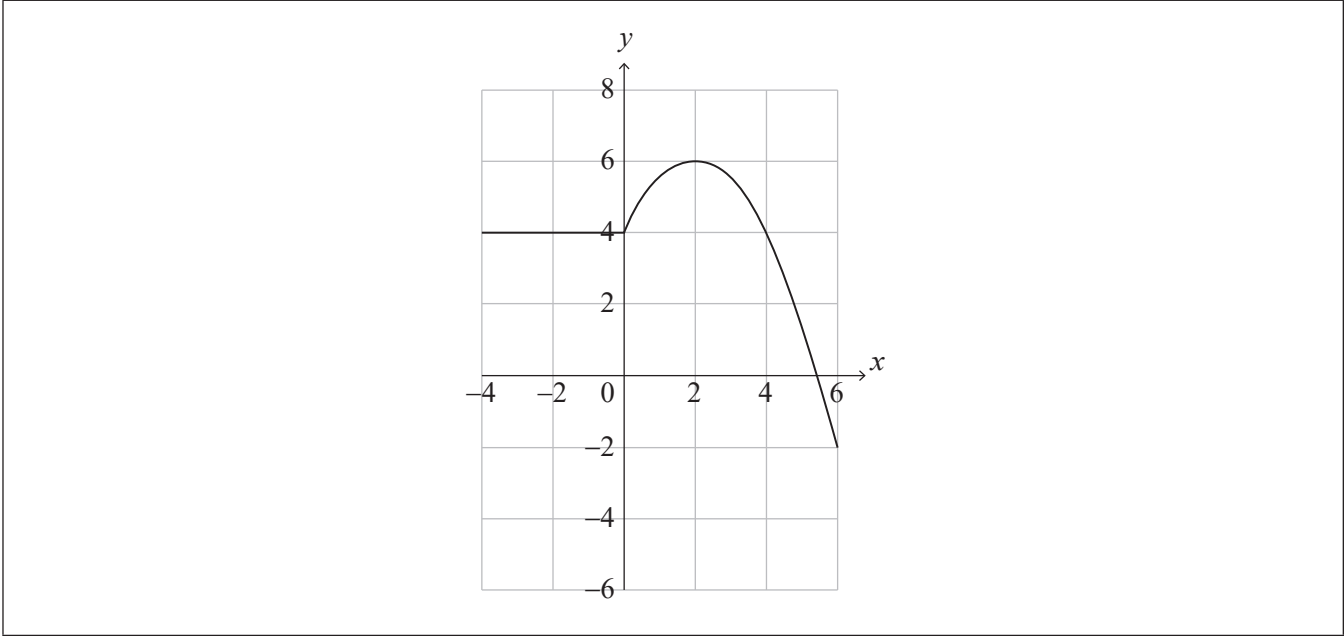
68 questions | questions

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The graph of $y = f(x)$ for $-4 \leq x \leq 6$ is shown in the following diagram.



- (a) Write down the value of
- (i) $f(2)$;

(ii) $(f \circ f)(2)$.
- [2]
- (b) Let $g(x) = \frac{1}{2}f(x) + 1$ for $-4 \leq x \leq 6$. On the axes above, sketch the graph of g .
- [3]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

The function f is defined by $f(x) = \frac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a) Find the value of p and the value of q . [2]

(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c) Find the x -coordinate of the point of inflexion. [2]

(d) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes. [5]

The function g is defined by $g(x) = \frac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\frac{2}{3}$.

(e) Find the equations of all the asymptotes on the graph of $y = g(x)$. [4]

(f) By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$. [4]



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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \frac{2x}{\sqrt{x^2}(x^2+1)}$ for $x \in \mathbb{R}$, $x \neq 0$.

(ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1} . [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]

References:

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2. [Maximum mark: 9]

(a) Write down the equation of

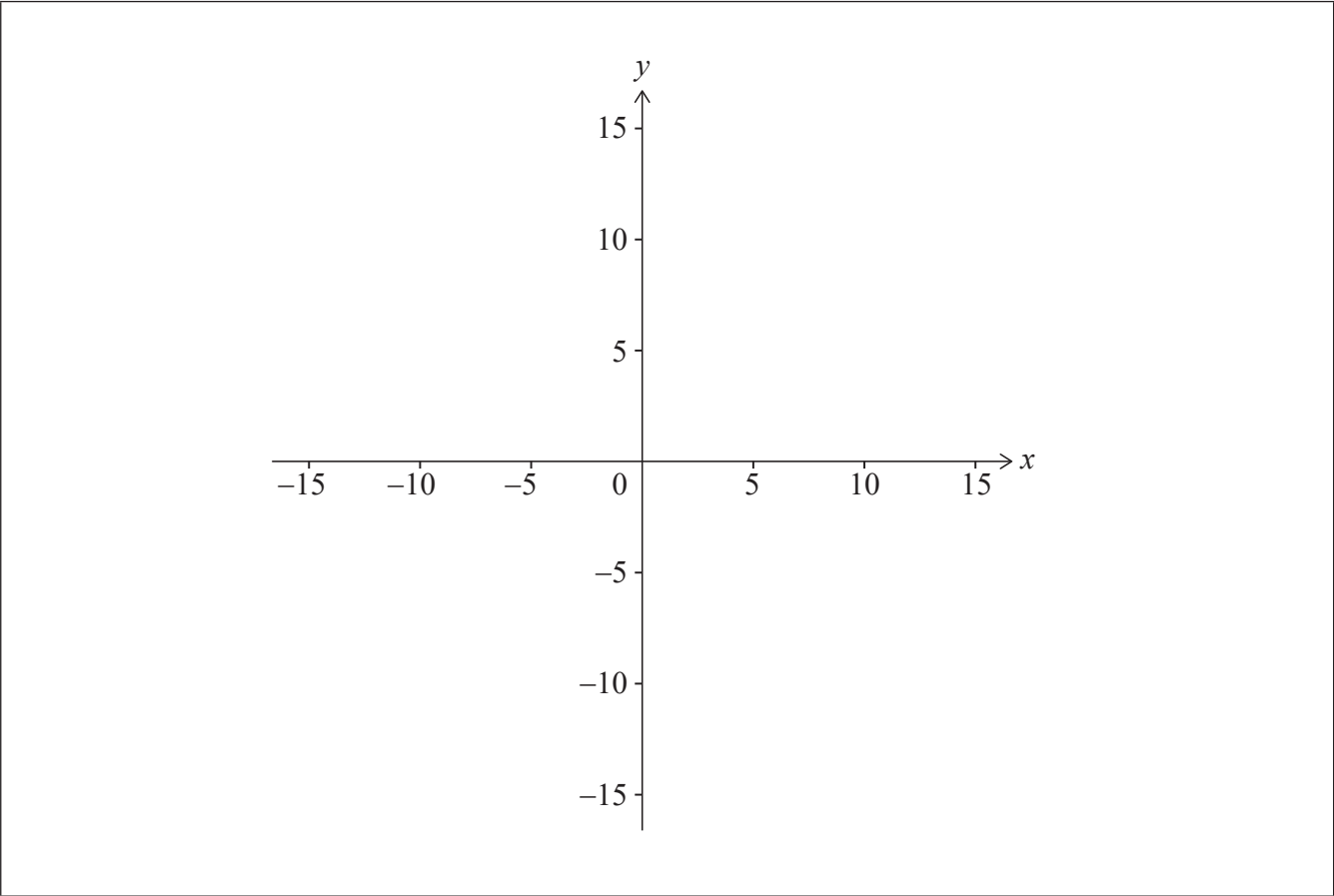
- (b) Find the coordinates where the graph of f crosses

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(Question 2 continued)

(c) Sketch the graph of f on the axes below. [1]



The function g is defined by $g(x) = \frac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of a . [4]

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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

7. [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p . [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a . [2]



Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

Consider the function $f(x) = \frac{x^2 - x - 12}{2x - 15}$, $x \in \mathbb{R}$, $x \neq \frac{15}{2}$.

(a) Find the coordinates where the graph of f crosses the

(i) x -axis;

(ii) y -axis.

[3]

(b) Write down the equation of the vertical asymptote of the graph of f .

[1]

(c) The oblique asymptote of the graph of f can be written as $y = ax + b$ where $a, b \in \mathbb{Q}$.

Find the value of a and the value of b .

[4]

(d) Sketch the graph of f for $-30 \leq x \leq 30$, clearly indicating the points of intersection with each axis and any asymptotes.

[3]

(e) (i) Express $\frac{1}{f(x)}$ in partial fractions.

(ii) Hence find the exact value of $\int_0^3 \frac{1}{f(x)} dx$, expressing your answer as a single logarithm.

[7]



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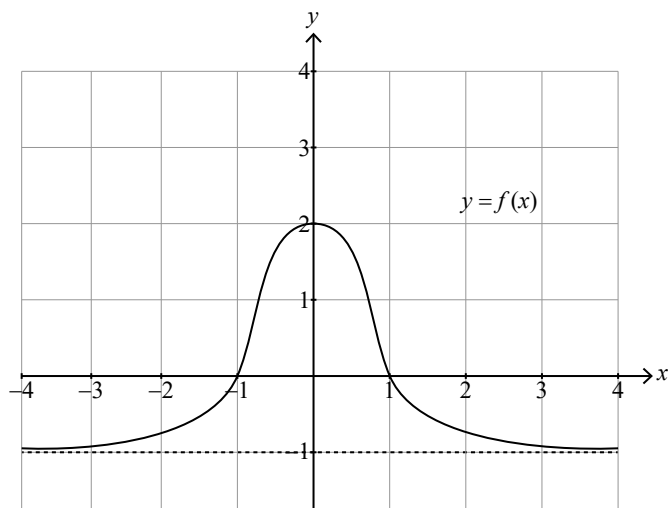
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Section A

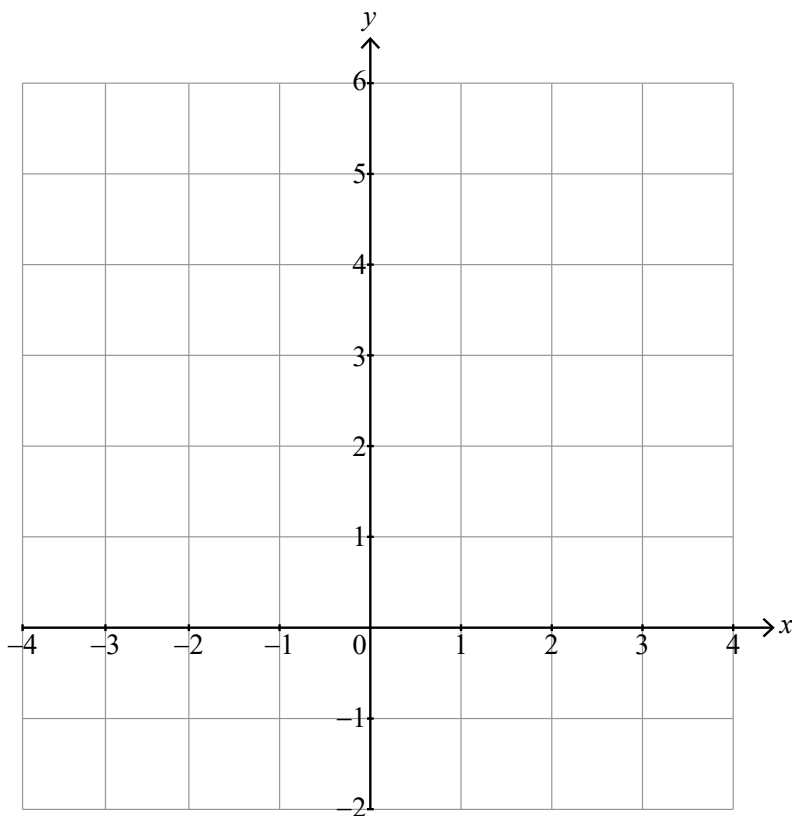
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x -axis at $x = -1$ and $x = 1$, and the y -axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



Turn over

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x+5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a . [3]

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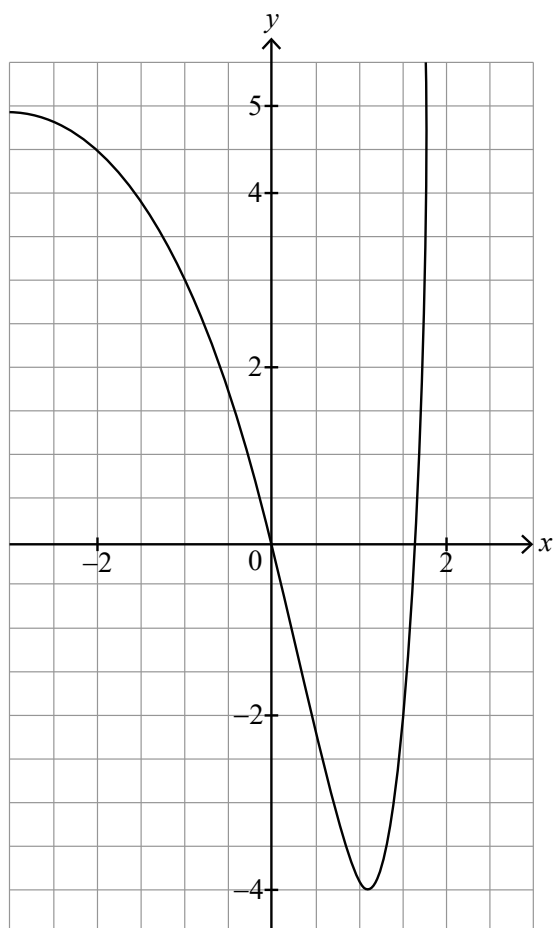


Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 8]

The function f is defined by $f(x) = e^{2x} - 6e^x + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



- (a) Find the largest value of a such that f has an inverse function. [3]
- (b) For this value of a , find an expression for $f^{-1}(x)$, stating its domain. [5]

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(Question 9 continued)

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2. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \leq x \leq 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

- (a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \leq x \leq 1$. [2]
- (b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n . Hence suggest an expression for odd values of n describing, in terms of n , the number of
- local maximum points;
 - local minimum points. [4]
- (c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \leq x \leq 1$. [2]
- (d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n . Hence suggest an expression for even values of n describing, in terms of n , the number of
- local maximum points;
 - local minimum points. [4]
- (e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos \frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n .

- (f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1) \arccos x)$.

- (g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x) \cos(\arccos x) - \sin(n \arccos x) \sin(\arccos x)$. [2]
- (h) Hence
- show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x)$, $n \in \mathbb{Z}^+$;
 - express $f_3(x)$ as a cubic polynomial. [5]

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

3. [Maximum mark: 8]

A function f is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R}$, $x \neq -1$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

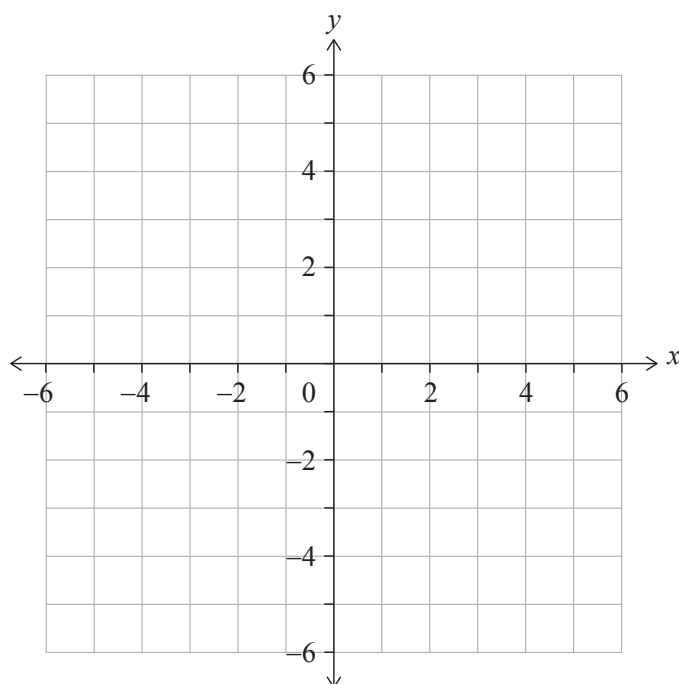
(ii) the horizontal asymptote.

[2]

(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.

[3]



(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$.

[1]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$.

[2]

(This question continues on the following page)





(Question 3 continued)

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 20]

A function f is defined by $f(x) = \frac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$.

- (a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x) = \frac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R}$, $x > 3$.

- (b) The inverse of g is g^{-1} .

(i) Show that $g^{-1}(x) = 1 + \frac{\sqrt{4x^2 + x}}{x}$.

- (ii) State the domain of g^{-1} . [7]

A function h is defined by $h(x) = \arctan \frac{x}{2}$, where $x \in \mathbb{R}$.

- (c) Given that $(h \circ g)(a) = \frac{\pi}{4}$, find the value of a .

Give your answer in the form $p + \frac{q}{2}\sqrt{r}$, where $p, q, r \in \mathbb{Z}^+$. [7]



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2. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where $p, q, r \in \mathbb{R}$, has roots α, β and γ .

- (a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma. \quad [3]$$

- (b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

- (c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

- (d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

- (e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

- (ii) Explain why the equation will have at least one real root for all values of q . [1]

(This question continues on the following page)

(Question 2 continued)

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where $p, q, r, s \in \mathbb{R}$, has roots α, β, γ and δ .

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

- (f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q . [3]
- (ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]
- (g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

- (h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]
- (ii) Write down the integer root of this equation. [1]
- (iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

References:

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8. [Maximum mark: 5]

(a) Consider the case where $k = \frac{\pi}{2}$.

[2]

[1]

[2]

[illegible]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 16]

Consider a three-digit code abc , where each of a , b and c is assigned one of the values 1, 2, 3, 4 or 5.

(a) Find the total number of possible codes

(i) assuming that each value can be repeated (for example, 121 or 444);

(ii) assuming that no value is repeated.

[4]

Let $P(x) = x^3 + ax^2 + bx + c$, where each of a , b and c is assigned one of the values 1, 2, 3, 4 or 5. Assume that no value is repeated.

Consider the case where $P(x)$ has a factor of $(x^2 + 3x + 2)$.

(b) (i) Find an expression for b in terms of a .

(ii) Hence show that the only way to assign the values is $a = 4$, $b = 5$ and $c = 2$.

(iii) Express $P(x)$ as a product of linear factors.

(iv) Hence or otherwise, sketch the graph of $y = P(x)$, clearly showing the coordinates of any intercepts with the axes.

[12]



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The population of a town t years after 1 January 2014 can be modelled by the function

Use this model to estimate the population of this town on 1 January 2041.

[illegible]

1. [Maximum mark: 7]

(a) Find the zero of $f(x)$. [2]

(b) For the graph of $y = f(x)$, write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

[illegible]

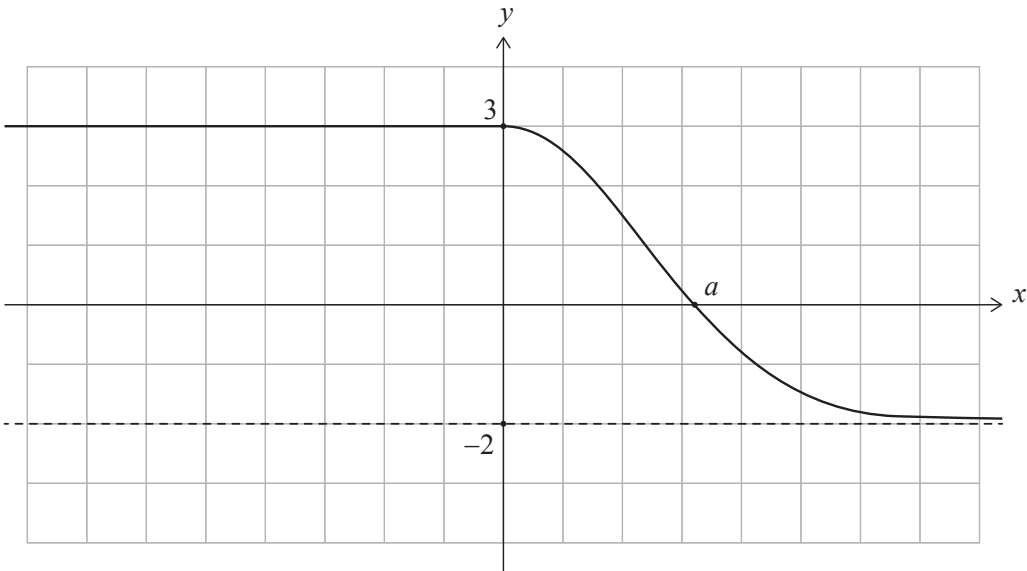
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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

8. [Maximum mark: 7]

Part of the graph of a function, f , is shown in the following diagram. The graph of $y = f(x)$ has a y -intercept at $(0, 3)$, an x -intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.



(This question continues on the following page)

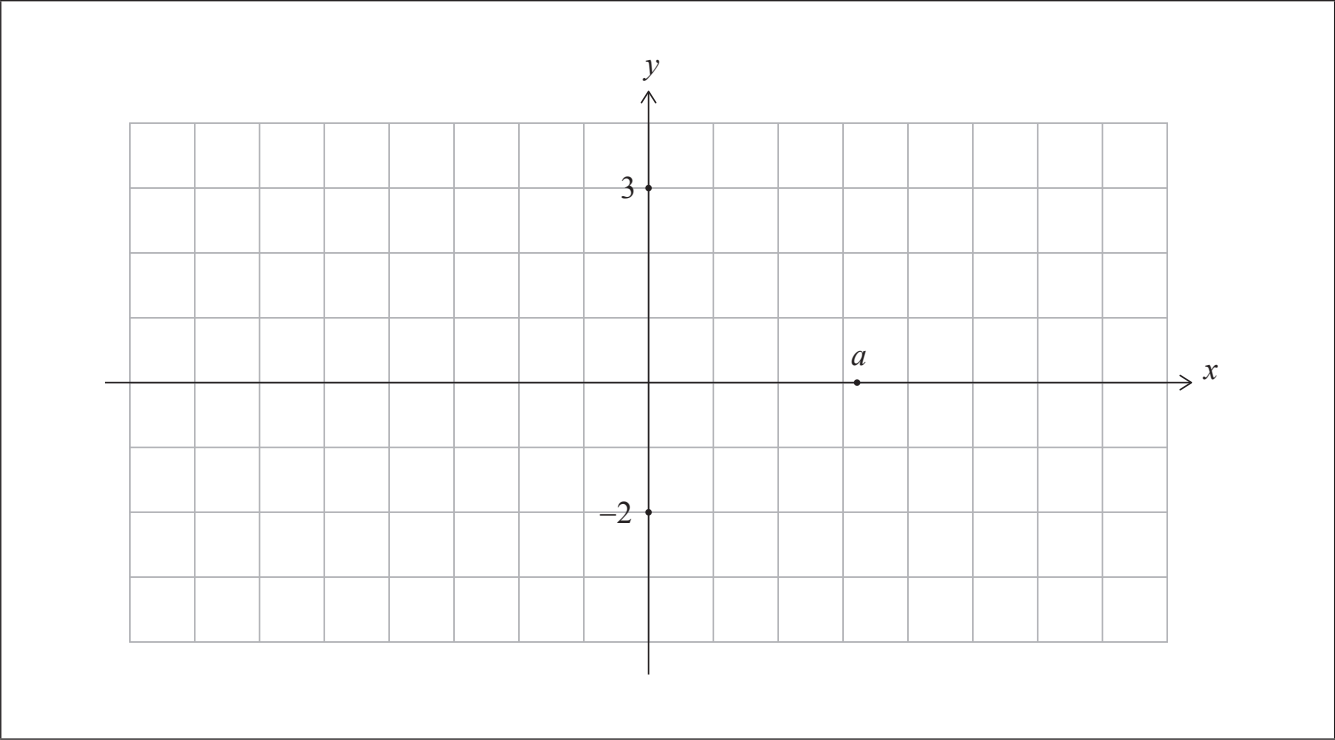


(Question 8 continued)

Consider the function $g(x) = |f(|x|)|$.

- (a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote.

[4]



- (b) Find the possible values of k such that $(g(x))^2 = k$ has exactly two solutions.

[3]

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2. [Maximum mark: 5]

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

(i) the vertical asymptote;

(ii) the horizontal asymptote.

[2]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(i) the y -axis;

(ii) the x -axis.

[2]

[illegible]

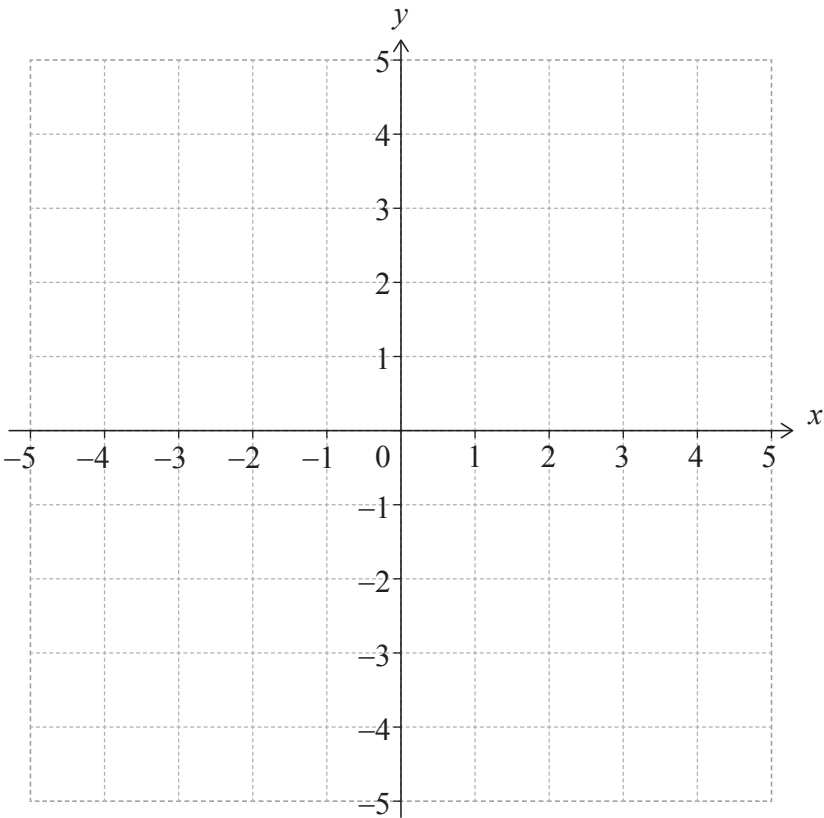
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(Question 2 continued)

- (c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b).

[1]



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8. [Maximum mark: 7]

(a) Determine the range of g . [4]

(b) Find the set of values of x such that $h(x) \leq 0$. [3]

[illegible]

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will not be marked.



1. [Maximum mark: 5]

(a) Write down an expression for $(g \circ f)(x)$. [2]

(b) Given that $(g \circ f)(2) = 10$, find the possible values of k . [3]

[illegible]

Turn over

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The functions f and g are defined by

$$f(x) = \ln(2x - 9), \text{ where } x > \frac{9}{2}$$

$$g(x) = 2 \ln x - \ln d, \text{ where } x > 0, d \in \mathbb{R}^+.$$

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of d . [9]

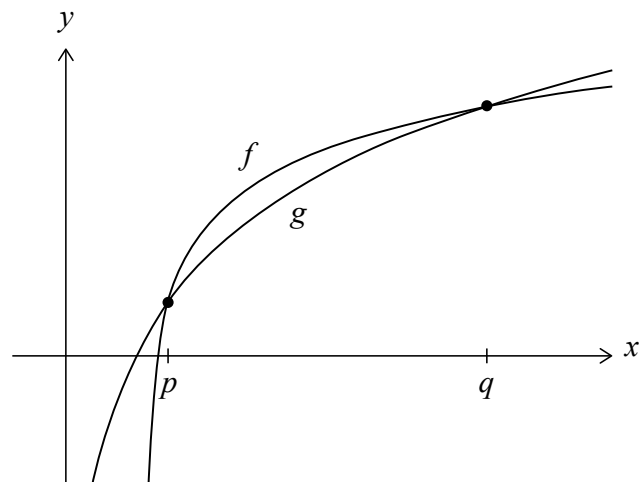
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(Question 10 continued)

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

- (c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{Z}^+$.

[5]



Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The functions f and g are defined by

$$f(x) = \ln(2x - 9), \text{ where } x > \frac{9}{2}$$

$$g(x) = 2 \ln x - \ln d, \text{ where } x > 0, d \in \mathbb{R}^+.$$

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b) (i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$.

(ii) Hence show that $d^2 - 9d > 0$.

(iii) Find the range of possible values of d . [9]

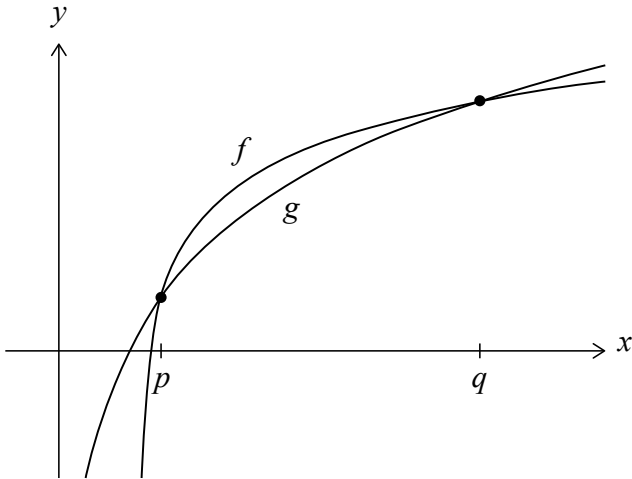
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(Question 10 continued)

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

- (c) In the case where $d = 10$, find the value of $q - p$. Express your answer in the form $a\sqrt{b}$, where $a, b \in \mathbb{Z}^+$.

[5]



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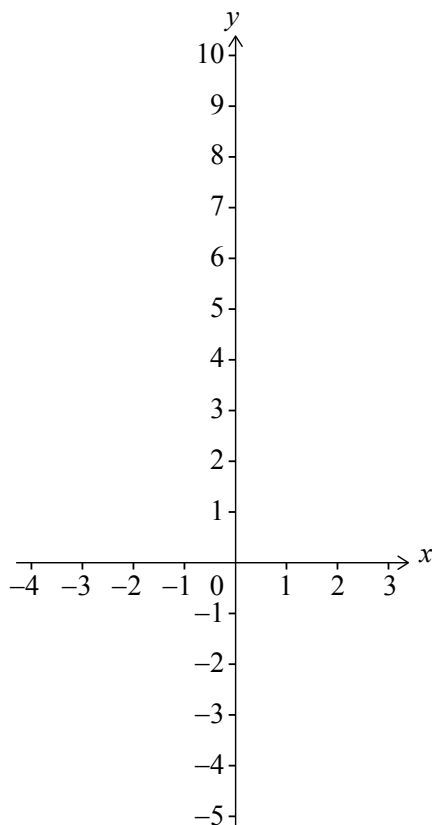
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of f for $-4 \leq x \leq 3$. [3]



The function g is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of g is obtained from the graph of f by a horizontal stretch with scale factor k , followed by a vertical translation of c units.

Find the value of k and the value of c . [2]

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(Question 2 continued)

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the function defined by $f(x) = \frac{x^2 - 14x + 24}{2x + 6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f . [1]

(b) Find the coordinates of the points where the graph of f crosses the x -axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of a and the value of b . [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f . [4]

(f) Solve the inequality $f(x) > x$. [4]



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Section A

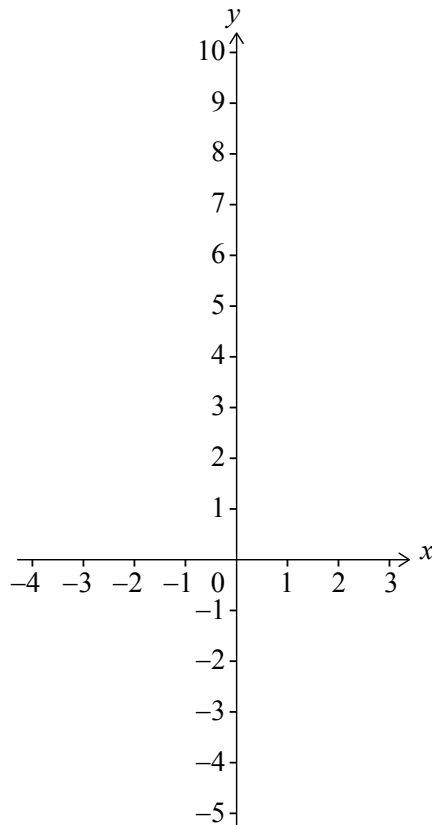
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

Consider the function $f(x) = e^x - 3x - 4$.

(a) On the following axes, sketch the graph of f for $-4 \leq x \leq 3$.

[3]



The function g is defined by $g(x) = e^{2x} - 6x - 7$.

(b) The graph of g is obtained from the graph of f by a horizontal stretch with scale factor k , followed by a vertical translation of c units.

Find the value of k and the value of c .

[2]

(This question continues on the following page)



(Question 2 continued)

[illegible]

Turn over

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 19]

Consider the function defined by $f(x) = \frac{x^2 - 14x + 24}{2x + 6}$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) State the equation of the vertical asymptote on the graph of f . [1]

(b) Find the coordinates of the points where the graph of f crosses the x -axis. [2]

The graph of f also has an oblique asymptote of the form $y = ax + b$, where $a, b \in \mathbb{Q}$.

(c) Find the value of a and the value of b . [4]

(d) Sketch the graph of f for $-50 \leq x \leq 50$, showing clearly the asymptotes and any intersections with the axes. [4]

(e) Find the range of f . [4]

(f) Solve the inequality $f(x) > x$. [4]



Turn over

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1. [Maximum mark: 24]

This question asks you to explore some properties of the family of curves

$y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

- (a) By systematically varying the value of b , or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x -axis intercepts. [2]
- (b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly
- (i) one x -axis intercept; [1]
 - (ii) three x -axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

- (c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly
- (i) two x -axis intercepts; [1]
 - (ii) one x -axis intercept; [1]
 - (iii) three x -axis intercepts. [1]

(This question continues on the following page)

(Question 1 continued)

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

- (d) Consider the case where the curve has exactly three x -axis intercepts. State whether each point of zero gradient is located above or below the x -axis. [1]
- (e) Show that the curve has a point of zero gradient at $P(0, b)$ and a point of zero gradient at $Q\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$. [5]
- (f) Consider the points P and Q for $a > 0$ and $b > 0$.
- (i) Find an expression for $\frac{d^2y}{dx^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]
- (ii) Determine whether each point is located above or below the x -axis. [1]
- (g) Consider the points P and Q for $a < 0$ and $b > 0$.
- (i) State whether P is a local maximum or a local minimum and whether it is above or below the x -axis. [1]
- (ii) State the conditions on a and b that determine when Q is below the x -axis. [1]
- (h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x -axis intercepts. [5]

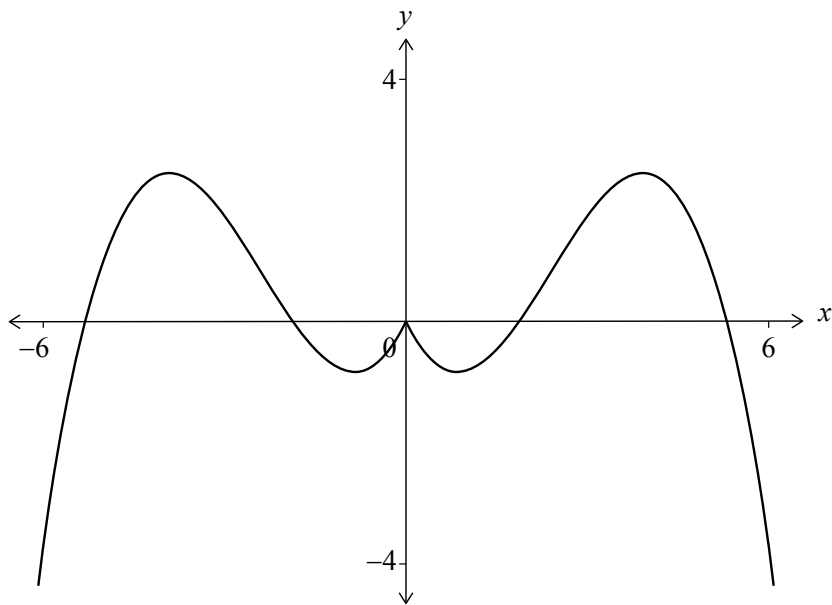
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Section A

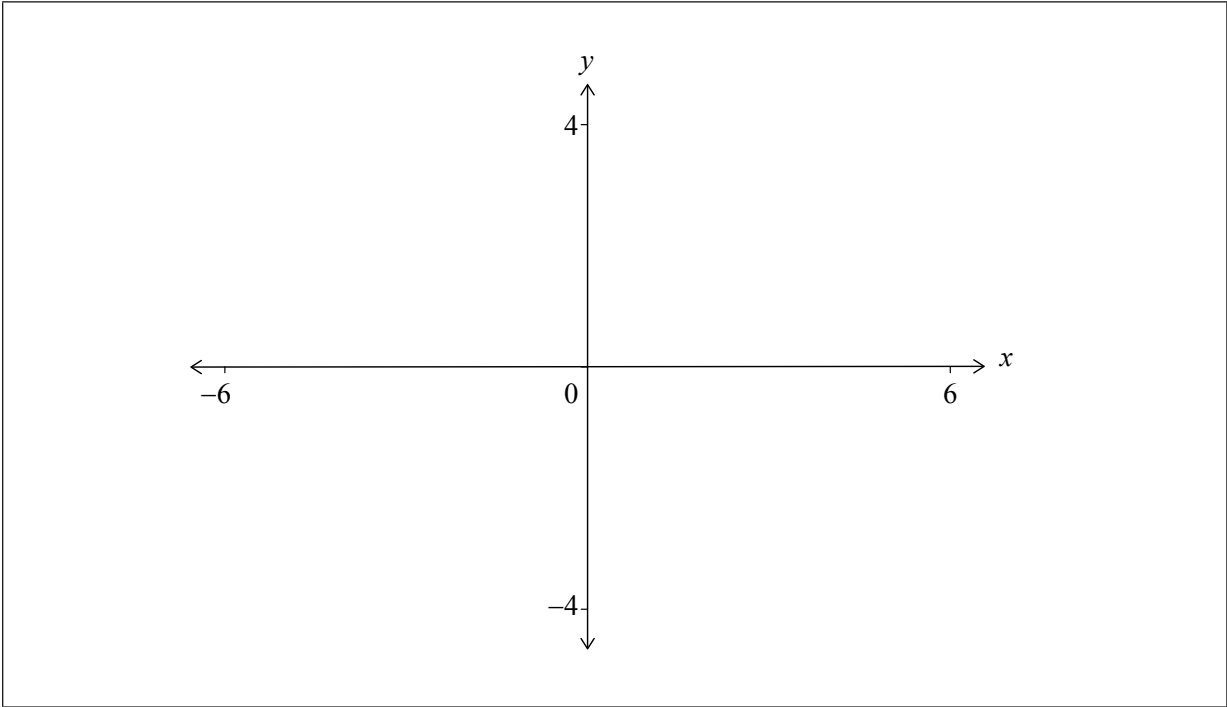
Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

9. [Maximum mark: 6]

The graph of $y = f(|x|)$ for $-6 \leq x \leq 6$ is shown in the following diagram.



(a) On the following axes, sketch the graph of $y = |f(|x|)|$ for $-6 \leq x \leq 6$. [2]



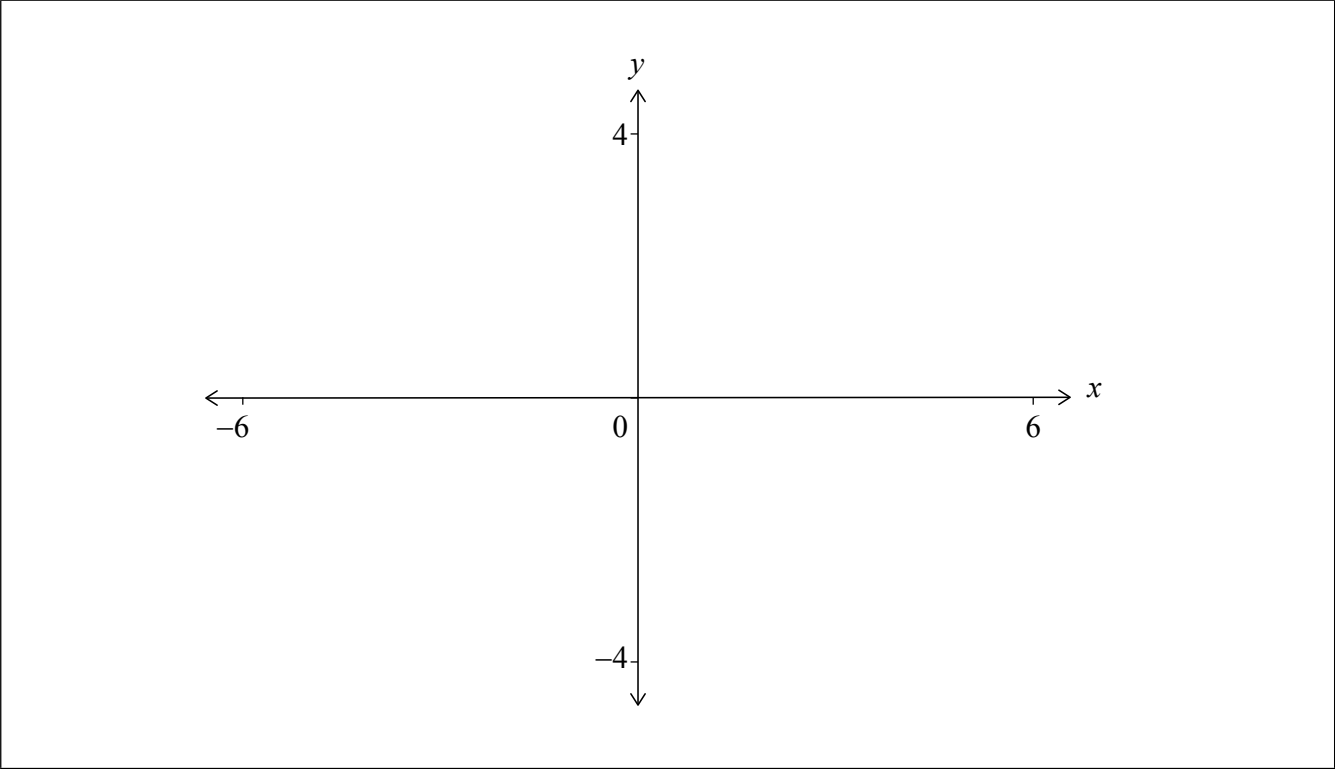
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(Question 9 continued)

It is given that f is an odd function.

- (b) On the following axes, sketch the graph of $y = f(x)$ for $-6 \leq x \leq 6$. [2]



It is also given that $\int_0^4 f(|x|) \, dx = 1.6$.

- (c) Write down the value of

(i) $\int_{-4}^0 f(x) \, dx$;

(ii) $\int_{-4}^4 (f(|x|) + f(x)) \, dx$. [2]

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Turn over

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Consider the function $f(x) = \frac{4x+2}{x-2}$, $x \neq 2$.

(a) Sketch the graph of $y = f(x)$. On your sketch, indicate the values of any axis intercepts and label any asymptotes with their equations. [5]

(b) Write down the range of f . [1]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p , where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$. [1]

(d) Find the value of b and the value of c . [3]

(e) Find the y -coordinate of the vertex of the graph of $y = g(x)$. [2]

(f) Find the product of the solutions of the equation $f(x) = g(x)$. [4]



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3. [Maximum mark: 7]

A has coordinates $(6, 0)$ and C has coordinates $(3, 3\sqrt{3})$.

- (a) (i) Write down the coordinates of the midpoint of $[AC]$.
- (ii) Hence or otherwise, find the equation of the line passing through the points O and B .
- (b) Given that $[OA]$ is perpendicular to $[AB]$, find the area of the quadrilateral OABC .

[illegible]

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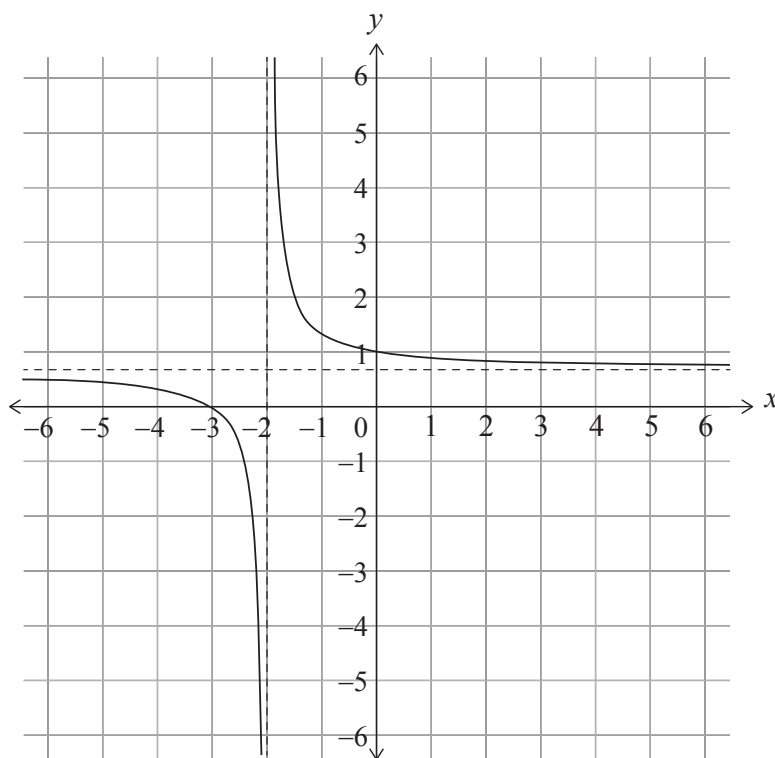
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

5. [Maximum mark: 8]

A function f is defined by $f(x) = \frac{2(x+3)}{3(x+2)}$, where $x \in \mathbb{R}$, $x \neq -2$.

The graph $y = f(x)$ is shown below.



(a) Write down the equation of the horizontal asymptote.

[1]

Consider $g(x) = mx + 1$, where $m \in \mathbb{R}$, $m \neq 0$.

(b) (i) Write down the number of solutions to $f(x) = g(x)$ for $m > 0$.

(ii) Determine the value of m such that $f(x) = g(x)$ has only one solution for x .

(iii) Determine the range of values for m , where $f(x) = g(x)$ has two solutions for $x \geq 0$.

[7]

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(Question 5 continued)

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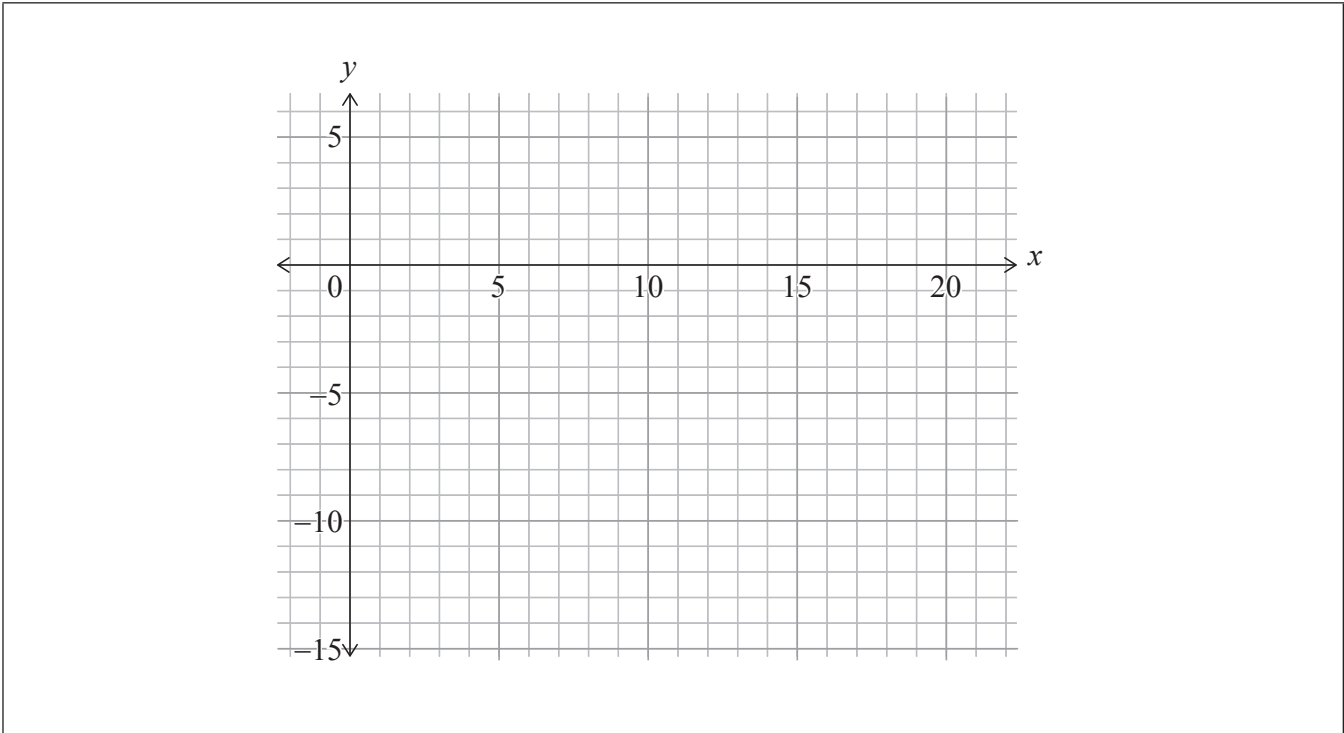
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 7]

Consider the function $f(x) = 11\sqrt{x} - 2x - 11$, where $0 \leq x \leq 20$.

- (a) Find the value of
 - (i) $f(0)$;
 - (ii) $f(20)$. [2]
- (b) Find the two roots of $f(x) = 0$. [2]
- (c) Sketch the graph of $y = f(x)$ on the following grid. [3]



(This question continues on the following page)



(Question 1 continued)

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Turn over

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

12. [Maximum mark: 21]

A curve C has equation $y = \frac{2x^2 + 6x - 3}{x + k}$, $x \in \mathbb{R}$, $x \neq -k$, where k is a real positive constant.

(a) Show that $\frac{dy}{dx} = \frac{2x^2 + 4kx + 6k + 3}{(x + k)^2}$. [4]

(b) Find the range of values of k for which a local minimum or maximum point exists. [4]

Consider the curve C , when $k = 2$.

(c) Write down the equation of the vertical asymptote. [1]

(d) Find the equation of the oblique asymptote. [4]

(e) Show that $\frac{dy}{dx} > 2$, for $x \in \mathbb{R}$, $x \neq -2$. [4]

(f) Sketch the curve C , showing clearly both asymptotes and the general behaviour of C as it approaches each asymptote. [You are not required to find any axes intercepts.] [4]



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2. [Maximum mark: 28]

A polynomial is said to be palindromic if the sequence of its coefficients remains the same in reverse. This question asks you to investigate some properties and solutions of palindromic polynomial equations.

In parts (a) and (b), consider quadratic equations of the form $ax^2 + bx + a = 0$, where $a \neq 0$.

The sequence of coefficients, $\{a, b, a\}$, remains the same in reverse.

The following table shows three palindromic quadratic equations and their sequence of coefficients.

Palindromic quadratic equation	Sequence of coefficients
$2x^2 - 5x + 2 = 0$	$\{2, -5, 2\}$
$x^2 + 4x + 1 = 0$	$\{1, 4, 1\}$
$x^2 + 1 = 0$	$\{1, 0, 1\}$

The quadratic equation $2x^2 - 5x + 2 = 0$ has roots 2 and $\frac{1}{2}$.

These roots form a “reciprocal pair”, since one root is the reciprocal of the other.

(a) (i) Determine the roots of $x^2 + 4x + 1 = 0$.

Give these roots in the form $s \pm \sqrt{t}$, where $s \in \mathbb{Z}$ and $t \in \mathbb{Z}^+$. [3]

(ii) Hence, or otherwise, show that these roots form a reciprocal pair. [2]

(b) Show that the complex roots of $x^2 + 1 = 0$ form a reciprocal pair. [2]

Let $p(x) = ax^2 + bx + a$, where $a \neq 0$.

(c) Verify that $p(x) = x^2 p\left(\frac{1}{x}\right)$ where $x \neq 0$. [2]

(This question continues on the following page)

(Question 2 continued)

In parts (d) and (e), you may assume the result that a polynomial, $p(x)$, of degree n is palindromic if and only if $p(x) = x^n p\left(\frac{1}{x}\right)$.

- (d) Use $p(x) = x^n p\left(\frac{1}{x}\right)$ to show that if $\alpha \neq 0$ is a root of $p(x) = 0$, then $\frac{1}{\alpha}$ is also a root. [2]

Let $f(x) = p(x)q(x)$, where p and q are palindromic polynomials of degree n and m respectively.

- (e) Show that f is a palindromic polynomial. [4]

Consider the palindromic polynomial $f(x) = x^4 + 2x^3 - x^2 + 2x + 1$.

This polynomial can be expressed in the form

$$f(x) = (x^2 + ux + 1)(x^2 + vx + 1), \text{ where } u, v \in \mathbb{Z} \text{ and } u < v.$$

- (f) By forming and solving an appropriate system of equations in u and v , determine the value of u and the value of v . [6]
- (g) Hence, find all the exact complex and purely real roots of $x^4 + 2x^3 - x^2 + 2x + 1 = 0$. [3]

Consider the palindromic polynomial equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_2 x^2 + a_1 x + 1 = 0, \text{ where } n \text{ is odd.}$$

- (h) Show that -1 is always a root of this equation. [4]

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 18]

The function f is defined by $f(x) = 5(x + 1)(x + 3)$, where $x \in \mathbb{R}$.

- (a) Write $f(x)$ in the form $a(x - h)^2 + k$, where $a, h, k \in \mathbb{Z}$. [4]
- (b) Sketch the graph of $y = f(x)$, showing the values of any intercepts with the axes and the coordinates of the vertex. [4]
- (c) Solve the inequality $f(x) \leq 40$. [4]

The function g is defined by $g(x) = \ln x$, where $x \in \mathbb{R}$, $x > 0$.

- (d) (i) Write down an expression for $(f \circ g)(x)$. [3]
- (ii) Solve the inequality $(f \circ g)(x) \leq 40$. [3]
- (e) Find the domain of $g \circ f$. [3]



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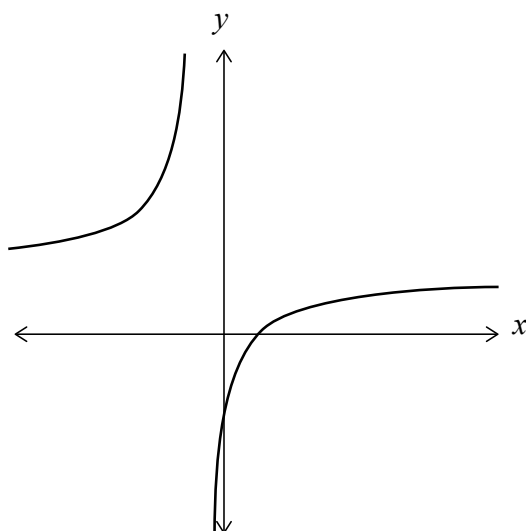
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The function f is defined by $f(x) = \frac{3x-2}{2x+1}$ for $x \in \mathbb{R}$, $x \neq -\frac{1}{2}$.

The following diagram shows part of the graph of $y = f(x)$.



(a) Write down the value of $f(0)$.

[1]

(b) Write down the equation of the horizontal asymptote.

[1]

The function g is defined by $g(x) = -f(x)$ for $x \geq 0$.

(c) Find the range of g .

[3]

(This question continues on the following page)



[illegible]

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Find the value of p and the value of q , where $p, q \in \mathbb{R}$.

[illegible]

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 17]

The function f is defined by $f(x) = 4^x$, where $x \in \mathbb{R}$.

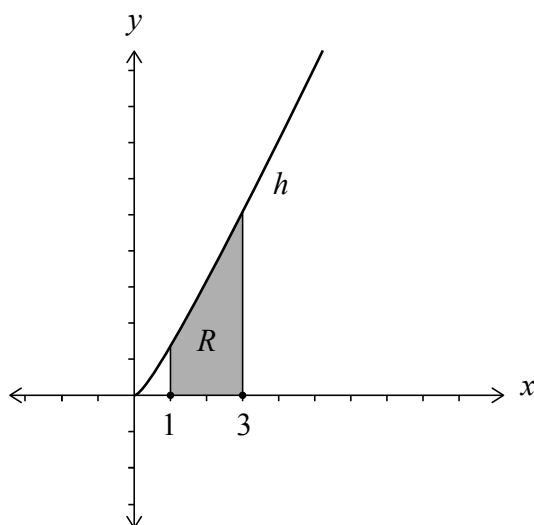
- (a) Find $f^{-1}(8)$. Express your answer in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. [3]

The function g is defined by $g(x) = 1 + \log_2 x$, where $x \in \mathbb{R}^+$.

- (b) (i) Find an expression for $g^{-1}(x)$.
 (ii) Describe a sequence of transformations that transforms the graph of $y = g^{-1}(x)$ to the graph of $y = f(x)$. [4]
 (c) Show that $(f \circ g)(x) = 4x^2$. [3]

The function h is defined by $h(x) = \frac{4x^2}{2x+1}$, $x \neq -\frac{1}{2}$.

The following diagram shows part of the graph of h . Let R be the region enclosed by the graph of h and the x -axis, between the lines $x = 1$ and $x = 3$.



- (d) (i) Show that $2x - 1 + \frac{1}{2x+1} = \frac{4x^2}{2x+1}$.
 (ii) Hence or otherwise, find the area of R , giving your answer in the form $p + q \ln r$, where $p, q, r \in \mathbb{Q}^+$. [7]



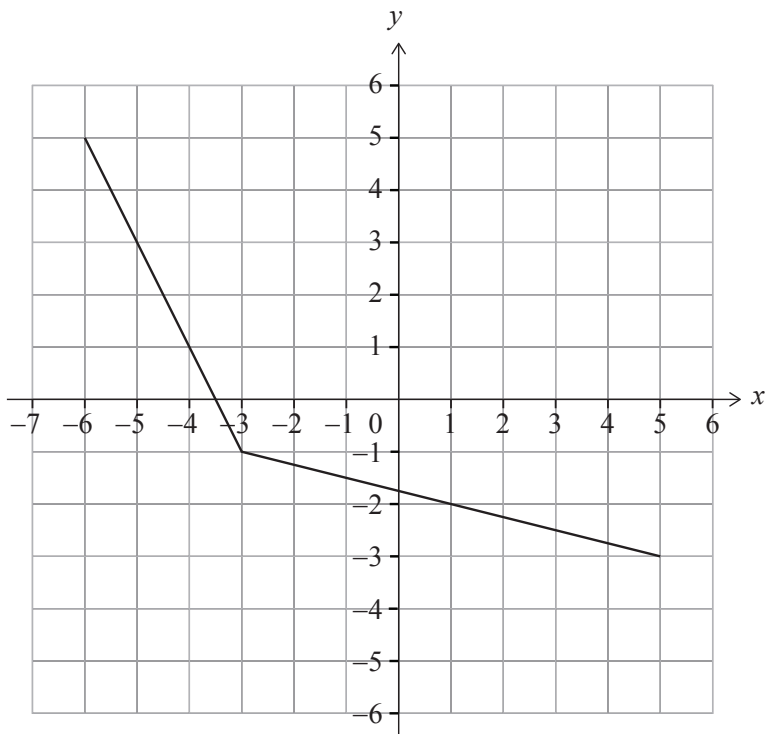
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Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

2. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$, for $-6 \leq x \leq 5$.



- (a) Write down the value of $f(-3)$.

[1]
- (b) State the domain of f^{-1} , the inverse function of f .

[1]
- (c) Find the value of x that satisfies $f^{-1}(2x - 7) = -3$.

[3]

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(b) Hence, find the values of m when the line is tangent to the curve. [5]

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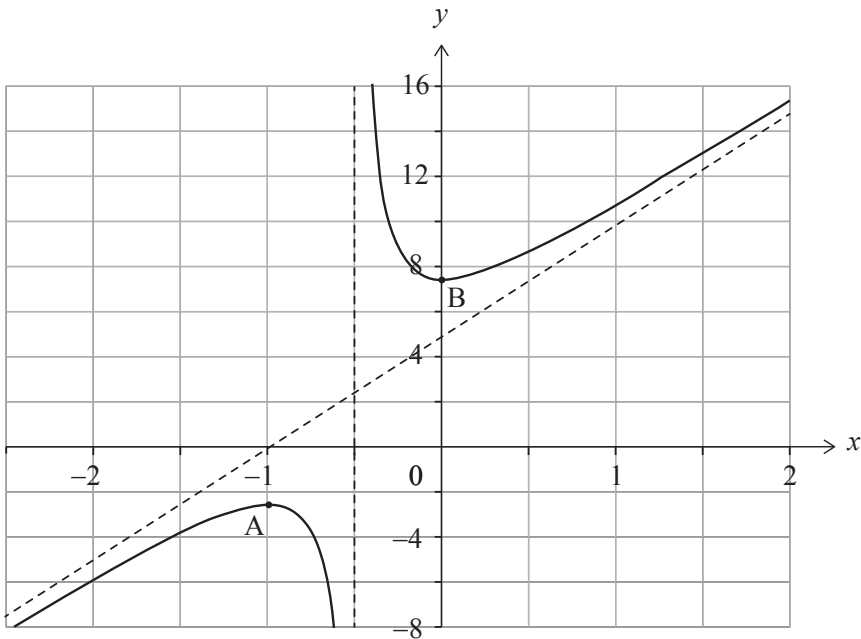
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

6. [Maximum mark: 7]

Consider the function f . The graph of f has a local maximum at $A\left(-1, -\frac{5}{2}\right)$, a local minimum at $B\left(0, \frac{15}{2}\right)$, a vertical asymptote at $x = -\frac{1}{2}$ and an oblique asymptote $y = 5x + 5$.

This information and part of the graph of f is shown in the following diagram.

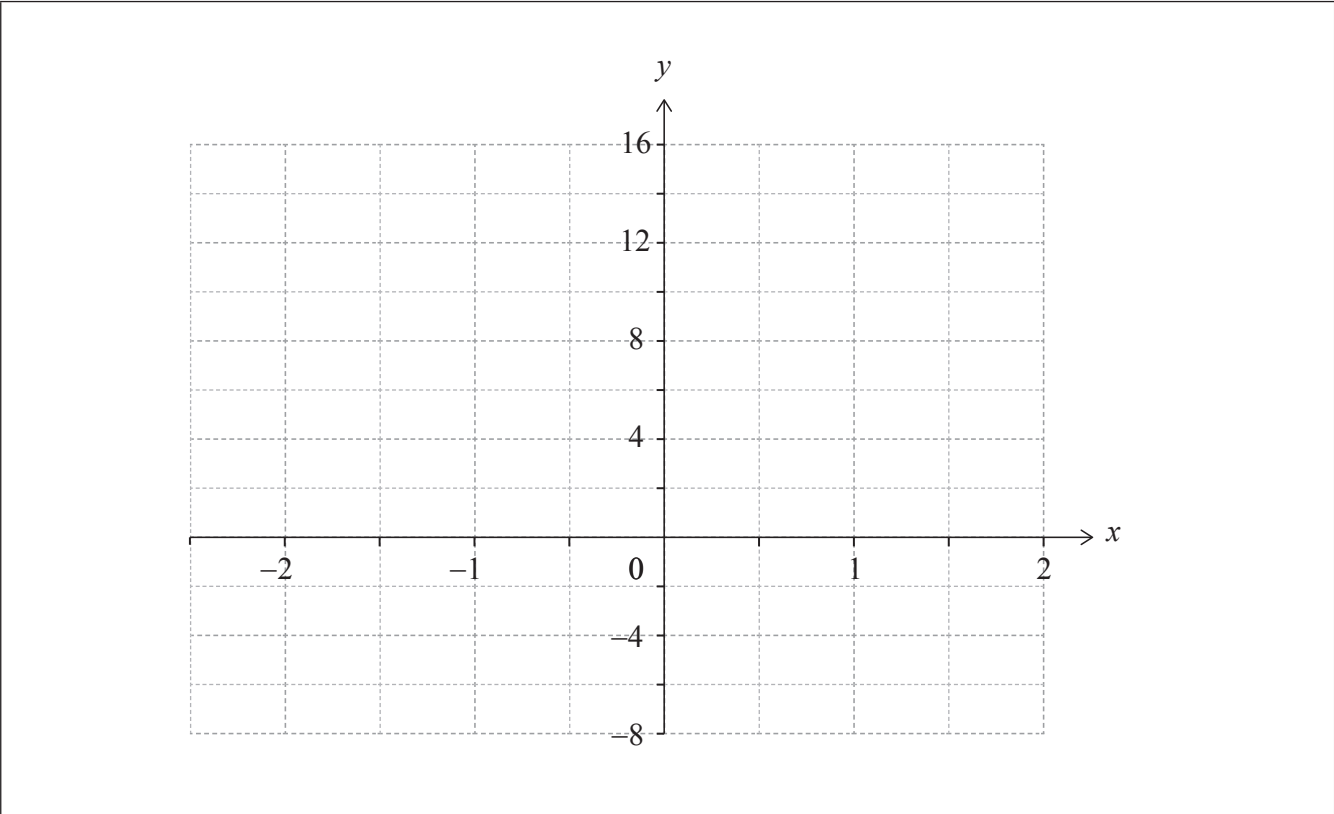


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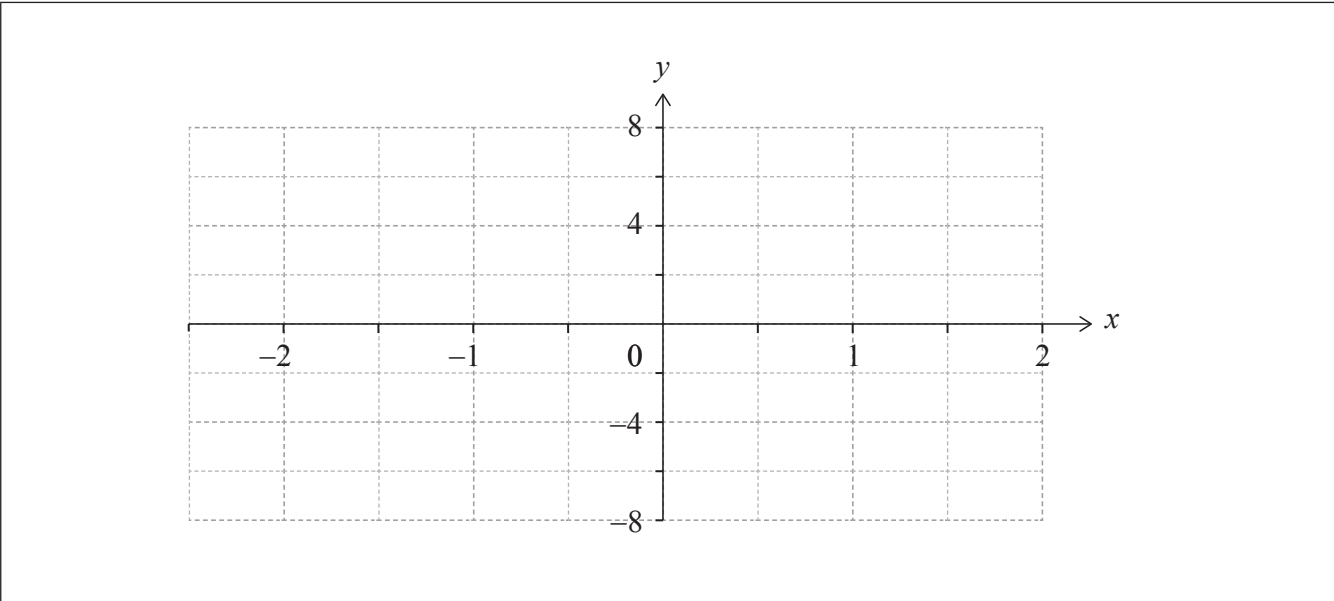


(Question 6 continued)

- (a) On the following grid, sketch the graph of $y = |f(x)|$, clearly indicating any asymptotes. [4]



- (b) On the following grid, sketch the graph of $y = \frac{15}{f(x)}$, clearly indicating any asymptotes and intercepts with the axes. [3]



1. [Maximum mark: 27]

This question asks you to explore self-composite linear functions, of the form $f(x) = mx + c$, for varying values of m .

A function composed with itself is called a self-composite function.

For a function f , the function composition with itself is given by $(f \circ f)(x) = f(f(x))$.

Let f^n denote the n th composition of f with itself where $f^n(x) = \underbrace{(f \circ f \circ \dots \circ f)}_{n \text{ times}}(x)$.

Hence, for example, $f^2(x) = (f \circ f)(x)$ and $f^3(x) = (f \circ f \circ f)(x) = f(f(f(x)))$.

Consider the linear function $f(x) = mx + c$, where $x \in \mathbb{R}$ and $m, c \in \mathbb{R}$.

(a) Show that

$$(i) \quad f^2(x) = m^2x + c(1 + m); \quad [3]$$

$$(ii) \quad f^3(x) = m^3x + c(1 + m + m^2). \quad [2]$$

(b) (i) Write down an expression for $f^4(x)$. [1]

(ii) Suggest a similar expression for $f^n(x)$, $n \in \mathbb{Z}^+$. [2]

(iii) By using your expression from part (b)(ii), or otherwise, find an expression in terms of n for $f^n(x)$ when $m = 1$. [3]

(c) For $m \neq 1$, use mathematical induction to prove that

$$f^n(x) = m^n x + c \left(\frac{1 - m^n}{1 - m} \right) \text{ for } n \in \mathbb{Z}^+. \quad [8]$$

(This question continues on the following page)

(Question 1 continued)

Consider $f^n(x) = m^n x + c \left(\frac{1-m^n}{1-m} \right)$ where $-1 < m < 1$.

- (d) As $n \rightarrow \infty$, the family of graphs $y = f^n(x)$ approaches the graph of a straight line, L . Determine the equation of L , giving your answer in terms of c and m . [4]

Consider $f^n(x) = m^n x + c \left(\frac{1-m^n}{1-m} \right)$ where $m = -1$.

- (e) (i) Show that $f^n(x) = -x + c$ when n is odd. [2]
- (ii) Find an expression for $f^n(x)$ when n is even. [2]

Turn over

4. [Maximum mark: 8]

$$f(x) = -x^2 + 4x + p \text{ and}$$

$$g(x) = x^2 + qx - 1,$$

The graph of $y = f(x)$ has a vertex at A and the graph of $y = g(x)$ has a vertex at B.

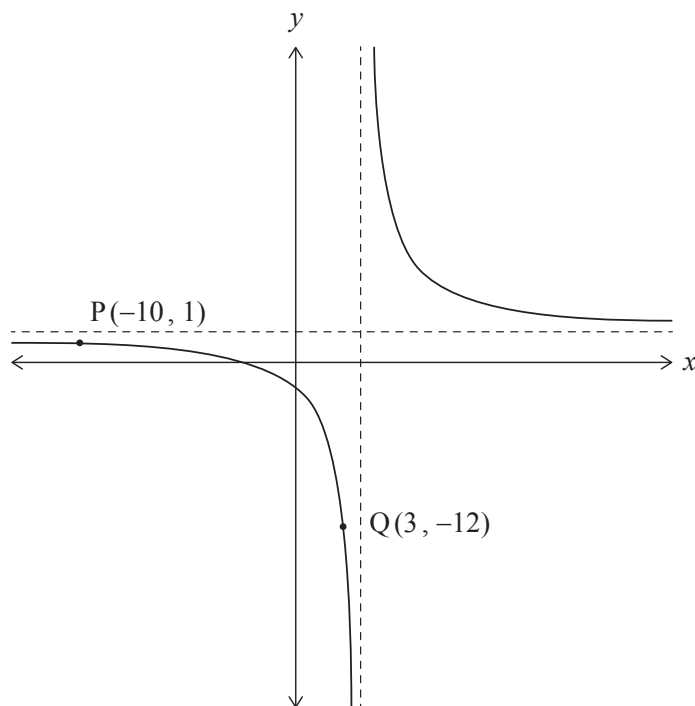
Points A and B both lie on the line $y = 2x - 1$.

- (a) Show that $p = -1$. [3]
- (b) Find the value of q . [5]

[illegible]

3. [Maximum mark: 8]

The graph passes through the points P(-10, 1) and Q(3, -12).



- (a) Determine the value of A and the value of B . [3]
- (b) Describe a sequence of transformations that would map the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{Ax + B}{x - 4}$. [5]

[illegible]

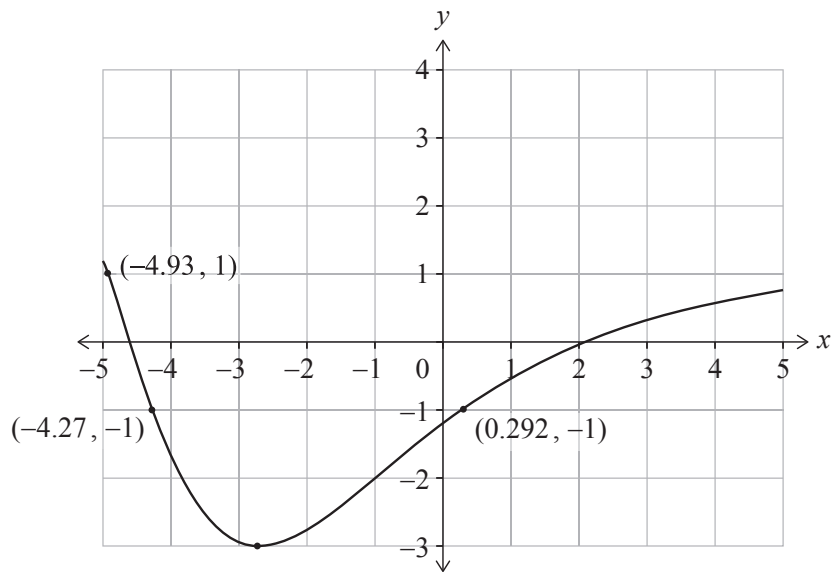
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

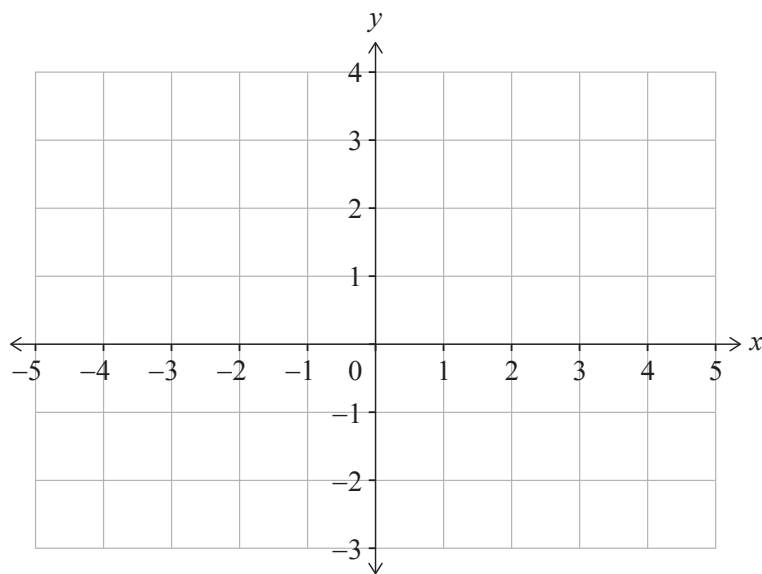
5. [Maximum mark: 8]

The graph of $y = f(x)$ for $-5 \leq x \leq 5$ is shown in the following diagram.

The curve passes through the points $(-4.93, 1)$, $(-4.27, -1)$ and $(0.292, -1)$.



- (a) On the following set of axes,
- (i) sketch the graph $y = f(|x|)$ showing the approximate intercepts with the axes;

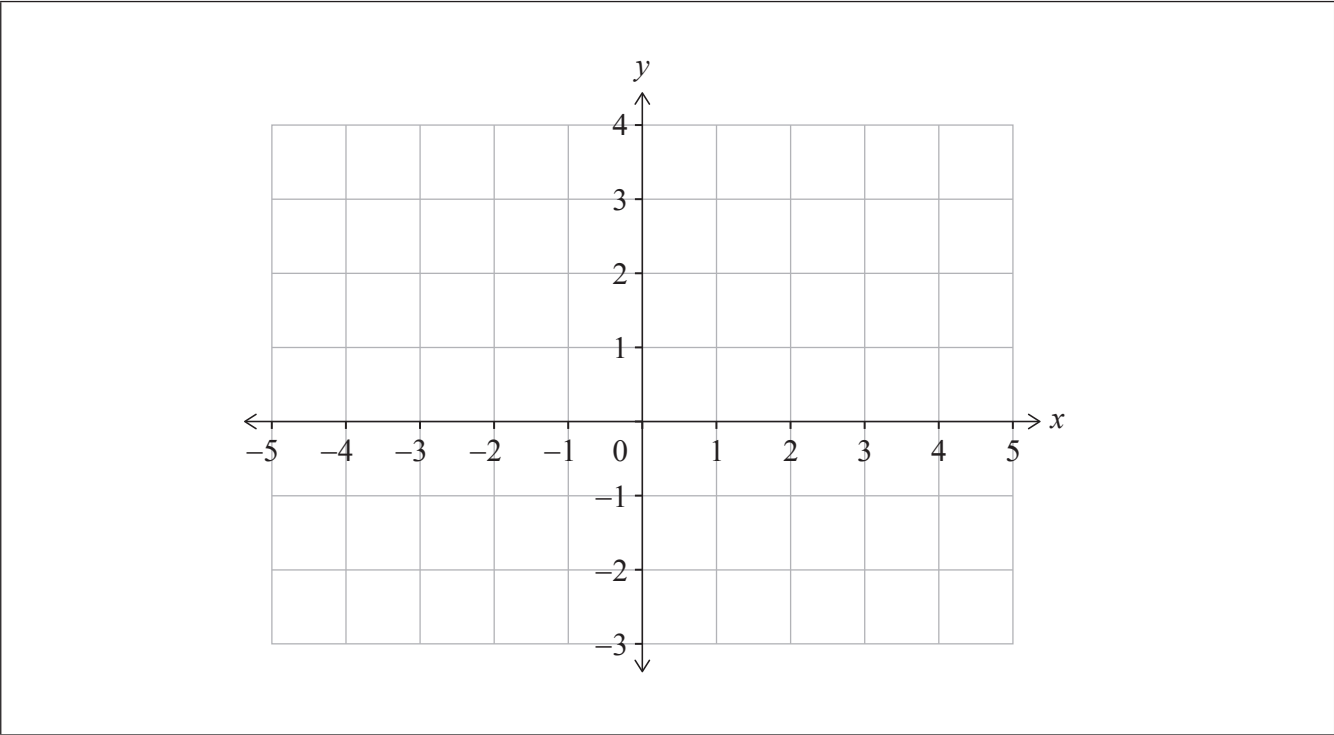


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(Question 5 continued)

(ii) sketch the graph $y = |f(x)|$ showing the approximate intercepts with the axes. [4]



(b) Solve the inequality $|f(x)| \geq (f(x))^2$. [4]

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Turn over

