APOLLO

Mathematics: applications and interpretation
HL Workbook

First assessment 2021
Whole questions arranged by syllabus topic.
Original question and scheme crops preserve source diagrams.

APOLLO

Number and algebra

54 questions

3. [Maximum mark: 8]

Charlie and Daniella each began a fitness programme. On day one, they both ran $500\,\text{m}$. On each subsequent day, Charlie ran $100\,\text{m}$ more than the previous day whereas Daniella increased her distance by 2% of the distance ran on the previous day.

(a) Calculate how far

(i) Charlie ran on day 20 of his fitness programme.

(ii) Daniella ran on day 20 of her fitness programme. [5]

On day n of the fitness programmes Daniella runs more than Charlie for the first time.

(b) Find the value of n. [3]

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Question 3 - working space

9. [Maximum mark: 8]

Consider $w = \mathrm{i}z + 1$, where $w, z \in \mathbb{C}$.

(a) Find w when

(i) $z = 2\mathrm{i}$.

(ii) $z = 1 + \mathrm{i}$. [3]

Point z on the Argand diagram can be transformed to point w by two transformations.

(b) Describe these two transformations and give the order in which they are applied. [3]

(c) Hence, or otherwise, find the value of z when $w = 2 - \mathrm{i}$. [2]

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Question 9 - working space

5. [Maximum mark: 5]

Roger buys a new laptop for himself at a cost of £495. At the same time, he buys his daughter Chloe a higher specification laptop at a cost of £2200.

It is anticipated that Roger's laptop will depreciate at a rate of 10% per year, whereas Chloe's laptop will depreciate at a rate of 15% per year.

(a) Estimate the value of Roger's laptop after 5 years. [2]

Roger and Chloe's laptops will have the same value k years after they were purchased.

(b) Find the value of k. [2]

(c) Comment on the validity of your answer to part (b). [1]

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Question 5 - working space

7. [Maximum mark: 6]

A meteorologist models the height of a hot air balloon launched from the ground. The model assumes the balloon travels vertically upwards and travels 450 metres in the first minute.

Due to the decrease in temperature as the balloon rises, the balloon will continually slow down. The model suggests that each minute the balloon will travel only 82% of the distance travelled in the previous minute.

(a) Find how high the balloon will travel in the first 10 minutes after it is launched. [3]

(b) The balloon is required to reach a height of at least 2520 metres.

Determine whether it will reach this height. [2]

(c) Suggest a limitation of the given model. [1]

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Question 7 - working space

12. [Maximum mark: 8]

It is given that $z_1 = 3\,\text{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2\,\text{cis}\left(\frac{n\pi}{16}\right)$, $n \in \mathbb{Z}^+$.

(a) In parts (a)(i) and (a)(ii), give your answers in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$.

(i) Find the value of $z_1^{\,3}$.

(ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$. [5]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]

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Question 12 - working space

1. [Maximum mark: 19]

Give your answers in parts (a), (d)(i), (e) and (f) to the nearest dollar.

Daisy invested $37\,000$ Australian dollars (AUD) in a fixed deposit account with an annual interest rate of 6.4% compounded **quarterly**.

(a) Calculate the value of Daisy's investment after 2 years. [3]

After m months, the amount of money in the fixed deposit account has appreciated to more than $50\,000$ AUD.

(b) Find the minimum value of m, where $m \in \mathbb{N}$. [4]

Daisy is saving to purchase a new apartment. The price of the apartment is $200\,000$ AUD.

Daisy makes an initial payment of 25% and takes out a loan to pay the rest.

(c) Write down the amount of the loan. [1]

The loan is for 10 years, compounded monthly, with equal monthly payments of 1700 AUD made by Daisy at the end of each month.

(d) For this loan, find

(i) the amount of interest paid by Daisy.

(ii) the annual interest rate of the loan. [5]

After 5 years of paying off this loan, Daisy decides to pay the **remainder** in one final payment.

(e) Find the amount of Daisy's final payment. [3]

(f) Find how much money Daisy saved by making one final payment after 5 years. [3]

Turn over

Question 1 - working space

5. [Maximum mark: 9]

In this question, give all answers correct to 2 decimal places.

Raul and Rosy want to buy a new house and they need a loan of 170 000 Australian dollars (AUD) from a bank. The loan is for 30 years and the annual interest rate for the loan is 3.8%, compounded monthly. They will pay the loan in fixed monthly instalments at the end of each month.

(a) Find the amount they will pay the bank each month. [3]

(b) (i) Find the amount Raul and Rosy will still owe the bank at the end of the first 10 years.

(ii) Using your answers to parts (a) and (b)(i), calculate how much interest they will have paid in total during the first 10 years. [6]

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Question 5 - working space

6. [Maximum mark: 5]

An infinite geometric sequence, with terms u_n, is such that $u_1 = 2$ and $\sum_{k=1}^{\infty} u_k = 10$.

(a) Find the common ratio, r, for the sequence. [2]

(b) Find the least value of n such that $u_n < \frac{1}{2}$. [3]

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Question 6 - working space

5. [Maximum mark: 13]

(a) Let $z = 1 - \mathrm{i}$.

(i) Plot the position of z on an Argand Diagram.

(ii) Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where a, $b \in \mathbb{R}$, giving the exact value of a and the exact value of b. [3]

(b) Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x-\frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(i) Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$.

(ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 < \alpha \leq \frac{\pi}{2}$. [6]

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_{A} and I_{B}. The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_{A}	12 amps	0
I_{B}	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_{T} can be expressed as $I_{\mathrm{A}} + I_{\mathrm{B}}$.

(c) (i) Find the maximum value of I_{T}.

(ii) Find the phase shift of I_{T}. [4]

Question 5 - working space

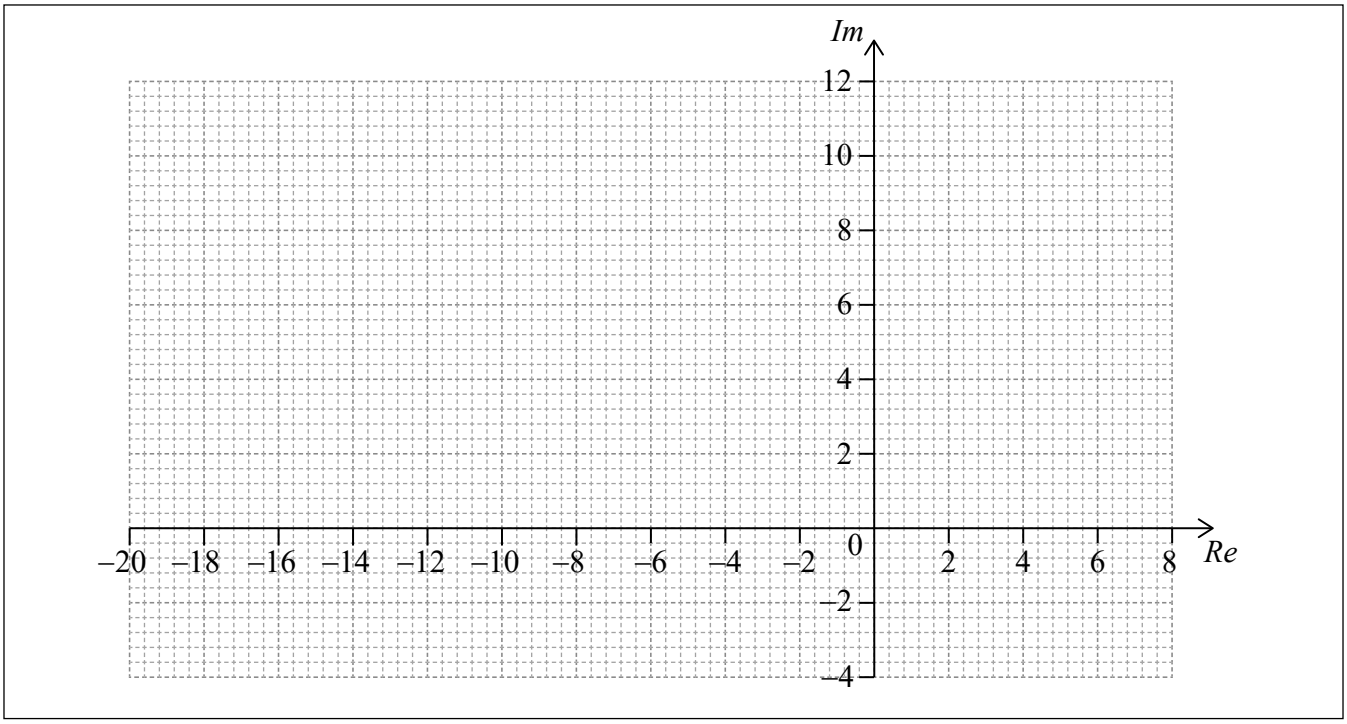
15. [Maximum mark: 7]

Let $w = ae^{\frac{\pi}{4}i}$, where $a \in \mathbb{R}^+$.

(a) For $a = 2$,

(i) find the values of w^2, w^3, and w^4;

(ii) draw w, w^2, w^3 and w^4 on the following Argand diagram. [5]



Let $z = \frac{w}{2 - i}$.

(b) Find the value of a for which successive powers of z lie on a circle. [2]

(This question continues on the following page)

(Question 15 continued)

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Question 15 - working space

18. [Maximum mark: 6]

The rate, A, of a chemical reaction at a fixed temperature is related to the concentration of two compounds, B and C, by the equation

$$A = kB^x C^y, \text{ where } x, y, k \in \mathbb{R}.$$

A scientist measures the three variables three times during the reaction and obtains the following values.

Experiment	A (mol l^{-1} s^{-1})	B (mol l^{-1})	C (mol l^{-1})
1	5.74	2.1	3.4
2	2.88	1.5	2.4
3	0.980	0.8	1.9

Find x, y and k.

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Question 18 - working space

3. [Maximum mark: 18]

In this question, give all answers to two decimal places.

Bryan decides to purchase a new car with a price of €$14\,000$, but cannot afford the full amount. The car dealership offers two options to finance a loan.

Finance option A:

A 6 year loan at a nominal annual interest rate of 14% **compounded quarterly**.
No deposit required and repayments are made each quarter.

(a) (i) Find the repayment made each quarter.

(ii) Find the total amount paid for the car.

(iii) Find the interest paid on the loan. [7]

Finance option B:

A 6 year loan at a nominal annual interest rate of $r\%$ **compounded monthly**. Terms of the loan require a 10% deposit and monthly repayments of €250.

(b) (i) Find the amount to be borrowed for this option.

(ii) Find the annual interest rate, r. [5]

(c) State which option Bryan should choose. Justify your answer. [2]

Bryan chooses option B. The car dealership invests the money Bryan pays as soon as they receive it.

(d) If they invest it in an account paying 0.4% interest per month and inflation is 0.1% per month, calculate the real amount of money the car dealership has received by the end of the 6 year period. [4]

Question 3 - working space

2. [Maximum mark: 5]

The ticket prices for a concert are shown in the following table.

Ticket Type	Price (in Australian dollars, $)
Adult	15
Child	10
Student	12

- A total of 600 tickets were sold.
- The total amount of money from ticket sales was $\$7816$.
- There were twice as many adult tickets sold as child tickets.

Let the number of adult tickets sold be x, the number of child tickets sold be y, and the number of student tickets sold be z.

(a) Write down three equations that express the information given above. [3]

(b) Find the number of each type of ticket sold. [2]

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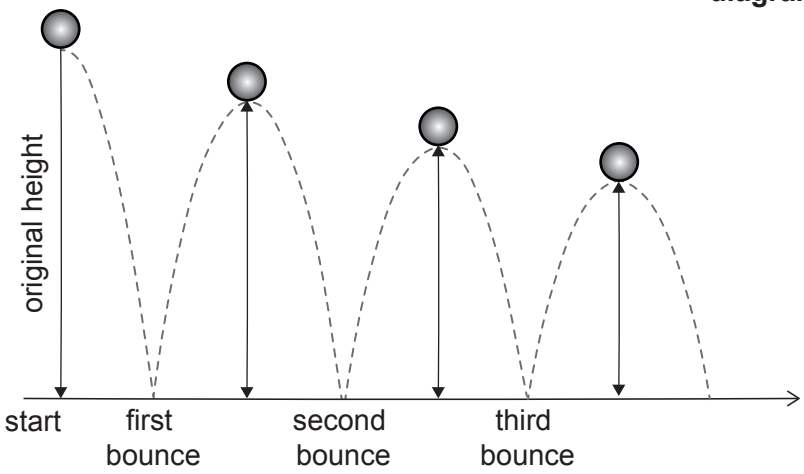


Question 2 - working space

5. [Maximum mark: 7]

A ball is dropped from a height of 1.8 metres and bounces on the ground. The maximum height reached by the ball, after each bounce, is 85% of the previous maximum height.

diagram not to scale



(a) Show that the maximum height reached by the ball after it has bounced for the sixth time is $68\,\text{cm}$, to the nearest cm. [2]

(b) Find the number of times, after the first bounce, that the maximum height reached is greater than $10\,\text{cm}$. [2]

(c) Find the total **vertical** distance travelled by the ball from the point at which it is dropped until the fourth bounce. [3]

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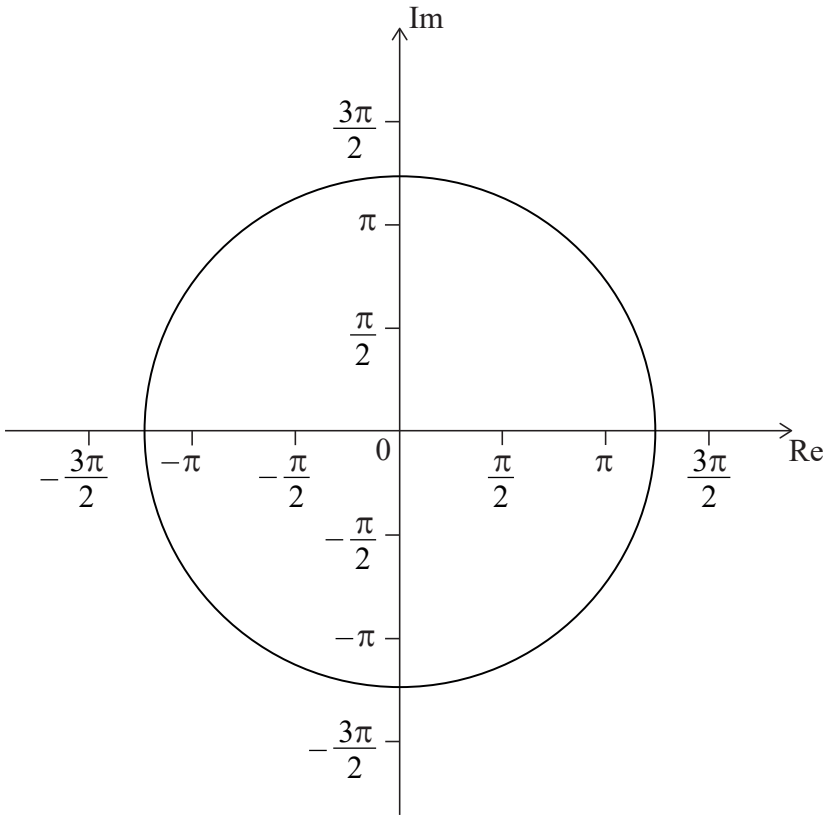
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Question 5 - working space

10. [Maximum mark: 7]

The following Argand diagram shows a circle centre 0 with a radius of 4 units.



A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$$z_\theta = \frac{1}{2}\theta e^{\theta i}, \text{ where } \theta \geq 0.$$

(a) Plot on the Argand diagram the points corresponding to

(i) $\theta = \frac{\pi}{2}$.

(ii) $\theta = \pi$.

(iii) $\theta = \frac{3\pi}{2}$. [3]

Consider the case where $|z_\theta| = 4$.

(b) (i) Find this value of θ.

(ii) For this value of θ, plot the approximate position of z_θ on the Argand diagram. [4]

(This question continues on the following page)

(Question 10 continued)

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Question 10 - working space

11. [Maximum mark: 5]

The matrix $\boldsymbol{M} = \begin{pmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{pmatrix}$ has eigenvalues -0.5 and 1.

(a) Find an eigenvector corresponding to the eigenvalue of 1. Give your answer in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, where $a, b \in \mathbb{Z}$. [3]

A switch has two states, A and B. Each second it either remains in the same state or moves according to the following rule: If it is in state A it will move to state B with a probability of 0.8 and if it is in state B it will move to state A with a probability of 0.7.

(b) Using your answer to (a), or otherwise, find the long-term probability of the switch being in state A. Give your answer in the form $\frac{c}{d}$, where $c, d \in \mathbb{Z}^+$. [2]

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Question 11 - working space

12. [Maximum mark: 8]

The strength of earthquakes is measured on the Richter magnitude scale, with values typically between 0 and 8 where 8 is the most severe.

The Gutenberg–Richter equation gives the average number of earthquakes per year, N, which have a magnitude of at least M. For a particular region the equation is

$$\log_{10} N = a - M, \text{ for some } a \in \mathbb{R}.$$

This region has an average of 100 earthquakes per year with a magnitude of at least 3.

(a) Find the value of a. [2]

The equation for this region can also be written as $N = \frac{b}{10^M}$.

(b) Find the value of b. [2]

Within this region the most severe earthquake recorded had a magnitude of 7.2.

(c) Find the average number of earthquakes in a year with a magnitude of at least 7.2. [1]

The number of earthquakes in a given year with a magnitude of at least 7.2 can be modelled by a Poisson distribution, with mean N. The number of earthquakes in one year is independent of the number of earthquakes in any other year.

Let Y be the number of years between the earthquake of magnitude 7.2 and the next earthquake of at least this magnitude.

(d) Find $\mathrm{P}(Y > 100)$. [3]

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Question 12 - working space

7. [Maximum mark: 5]

The sum of an infinite geometric sequence is 9.

The first term is 4 more than the second term.

Find the third term. Justify your answer.

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Question 7 - working space

11. [Maximum mark: 6]

Juliana plans to invest money for 10 years in an account paying 3.5% interest, compounded annually. She expects the annual inflation rate to be 2% per year throughout the 10-year period.

Juliana would like her investment to be worth a real value of $\$4000$, compared to current values, at the end of the 10-year period. She is considering two options.

Option 1: Make a one-time investment at the start of the 10-year period.

Option 2: Invest $\$1000$ at the start of the 10-year period and then invest $\$x$ into the account at the end of each year (including the first and last years).

(a) For option 1, determine the minimum amount Juliana would need to invest. Give your answer to the nearest dollar. [3]

(b) For option 2, find the minimum value of x that Juliana would need to invest each year. Give your answer to the nearest dollar. [3]

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Question 11 - working space

13. [Maximum mark: 5]

An electric circuit has two power sources. The voltage, V_1, provided by the first power source, at time t, is modelled by

$$V_1 = \mathrm{Re}(2\mathrm{e}^{3t\mathrm{i}}).$$

The voltage, V_2, provided by the second power source is modelled by

$$V_2 = \mathrm{Re}(5\mathrm{e}^{(3t+4)\mathrm{i}}).$$

The total voltage in the circuit, V_T, is given by

$$V_T = V_1 + V_2.$$

(a) Find an expression for V_T in the form $A\cos(Bt + C)$, where A, B and C are real constants. [4]

(b) Hence write down the maximum voltage in the circuit. [1]

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Question 13 - working space

1. [Maximum mark: 13]

Scott purchases food for his dog in large bags and feeds the dog the same amount of dog food each day. The amount of dog food left in the bag at the end of each day can be modelled by an arithmetic sequence.

On a particular day, Scott opened a new bag of dog food and fed his dog. By the end of the third day there were 115.5 cups of dog food remaining in the bag and at the end of the eighth day there were 108 cups of dog food remaining in the bag.

(a) Find the number of cups of dog food

(i) fed to the dog per day;

(ii) remaining in the bag at the end of the first day. [4]

(b) Calculate the number of days that Scott can feed his dog with one bag of food. [2]

In 2021, Scott spent $\$625$ on dog food. Scott expects that the amount he spends on dog food will increase at an annual rate of 6.4%.

(c) Determine the amount that Scott expects to spend on dog food in 2025. Round your answer to the nearest dollar. [3]

(d) (i) Calculate the value of $\sum_{n=1}^{10}(625 \times 1.064^{(n-1)})$.

(ii) Describe what the value in part (d)(i) represents in this context. [3]

(e) Comment on the appropriateness of modelling this scenario with a geometric sequence. [1]

Question 1 - working space

7. [Maximum mark: 7]

On 1 December 2022, Laviola invests 800 euros (EUR) into a savings account which pays a nominal annual interest rate of 7.5% compounded monthly. At the end of each month, Laviola deposits an additional EUR 500 into the savings account.

At the end of k months, Laviola will have saved enough money to withdraw EUR $10\,000$.

(a) Find the smallest possible value of k, for $k \in \mathbb{Z}^+$. [4]

(b) For this value of k, find the interest earned in the savings account. Express your answer correct to the nearest EUR. [3]

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Question 7 - working space

1. [Maximum mark: 5]

On 1 January 2022, Mina deposited $\$1000$ into a bank account with an annual interest rate of 4%, compounded monthly. At the end of January, and the end of every month after that, she deposits $\$100$ into the same account.

(a) Calculate the amount of money in her account at the start of 2024. Give your answer to two decimal places. [3]

(b) Find how many complete months, counted from 1 January 2022, it will take for Mina to have more than $\$5000$ in her account. [2]

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Question 1 - working space

12. [Maximum mark: 5]

Two AC (alternating current) electrical sources with the same frequencies are combined. The voltages from these sources can be expressed as $V_1 = 6\sin(at + 30°)$ and $V_2 = 6\sin(at + 90°)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V\sin(at + \theta°)$.

Determine the value of V and the value of θ.

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Question 12 - working space

4. [Maximum mark: 6]

Angel has $\$520$ in his savings account. Angel considers investing the money for 5 years with a bank. The bank offers an annual interest rate of 1.2% compounded quarterly.

(a) Calculate the amount of money Angel would have at the end of 5 years with the bank. Give your answer correct to two decimal places. [3]

Instead of investing the money, Angel decides to buy a phone that costs $\$520$. At the end of 5 years, the phone will have a value of $\$30$. It may be assumed that the depreciation rate per year is constant.

(b) Calculate the annual depreciation rate of the phone. [3]

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Question 4 - working space

11. [Maximum mark: 6]

Two AC (alternating current) electrical sources of equal frequencies are combined.

The voltage of the first source is modelled by the equation $V = 30\sin(t + 60^\circ)$.

The voltage of the second source is modelled by the equation $V = 60\sin(t + 10^\circ)$.

(a) Determine the maximum voltage of the combined sources. [2]

(b) Using your graphic display calculator, find a suitable equation for the combined voltages, giving your answer in the form $V = V_0\sin(at + b)$, where a, b and V_0 are constants, $a > 0$ and $0^\circ \leq b < 180^\circ$. [4]

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Question 11 - working space

1. [Maximum mark: 7]

The growth of a particular type of seashell is being studied by Manon. At the end of each month Manon records the increase in the width of a seashell since the end of the previous month.

She models the monthly increase in the width of the seashell by a geometric sequence with common ratio 0.8. In the first month, the width of the seashell increases by $4\,\text{mm}$.

(a) Find by how much the width of the seashell will increase during the third month, according to her model. [2]

(b) Find the total increase in the width of the seashell, predicted by Manon's model, during the first year. [2]

Manon's seashell had a width of $25\,\text{mm}$ at the beginning of the first month.

(c) Find the maximum possible width of the seashell, predicted by Manon's model. [3]

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Question 1 - working space

8. [Maximum mark: 6]

Given $z = \sqrt{3} - \mathrm{i}$.

(a) Write z in the form $z = r\mathrm{e}^{\theta \mathrm{i}}$, where $r \in \mathbb{R}^+$, $-\pi < \theta \leq \pi$. [2]

Let $z_1 = \mathrm{e}^{2t\mathrm{i}}$ and $z_2 = 2\mathrm{e}^{\left(2t - \frac{\pi}{6}\right)\mathrm{i}}$.

(b) Find $\mathrm{Im}(z_1 + z_2)$ in the form $p\sin(2t + q)$, where $p > 0$, $t \in \mathbb{R}$ and $-\pi \leq q \leq \pi$. [4]

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Question 8 - working space

3. [Maximum mark: 16]

Tiffany wants to buy a house for a price of $285\,000$ US Dollars (USD). She goes to a bank to get a loan to buy the house. To be eligible for the loan, Tiffany must make an initial down payment equal to $15\,\%$ of the price of the house.

The bank offers her a 30-year loan for the remaining balance, with a $4\,\%$ nominal interest rate per annum, compounded monthly. Tiffany will pay the loan in fixed payments at the end of each month.

(a) (i) Find the original amount of the loan after the down payment is paid. Give the exact answer.

(ii) Calculate Tiffany's monthly payment for this loan, to two decimal places. [5]

(b) Using your answer from part (a)(ii), calculate the total amount Tiffany will pay over the life of the loan, to the nearest dollar. Do **not** include the initial down payment. [2]

Tiffany would like to repay the loan faster and increases her payments such that she pays $1300\,\text{USD}$ each month.

(c) Find the total number of monthly payments she will need to make to pay off the loan. [2]

This strategy will result in Tiffany's final payment being less than $1300\,\text{USD}$.

(d) Determine the amount of Tiffany's final payment, to two decimal places. [4]

(e) Hence, determine the total amount Tiffany will save, to the nearest dollar, by making the higher monthly payments. [3]

Question 3 - working space

1. [Maximum mark: 6]

Maan deposited $\$100\,000$ into a savings account with a nominal annual interest rate of $I\,\%$ **compounded monthly**. At the end of the eighth year, the amount in the account had increased to $\$150\,000$.

(a) Find the value of I. [3]

Maan withdraws the $\$150\,000$ and places it in an annuity, earning a nominal annual interest rate of $6.1\,\%$ **compounded monthly**. At the end of each month, Maan will receive a payment of $\$1000$.

(b) Find the amount of money remaining in the annuity at the end of 10 years. Express your answer to the nearest dollar. [3]

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Question 1 - working space

6. [Maximum mark: 7]

On 1 January 2025, the Faber Car Company will release a new car to global markets. The company expects to sell 40 cars in January 2025. The number of cars sold each month can be modelled by a geometric sequence where $r = 1.1$.

(a) Use this model to find the number of cars that will be sold in December 2025. [2]

(b) Use this model to find the total number of cars that will be sold in the year

(i) 2025.

(ii) 2026. [5]

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Question 6 - working space

13. [Maximum mark: 6]

Let $z_1 = 4 + 5\mathrm{i}$.

(a) (i) Find $|z_1|$.

(ii) Find $\arg(z_1)$. [2]

Let $z_2 = 3\mathrm{e}^{2\mathrm{i}}$.

(b) Find the area of the triangle on an Argand diagram with vertices 0, z_1 and z_2. [4]

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Question 13 - working space

1. [Maximum mark: 9]

Imani invests $\$3000$ in a bank that pays a nominal annual interest rate of 1.25% compounded monthly.

(a) Calculate the amount of money Imani will have in the bank at the end of 6 years. Give your answer correct to two decimal places. [3]

(b) Calculate the number of months it takes until Imani has at least $\$3550$ in the bank. [2]

Imani uses the $\$3550$ as a partial payment for a used car costing $\$22\,000$. For the remainder she takes out a loan from a bank.

(c) Write down the amount of money that Imani takes out as a loan. [1]

The loan is for 8 years and the nominal annual interest rate is 12.6% compounded monthly. Imani will pay the loan in fixed monthly instalments at the end of each month.

(d) Calculate the amount, correct to the nearest dollar, that Imani will have to pay the bank each month. [3]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

5. [Maximum mark: 8]

The annual growth of a tree is $80\,\%$ of its growth during the previous year.

This year the tree is $42\,\text{m}$ in height and one year ago its height was $37\,\text{m}$.

(a) Calculate the annual growth of the tree in the coming year. [2]

(b) Calculate the height of the tree 6 years from now. Give your answer correct to the nearest cm. [4]

If the tree continues to follow this pattern of growth, its height will never exceed k metres.

(c) Find the smallest possible value of k. [2]

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Question 5 - working space

15. [Maximum mark: 8]

Consider the complex number $z = -1 + \mathrm{i}$.

(a) Express z in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $-\pi < \theta \leq \pi$. [2]

A and B are the points on the Argand diagram that represent the complex numbers z and z^2, respectively.

A is mapped onto B by the composition of a rotation and an enlargement.

(b) (i) Describe fully this mapping of A onto B, stating the scale factor of the enlargement and the angle of rotation.

(ii) Find and simplify a matrix that maps A onto B. [5]

(c) Find the smallest positive integer, n, for which z^n is real and positive. [1]

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Question 15 - working space

4. [Maximum mark: 5]

On 1 January in a particular year, Eva invests $\$25\,000$ in a new bank account. The account earns 5% simple interest, on the original $\$25\,000$, at the start of each subsequent year.

The amounts in the account at the start of each year form an arithmetic sequence.

(a) Find the common difference of this sequence. [2]

After k complete years, the amount in Eva's account will be greater than $\$44\,000$ for the first time.

(b) Find the value of k. [3]

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Question 4 - working space

7. [Maximum mark: 6]

When Daniel retires, he invests $\$400\,000$ in an annuity fund that earns interest at a nominal rate of 4.5% per year, compounded monthly.

Daniel then withdraws $\$3600$ at the end of every month to pay for his living expenses.

(a) Find how much is in the annuity fund after 5 years. [3]

(b) Calculate how many times Daniel is able to make these withdrawals. [3]

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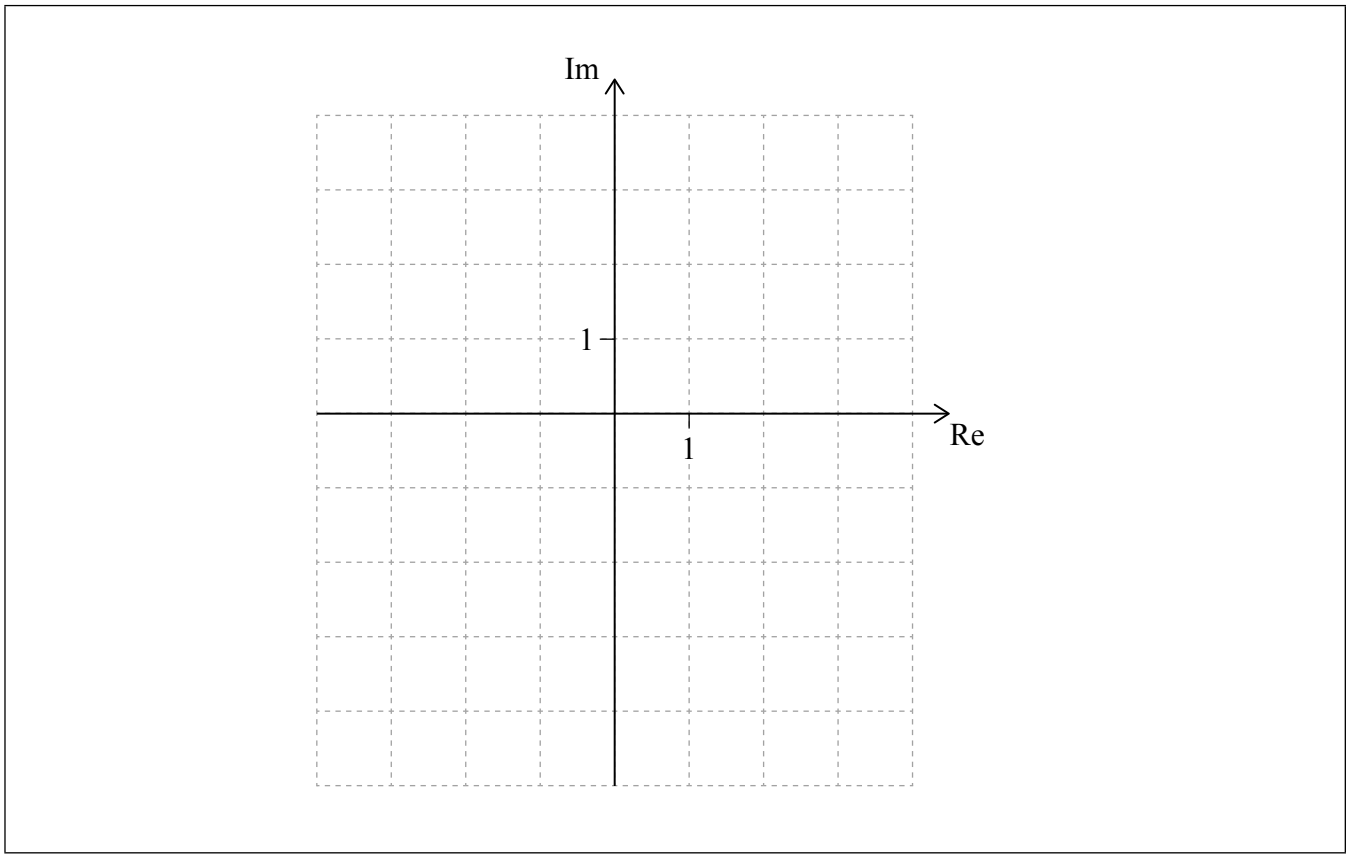
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Question 7 - working space

9. [Maximum mark: 5]

Let $z = 2 - 3\mathrm{i}$.

(a) Plot z on the Argand diagram. [1]



z can be written in the form $r\,\mathrm{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Find the value of

(i) r

(ii) θ. [2]

(c) Find the value of $z\mathrm{i}$ in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$. [1]

$z\mathrm{i}$ can be obtained from z by a geometric transformation.

(d) By plotting $z\mathrm{i}$ on the Argand diagram, or otherwise, describe fully this transformation. [1]

(This question continues on the following page)

(Question 9 continued)

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Question 9 - working space

11. [Maximum mark: 8]

Let $\boldsymbol{M} = \begin{pmatrix} -4 & 2 \\ -3 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of $\boldsymbol{M}$. [3]

$\boldsymbol{M}$ can be written in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(b) (i) Write down $\boldsymbol{D}$.

(ii) Find $\boldsymbol{P}$. [5]

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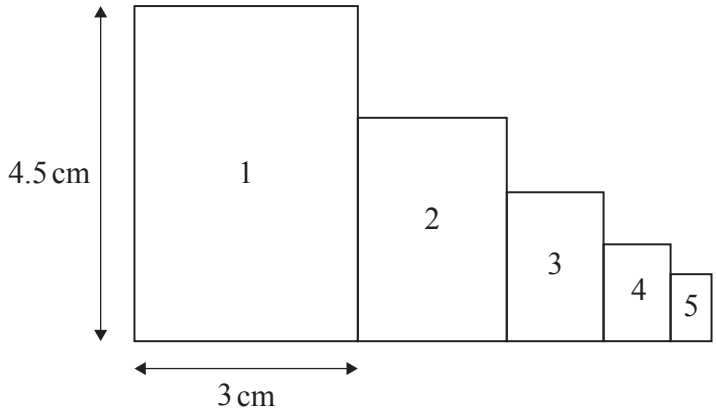
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Question 11 - working space

3. [Maximum mark: 15]

Ayaka is creating a design made from a sequence of rectangles. The diagram shows part of her design, using 5 rectangles.

diagram not to scale



The first rectangle has the following dimensions: height $4.5\,\text{cm}$ and width $3\,\text{cm}$.

The dimensions of each subsequent rectangle are $\frac{2}{3}$ of the dimensions of the previous rectangle.

(a) Calculate the width of the 5th rectangle. [2]

(b) Calculate the total width of the design that uses 5 rectangles. [2]

Ayaka continues to add rectangles to her design.

(c) Find the smallest total width that her design cannot exceed. [3]

The width of Ayaka's final design must be at least $8.5\,\text{cm}$ and use the least number of rectangles.

(d) Find the total number of rectangles in her final design. [3]

The decreasing **areas** of the rectangles form a geometric sequence.

(e) Find the common ratio for this sequence of areas. [2]

(f) Find the total area of Ayaka's final design. [3]

Turn over

Question 3 - working space

3. [Maximum mark: 8]

Raul wants to buy a new truck for $42\,695$ US dollars (USD). Raul is offered two different loan options to pay for the truck. For both options, Raul will pay back the loan over a period of 6 years.

The two options are shown in the following table.

Loan option	**Size of loan (% purchase price)**	**Nominal annual interest rate (compounded monthly)**
A	$100\,\%$ of purchase price (after no initial payment)	$7.15\,\%$
B	$80\,\%$ of purchase price (after a $20\,\%$ initial payment from Raul)	$6.40\,\%$

For both loan options, Raul will make monthly payments, at the end of each month, until the loan is completely paid off.

(a) Calculate the amount of Raul's monthly payment, to two decimal places, if he chooses

(i) option A.

(ii) option B. [6]

(b) State a reason why Raul might choose

(i) option A.

(ii) option B. [2]

(This question continues on the following page)

(Question 3 continued)

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Question 3 - working space

10. [Maximum mark: 9]

Consider the function $f(x) = \ln(2 - x) + \ln(-2 - x) - 2\ln(-x)$, where $x < -2$.

The function $f(x)$ can be rewritten in the form $\ln(g(x))$.

(a) Find an expression for $g(x)$. [2]

(b) Write down the range of $f^{-1}(x)$. [2]

The function $f^{-1}(x)$ can be written in the form $\frac{a}{\sqrt{1 - e^x}}$.

(c) Find the value of a. [5]

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Question 10 - working space

11. [Maximum mark: 8]

Nanthana is completing an exploration on multiplication using complex numbers. She considers a geometric sequence where $u_1 = 9$ and $r = \frac{2}{3} + \frac{2}{3}\mathrm{i}$.

(a) Write down the value of

(i) u_2.

(ii) u_3. [2]

Nanthana claims that the sequence $|u_1|, |u_2|, |u_3|, \ldots$ also forms a geometric sequence.

(b) Show that Nanthana's claim is correct, stating the exact value of the common ratio for this sequence. [4]

(c) Hence, find the sum of the infinite sequence $|u_1|, |u_2|, |u_3|, \ldots$. [2]

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Question 11 - working space

14. [Maximum mark: 6]

Consider the following functions:

$$L_1(x) = 14.8\sin(0.35x + 1.27)$$

$$L_2(x) = 18.5\sin(0.35x - 2.36)$$

(a) Write down both functions in the form $\mathrm{Im}(a\mathrm{e}^{bx\mathrm{i}}\,\mathrm{e}^{c\mathrm{i}})$. [2]

The sum of $L_1(x)$ and $L_2(x)$ can be expressed in the form $L_3(x) = q\sin(0.35x + r)$, where $0 \le r < 2\pi$.

(b) Use your answer to part (a) to find the value of q and of r. [4]

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Question 14 - working space

1. [Maximum mark: 7]

Give answers to this question correct to two decimal places.

Pierre invests 1500 euros (EUR) at the end of each month for 10 years into a savings plan that pays a nominal annual interest rate of 3.6% compounded monthly.

(a) Calculate the value of Pierre's savings plan at the end of the 10 years. [3]

At the end of the 10 years, Pierre withdraws $100\,000\,\text{EUR}$ from the savings plan to use as a deposit on a house.

Pierre invests the remainder into another account for 15 years at a nominal annual interest rate of 4.5% compounded quarterly.

(b) Calculate the amount in Pierre's account at the end of this time. [4]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

5. [Maximum mark: 6]

A speed camera is used to determine whether a car is exceeding a speed limit of $8.3\,\text{m}\,\text{s}^{-1}$.

An exact distance of $10\,\text{m}$ is marked out.

The car travels this $10\,\text{m}$ distance in 1.2 seconds, measured to the nearest 0.1 second.

Determine whether it is certain that the car was exceeding the speed limit of $8.3\,\text{m}\,\text{s}^{-1}$.

Justify your answer.

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Question 5 - working space

16. [Maximum mark: 7]

An electrical engineer models a circuit using the equation

$$z^2 + 2tz + 8t = 0$$

where t is the time in seconds and $0 \leq t \leq 2$.

(a) When $t = 1$, find the value of z which satisfies $\frac{\pi}{2} < \arg z < \pi$. Give your answer in the form $a + bi$. [2]

The power in the circuit is given by $|z|^2$.

(b) Find the value of t in the interval $0 \leq t \leq 2$ for which the power is maximized. [5]

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Question 16 - working space

2. [Maximum mark: 5]

The loudness of a sound, L, measured in decibels (dB) is determined by the intensity of the sound, I, measured in watts per square metre (Wm^{-2}). The relationship between loudness and intensity can be expressed using the logarithmic function

$$L = 10\log_{10}\left(\frac{I}{I_0}\right),\ I > 0$$

where I_0 is the reference intensity (the intensity of the least audible sound to the human ear).

The reference intensity I_0 is $10^{-12}\,\mathrm{Wm}^{-2}$.

The intensity of sound on a busy street is $10^{-5}\,\mathrm{Wm}^{-2}$.

(a) Calculate the loudness of the sound. [2]

The sound of a jet engine reaches a loudness of $185\,\mathrm{dB}$.

(b) Determine the intensity of its sound. Give your answer in the form $a \times 10^k$ where $1 \leq a < 10$, $k \in \mathbb{Z}$. [3]

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Question 2 - working space

5. [Maximum mark: 8]

Sofia takes out a loan for $50\,000$ euros. She pays a nominal interest rate of 4.2% per year, compounded monthly. She must pay back the loan in 25 years through regular monthly payments, made at the end of each month.

(a) Find the amount Sofia must pay back each month. Give your answer correct to two decimal places. [2]

At the end of eight years, Sofia wins the lottery and pays back the remainder of the loan early in one single payment.

(b) (i) Find how much Sofia pays back in this single payment. Give your answer correct to two decimal places.

(ii) Find the amount of money saved by Sofia by paying back the loan early. [6]

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Question 5 - working space

1. [Maximum mark: 16]

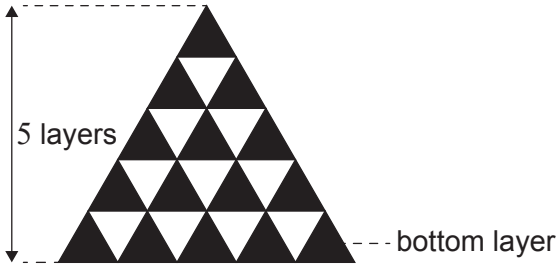
Thai cushions are designed with a triangular cross-section and are made from layers of smaller cushions. These cushions can be modelled as triangular prisms.

This is shown in the diagram.

diagram not to scale



Thai cushion with 4 layers



Cross-section of Thai cushion with 5 layers

(a) Write down the number of triangular prisms in the bottom layer of the cushion with

(i) 4 layers.

(ii) 5 layers. [2]

Mayumi notices that the number of triangular prisms in the bottom layer of the cushions forms an arithmetic sequence.

(b) (i) Write down the common difference of this sequence.

(ii) Find an expression for the number of triangular prisms in the bottom layer of a cushion with n layers. [3]

(This question continues on the following page)

(Question 1 continued)

Mayumi wants to extend this design to create a cushion with 9 layers.

(c) (i) Find the number of triangular prisms in the bottom layer of Mayumi's cushion.

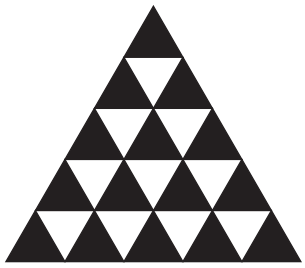
(ii) Calculate the **total** number of triangular prisms in Mayumi's cushion. [3]

(d) Find an expression for the **total** number of triangular prisms in a cushion with n layers, giving your answer in its simplest form. [2]

The cross-section of the cushion consists of black triangles and white triangles.



This cushion with 4 layers has a total of 6 white triangles.



This cushion with 5 layers has 4 white triangles in its bottom layer.

(e) Write down the total number of black triangles in a cushion with 4 layers. [1]

The number of black triangles in each layer forms an arithmetic sequence.

(f) Find and simplify an expression for the total number of black triangles in a cushion with n layers. [2]

The total number of white triangles in a cushion with n layers is $\frac{n(n-1)}{2}$.

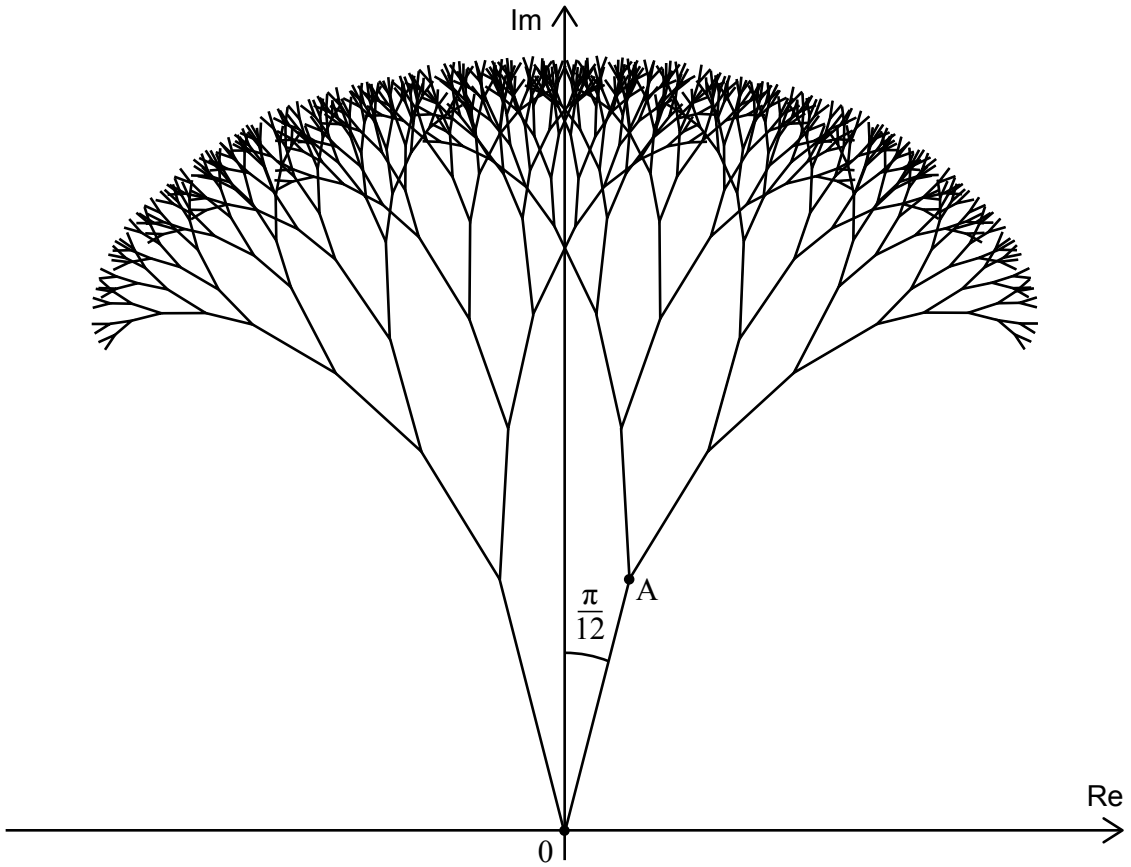
(g) Using both the given expression and your answer to part (f), find and simplify an expression for the total number of black and white triangles in a cushion with n layers. [3]

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Question 1 - working space

7. [Maximum mark: 17]

A fractal tree is shown on the following Argand diagram.



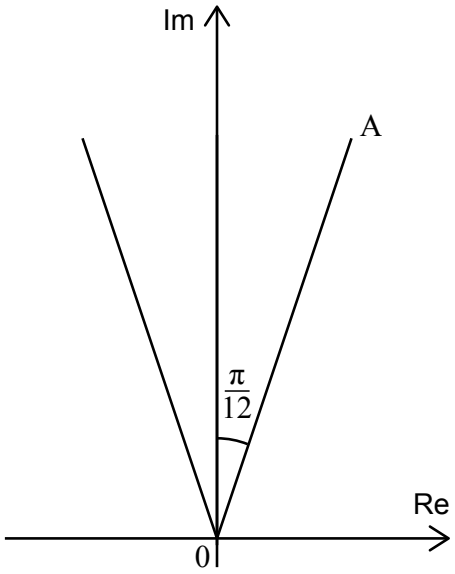
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(Question 7 continued)

The tree is formed in the following manner.

The line segment joining 0 to $5i$ is rotated clockwise by $\frac{\pi}{12}$ about 0 to form the first branch and then anti-clockwise by $\frac{\pi}{12}$ about 0 to form the second branch. The end of one of these branches is shown as A in **Diagram 1**.

Diagram 1



(a) Find the complex number represented by the point A, giving your answer in exponential form. [3]

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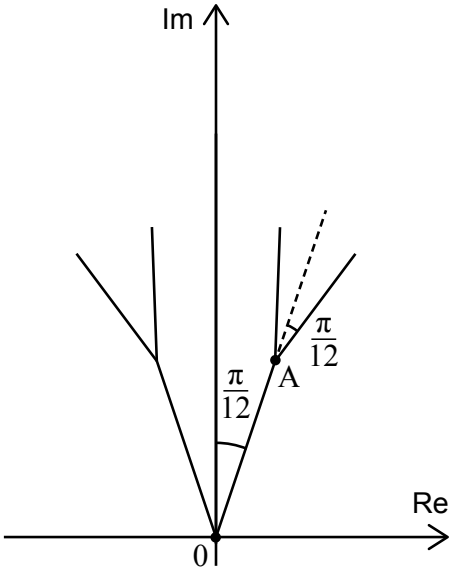
Turn over

(Question 7 continued)

Subsequent branches are created by adding two branches to the end of each branch created in the previous stage.

Each new branch is 70% the length of the previous one. One new branch is rotated $\frac{\pi}{12}$ clockwise from the direction of the previous branch and the other is rotated anticlockwise. This is shown in **Diagram 2**.

Diagram 2



This process is continued using the end of all the branches formed in the previous stage as the centres of rotation for the next stage.

(This question continues on the following page)

(Question 7 continued)

The trees after the 3rd and 10th stage are shown in **Diagrams 3** and **4** respectively.

Diagram 3

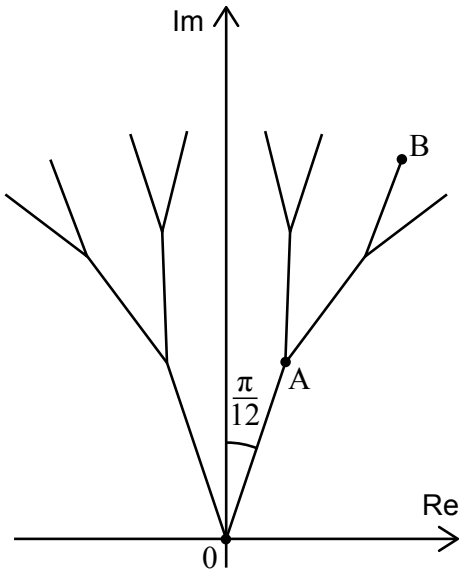
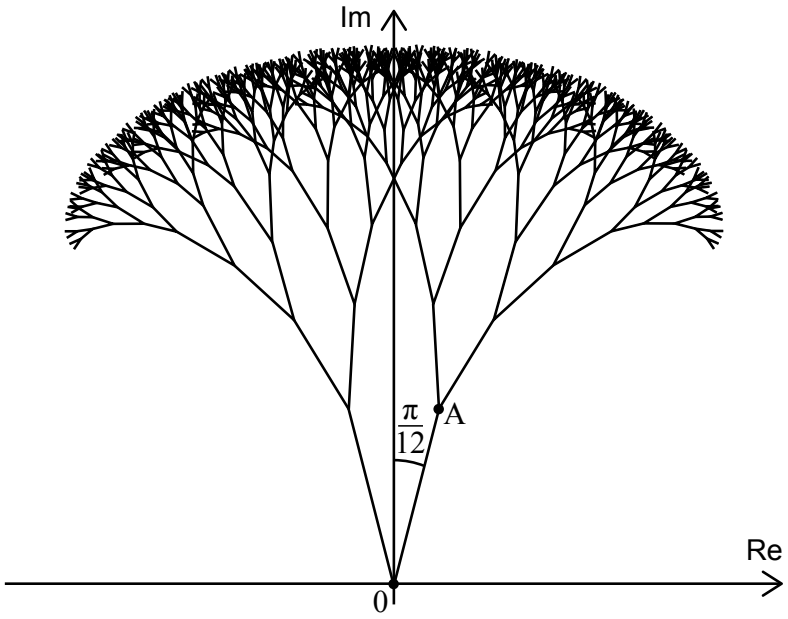


Diagram 4



(b) (i) Show that the total length of all branches in the tree after the second stage (**Diagram 2**) is equal to 24.

(ii) Find the total length of all branches after 10 stages (**Diagram 4**).

(iii) If this process continues indefinitely, state whether there is a limit to the total length of all the branches. Justify your answer. [9]

Let b be the complex number represented by the point B in **Diagram 3**.

(c) (i) Find an expression for b as the sum of three complex numbers, each written in exponential form.

(ii) Hence write down b in the form $a + b\mathrm{i}, a, b \in \mathbb{R}$. [5]

Question 7 - working space

2. [Maximum mark: 5]

A bakery sells egg tarts in boxes. Each box contains 6 tarts. On a particular morning, the bakery has 400 freshly baked tarts. The number of tarts **remaining**, after n boxes have been sold, forms an arithmetic sequence, $u_n = u_1 + (n-1)d$.

(a) Write down the value of d. [1]

(b) Find the value of u_1. [1]

(c) Find the greatest number of boxes the bakery can sell that morning. [3]

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Question 2 - working space

5. [Maximum mark: 5]

Two complex numbers are $z = 2 + a\mathrm{i}$ and $w = b + 4\mathrm{i}$, where $a, b \in \mathbb{R}$.

(a) Find an expression for the real part of z^2, in terms of a. [2]

(b) Find the value of a and of b, given that $z^2 + 2w = 3 + 6\mathrm{i}$. [3]

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Question 5 - working space

7. [Maximum mark: 18]

The Fibonacci sequence has the initial values $0, 1, 1, 2, 3, 5, 8, \ldots$

To find the next number in the sequence, the previous two numbers are added. This can be written as $f_{n+1} = f_n + f_{n-1}$, together with $f_1 = 0$ and $f_2 = 1$.

Consecutive pairs in the sequence can be written as a matrix, $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$, and used to produce a new sequence of matrices:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \ldots$$

(a) Write down the next two terms in this sequence of matrices. [2]

(b) Show that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$. [2]

(c) Show that the characteristic equation of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - \lambda - 1 = 0$. [2]

The eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ are λ_1 and λ_2, and a corresponding set of eigenvectors is $\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$.

(d) Using part (c), and without finding the value of λ_1, verify that $\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ satisfies the equation for an eigenvector,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}.$$ [2]

(This question continues on the following page)

(Question 7 continued)

It is given that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}$ can be written in the form $\boldsymbol{PD}^{n-1}\boldsymbol{P}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(e) Write down, in terms of λ_1 and λ_2,

(i) $\boldsymbol{D}$;

(ii) $\boldsymbol{P}$;

(iii) $\boldsymbol{P}^{-1}$. [3]

(f) Hence, by multiplying the matrices $\boldsymbol{PD}^{n-1}\boldsymbol{P}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, show that

$$f_n = \frac{1}{\lambda_1 - \lambda_2}\left((\lambda_1)^{n-1} - (\lambda_2)^{n-1}\right).$$ [4]

(g) (i) Use part (c) to find the exact value of λ_1 and λ_2.

(ii) Hence, write down an expression for f_n in terms of n. [3]

Question 7 - working space

APOLLO

Functions

51 questions

4. [Maximum mark: 6]

Professor Wei observed that students have difficulty remembering the information presented in his lectures.

He modelled the percentage of information retained, R, by the function $R(t) = 100\,\mathrm{e}^{-pt}$, $t \geq 0$, where t is the number of days after the lecture.

He found that 1 day after a lecture, students had forgotten $50\,\%$ of the information presented.

(a) Find the value of p. [2]

(b) Use this model to find the percentage of information retained by his students 36 hours after Professor Wei's lecture. [2]

Based on his model, Professor Wei believes that his students will always retain some information from his lecture.

(c) State a mathematical reason why Professor Wei might believe this. [1]

(d) Write down one possible limitation of the **domain** of the model. [1]

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Question 4 - working space

17. [Maximum mark: 7]

The graph of the function $f(x) = \ln x$ is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ so that it then passes through the points $(0, 1)$ and $(e^3, 1 + \ln 2)$.

Find the value of a and the value of b.

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References:

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Question 17 - working space

2. [Maximum mark: 7]

A function is defined by $f(x) = 2 - \frac{12}{x+5}$ for $-7 \leq x \leq 7$, $x \neq -5$.

(a) Find the range of f. [3]

(b) Find an expression for the inverse function $f^{-1}(x)$. The domain is not required. [3]

(c) Write down the range of $f^{-1}(x)$. [1]

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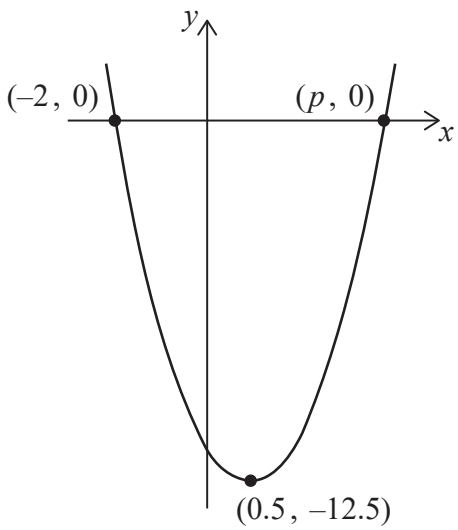
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Question 2 - working space

6. [Maximum mark: 7]

Consider the function $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is shown in the diagram. The vertex of the graph has coordinates $(0.5, -12.5)$. The graph intersects the x-axis at two points, $(-2, 0)$ and $(p, 0)$.

diagram not to scale



(a) Find the value of p. [1]

(b) Find the value of

(i) a.

(ii) b.

(iii) c. [5]

(c) Write down the equation of the axis of symmetry of the graph. [1]

(This question continues on the following page)

(Question 6 continued)

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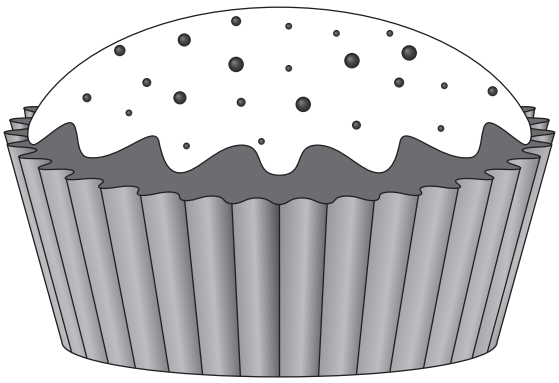
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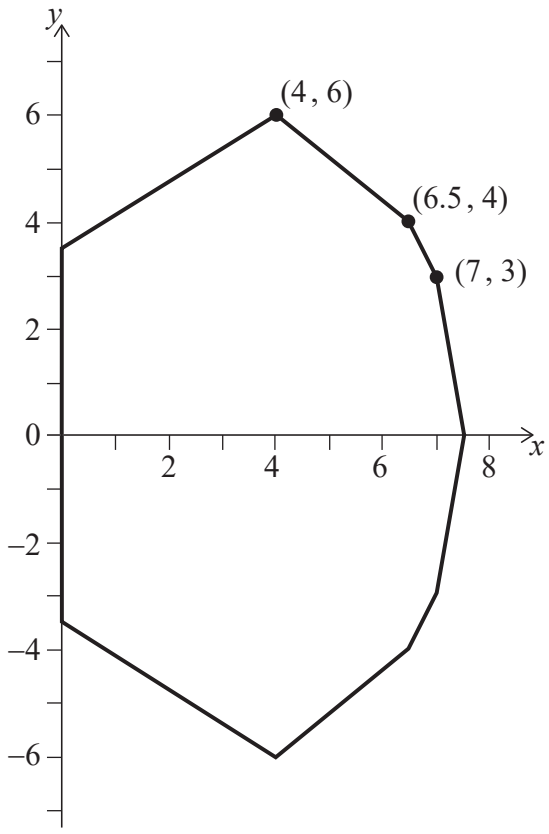
Question 6 - working space

4. [Maximum mark: 14]

Charlotte decides to model the shape of a cupcake to calculate its volume.



From rotating a photograph of her cupcake she estimates that its cross-section passes through the points $(0, 3.5)$, $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$, where all units are in centimetres. The cross-section is symmetrical in the x-axis, as shown below:



She models the section from $(0, 3.5)$ to $(4, 6)$ as a straight line.

(a) Find the equation of the line passing through these two points. [2]

(This question continues on the following page)

(Question 4 continued)

Charlotte models the section of the cupcake that passes through the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$ with a quadratic curve.

(b) (i) Find the equation of the least squares regression quadratic curve for these four points.

(ii) By considering the gradient of this curve when $x = 4$, explain why it may not be a good model. [3]

Charlotte thinks that a quadratic with a maximum point at $(4, 6)$ and that passes through the point $(7.5, 0)$ would be a better fit.

(c) Find the equation of the new model. [4]

Believing this to be a better model for her cupcake, Charlotte finds the volume of revolution about the x-axis to estimate the volume of the cupcake.

(d) (i) Write down an expression for her estimate of the volume as a sum of two integrals.

(ii) Find the value of Charlotte's estimate. [5]

Turn over

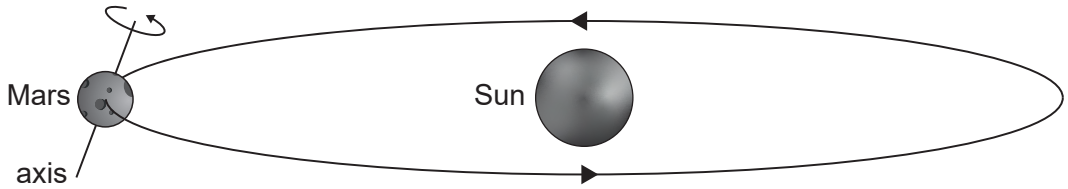
Question 4 - working space

1. [Maximum mark: 27]

A suitable site for the landing of a spacecraft on the planet Mars is identified at a point, A. The shortest time from sunrise to sunset at point A must be found.

Radians should be used throughout this question. All values given in the question should be treated as exact.

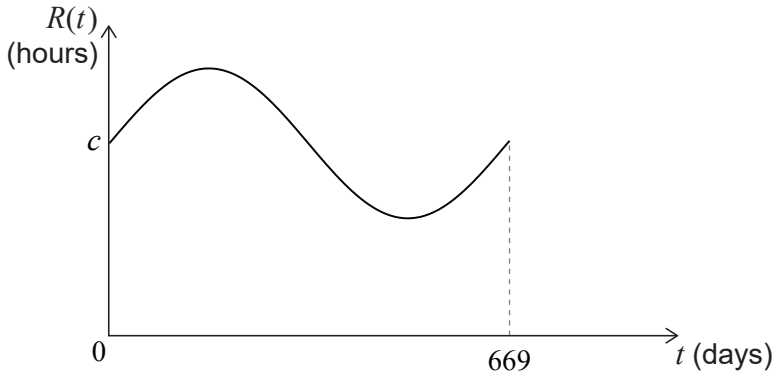
Mars completes a full orbit of the Sun in 669 Martian days, which is one Martian year.



On day t, where $t \in \mathbb{Z}$, the length of time, in hours, from the start of the Martian day until sunrise at point A can be modelled by a function, $R(t)$, where

$$R(t) = a\sin(bt) + c,\ t \in \mathbb{R}.$$

The graph of R is shown for one Martian year.



(a) Show that $b \approx 0.00939$. [2]

Mars completes a full rotation on its axis in 24 hours and 40 minutes.

(b) Find the angle through which Mars rotates on its axis each hour. [3]

(This question continues on the following page)

(Question 1 continued)

The time of sunrise on Mars depends on the angle, δ, at which it tilts towards the Sun. During a Martian year, δ varies from -0.440 to 0.440 radians.

The angle, ω, through which Mars rotates on its axis from the start of a Martian day to the moment of sunrise, at point A, is given by $\cos\omega = 0.839\tan\delta,\ 0 \le \omega \le \pi$.

(c) (i) Show that the maximum value of $\omega = 1.98$, correct to three significant figures. [3]

(ii) Find the minimum value of ω. [1]

(d) Use your answers to parts (b) and (c) to find

(i) the maximum value of $R(t)$; [2]

(ii) the minimum value of $R(t)$. [1]

(e) Hence show that $a = 1.6$, correct to two significant figures. [2]

(f) Find the value of c. [2]

Let $S(t)$ be the length of time, in hours, from the start of the Martian day until **sunset** at point A on day t. $S(t)$ can be modelled by the function

$$S(t) = 1.5\sin(0.00939t + 2.83) + 18.65.$$

The length of time between sunrise and sunset at point A, $L(t)$, can be modelled by the function

$$L(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t) + d.$$

(g) Find the value of d. [2]

Let $f(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t)$ and hence $L(t) = f(t) + d$.

$f(t)$ can be written in the form $\mathrm{Im}(z_1 - z_2)$, where z_1 and z_2 are complex functions of t.

(h) (i) Write down z_1 and z_2 in exponential form, with a constant modulus. [3]

(ii) Hence or otherwise find an equation for L in the form $L(t) = p\sin(qt + r) + d$, where $p,\ q,\ r,\ d \in \mathbb{R}$. [4]

(iii) Find, in hours, the shortest time from sunrise to sunset at point A that is predicted by this model. [2]

Turn over

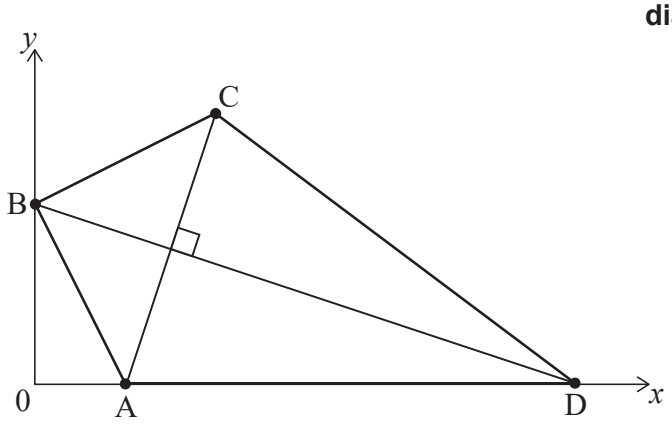
Question 1 - working space

1. [Maximum mark: 6]

Dilara is designing a kite ABCD on a set of coordinate axes in which one unit represents $10\,\text{cm}$.

The coordinates of A, B and C are $(2,\ 0)$, $(0,\ 4)$ and $(4,\ 6)$ respectively. Point D lies on the x-axis. $[\text{AC}]$ is perpendicular to $[\text{BD}]$. This information is shown in the following diagram.

diagram not to scale



(a) Find the gradient of the line through A and C. [2]

(b) Write down the gradient of the line through B and D. [1]

(c) Find the equation of the line through B and D. Give your answer in the form $ax + by + d = 0$, where a, b and d are integers. [2]

(d) Write down the x-coordinate of point D. [1]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

3. [Maximum mark: 7]

Let the function $h(x)$ represent the height in centimetres of a cylindrical tin can with diameter $x\,\text{cm}$.

$$h(x) = \frac{640}{x^2} + 0.5 \text{ for } 4 \leq x \leq 14.$$

(a) Find the range of h. [3]

The function h^{-1} is the inverse function of h.

(b) (i) Find $h^{-1}(10)$.

(ii) In the context of the question, interpret your answer to part (b)(i).

(iii) Write down the range of h^{-1}. [4]

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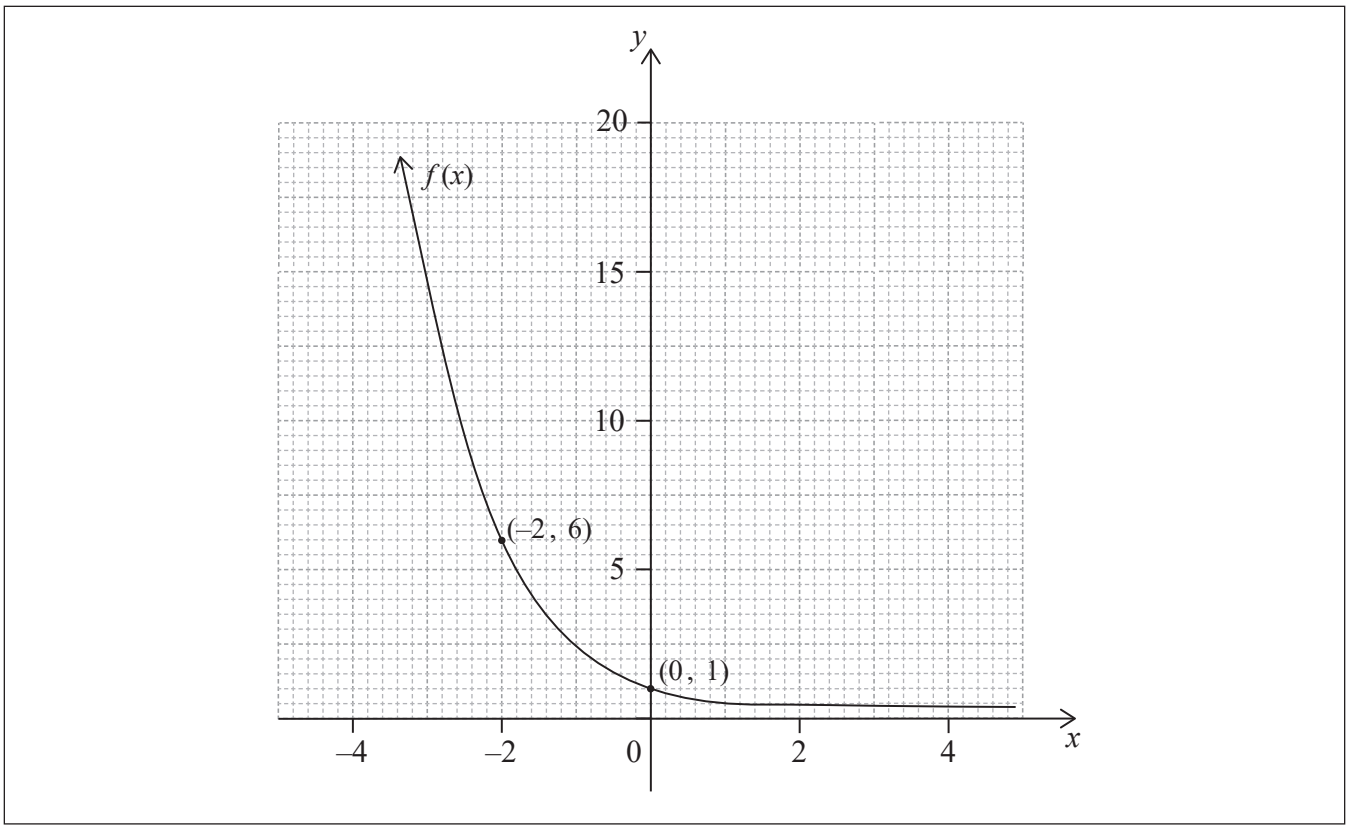
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Question 3 - working space

10. [Maximum mark: 4]

The graph of $y = f(x)$ is given on the following set of axes. The graph passes through the points $(-2, 6)$ and $(0, 1)$, and has a horizontal asymptote at $y = 0$.



Let $g(x) = 2f(x-2) + 4$.

(a) Find $g(0)$. [2]

(b) On the same set of axes draw the graph of $y = g(x)$, showing any intercepts and asymptotes. [2]

(This question continues on the following page)

(Question 10 continued)

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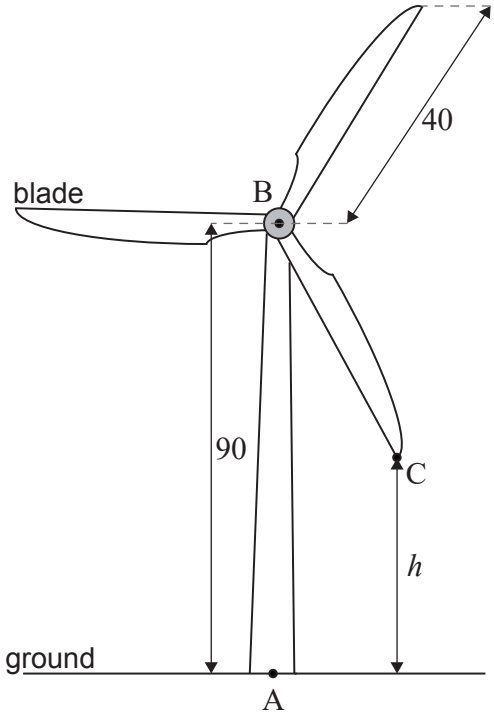
Question 10 - working space

2. [Maximum mark: 20]

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point A is on the base of the tower directly below point B at the top of the tower. The height of the tower, AB, is $90\,\mathrm{m}$. The blades of the turbine are centred at B and are each of length $40\,\mathrm{m}$. This is shown in the following diagram.

diagram not to scale



The end of one of the blades of the turbine is represented by point C on the diagram. Let h be the height of C above the ground, measured in metres, where h varies as the blade rotates.

(a) Find the

(i) maximum value of h.

(ii) minimum value of h. [2]

The blades of the turbine complete 12 rotations per minute under normal conditions, moving at a constant rate.

(b) (i) Find the time, in seconds, it takes for the blade $[\mathrm{BC}]$ to make one complete rotation under these conditions.

(ii) Calculate the angle, in degrees, that the blade $[\mathrm{BC}]$ turns through in one second. [3]

(This question continues on the following page)

(Question 2 continued)

The height, h, of point C can be modelled by the following function. Time, t, is measured from the instant when the blade [BC] first passes [AB] and is measured in seconds.

$$h(t) = 90 - 40\cos(72t^\circ),\ t \geq 0$$

(c) (i) Write down the amplitude of the function.

(ii) Find the period of the function. [2]

(d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]

(e) (i) Find the height of C above the ground when $t = 2$.

(ii) Find the time, in seconds, that point C is above a height of $100\,\text{m}$, during each complete rotation. [5]

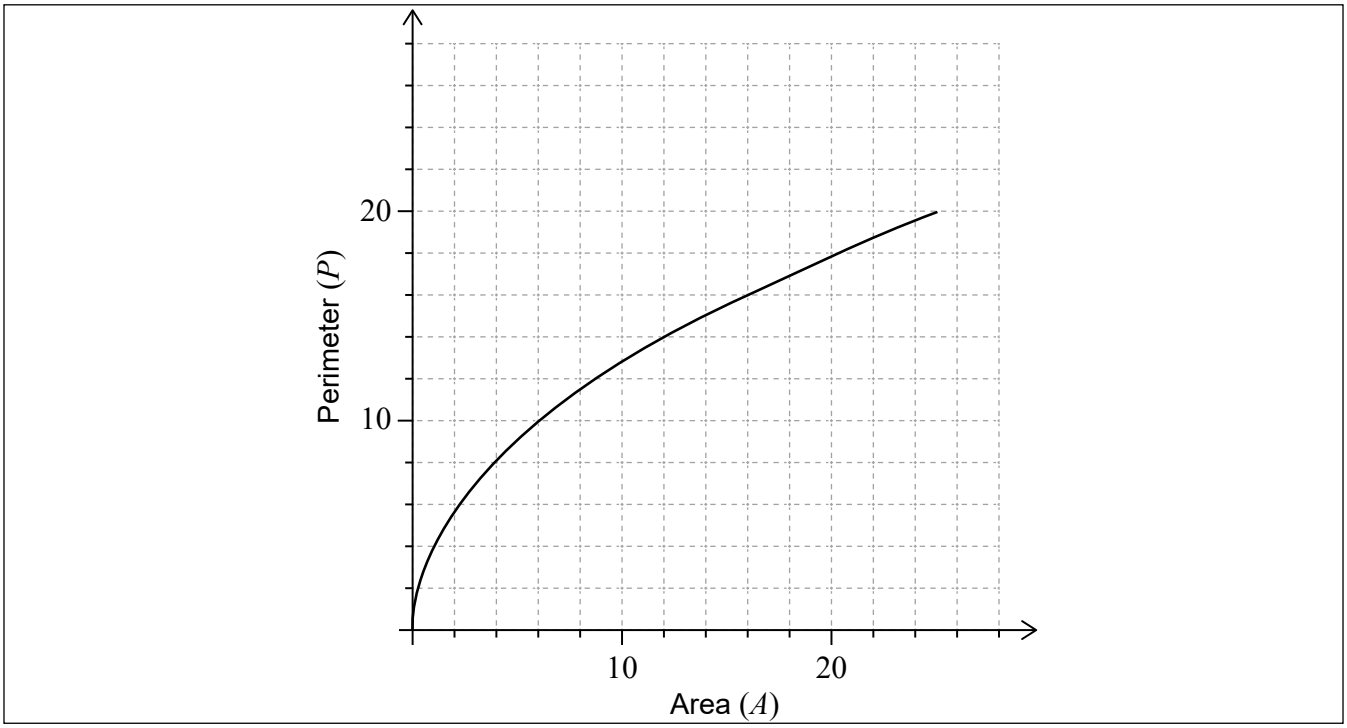
The wind speed increases and the blades rotate faster, but still at a constant rate.

(f) Given that point C is now higher than $110\,\text{m}$ for 1 second during each complete rotation, find the time for one complete rotation. [5]

Question 2 - working space

2. [Maximum mark: 5]

The perimeter of a given square P can be represented by the function $P(A) = 4\sqrt{A}$, $A \geq 0$, where A is the area of the square. The graph of the function P is shown for $0 \leq A \leq 25$.



(a) Write down the value of $P(25)$. [1]

(b) On the axes above, draw the graph of the inverse function, P^{-1}. [3]

(c) In the context of the question, explain the meaning of $P^{-1}(8) = 4$. [1]

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Question 2 - working space

3. [Maximum mark: 6]

Professor Vinculum investigated the migration season of the Bulbul bird from their natural wetlands to a warmer climate.

He found that during the migration season their population, P could be modelled by $P = 1350 + 400(1.25)^{-t}$, $t \geq 0$, where t is the number of days since the start of the migration season.

(a) Find the population of the Bulbul birds,

(i) at the start of the migration season.

(ii) in the wetlands after 5 days. [3]

(b) Calculate the time taken for the population to decrease below 1400. [2]

(c) According to this model, find the smallest possible population of Bulbul birds during the migration season. [1]

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Question 3 - working space

1. [Maximum mark: 5]

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

(a) Write down the height of the ball above the ground at the instant it is hit by the bat. [1]

(b) Find the value of t when the ball hits the ground. [2]

(c) State an appropriate domain for t in this model. [2]

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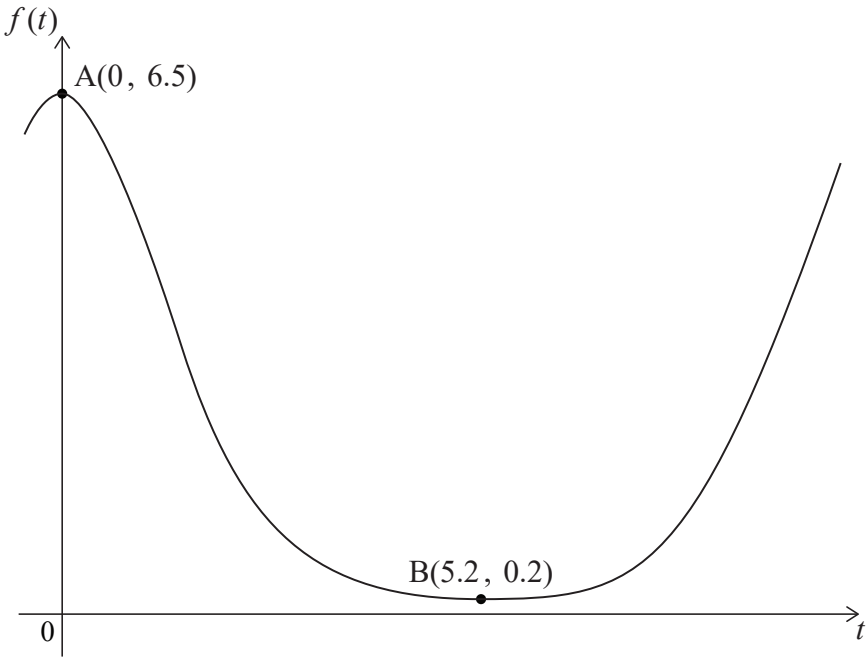
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Question 1 - working space

17. [Maximum mark: 8]

A function f is of the form $f(t) = pe^{q\cos(rt)}$, $p, q, r \in \mathbb{R}^+$. Part of the graph of f is shown.



The points A and B have coordinates $A(0, 6.5)$ and $B(5.2, 0.2)$, and lie on f.

The point A is a local maximum and the point B is a local minimum.

Find the value of p, of q and of r.

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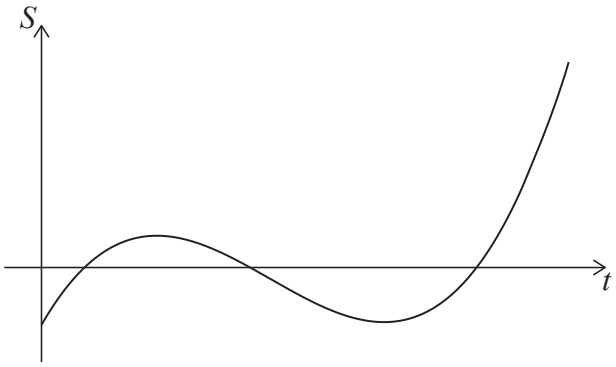
References:

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Question 17 - working space

5. [Maximum mark: 8]

The graph below shows the average savings, S thousand dollars, of a group of university graduates as a function of t, the number of years after graduating from university.



(a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario. [1]

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where a, b, c and d are real constants.

The graph of the model must pass through the following four points.

t	0	1	2	3
S	−5	3	−1	−5

(b) (i) Write down the value of d.

(ii) Write down three simultaneous equations for a, b and c.

(iii) Hence, or otherwise, find the values of a, b and c. [4]

A negative value of S indicates that a graduate is expected to be in debt.

(c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university. [3]

(This question continues on the following page)

(Question 5 continued)

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Question 5 - working space

10. [Maximum mark: 5]

The function $f(x) = \ln\left(\frac{1}{x-2}\right)$ is defined for $x > 2$, $x \in \mathbb{R}$.

(a) Find an expression for $f^{-1}(x)$. You are not required to state a domain. [3]

(b) Solve $f(x) = f^{-1}(x)$. [2]

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Question 10 - working space

4. [Maximum mark: 15]

A student investigating the relationship between chemical reactions and temperature finds the Arrhenius equation on the internet.

$$k = A\mathrm{e}^{-\frac{c}{T}}$$

This equation links a variable k with the temperature T, where A and c are positive constants and $T > 0$.

(a) Show that $\frac{\mathrm{d}k}{\mathrm{d}T}$ is always positive. [3]

(b) Given that $\lim_{T\to\infty} k = A$ and $\lim_{T\to 0} k = 0$, sketch the graph of k against T. [3]

The Arrhenius equation predicts that the graph of $\ln k$ against $\frac{1}{T}$ is a straight line.

(c) Write down

(i) the gradient of this line in terms of c;

(ii) the y-intercept of this line in terms of A. [4]

The following data are found for a particular reaction, where T is measured in Kelvin and k is measured in $\mathrm{cm}^3\ \mathrm{mol}^{-1}\ \mathrm{s}^{-1}$:

T	k
590	5×10^{-4}
600	6×10^{-4}
610	10×10^{-4}
620	14×10^{-4}
630	20×10^{-4}
640	29×10^{-4}
650	36×10^{-4}

(d) Find the equation of the regression line for $\ln k$ on $\frac{1}{T}$. [2]

(e) Find an estimate of

(i) c;

(ii) A.

It is not required to state units for these values. [3]

Turn over

Question 4 - working space

2. [Maximum mark: 5]

Celeste heated a cup of coffee and then let it cool to room temperature. Celeste found the coffee's temperature, T, measured in $^{\circ}\mathrm{C}$, could be modelled by the following function,

$$T(t) = 71\,\mathrm{e}^{-0.0514t} + 23\,,\ t \geq 0\,,$$

where t is the time, in minutes, after the coffee started to cool.

(a) Find the coffee's temperature 16 minutes after it started to cool. [2]

(b) Write down the room temperature. [1]

(c) Given that $T^{-1}(50) = k$, find the value of k. [2]

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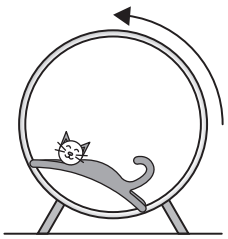
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Question 2 - working space

6. [Maximum mark: 6]

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, h cm, of a fixed point, P, on the wheel can be modelled by $h(t) = a\sin(bt) + c$ where t is the time in seconds and a, b, $c \in \mathbb{R}^+$.



When $t = 0$, point P is at a height of 78 cm.

(a) Write down the value of c. [1]

When $t = 4$, point P first reaches its maximum height of 143 cm.

(b) Find the value of

(i) a.

(ii) b. [3]

(c) Write down the minimum height of point P. [1]

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

(d) Write down the new value of b. [1]

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Question 6 - working space

10. [Maximum mark: 7]

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m, of another star can be modelled as a function of its brightness, b, relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5 \log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

(a) Find the magnitude of Acubens. [2]

Ceres has a magnitude of 7 and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

The star Proxima Centauri has a greater magnitude than the planet Neptune. The difference in their magnitudes is 3.2.

(c) Find how many times brighter Neptune is compared to Proxima Centauri. [3]

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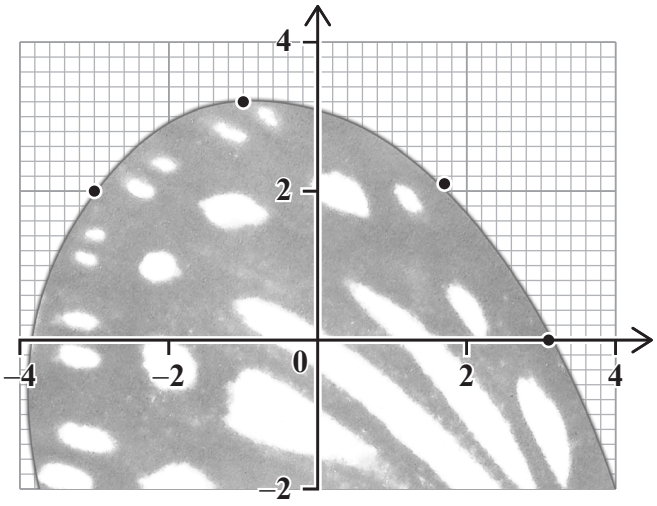
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Question 10 - working space

11. [Maximum mark: 5]

Gloria wants to model the curved edge of a butterfly wing. She inserts a photo of the wing into her graphing software and finds the coordinates of four points on the edge of the wing.



x	y
–3	2
–1	3.2
1.7	2.1
3.1	0

Gloria thinks a cubic curve will be a good model for the butterfly wing.

(a) Find the equation of the cubic regression curve for this data. [2]

For the photo of a second butterfly wing, Gloria finds the equation of the regression curve is $y = 0.0083x^3 - 0.075x^2 - 0.58x + 2.2$.

Gloria realizes that her photo of the second butterfly is an enlargement of the life-size butterfly, scale factor 2 and centred on $(0, 0)$.

(b) Find the equation of the cubic curve that models the life-size wing. [3]

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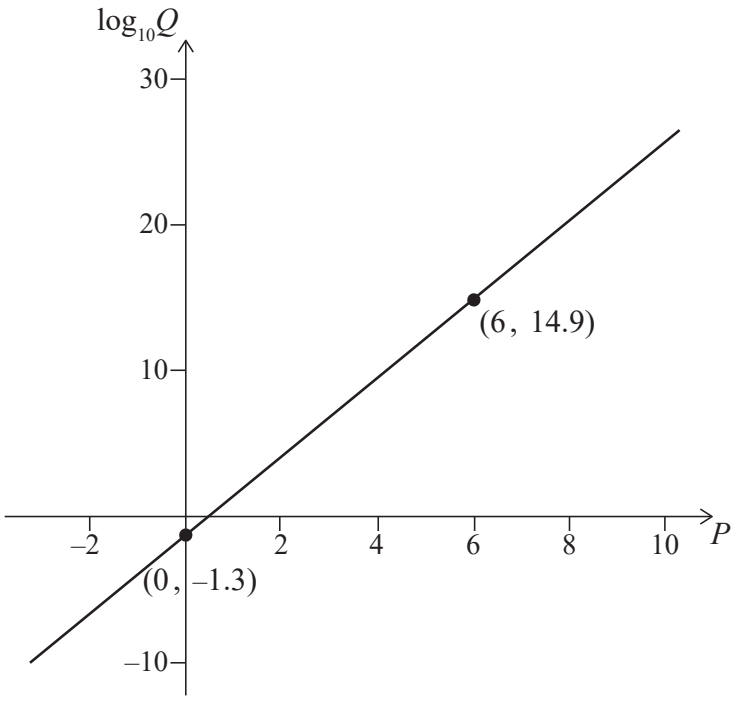
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Question 11 - working space

13. [Maximum mark: 6]

Gen is investigating the relationship between two sets of data, labelled P and Q, that she collected. She created a scatter plot with P on the x-axis and $\log_{10} Q$ on the y-axis. Gen noticed that the points had a strong linear correlation, so she drew a line of best fit, as shown in the diagram. The line passes through the points $(0, -1.3)$ and $(6, 14.9)$.



(a) Find an equation for Q in terms of P. [3]

Gen also investigates the relationship between the same data, Q, and some new data, R. She believes that the data can be modelled by $Q = a \ln R + b$ and she decides to create a scatter plot to verify her belief.

(b) State what expression Gen should plot on each axis to verify her belief. [1]

The scatter plot has a linear relationship and Gen finds $a = 4.3$ and $b = 12.1$.

(c) Find an equation for P in terms of R. [2]

(This question continues on the following page)

(Question 13 continued)

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Question 13 - working space

17. [Maximum mark: 8]

The time of sunrise, R hours after midnight, in Taipei can be modelled by

$$R = 1.08\cos(0.0165t + 0.413) + 4.94,$$

where t is the day of the year 2021 (for example, $t = 2$ represents 2 January 2021).

The time of sunset, S hours after midnight, in Taipei can be modelled by

$$S = 1.15\cos(0.0165t - 2.97) + 18.9.$$

The number of daylight hours, D, in Taipei during 2021 can be modelled by

$$D = a\cos(0.0165t + b) + c.$$

(a) Find the value of a, of b and of c. [6]

(b) Hence, or otherwise, find the largest number of daylight hours in Taipei during 2021 and the day of the year on which this occurs. [2]

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References:

11. Fleur, 2019. *photo-1560263816-d704d83cce0f.* [image online] Available at: https://unsplash.com/photos/SE2zTdS1MNo [Accessed 8 February 2022]. Source adapted.

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Question 17 - working space

6. [Maximum mark: 7]

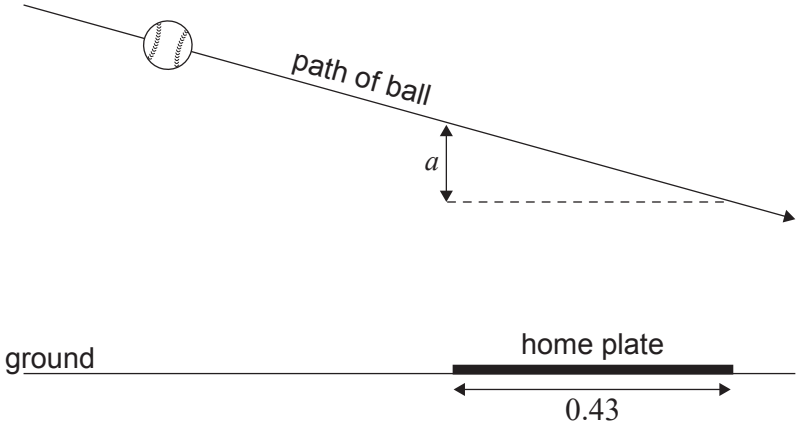
In a baseball game, Sakura is the batter standing beside home plate. The ball is thrown towards home plate along a path that can be modelled by the following function.

$$y = -0.045x + 2$$

In this model, x is the horizontal distance of the ball from the point the ball is thrown and y is the vertical height of the ball above the ground. Both measured in metres.

The outcome of the throw is called a strike if the height of the ball is between $0.53\,\text{m}$ and $1.24\,\text{m}$ at some point while it travels over home plate. The length of home plate is $0.43\,\text{m}$.

diagram not to scale



When the ball reaches the front of home plate, the height of the ball above the ground is $1.25\,\text{m}$. The height of the ball changes by a metres as the ball travels over the length of home plate.

(a) (i) Find the value of a.

(ii) Justify why this throw is a strike. [4]

On the next throw, Sakura hits the ball towards a wall that is 5 metres high. The horizontal distance of the wall from the point where the ball was hit is 96 metres. The path of the ball after it is hit can be modelled by the function $h(d)$.

$$h(d) = -0.01d^2 + 1.04d + 0.66, \text{ for } h,\ d > 0$$

In this model, h is the height of the ball above the ground and d is the horizontal distance of the ball from the point where it was hit. Both h and d are measured in metres.

(b) Determine whether the ball will go over the wall. Justify your answer. [3]

(This question continues on the following page)

(Question 6 continued)

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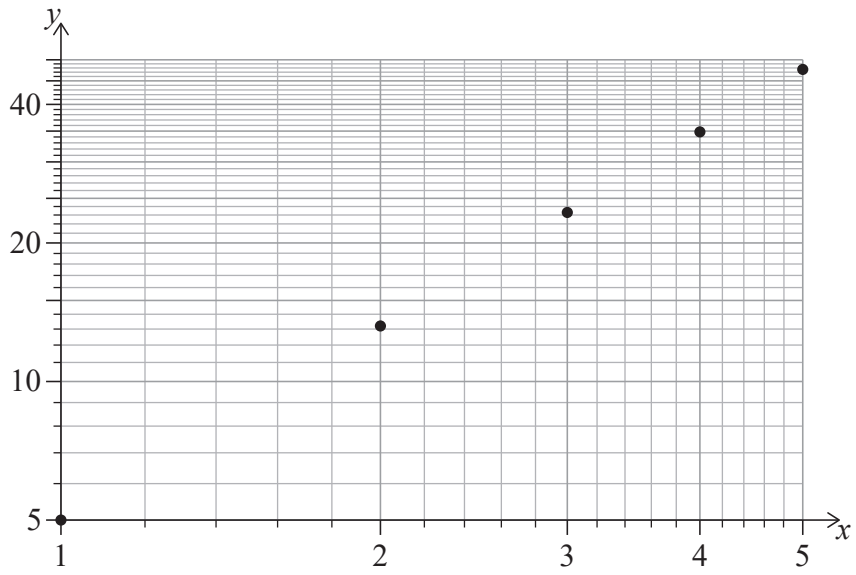
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Question 6 - working space

14. [Maximum mark: 6]

Petra examines two quantities, x and y, and plots data points on a log-log graph.



She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2, 13.1951)$ and $(4, 34.822)$, find the equation of the relationship connecting x and y. Your final answer should not include logarithms.

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Question 14 - working space

1. [Maximum mark: 7]

A player throws a basketball. The height of the basketball is modelled by

$$h(t) = -4.75t^2 + 8.75t + 1.5,\ t \geq 0,$$

where h is the height of the basketball above the ground, in metres, and t is the time, in seconds, after it was thrown.

(a) Find how long it takes for the basketball to reach its maximum height. [2]

(b) Assuming that no player catches the basketball, find how long it would take for the basketball to hit the ground. [2]

Another player catches the basketball when it is at a height of 1.2 metres.

(c) Find the value of t when this player catches the basketball. [2]

(d) Write down one limitation of using $h(t)$ to model the height of the basketball. [1]

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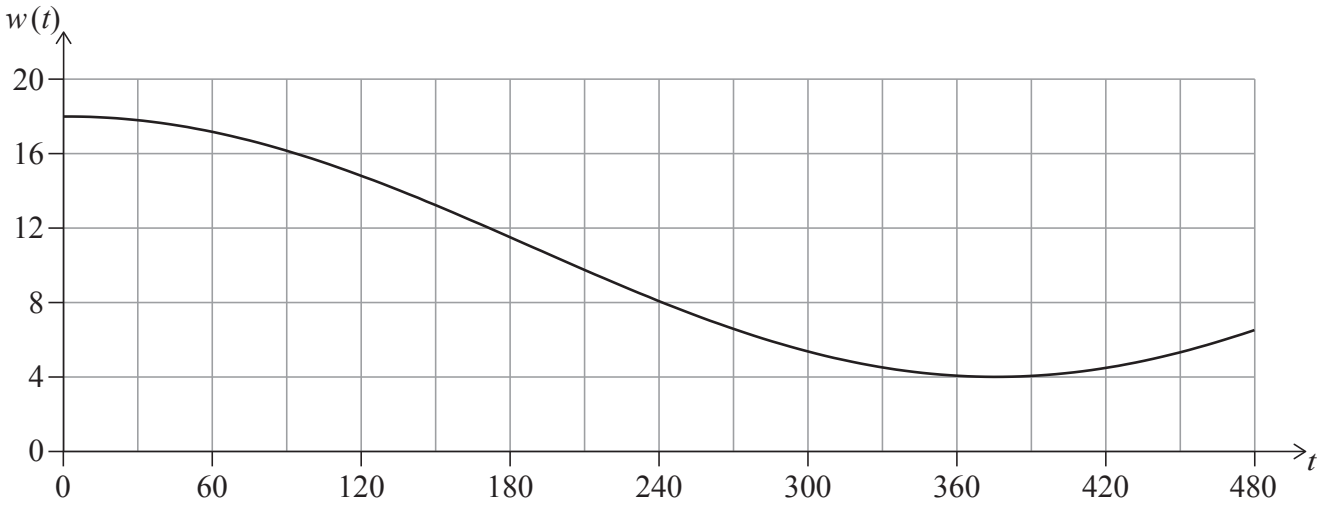
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Question 1 - working space

2. [Maximum mark: 15]

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a\cos(bt^\circ) + d$ where t is the length of time, in minutes, after $06{:}00$.

On 20 January, the first high tide occurs at $06{:}00$, at which time the depth of water is $18\,\text{m}$. The following low tide occurs at $12{:}15$ when the depth of water is $4\,\text{m}$. This is shown in the diagram.



(a) Find the value of a. [2]

(b) Find the value of d. [2]

(c) Find the period of the function in minutes. [3]

(d) Find the value of b. [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least $6\,\text{m}$.

(e) Find the latest time before $12{:}00$, to the nearest minute, that Naomi can enter the harbour. [4]

(f) Find the length of time (in minutes) between $06{:}00$ and $15{:}00$ on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

Question 2 - working space

2. [Maximum mark: 12]

A scientist is conducting an experiment on the growth of a certain species of bacteria.

The population of the bacteria, P, can be modelled by the function

$$P(t) = 1200 \times k^t,\ t \geq 0,$$

where t is the number of hours since the experiment began, and k is a positive constant.

(a) (i) Write down the value of $P(0)$.

(ii) Interpret what this value means in this context. [2]

3 hours after the experiment began, the population of the bacteria is $18\,750$.

(b) Find the value of k. [2]

(c) Find the population of the bacteria 1 hour and 30 minutes after the experiment began. [2]

The scientist conducts a second experiment with a different species of bacteria.
The population of this bacteria, S, can be modelled by the function

$$S(t) = 5000 \times 1.65^t,\ t \geq 0,$$

where t is the number of hours since both experiments began.

(d) Find the value of t when the two populations of bacteria are equal. [2]

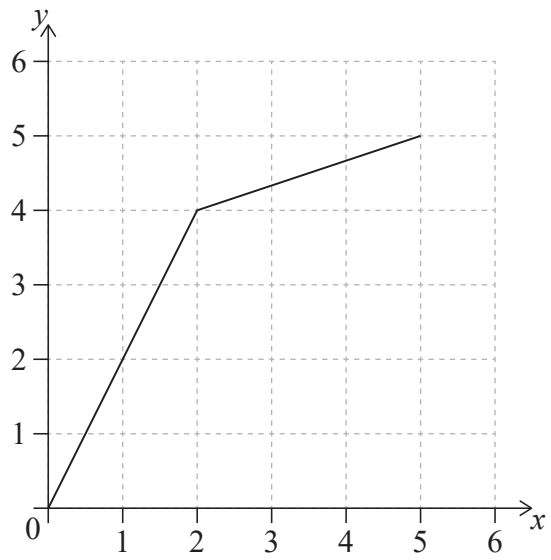
It takes 2 hours and m minutes for the number of bacteria in the second experiment to reach $19\,000$.

(e) Find the value of m, giving your answer as an integer value. [4]

Question 2 - working space

2. [Maximum mark: 7]

The graph of the function f is given in the following diagram.



(a) Write down $f(2)$. [1]

(b) On the axes, sketch $y = f^{-1}(x)$. [2]

The function g is defined as $g(x) = 3x - 1$.

(c) Find an expression for $g^{-1}(x)$. [2]

(d) Find a value of x where $f^{-1}(x) = g^{-1}(x)$. [2]

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Question 2 - working space

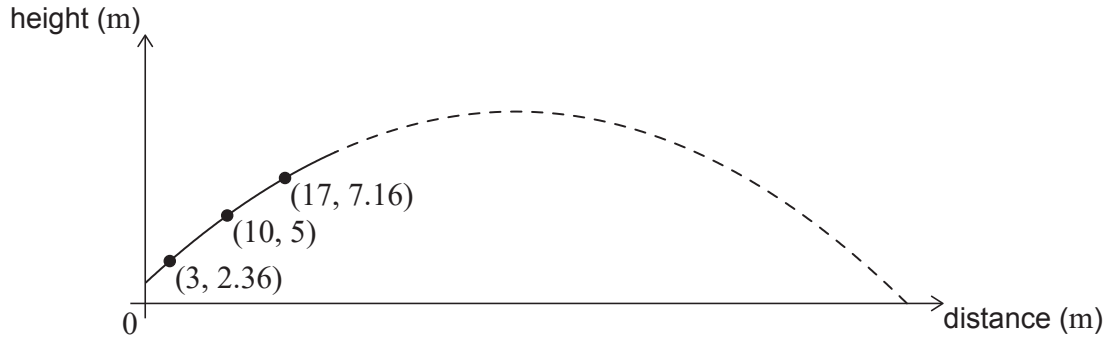
5. [Maximum mark: 9]

A sports player on a horizontal athletic field hits a ball. The height of the ball above the field, in metres, after it is hit can be modelled using a quadratic function of the form $f(x) = ax^2 + bx + c$, where x represents the horizontal distance, in metres, that the ball has travelled from the player.

A specialized camera tracks the initial path of the ball after it is hit by the player. The camera records that the ball travels through the three points $(3, 2.36)$, $(10, 5)$ and $(17, 7.16)$, as shown in **Diagram 1**.

diagram not to scale

Diagram 1



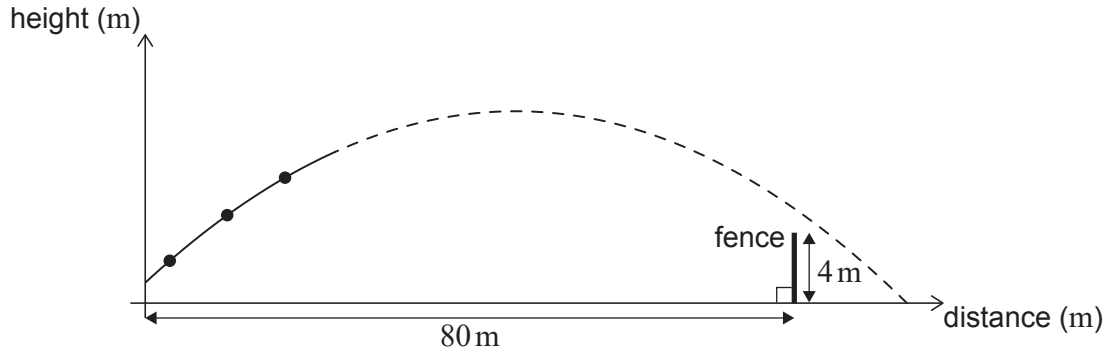
(a) Use the coordinates $(3, 2.36)$ to write down an equation in terms of a, b, and c. [1]

(b) Use your answer to part (a) and two similar equations to find the equation of the quadratic model for the height of the ball. [3]

A 4-metre-high fence is 80 metres from where the player hit the ball, as shown in **Diagram 2**.

diagram not to scale

Diagram 2



(c) Show that the model predicts that the ball will go over the fence. [3]

(d) Find the horizontal distance that the ball will travel, from the player until it first hits the ground, according to this model. [2]

(This question continues on the following page)

(Question 5 continued)

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Question 5 - working space

7. [Maximum mark: 8]

When studying big cats, researchers use a model in which the mass (m kilograms) of an animal is directly proportional to the cube of its shoulder height (h metres).



Cheetah



Lion

A cheetah has a mass of $64\,\text{kg}$ and shoulder height of 0.8 metres.

(a) (i) Use the model to find an expression for m in terms of h.

(ii) Hence find the mass of a different cheetah, with a shoulder height of 0.75 metres. [4]

'Rubner's law' states that the energy needs of an animal (E) are directly proportional to the square of h.

The energy needs of a lion of mass $220\,\text{kg}$ are k times the energy needs of a cheetah of mass $64\,\text{kg}$.

(b) Find the value of k. [4]

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Question 7 - working space

14. [Maximum mark: 4]

Two model cars are driving along the same straight track. Car A leaves point O at $t = 0$ and its displacement (x_A metres) from O at t seconds is given by the equation

$$x_A = 4\sqrt{t}, \text{ for } 0 \leq t \leq 9.$$

A faster car, B, leaves O at $t = 3$.

After starting its motion, the time taken by car B to reach any point on the track is half the time that was taken by A.

(a) Write down the equation for the displacement of B, x_B, in terms of t. [2]

(b) Find the value of t at which B catches up with A. [2]

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Question 14 - working space

3. [Maximum mark: 7]

The pH scale is a measure of the acidity of a solution. Its value is given by the formula

$$\text{pH} = -\log_{10}[\text{H}^+],$$

where $[\text{H}^+]$ is the concentration of hydrogen ions in the solution (measured in moles per litre).

(a) Calculate the pH value if the concentration of hydrogen ions is 0.0003. [2]

The pH of milk is 6.6.

(b) Calculate the concentration of hydrogen ions in milk. [2]

The strength of an acid is measured by its concentration of hydrogen ions.

A lemon has a pH value of 2 and a tomato has a pH value of 4.5.

(c) Calculate how many times stronger the acid in a lemon is when compared to the acid in a tomato. [3]

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Question 3 - working space

2. [Maximum mark: 6]

Radioactive carbon is a material that decays over time.

The mass, $m(t)$ (in nanograms), of radioactive carbon in a fossil of a plant, after t years, can be modelled by the function

$$m(t) = 120\mathrm{e}^{-0.000121t}$$

where t is the time since the plant died.

(a) Write down the initial mass of the radioactive carbon. [1]

(b) Find the mass of the radioactive carbon after $20\,000$ years. [2]

(c) Calculate the smallest number of complete years it takes for more than half the sample to decay. [3]

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Question 2 - working space

6. [Maximum mark: 4]

A museum has an annual membership fee of $\$200$, which includes 10 free visits. Any additional visits are charged at $\$30$ each. The total cost, $\$C$, of n visits during the year can be modelled by

$$C(n) = \begin{cases} 200, & n < p \\ an + b, & n \geq p \end{cases}, \text{ where } a, b, p, n \in \mathbb{Z}.$$

(a) Write down the value of

(i) a

(ii) p. [2]

(b) Find the value of b. [2]

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Question 6 - working space

8. [Maximum mark: 7]

The amount of daylight, L (in hours), in London in 2024 can be modelled by

$$L = a\sin\left(b\left(t - c\right)\right) + d,$$

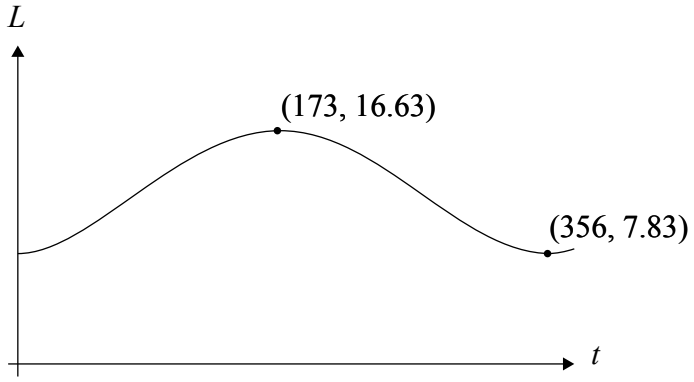
where $a, b, c, d > 0$ and t is the day of the year.

For example, day $1 = 1$ January, day $2 = 2$ January, and so on.

The maximum value of L is 16.63 hours on day 173 (21 June 2024).

The minimum value of L is 7.83 hours on day 356 (21 December 2024).

This information is shown in the following diagram.



Find the value of

(a) d [2]

(b) a [1]

(c) b [2]

(d) c. [2]

(This question continues on the following page)

(Question 8 continued)

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Question 8 - working space

10. [Maximum mark: 8]

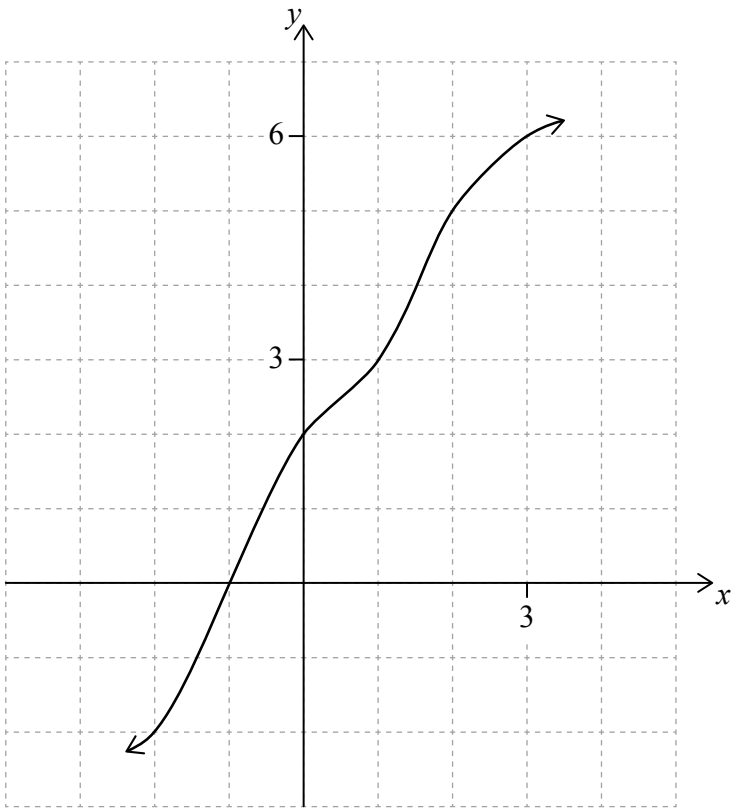
The function f is defined by $f(x) = 4\ln(2x - 3)$, where $x > \frac{3}{2}$.

The graph of $y = f(x)$ is obtained from the graph of $y = \ln x$ by a sequence of three transformations.

(a) Describe fully these transformations, including the order in which they occur. [3]

The graph of $y = g(x)$ is shown.

The graph passes through the points $(-2, -2)$, $(-1, 0)$, $(0, 2)$, $(1, 3)$, $(2, 5)$ and $(3, 6)$.



(b) Find $(f \circ g)(2)$. [2]

(c) Solve $(f \circ g)(x) = 2\ln 9$. [3]

(This question continues on the following page)

(Question 10 continued)

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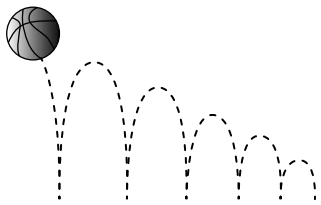
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Question 10 - working space

12. [Maximum mark: 7]

Andreas drops a ball and records a video of the ball bouncing.



He uses the video to find the maximum height, in metres, after each of the first four bounces. His results are shown in the table.

Bounce number, n	Maximum height, h
1	0.613
2	0.514
3	0.439
4	0.377

Andreas thinks the maximum height can be modelled by the function

$$h(n) = 0.613\left(\frac{0.514}{0.613}\right)^{n-1}, \text{ where } n \in \mathbb{Z}^+.$$

(a) Complete the following table. [2]

Bounce number, n	Maximum height, h, according to the model
1	0.613
2	0.514
3	
4	

(b) Hence, calculate the sum of square residuals (SS_{res}). [2]

Andreas' friend thinks a better model for h could be found using an exponential least squares regression curve.

(c) (i) Find the equation of this model.

(ii) Use this model to estimate the height from which the ball was originally dropped. [3]

(This question continues on the following page)

(Question 12 continued)

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Question 12 - working space

13. [Maximum mark: 6]

In 2008, there were no beavers in Scotland, as they had become extinct. In 2009, beavers were introduced from Norway. Surveys were carried out in the region of Tayside to monitor the beaver population.

Figure 1: A beaver



It is believed that the population, P, grows according to a logistic model

$$P = \frac{L}{1 + C\mathrm{e}^{-0.28t}},$$

where t is the number of years after 2008, $t \geq 1$.

A survey in 2017 estimated the population to be 433.

A survey in 2020 estimated the population to be 954.

Find the carrying capacity, L, for the number of beavers in Tayside, according to this model.

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Question 13 - working space

2. [Maximum mark: 7]

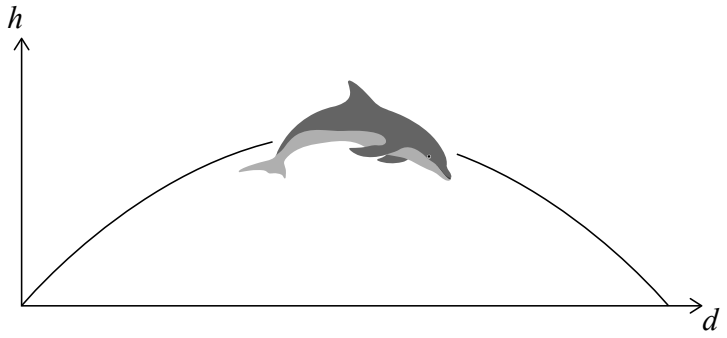
A dolphin jumps out of water. The path of its jump can be modelled by a quadratic function using the following variables:

d is the horizontal distance from the point where the dolphin leaves the water

h is the height of the dolphin above the surface of the water.

All distances are measured in metres, from the point where the dolphin leaves the water.

diagram not to scale



On a particular jump, the dolphin first reaches a height of $5.1\,\text{m}$ above the surface of the water when the horizontal distance is $3\,\text{m}$ and again when the horizontal distance is $8.5\,\text{m}$.

(a) Write down the equation of the axis of symmetry of the quadratic function. [2]

The quadratic function that models the dolphin's jump is $h(d) = -0.2d^2 + bd$, where $\text{d} > 0$ and b is a constant.

(b) Calculate the value of b. [2]

(c) Find the horizontal distance when the dolphin re-enters the water. [2]

(d) Explain what a negative h-value would mean in this context. [1]

(This question continues on the following page)

(Question 2 continued)

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Question 2 - working space

4. [Maximum mark: 7]

A manufacturer designs hats that businesses can order.

For an order of 15 hats or fewer, the price per hat is p euros (EUR). For larger orders, the price of **each** hat after the first 15 ordered is reduced by $5\,\text{EUR}$.

$C(n)$ represents the total cost of purchasing n hats from the manufacturer.

(a) Write down an expression in terms of p for the total cost of ordering 15 hats. [1]

(b) Write down an equation for $C(n)$, in terms of p and n, for the total cost of ordering n hats when $n > 15$. [3]

A company decides to order 100 hats. The mean price per hat for this order is $25.73\,\text{EUR}$.

(c) (i) Calculate the exact total cost of the order.

(ii) Hence, determine the value of p, correct to 2 decimal places. [3]

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Question 4 - working space

6. [Maximum mark: 6]

Walkea Tree is a furniture store. They offer a 30% discount on all chairs priced over 250 Canadian dollars (CAD).

A different furniture store, LuxeCraft, offer customers a reduction of $150\,\text{CAD}$ on all chairs priced over $250\,\text{CAD}$.

(a) Given that x is the original price of a chair and $x > 250$,

(i) write down a function $f(x)$ that models the final price of a chair at Walkea Tree, after the discount is applied.

(ii) write down a function $g(x)$ that models the final price of a chair at LuxeCraft, after the reduction is applied. [2]

A third store, Opulent Furnishings, decides to offer both a 30% discount **and** a reduction of $150\,\text{CAD}$ on all their chairs priced over $250\,\text{CAD}$.

(b) Describe the meaning of $(f \circ g)(x)$, in context. [2]

(c) State, with justification, whether $(f \circ g)(x)$ or $(g \circ f)(x)$ is better for the **customer** when they buy a chair with an original price of $450\,\text{CAD}$. [2]

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Question 6 - working space

10. [Maximum mark: 5]

Consider the following function, $f(x)$, defined on the domain of integers from 0 to 4 inclusive.

x	0	1	2	3	4
$f(x)$	3	1	0	4	2

(a) Solve $f(x) = 4$. [1]

(b) Solve $f(x) = x$. [1]

(c) Complete the following table.

x	0	1	2	3	4
$f^{-1}(x)$					

[3]

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Question 10 - working space

11. [Maximum mark: 6]

A biologist believes that there is a relationship between the possible population size of a group of birds (p thousand) and the population of a colony of wasps (w thousand). Based on her research she believes that the relationship is

$$w = p^3 - 4p^2 + 3p.$$

(a) When $w = 0$, find the possible values of p. [2]

(b) Determine the positive values of w for which there is only one positive value of p. [4]

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Question 11 - working space

10. [Maximum mark: 8]

A satellite orbiting the Earth experiences a gravitational force, $F(\mathrm{N})$, which varies inversely as the square of its distance, d (kilometres), from the centre of the Earth.

When $d = 7000$, $F = 40\,000$.

(a) (i) Find an equation for F in terms of d.

(ii) Find the value of F when d is $3500\,\mathrm{km}$. [4]

(b) (i) Identify which **two** of the following graphs would form a straight line.

A. F against d

B. F against d^2

C. F against d^{-2}

D. $\log_{10} F$ against $\log_{10} d$

E. $\log_{10} F$ against d

F. F against $\log_{10} d$

(ii) For each of these two graphs, find the value of the gradient of the line formed. [4]

(This question continues on the following page)

(Question 10 continued)

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Question 10 - working space

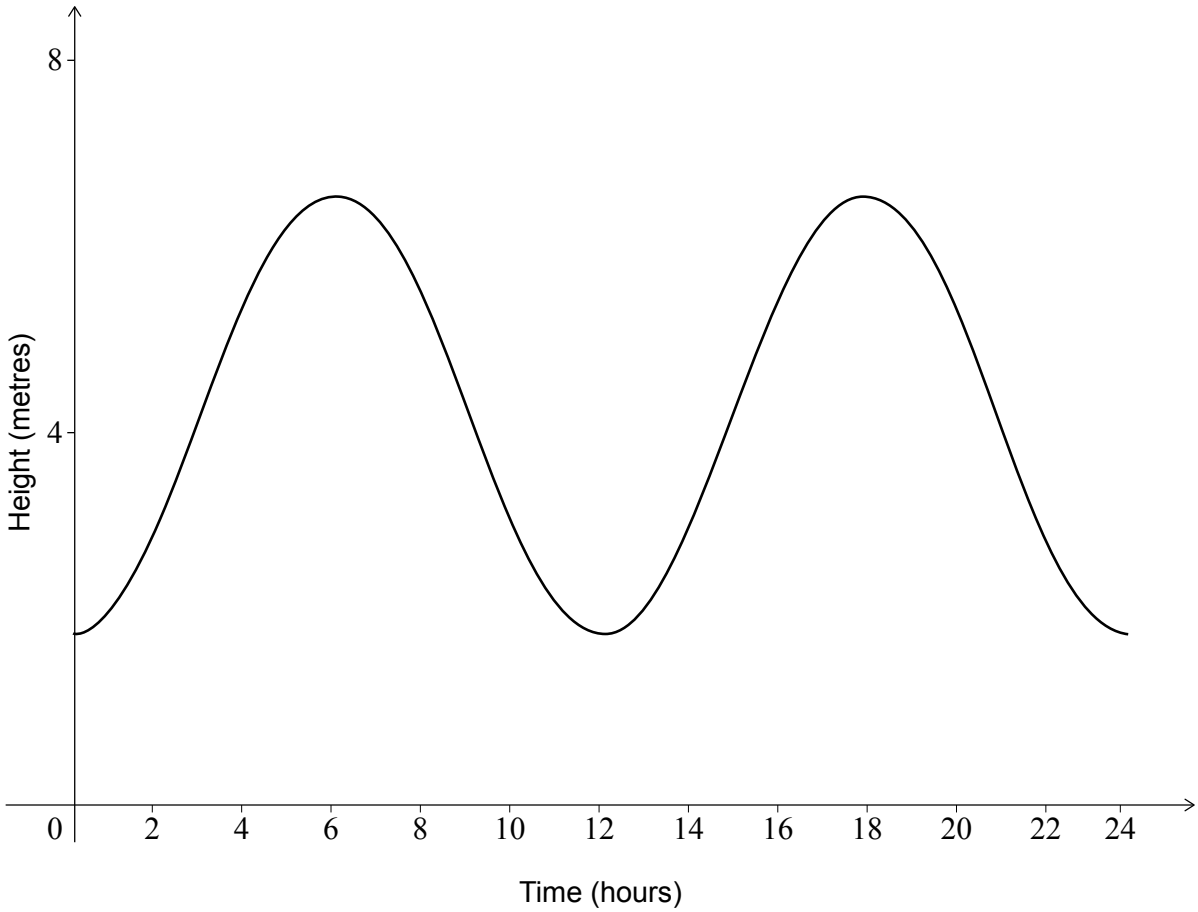
2. [Maximum mark: 15]

On a particular day the height of the tide, h, in metres, at Albion harbour can be modelled by the function

$$h(t) = -2.5\cos(bt°) + 4.5, \text{ where } b \in \mathbb{R}, 0 \leq t \leq 24$$

and t represents the number of hours after midnight.

The graph of h is shown in the following diagram.



(a) Show that the value of b is 30. [1]

(b) Find the height of the tide when $t = 5$. [2]

(c) Write down

(i) the amplitude of h.

(ii) the equation of the principal axis. [3]

(This question continues on the following page)

(Question 2 continued)

Boats can only leave or return to Albion harbour when the height of the tide is greater than $2.65\,\text{m}$.

Robin knows that due to local weather conditions, there could be possible errors in the height of the tide, h, predicted by the model. The actual height may vary by up to $10\,\text{cm}$ above and below the value of h.

Robin wants to leave the harbour to go fishing as soon as possible after the time is 12:00.

(d) Use the above information to determine the earliest time that it could be possible for Robin to leave the harbour. Give your answer to the nearest minute. [4]

The boat will take 15 minutes to travel from the harbour to the fishing site. Robin intends to return to the harbour on the same day. He wants to be certain he will be able to get back into the harbour.

(e) Determine the maximum possible length of time he could spend at the fishing site, in hours, and still be certain he will be able to enter the harbour on his return. [5]

Turn over

Question 2 - working space

2. [Maximum mark: 27]

George is researching the growth in the number of electric vehicles (EVs) in the European Union in order to investigate some of the difficulties that might arise if the target of banning sales of all petrol and diesel cars in 2035 is to be met.

George begins his research by predicting how many electric vehicles (EVs) will be in the European Union in 2035.

The number of EVs in the European Union, N, measured in millions, is shown in the following table. The time t is measured in years from the **beginning** of 2016 where $t \in \mathbb{R}$.

End of year	**2016**	**2017**	**2018**	**2019**	**2020**	**2021**	**2022**
t	1	2	3	4	5	6	7
N	0.2	0.3	0.4	0.6	1.1	1.9	3.1

George models this data set using the logistic function $N = \frac{310}{1 + C\mathrm{e}^{-kt}}$, where $C, k \in \mathbb{R}^+$.

(a) In context, explain the significance of 310 in George's logistic function. [1]

(b) (i) Use the value of N at $t = 1$ to show that $C = 1549\mathrm{e}^{k}$. [2]

(ii) Use the value of N at $t = 7$ to find a second expression for C. [3]

(c) Use your answers to part (b) to find a value for

(i) k. [2]

(ii) C. [1]

George uses his model to predict values of N at the end of the years 2017 to 2021. These values are shown in the following table, correct to one decimal place.

End of year	**2017**	**2018**	**2019**	**2020**	**2021**
t	2	3	4	5	6
N	0.3	0.4	0.6	1.1	1.9
Predicted values	0.3	0.5	a	1.2	2.0

(d) Calculate the value of a correct to one decimal place. [2]

(e) Using the value of a correct to one decimal place, find the sum of square residuals, SS_{res}, when using this model to predict the values of N. [2]

(This question continues on the following page)

(Question 2 continued)

As a measure of how well the model fits the data, George uses the error function, E,

where $E = \sqrt{\frac{SS_{res}}{n}}$, and n is the number of predictions made using the model.

George decides he will use this model if E is less than 0.25.

(f) By finding the value of E for George's model, show that George will decide that the model can be used. [3]

(g) Use the model to predict the number of EVs in the European Union at the end of 2035. [2]

One of the main difficulties in meeting the European Union's target is the provision of sufficient public charging points.

The European Union estimates that 80% of car owners will be able to charge their cars at home and the remaining 20% will require public charging points. It is planned to have one public charging point for every 10 cars that require them.

(h) Use George's model to find an expression for the total number of public charging points required at time t. [2]

At the end of 2020 there were 0.22 million public charging points in the European Union and at the end of 2022 there were 0.54 million.

(i) Find the average number of public charging points installed per year during this period. [1]

George assumes that the number of public charging points will continue to increase linearly, at the same rate, until 2035.

(j) Determine the value of t, where $t \geq 5$, at which the number of charging points will first be insufficient to meet demand, and hence the year in which this occurs. [6]

Question 2 - working space

6. [Maximum mark: 7]

At 12:05 pm, Navam starts draining water from a small reservoir into an empty pond.

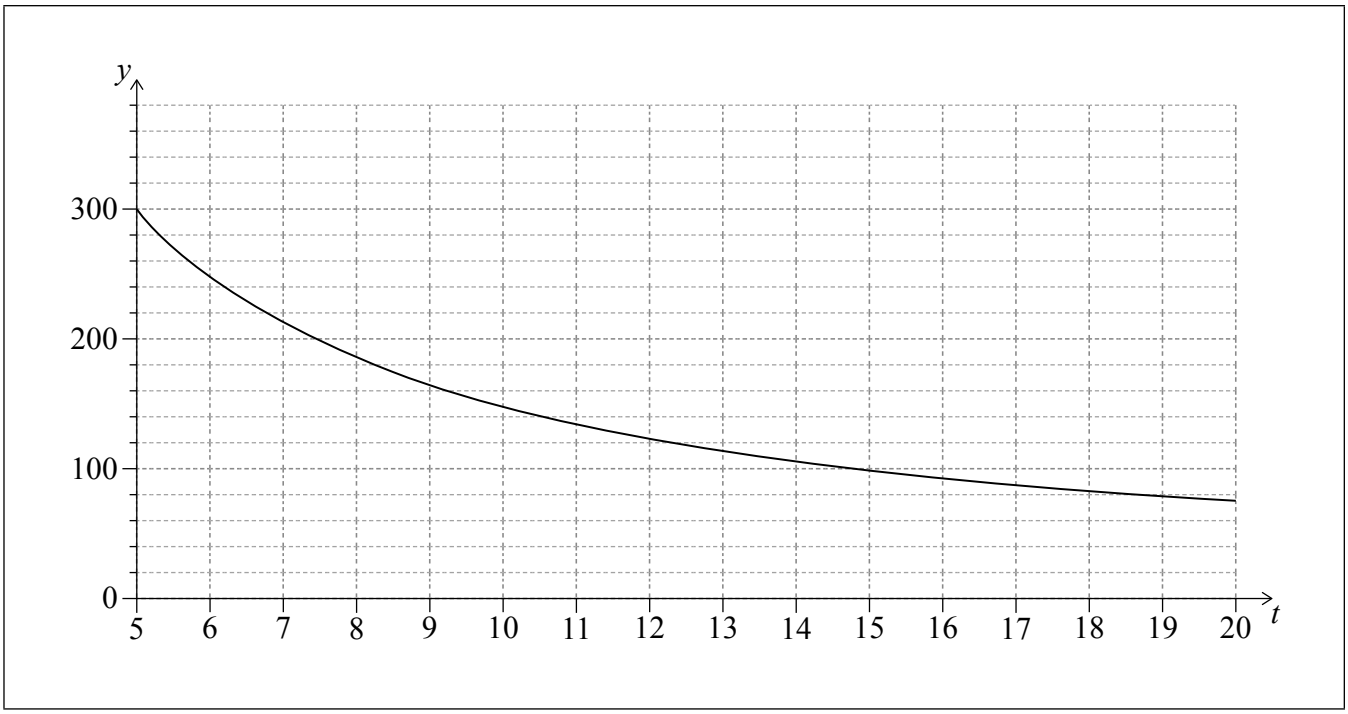
He controls the rate the water is drained so that the volume of water **remaining** in the reservoir, $V\,\text{m}^3$, varies inversely with the time, t, where t is the number of minutes after 12:00 pm.

At 12:05 pm, the volume of water in the reservoir is $300\,\text{m}^3$.

(a) Show that $V = \frac{1500}{t}$. [2]

(b) Find the value of t when the volume of water remaining in the reservoir equals the volume of water in the pond. [2]

The following diagram shows part of the graph of $y = V(t)$.



(c) On the same diagram, sketch the graph of $y = P(t)$, where P is the volume of water in the pond. [2]

(d) Write down an expression for $P(t)$. [1]

(This question continues on the following page)

(Question 6 continued)

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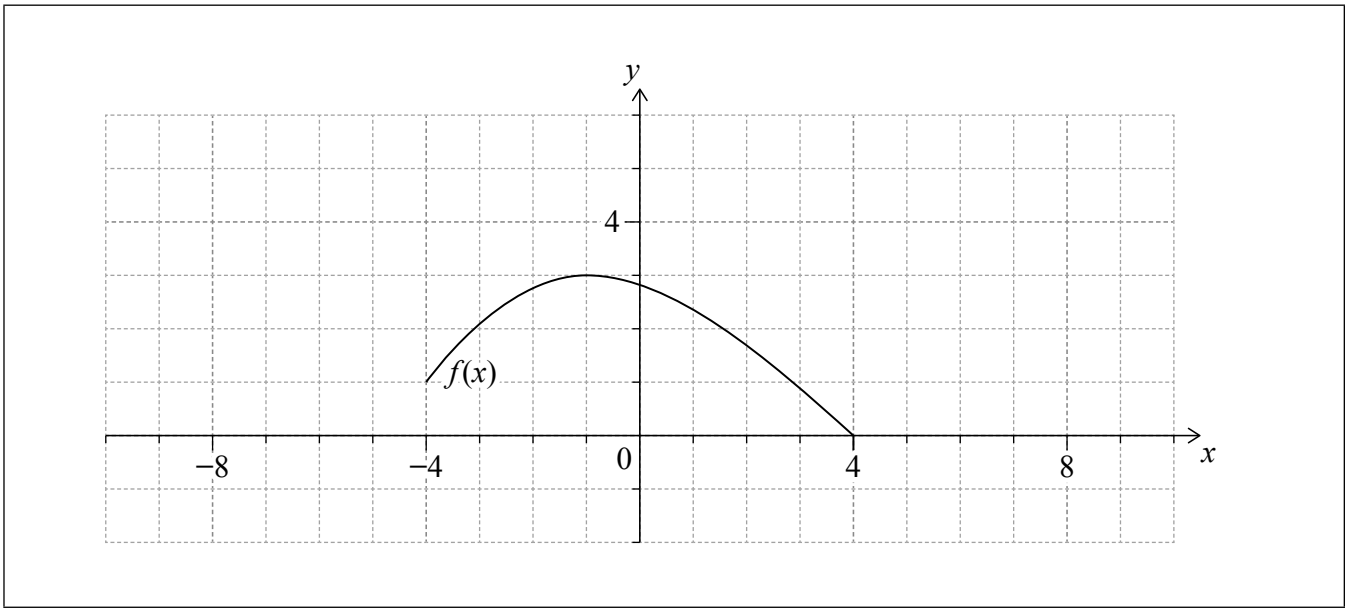
Question 6 - working space

8. [Maximum mark: 4]

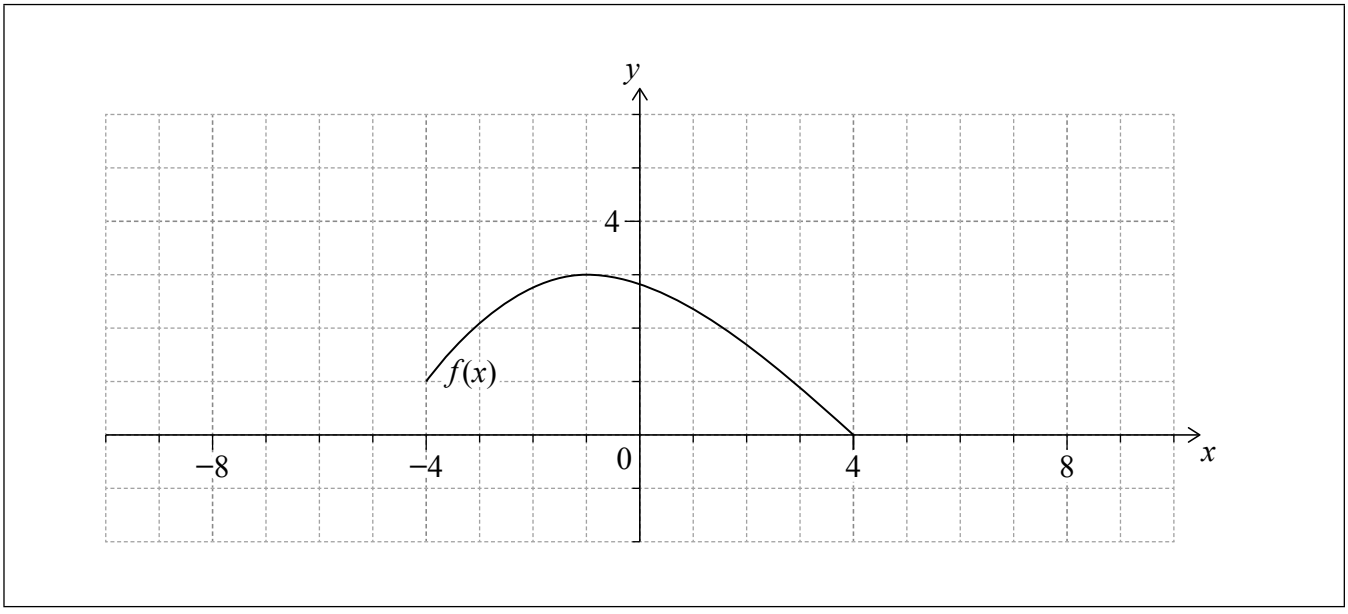
The function f has domain $-4 \leq x \leq 4$.

The graph of $y = f(x)$ has endpoints $(-4, 1)$ and $(4, 0)$, and a local maximum at $(-1, 3)$. This is shown in each of the following diagrams.

(a) On the axes provided, sketch the graph of $y = 2f(x) - 1$. [2]



(b) On the axes provided, sketch the graph of $y = f(2x - 2)$. [2]



Question 8 - working space

3. [Maximum mark: 16]

Jenny designs a flexible suspension bridge, spanning 10 metres over the Swinburne Creek.

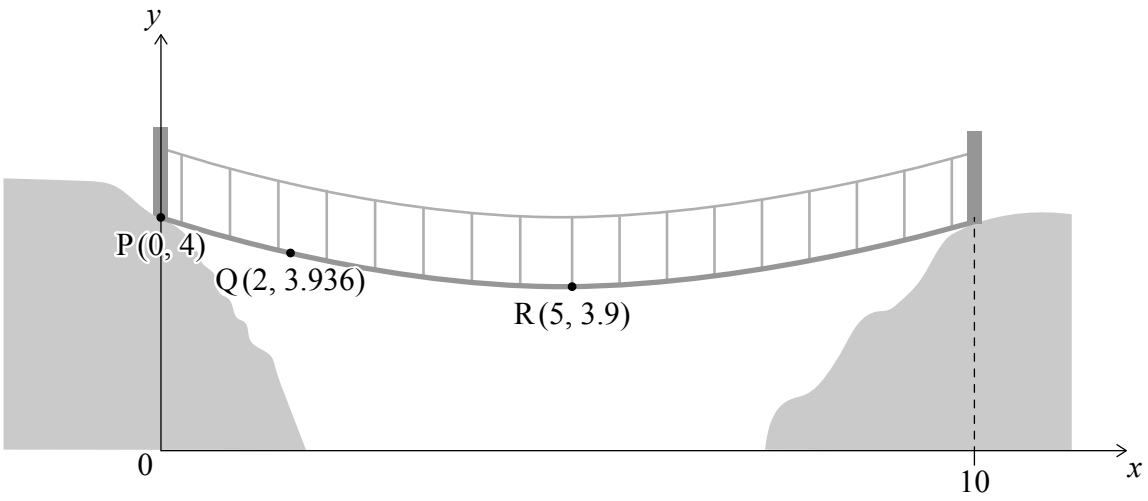
Jenny believes that when no people are on the bridge, the height, y, of the walkway on the bridge above the creek can be modelled by

$$y = ax^2 + bx + c \text{ for } 0 \le x \le 10.$$

She uses a coordinate system where the origin is directly below one side of the bridge, as shown in the following diagram. All distances are measured in metres.

According to this model, the points $\mathrm{P}(0, 4)$, $\mathrm{Q}(2, 3.936)$ and $\mathrm{R}(5, 3.9)$ will be located on the walkway of the bridge. Point R is the lowest point of the bridge.

diagram not to scale



(a) Write down the value of c, the y-intercept of the model. [1]

Using point Q, Jenny finds the equation $-0.064 = 4a + 2b$.

(b) (i) Find another equation in terms of a and b.

(ii) Hence or otherwise, find the value of a and the value of b. [4]

(c) For Jenny's model find

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$;

(ii) the steepest gradient of the walkway of bridge. [4]

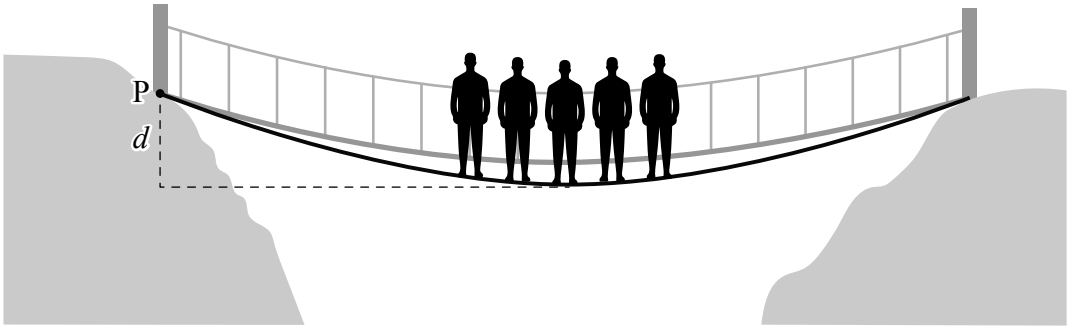
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(Question 3 continued)

The deflection of the bridge, d, is the vertical distance of the walkway below P at the midpoint of the walkway.

After the bridge is built, Jenny explores the relationship between the number of people standing on the bridge and its deflection.

diagram not to scale



Jenny collected data on the number of people, n, on the bridge and the deflection, d metres, as shown in the following table.

Number of people, n	1	2	3	4	5	6
Deflection, d (m)	0.215	0.233	0.250	0.269	0.297	0.305

(d) For this data,

(i) find the value of Pearson's product-moment correlation coefficient, r;

(ii) comment on the strength of the correlation. [3]

The equation of the regression line for this data is $d = 0.01889n + 0.1954$ correct to four significant figures.

(e) Interpret, in context, the meaning of 0.01889 in the equation of the regression line. [1]

(f) Determine whether Jenny's **model** for the height of the walkway or the equation of the regression line gives the smaller estimate of the deflection. Justify your answer. [3]

Question 3 - working space

4. [Maximum mark: 14]

Peter likes to exercise and walks for a maximum of 12 hours each day. The distance he walks in one day, d kilometres, is modelled by

$$d = 3\ln(5t + 1), \text{ for } 0 \leq t \leq 12$$

where t is the time he spends walking, in hours, on that day.

(a) Find the time it takes for Peter to walk $6\,\text{km}$ in one day. [2]

Peter uses a motivational fitness app, and earns points each day for walking. The number of points earned, p, is given by

$$p = 20(1 - e^{-0.1d}), \text{ for } p \in \mathbb{R}.$$

(b) Find the number of points Peter earns for walking $6\,\text{km}$. [1]

(c) Find an expression for p as a function of t, giving your answer in the form

$$p = 20\left(1 - \frac{1}{(5t+1)^n}\right),$$

where n is a number to be determined. [4]

(d) Hence or otherwise, find the number of points Peter earns by walking for 2.5 hours. [1]

(e) Find the greatest number of points Peter can earn in one day. [2]

The fitness app introduces bonus points, b, for the total distance walked in one week, D, using

$$b = 50\left(1 - \frac{1}{D+1}\right).$$

Peter decides to walk for the same length of time, T hours, each day.

He wants to earn a total of 120 points in one week (7 days).

(f) Find the value of T. [4]

Question 4 - working space

APOLLO

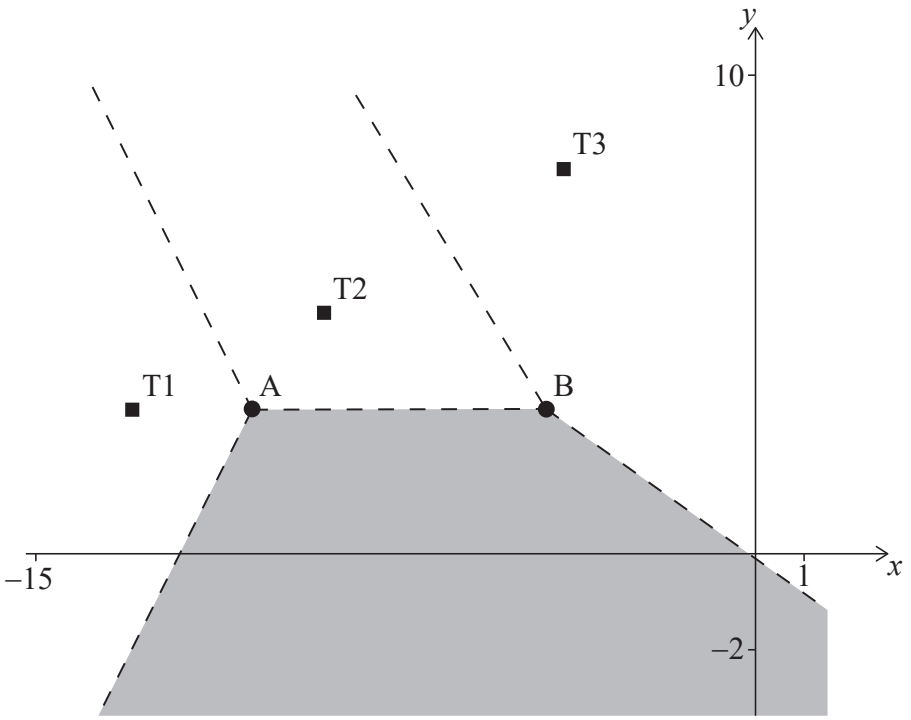
Geometry and trigonometry

101 questions

2. [Maximum mark: 6]

The Voronoi diagram below shows three identical cellular phone towers, T1, T2 and T3. A fourth cellular phone tower, T4 is located in the shaded region. The dashed lines in the diagram below represent the edges in the Voronoi diagram.

Horizontal scale: 1 unit represents 1 km.
Vertical scale: 1 unit represents 1 km.



Tim stands inside the shaded region.

(a) Explain why Tim will receive the strongest signal from tower T4. [1]

Tower T2 has coordinates $(-9, 5)$ and the edge connecting vertices A and B has equation $y = 3$.

(b) Write down the coordinates of tower T4. [2]

Tower T1 has coordinates $(-13, 3)$.

(c) Find the gradient of the edge of the Voronoi diagram between towers T1 and T2. [3]

(This question continues on the following page)

(Question 2 continued)

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Question 2 - working space

5. [Maximum mark: 6]

A garden has a triangular sunshade suspended from three points $\mathrm{A}(2, 0, 2)$, $\mathrm{B}(8, 0, 2)$ and $\mathrm{C}(5, 4, 3)$, relative to an origin in the corner of the garden. All distances are measured in metres.

(a) (i) Find $\overrightarrow{\mathrm{CA}}$.

(ii) Find $\overrightarrow{\mathrm{CB}}$. [2]

(b) Find $\overrightarrow{\mathrm{CA}} \times \overrightarrow{\mathrm{CB}}$. [2]

(c) Hence find the area of the triangle ABC. [2]

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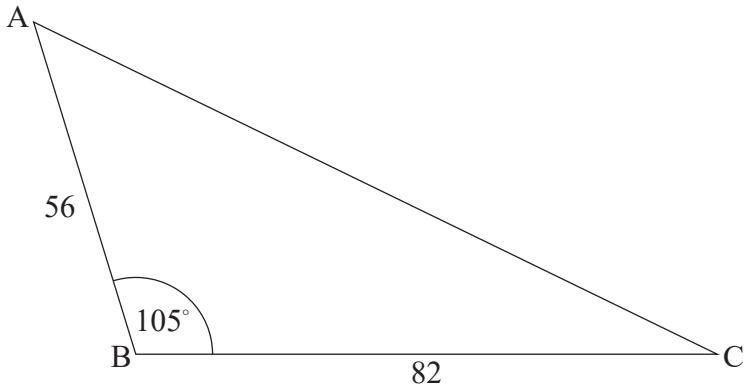
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Question 5 - working space

6. [Maximum mark: 5]

A triangular field ABC is such that $AB = 56\,\text{m}$ and $BC = 82\,\text{m}$, each measured correct to the nearest metre, and the angle at B is equal to 105°, measured correct to the nearest 5°.

diagram not to scale



Calculate the maximum possible area of the field.

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Question 6 - working space

10. [Maximum mark: 7]

An engineer plans to visit six oil rigs (A–F) in the Gulf of Mexico, starting and finishing at A. The travelling time, in minutes, between each of the rigs is shown in the table.

	A	**B**	**C**	**D**	**E**	**F**
A		55	63	79	87	93
B	55		46	58	88	92
C	63	46		87	77	66
D	79	58	87		23	70
E	87	88	77	23		47
F	93	92	66	70	47	

The data above can be represented by a graph G.

(a) (i) Use Prim's algorithm to find the weight of the minimum spanning tree of the subgraph of G obtained by deleting A and starting at B. List the order in which the edges are selected.

(ii) Hence find a lower bound for the travelling time needed to visit all the oil rigs. [6]

(b) Describe how an improved lower bound might be found. [1]

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Question 10 - working space

13. [Maximum mark: 7]

A submarine is located in a sea at coordinates $(0.8, 1.3, -0.3)$ relative to a ship positioned at the origin O. The x direction is due east, the y direction is due north and the z direction is vertically upwards.

All distances are measured in kilometres.

The submarine travels with direction vector $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$.

(a) Assuming the submarine travels in a straight line, write down an equation for the line along which it travels. [2]

The submarine reaches the surface of the sea at the point P.

(b) (i) Find the coordinates of P.

(ii) Find OP. [5]

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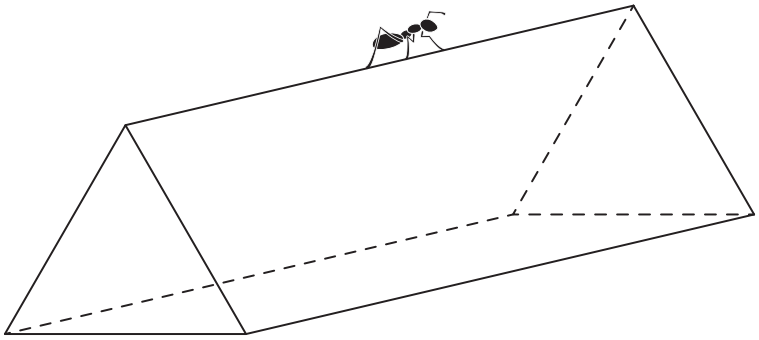
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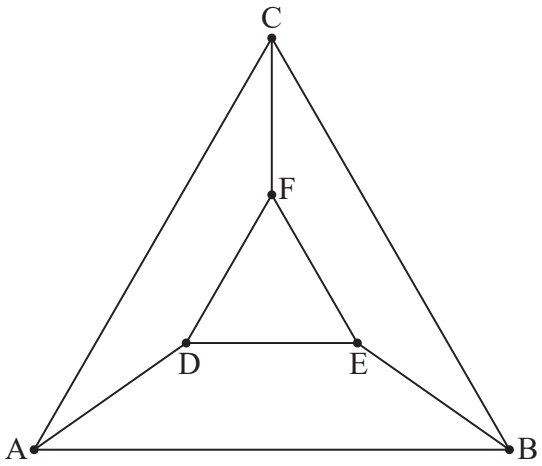
Question 13 - working space

16. [Maximum mark: 5]

An ant is walking along the edges of a wire frame in the shape of a triangular prism.



The vertices and edges of this frame can be represented by the graph below.



(a) Write down the adjacency matrix, $\boldsymbol{M}$, for this graph. [3]

(b) Find the number of ways that the ant can start at the vertex A, and walk along exactly 6 edges to return to A. [2]

(This question continues on the following page)

(Question 16 continued)

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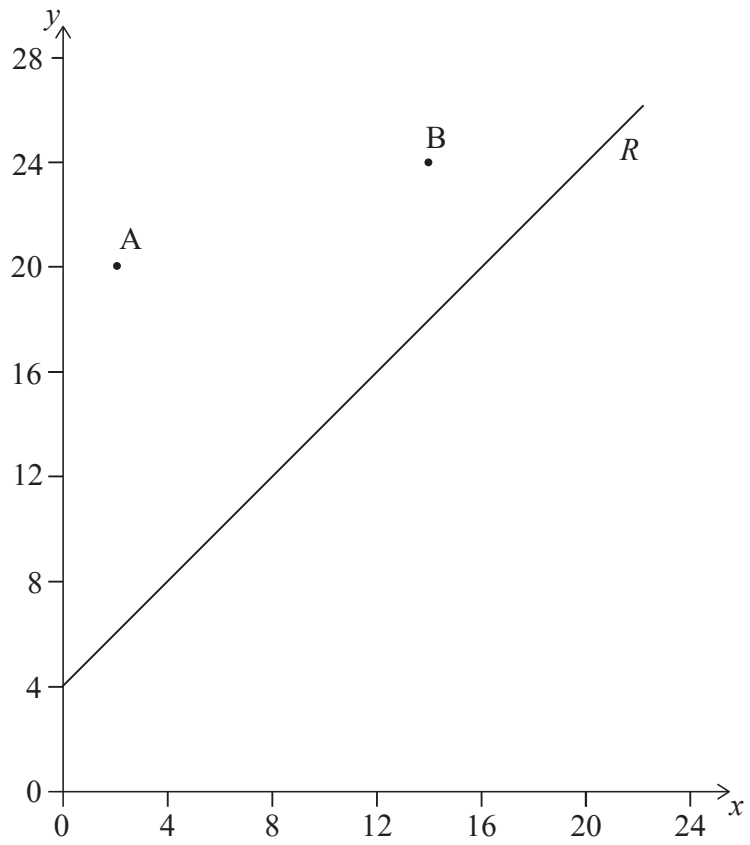
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Question 16 - working space

1. [Maximum mark: 7]

Two schools are represented by points $\mathrm{A}(2,\ 20)$ and $\mathrm{B}(14,\ 24)$ on the graph below. A road, represented by the line R with equation $-x + y = 4$, passes near the schools. An architect is asked to determine the location of a new bus stop on the road such that it is the same distance from the two schools.



(a) Find the equation of the perpendicular bisector of $[\mathrm{AB}]$. Give your equation in the form $y = mx + c$. [5]

(b) Determine the coordinates of the point on R where the bus stop should be located. [2]

(This question continues on the following page)

(Question 1 continued)

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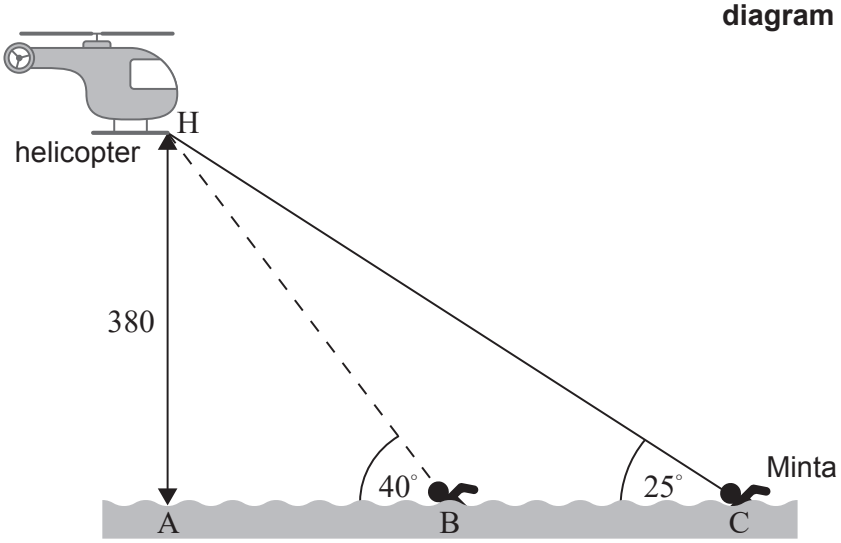
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Question 1 - working space

4. [Maximum mark: 6]

The diagram below shows a helicopter hovering at point H, $380\,\text{m}$ vertically above a lake. Point A is the point on the surface of the lake, directly below the helicopter.

diagram not to scale



Minta is swimming at a constant speed in the direction of point A. Minta observes the helicopter from point C as she looks upward at an angle of 25°. After 15 minutes, Minta is at point B and she observes the same helicopter at an angle of 40°.

(a) Find the distance from A to C. [2]

(b) Find the distance from B to C. [3]

(c) Find Minta's speed, in metres per hour. [1]

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Question 4 - working space

8. [Maximum mark: 7]

Two lines L_1 and L_2 are given by the following equations, where $p \in \mathbb{R}$.

$$L_1 : r = \begin{pmatrix} 2 \\ p+9 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} p \\ 2p \\ 4 \end{pmatrix}$$

$$L_2 : r = \begin{pmatrix} 14 \\ 7 \\ p+12 \end{pmatrix} + \mu \begin{pmatrix} p+4 \\ 4 \\ -7 \end{pmatrix}$$

It is known that L_1 and L_2 are perpendicular.

(a) Find the possible value(s) for p. [3]

(b) In the case that $p < 0$, determine whether the lines intersect. [4]

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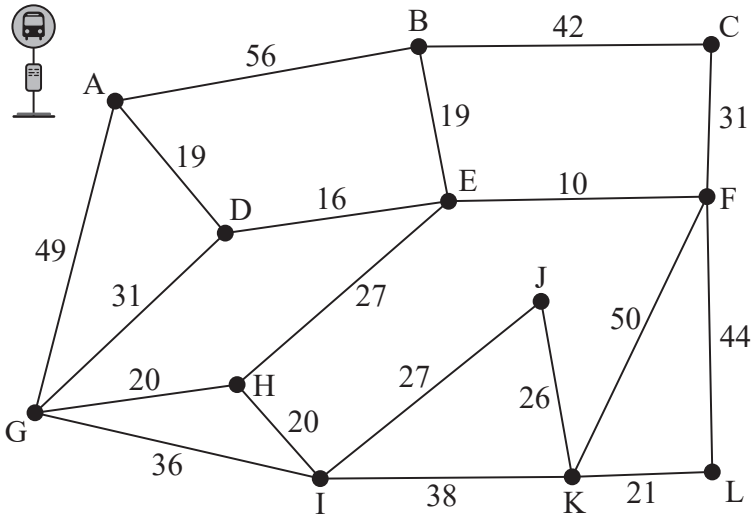
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Question 8 - working space

11. [Maximum mark: 7]

The diagram below shows a network of roads in a small village with the weights indicating the distance of each road, in metres, and junctions indicated with letters.



Musab is required to deliver leaflets to every house on each road. He wishes to minimize his total distance.

(a) Musab starts and finishes from the village bus-stop at A. Determine the total distance Musab will need to walk. [5]

Instead of having to catch the bus to the village, Musab's sister offers to drop him off at any junction and pick him up at any other junction of his choice.

(b) Explain which junctions Musab should choose as his starting and finishing points. [2]

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Question 11 - working space

14. [Maximum mark: 7]

A geometric transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix}$ is defined by

$$T: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

(a) Find the coordinates of the image of the point $(6, -2)$. [2]

(b) Given that $T: \begin{pmatrix} p \\ q \end{pmatrix} \mapsto 2\begin{pmatrix} p \\ q \end{pmatrix}$, find the value of p and the value of q. [3]

(c) A triangle L with vertices lying on the xy plane is transformed by T.

Explain why both L and its image will have exactly the same area. [2]

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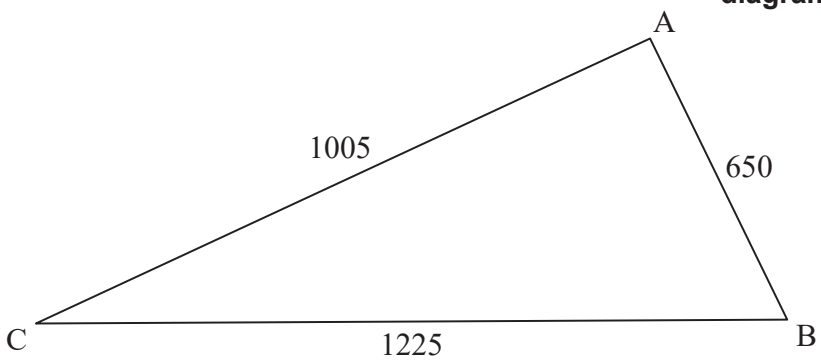
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Question 14 - working space

1. [Maximum mark: 15]

A farmer owns a field in the shape of a triangle ABC such that $AB = 650\,\text{m}$, $AC = 1005\,\text{m}$ and $BC = 1225\,\text{m}$.

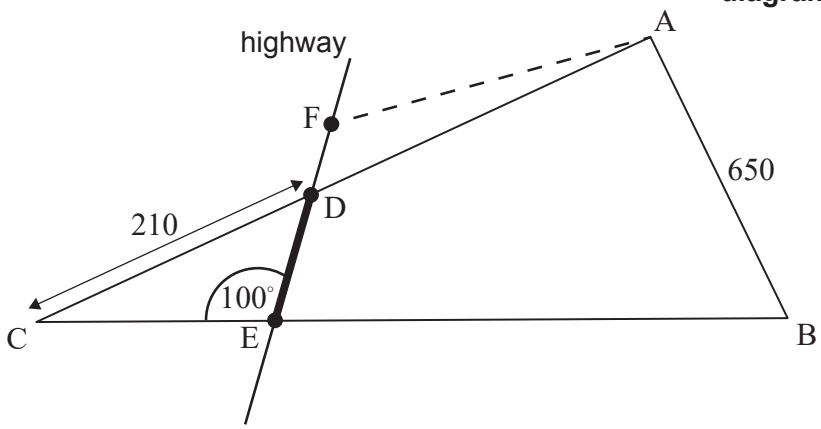
diagram not to scale



(a) Find the size of $A\hat{C}B$. [3]

The local town is planning to build a highway that will intersect the borders of the field at points D and E, where $DC = 210\,\text{m}$ and $C\hat{E}D = 100^\circ$, as shown in the diagram below.

diagram not to scale



(b) Find DE. [3]

The town wishes to build a carpark here. They ask the farmer to exchange the part of the field represented by triangle DCE. In return the farmer will get a triangle of equal area ADF, where F lies on the same line as D and E, as shown in the diagram above.

(c) Find the area of triangle DCE. [5]

(d) Estimate DF. You may assume the highway has a width of zero. [4]

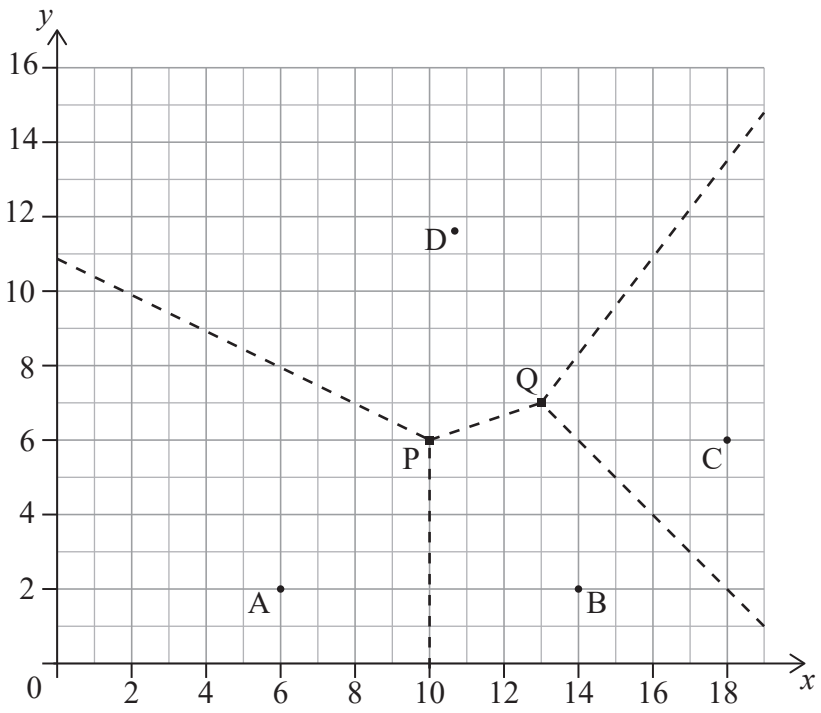
Question 1 - working space

4. [Maximum mark: 6]

There are four stations used by the fire wardens in a national forest.

On the following Voronoi diagram, the coordinates of the stations are $A(6, 2)$, $B(14, 2)$, $C(18, 6)$ and $D(10.8, 11.6)$ where distances are measured in kilometres.

The dotted lines represent the boundaries of the regions patrolled by the fire warden at each station. The boundaries meet at $P(10, 6)$ and $Q(13, 7)$.



To reduce the areas of the regions that the fire wardens patrol, a new station is to be built within the quadrilateral $ABCD$. The new station will be located so that it is as far as possible from the nearest existing station.

(a) Show that the new station should be built at P. [3]

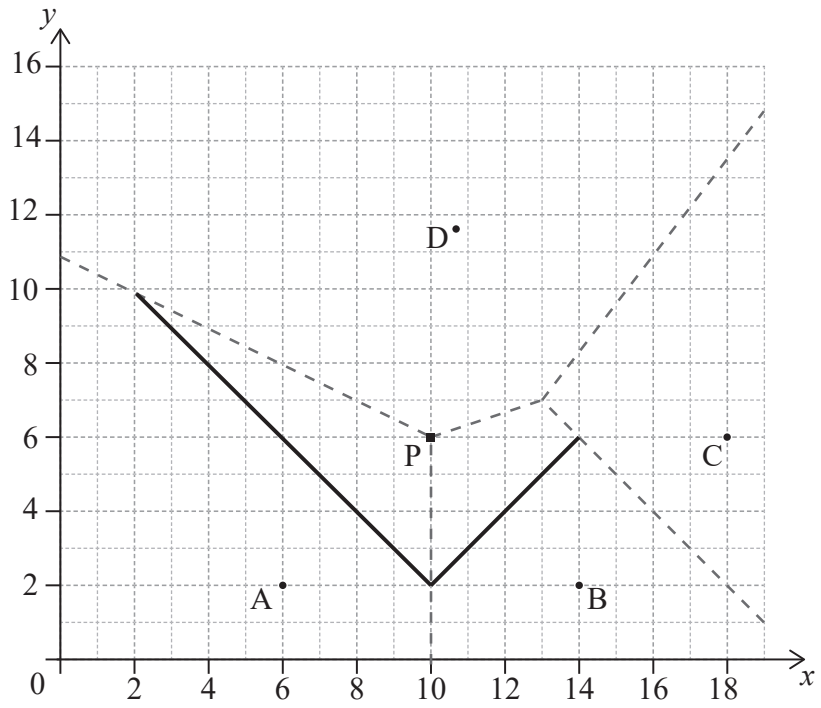
The Voronoi diagram is to be updated to include the region around the new station at P. The edges defined by the perpendicular bisectors of $[AP]$ and $[BP]$ have been added to the following diagram.

(b) (i) Write down the equation of the perpendicular bisector of $[PC]$.

(ii) Hence draw the missing boundaries of the region around P on the following diagram. [3]

(This question continues on the following page)

(Question 4 continued)



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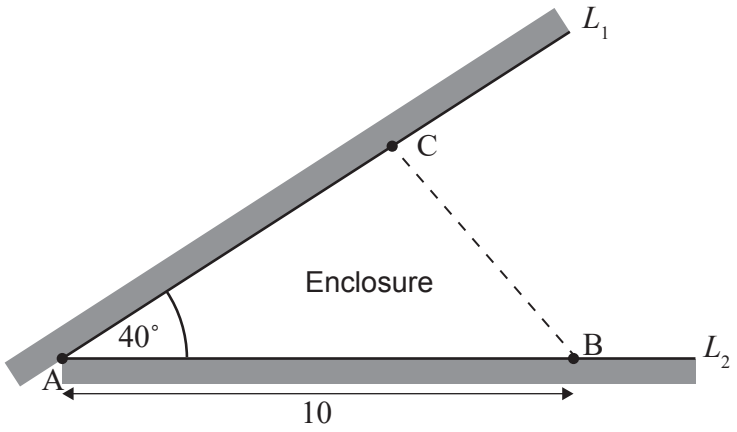
Question 4 - working space

11. [Maximum mark: 6]

The following diagram shows a corner of a field bounded by two walls defined by lines L_1 and L_2. The walls meet at a point A, making an angle of 40°.

Farmer Nate has $7\,\text{m}$ of fencing to make a triangular enclosure for his sheep. One end of the fence is positioned at a point B on L_2, $10\,\text{m}$ from A. The other end of the fence will be positioned at some point C on L_1, as shown on the diagram.

diagram not to scale



He wants the enclosure to take up as little of the current field as possible.

Find the minimum possible area of the triangular enclosure ABC. [6]

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Question 11 - working space

16. [Maximum mark: 9]

A ship S is travelling with a constant velocity, $\boldsymbol{v}$, measured in kilometres per hour, where

$$\boldsymbol{v} = \begin{pmatrix} -12 \\ 15 \end{pmatrix}.$$

At time $t = 0$ the ship is at a point $\mathrm{A}(300,\ 100)$ relative to an origin O, where distances are measured in kilometres.

(a) Find the position vector $\overrightarrow{\mathrm{OS}}$ of the ship at time t hours. [1]

A lighthouse is located at a point $(129,\ 283)$.

(b) Find the value of t when the ship will be closest to the lighthouse. [6]

An alarm will sound if the ship travels within 20 kilometres of the lighthouse.

(c) State whether the alarm will sound. Give a reason for your answer. [2]

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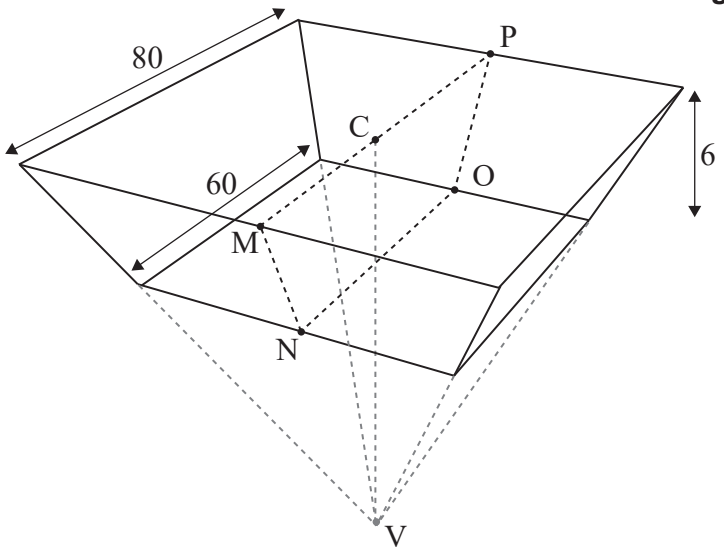
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Question 16 - working space

1. [Maximum mark: 14]

A large water reservoir is built in the form of part of an upside-down right pyramid with a horizontal square base of length 80 metres. The point C is the centre of the square base and point V is the vertex of the pyramid.

diagram not to scale



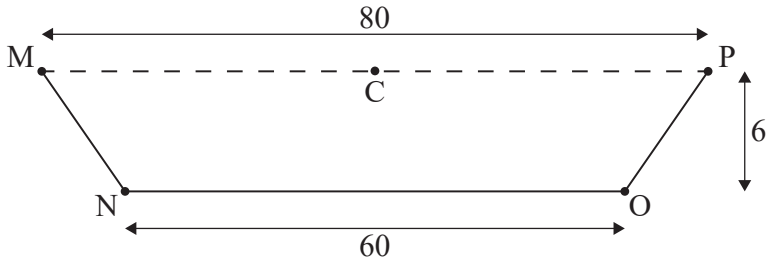
The bottom of the reservoir is a square of length 60 metres that is parallel to the base of the pyramid, such that the depth of the reservoir is 6 metres as shown in the diagram.

(This question continues on the following page)

(Question 1 continued)

The second diagram shows a vertical cross section, MNOPC, of the reservoir.

diagram not to scale



(a) Find the angle of depression from M to N. [2]

(b) (i) Find CV.

(ii) Hence or otherwise, show that the volume of the reservoir is $29\,600\,\text{m}^3$. [5]

Every day $80\,\text{m}^3$ of water from the reservoir is used for irrigation.

Joshua states that, if no other water enters or leaves the reservoir, then when it is full there is enough irrigation water for at least one year.

(c) By finding an appropriate value, determine whether Joshua is correct. [2]

To avoid water leaking into the ground, the five interior sides of the reservoir have been painted with a watertight material.

(d) Find the area that was painted. [5]

Question 1 - working space

4. [Maximum mark: 18]

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x-y-axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$.
- a rotation clockwise of $\frac{\pi}{3}$ radians about O.

All the movements are performed in the listed order.

(a) (i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation.

(ii) Find a single matrix $\boldsymbol{P}$ that defines a transformation that represents the overall change in position.

(iii) Find $\boldsymbol{P}^2$.

(iv) Hence state what the value of $\boldsymbol{P}^2$ indicates for the possible movement of the drone. [12]

(b) Three drones are initially positioned at the points A, B and C. After performing the movements listed above, the drones are positioned at points A', B' and C' respectively.

Show that the area of triangle ABC is equal to the area of triangle $\mathrm{A'B'C'}$. [2]

(c) Find a single transformation that is equivalent to the three transformations represented by matrix $\boldsymbol{P}$. [4]

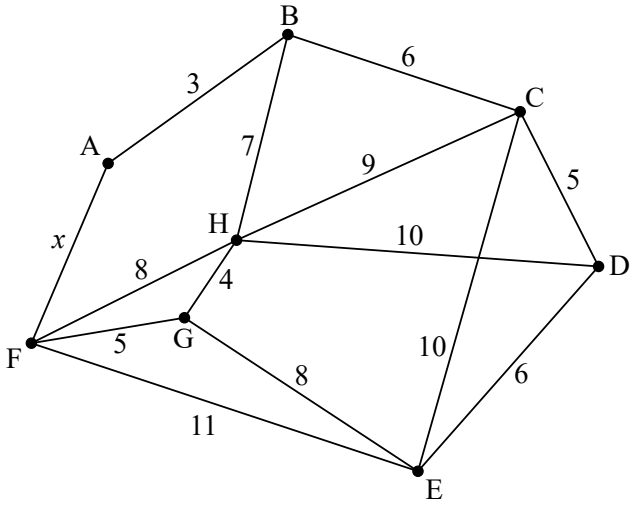
Turn over

Question 4 - working space

1. [Maximum mark: 25]

This question explores how graph algorithms can be applied to a graph with an unknown edge weight.

Graph W is shown in the following diagram. The vertices of W represent tourist attractions in a city. The weight of each edge represents the travel time, to the nearest minute, between two attractions. The route between A and F is currently being resurfaced and this has led to a variable travel time. For this reason, AF has an unknown travel time x minutes, where $x \in \mathbb{Z}^+$.



(a) Write down a Hamiltonian cycle in W. [1]

Daniel plans to visit all the attractions, starting and finishing at A. He wants to minimize his travel time.

To find a lower bound for Daniel's travel time, vertex A and its adjacent edges are first deleted.

(b) (i) Use Prim's algorithm, starting at vertex B, to find the weight of the minimum spanning tree of the remaining graph. You should indicate clearly the order in which the algorithm selects each edge. [5]

(ii) Hence, for the case where $x < 9$, find a lower bound for Daniel's travel time, in terms of x. [2]

(This question continues on the following page)

(Question 1 continued)

Daniel makes a table to show the minimum travel time between each pair of attractions.

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
A		3	9	14	19 **or** $(11+x)$	18 **or** x	14 **or** $(5+x)$	10 **or** $(8+x)$
B			6	11	16 **or** $(14+x)$	15 **or** $(3+x)$	11 **or** $(8+x)$	7
C				5	10	17 **or** $(9+x)$	p	9
D					6	q **or** $(r+x)$	14	10
E						11	8	12
F							5	8
G								4
H								

(c) Write down the value of

(i) p; [1]

(ii) q; [1]

(iii) r. [1]

To find an upper bound for Daniel's travel time, the nearest neighbour algorithm is used, starting at vertex A.

(d) Consider the case where $x = 3$.

(i) Use the nearest neighbour algorithm to find two possible cycles. [3]

(ii) Find the best upper bound for Daniel's travel time. [2]

(e) Consider the case where $x > 3$.

(i) Find the least value of x for which the edge AF will definitely not be used by Daniel. [2]

(ii) Hence state the value of the upper bound for Daniel's travel time for the value of x found in part (e)(i). [2]

The tourist office in the city has received complaints about the lack of cleanliness of some routes between the attractions. Corinne, the office manager, decides to inspect all the routes between all the attractions, starting and finishing at H. The sum of the weights of all the edges in graph W is $(92+x)$.

Corinne inspects all the routes as quickly as possible and takes 2 hours.

(f) Find the value of x during Corinne's inspection. [5]

Turn over

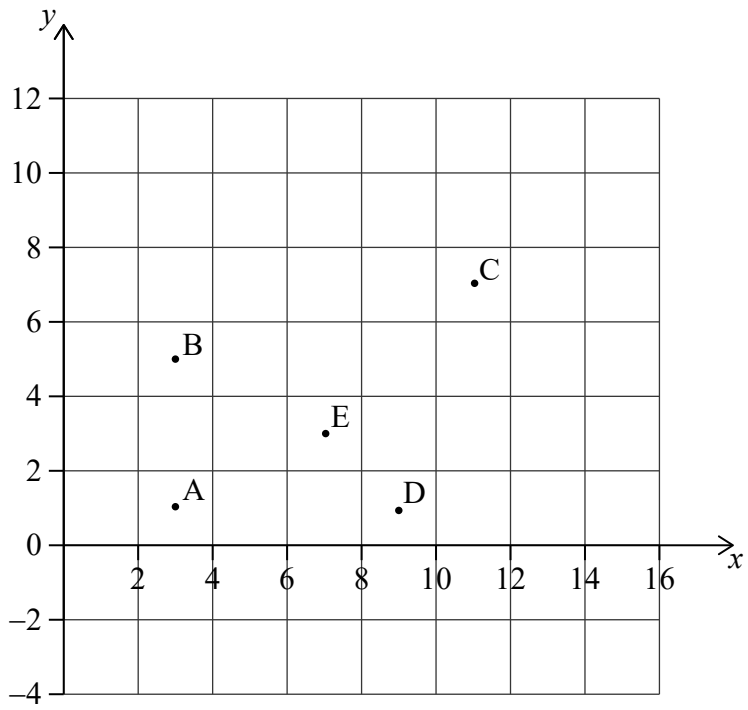
Question 1 - working space

4. [Maximum mark: 6]

Points $A(3, 1)$, $B(3, 5)$, $C(11, 7)$, $D(9, 1)$ and $E(7, 3)$ represent snow shelters in the Blackburn National Forest. These snow shelters are illustrated in the following coordinate axes.

Horizontal scale: 1 unit represents $1\,\text{km}$.

Vertical scale: 1 unit represents $1\,\text{km}$.

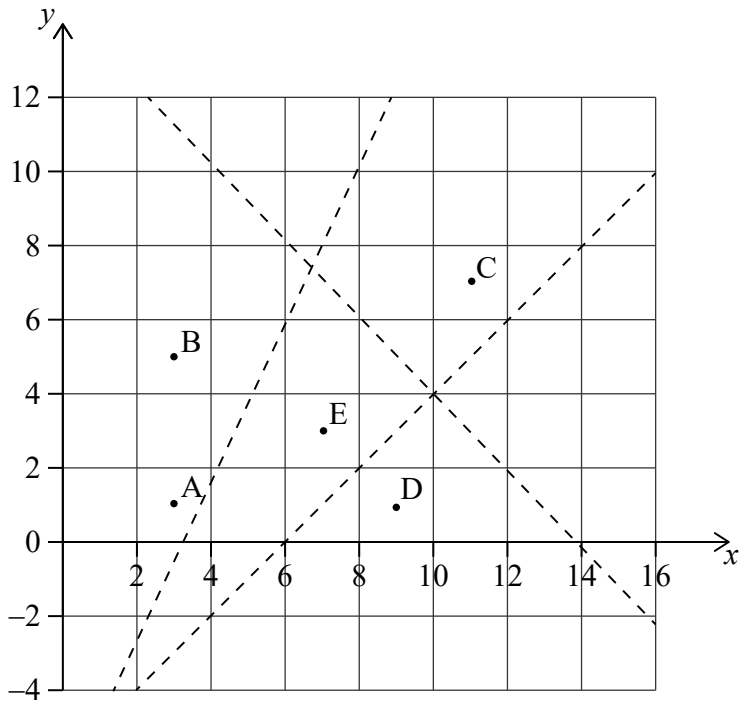


(a) Calculate the gradient of the line segment AE. [2]

(This question continues on the following page)

(Question 4 continued)

The Park Ranger draws three straight lines to form an incomplete Voronoi diagram.



(b) Find the equation of the line which would complete the Voronoi cell containing site E. Give your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$. [3]

(c) In the context of the question, explain the significance of the Voronoi cell containing site E. [1]

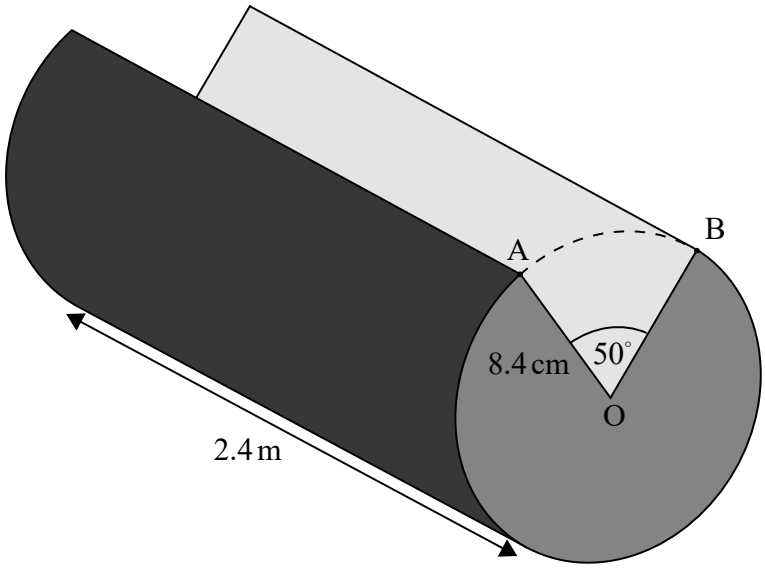
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Question 4 - working space

5. [Maximum mark: 5]

Helen is building a cabin using cylindrical logs of length $2.4\,\text{m}$ and radius $8.4\,\text{cm}$. A wedge is cut from one log and the cross-section of this log is illustrated in the following diagram.

diagram not to scale



(a) Find 50° in radians. [1]

(b) Find the volume of this log. [4]

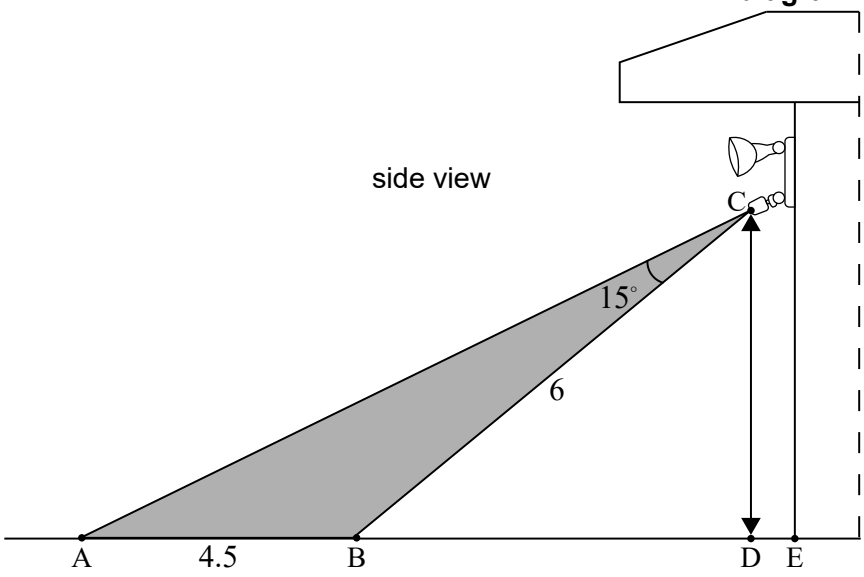
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Question 5 - working space

8. [Maximum mark: 8]

Ollie has installed security lights on the side of his house that are activated by a sensor. The sensor is located at point C directly above point D. The area covered by the sensor is shown by the shaded region enclosed by triangle ABC. The distance from A to B is $4.5\,\text{m}$ and the distance from B to C is $6\,\text{m}$. Angle $A\hat{C}B$ is 15°.

diagram not to scale



(a) Find $C\hat{A}B$. [3]

Point B on the ground is $5\,\text{m}$ from point E at the entrance to Ollie's house. He is $1.8\,\text{m}$ tall and is standing at point D, below the sensor. He walks towards point B.

(b) Find the distance Ollie is **from the entrance to his house** when he first activates the sensor. [5]

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Turn over

Question 8 - working space

11. [Maximum mark: 6]

A particle P moves with velocity $\boldsymbol{v} = \begin{pmatrix} -15 \\ 2 \\ 4 \end{pmatrix}$ in a magnetic field, $\boldsymbol{B} = \begin{pmatrix} 0 \\ d \\ 1 \end{pmatrix}, d \in \mathbb{R}$.

(a) Given that $\boldsymbol{v}$ is perpendicular to $\boldsymbol{B}$, find the value of d. [2]

The force, $\boldsymbol{F}$, produced by P moving in the magnetic field is given by the vector equation $\boldsymbol{F} = a\boldsymbol{v} \times \boldsymbol{B}$, $a \in \mathbb{R}^+$.

(b) Given that $\left|\boldsymbol{F}\right| = 14$, find the value of a. [4]

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Question 11 - working space

4. [Maximum mark: 14]

An aircraft's position is given by the coordinates (x, y, z), where x and y are the aircraft's displacement east and north of an airport, and z is the height of the aircraft above the ground. All displacements are given in kilometres.

The velocity of the aircraft is given as $\begin{pmatrix} -150 \\ -50 \\ -20 \end{pmatrix} \text{km h}^{-1}$.

At $13{:}00$ it is detected at a position $30\,\text{km}$ east and $10\,\text{km}$ north of the airport, and at a height of $5\,\text{km}$. Let t be the length of time in hours from $13{:}00$.

(a) Write down a vector equation for the displacement, $\boldsymbol{r}$ of the aircraft in terms of t. [2]

(b) If the aircraft continued to fly with the velocity given

(i) verify that it would pass directly over the airport;

(ii) state the height of the aircraft at this point;

(iii) find the time at which it would fly directly over the airport. [4]

When the aircraft is $4\,\text{km}$ above the ground it continues to fly on the same bearing but adjusts the angle of its descent so that it will land at the point $(0, 0, 0)$.

(c) (i) Find the time at which the aircraft is $4\,\text{km}$ above the ground.

(ii) Find the direct distance of the aircraft from the airport at this point. [5]

(d) Given that the velocity of the aircraft, after the adjustment of the angle of descent, is $\begin{pmatrix} -150 \\ -50 \\ a \end{pmatrix} \text{km h}^{-1}$, find the value of a. [3]

Turn over

Question 4 - working space

5. [Maximum mark: 17]

The following table shows the costs in US dollars (US$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

(a) Show the direct flights between the cities as a graph. [2]

(b) Write down the adjacency matrix for this graph. [2]

(c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]

(d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

The following table shows the least cost to travel between the cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

(e) Find the values of a and b. [2]

A travelling salesman has to visit each of the cities, starting and finishing at city A.

(f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]

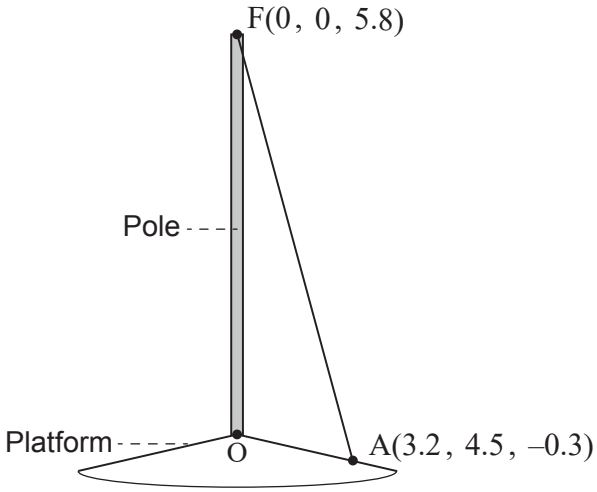
(g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

Question 5 - working space

6. [Maximum mark: 8]

A vertical pole stands on a sloped platform. The bottom of the pole is used as the origin, O, of a coordinate system in which the top, F, of the pole has coordinates $(0, 0, 5.8)$. All units are in metres.

diagram not to scale



The pole is held in place by ropes attached at F.

One of these ropes is attached to the platform at point $A(3.2, 4.5, -0.3)$. The rope forms a straight line from A to F.

(a) Find $\vec{AF}$. [1]

(b) Find the length of the rope. [2]

(c) Find $F\hat{A}O$, the angle the rope makes with the platform. [5]

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Question 6 - working space

13. [Maximum mark: 5]

At $1{:}00\ \text{pm}$ a ship is $1\,\text{km}$ east and $4\,\text{km}$ north of a harbour. A coordinate system is defined with the harbour at the origin. The position vector of the ship at $1{:}00\ \text{pm}$ is given by $\begin{pmatrix}1\\4\end{pmatrix}$.

The ship has a constant velocity of $\begin{pmatrix}1.2\\-0.6\end{pmatrix}$ kilometres per hour (km h^{-1}).

(a) Write down an expression for the position vector $\boldsymbol{r}$ of the ship, t hours after $1{:}00\ \text{pm}$. [1]

(b) Find the time at which the bearing of the ship from the harbour is 045°. [4]

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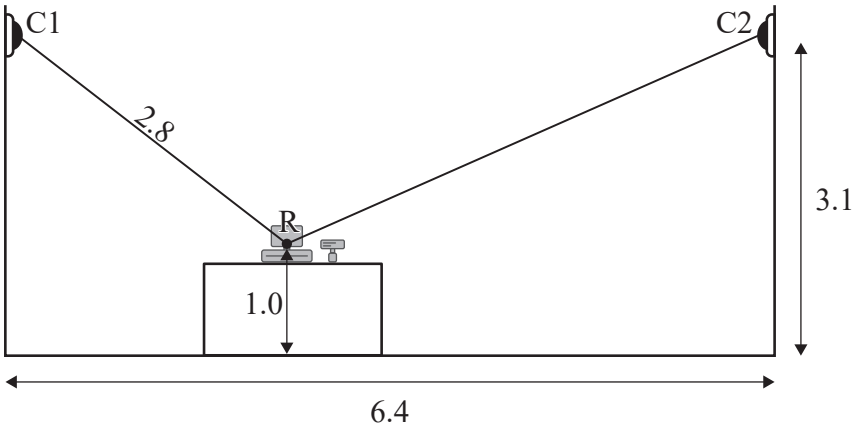
Question 13 - working space

2. [Maximum mark: 8]

The owner of a convenience store installs two security cameras, represented by points $C1$ and $C2$. Both cameras point towards the centre of the store's cash register, represented by the point R.

The following diagram shows this information on a cross-section of the store.

diagram not to scale



The cameras are positioned at a height of $3.1\,\text{m}$, and the horizontal distance between the cameras is $6.4\,\text{m}$. The cash register is sitting on a counter so that its centre, R, is $1.0\,\text{m}$ above the floor.

The distance from Camera 1 to the centre of the cash register is $2.8\,\text{m}$.

(a) Determine the angle of depression from Camera 1 to the centre of the cash register. Give your answer in degrees. [2]

(b) Calculate the distance from Camera 2 to the centre of the cash register. [4]

(c) Without further calculation, determine which camera has the largest angle of depression to the centre of the cash register. Justify your response. [2]

(This question continues on the following page)

(Question 2 continued)

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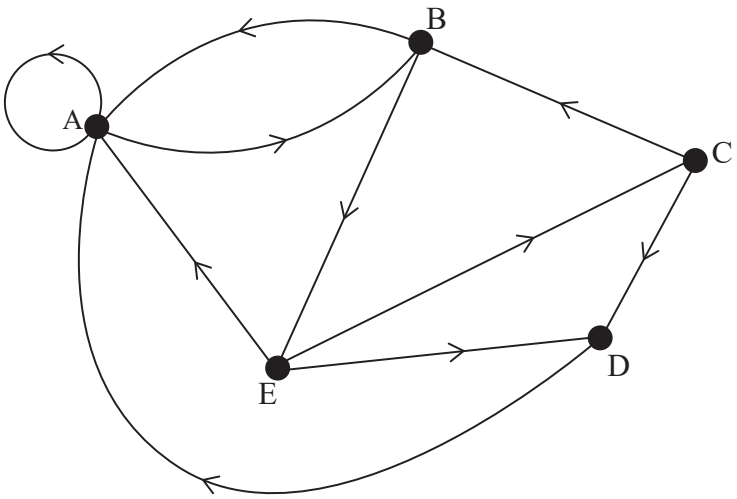
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Question 2 - working space

6. [Maximum mark: 5]

Consider the following directed network.



(a) Write down the adjacency matrix for this network. [2]

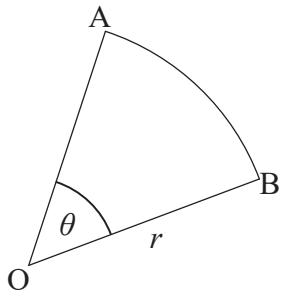
(b) Determine the number of different walks of length 5 that start and end at the same vertex. [3]

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Question 6 - working space

8. [Maximum mark: 8]

The diagram shows a sector, OAB, of a circle with centre O and radius r, such that $\text{A}\hat{\text{O}}\text{B} = \theta$.



Sam measured the value of r to be $2\,\text{cm}$ and the value of θ to be 30°.

(a) Use Sam's measurements to calculate the area of the sector. Give your answer to four significant figures. [2]

It is found that Sam's measurements are accurate to only one significant figure.

(b) Find the upper bound and lower bound of the area of the sector. [3]

(c) Find, with justification, the largest possible percentage error if the answer to part (a) is recorded as the area of the sector. [3]

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Question 8 - working space

15. [Maximum mark: 7]

The equation of the line $y = mx + c$ can be expressed in vector form $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$.

(a) Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in terms of m and/or c. [2]

The matrix $\boldsymbol{M}$ is defined by $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$.

(b) Find the value of $\det \boldsymbol{M}$. [1]

The line $y = mx + c$ (where $m \neq -2$) is transformed into a new line using the transformation described by matrix $\boldsymbol{M}$.

(c) Show that the equation of the resulting line does not depend on m or c. [4]

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Question 15 - working space

16. [Maximum mark: 7]

The position vector of a particle, P, relative to a fixed origin O at time t is given by

$$\vec{\mathrm{OP}} = \begin{pmatrix} \sin(t^2) \\ \cos(t^2) \end{pmatrix}.$$

(a) Find the velocity vector of P. [2]

(b) Show that the acceleration vector of P is never parallel to the position vector of P. [5]

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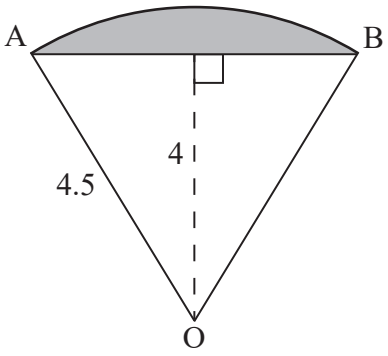
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Question 16 - working space

2. [Maximum mark: 17]

A sector of a circle, centre O and radius $4.5\,\text{m}$, is shown in the following diagram.

diagram not to scale

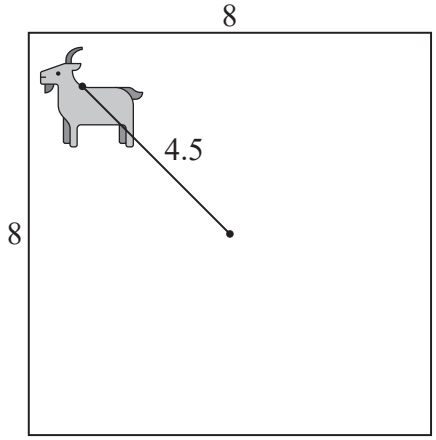


(a) (i) Find the angle $\text{A}\hat{\text{O}}\text{B}$.

(ii) Find the area of the shaded segment. [8]

A square field with side $8\,\text{m}$ has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to $4.5\,\text{m}$ from the post.

diagram not to scale



(b) Find the area of the field that can be reached by the goat. [4]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{\text{d}V}{\text{d}t} = 0.3\,t\text{e}^{-t}$.

(c) Find the value of t at which the goat is eating grass at the greatest rate. [2]

The goat is tied in the field for 8 hours.

(d) Find the total volume of grass eaten by the goat during this time. [3]

Question 2 - working space

7. [Maximum mark: 18]

A transformation, T, of a plane is represented by $\boldsymbol{r}' = \boldsymbol{P}\boldsymbol{r} + \boldsymbol{q}$, where $\boldsymbol{P}$ is a 2×2 matrix, $\boldsymbol{q}$ is a 2×1 vector, $\boldsymbol{r}$ is the position vector of a point in the plane and $\boldsymbol{r}'$ the position vector of its image under T.

The triangle OAB has coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$. Under T, these points are transformed to $(0, 1)$, $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ respectively.

(a) (i) By considering the image of $(0, 0)$, find $\boldsymbol{q}$.

(ii) By considering the image of $(1, 0)$ and $(0, 1)$, show that [6]

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$\boldsymbol{P}$ can be written as $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{S}$, where $\boldsymbol{S}$ and $\boldsymbol{R}$ are matrices.

$\boldsymbol{S}$ represents an enlargement with scale factor 0.5, centre $(0, 0)$.

$\boldsymbol{R}$ represents a rotation about $(0, 0)$.

(b) Write down the matrix $\boldsymbol{S}$. [1]

(c) (i) Use $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{S}$ to find the matrix $\boldsymbol{R}$.

(ii) Hence find the angle and direction of the rotation represented by $\boldsymbol{R}$. [7]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre (a, b), followed by a rotation about the same centre (a, b).

(d) (i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(ii) Find the value of a and the value of b. [4]

References:

2. mynamepong, n.d. Goat [image online] Available at: https://thenounproject.com/term/goat/1761571/ This file is licensed under the Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0) https://creativecommons.org/licenses/by-sa/3.0/deed.en [Accessed 22 April 2010] Source adapted.

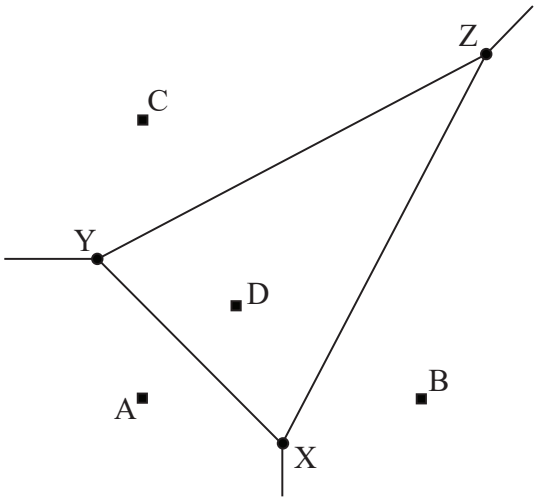
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Question 7 - working space

3. [Maximum mark: 18]

The Voronoi diagram below shows four supermarkets represented by points with coordinates $\mathrm{A}(0,\ 0)$, $\mathrm{B}(6,\ 0)$, $\mathrm{C}(0,\ 6)$ and $\mathrm{D}(2,\ 2)$. The vertices X, Y, Z are also shown. All distances are measured in kilometres.

diagram not to scale



(a) Find the midpoint of $[\mathrm{BD}]$. [2]

(b) Find the equation of (XZ). [4]

The equation of (XY) is $y = 2 - x$ and the equation of (YZ) is $y = 0.5x + 3.5$.

(c) Find the coordinates of X. [3]

The coordinates of Y are $(-1,\ 3)$ and the coordinates of Z are $(7,\ 7)$.

(d) Determine the exact length of $[\mathrm{YZ}]$. [2]

(e) Given that the exact length of $[\mathrm{XY}]$ is $\sqrt{32}$, find the size of $\mathrm{X\hat{Y}Z}$ in degrees. [4]

(f) Hence find the area of triangle XYZ. [2]

A town planner believes that the larger the area of the Voronoi cell XYZ, the more people will shop at supermarket D.

(g) State one criticism of this interpretation. [1]

Question 3 - working space

6. [Maximum mark: 21]

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

(a) (i) Find the initial speed of the ball.

(ii) Find the angle of elevation of the ball as it is launched. [4]

(b) Find the maximum height reached by the ball. [3]

(c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]

(d) For the path of the ball, find an expression for y in terms of x. [3]

An archer releases an arrow from the point $(0, 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed $60\,\text{m s}^{-1}$ and an angle of elevation of 10°.

(e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]

(f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

Turn over

Question 6 - working space

2. [Maximum mark: 27]

This question is about a metropolitan area council planning a new town and the location of a new toxic waste dump.

A metropolitan area in a country is modelled as a square. The area has four towns, located at the corners of the square. All units are in kilometres with the x-coordinate representing the distance east and the y-coordinate representing the distance north from the origin at $(0, 0)$.

- Edison is modelled as being positioned at $\mathrm{E}(0, 40)$.
- Fermitown is modelled as being positioned at $\mathrm{F}(40, 40)$.
- Gaussville is modelled as being positioned at $\mathrm{G}(40, 0)$.
- Hamilton is modelled as being positioned at $\mathrm{H}(0, 0)$.

(a) The model assumes that each town is positioned at a single point. Describe possible circumstances in which this modelling assumption is reasonable. [1]

(b) Sketch a Voronoi diagram showing the regions within the metropolitan area that are closest to each town. [1]

The metropolitan area council decides to build a new town called Isaacopolis located at $\mathrm{I}(30, 20)$.

A new Voronoi diagram is to be created to include Isaacopolis. The equation of the perpendicular bisector of $[\mathrm{IE}]$ is $y = \frac{3}{2}x + \frac{15}{2}$.

(c) (i) Find the equation of the perpendicular bisector of $[\mathrm{IF}]$. [4]

(ii) Given that the coordinates of one vertex of the new Voronoi diagram are $(20, 37.5)$, find the coordinates of the other two vertices within the metropolitan area. [4]

(iii) Sketch this new Voronoi diagram showing the regions within the metropolitan area which are closest to each town. [2]

The metropolitan area is divided into districts based on the Voronoi regions found in part (c).

(d) A car departs from a point due north of Hamilton. It travels due east at constant speed to a destination point due North of Gaussville. It passes through the Edison, Isaacopolis and Fermitown districts. The car spends 30% of the travel time in the Isaacopolis district.

Find the distance between Gaussville and the car's destination point. [4]

(This question continues on the following page)

(Question 2 continued)

A toxic waste dump needs to be located within the metropolitan area. The council wants to locate it as far as possible from the nearest town.

(e) (i) Find the location of the toxic waste dump, given that this location is not on the edge of the metropolitan area. [4]

(ii) Make one possible criticism of the council's choice of location. [1]

(f) The toxic waste dump, T, is connected to the towns via a system of sewers.

The connections are represented in the following matrix, $\boldsymbol{M}$, where the order of rows and columns is $(\mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{I}, \mathrm{T})$.

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

A leak occurs from the toxic waste dump and travels through the sewers. The pollution takes one day to travel between locations that are directly connected.

The digit 1 in $\boldsymbol{M}$ represents a direct connection. The values of 1 in the leading diagonal of $\boldsymbol{M}$ mean that once a location is polluted it will stay polluted.

(i) Find which town is last to be polluted. Justify your answer. [3]

(ii) Write down the number of days it takes for the pollution to reach the last town. [1]

(iii) A sewer inspector needs to plan the shortest possible route through each of the connections between different locations. Determine an appropriate start point and an appropriate end point of the inspection route.

Note that the fact that each location is connected to itself does not correspond to a sewer that needs to be inspected. [2]

References:

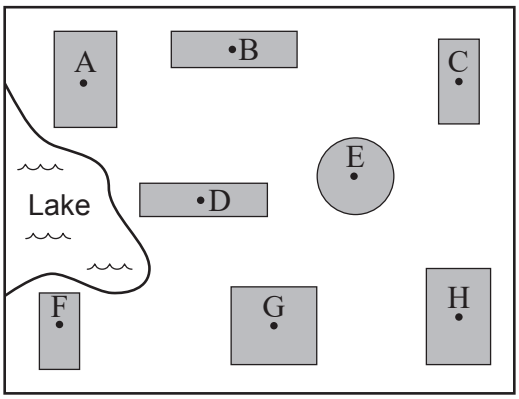
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Question 2 - working space

2. [Maximum mark: 28]

This question compares possible designs for a new computer network between multiple school buildings, and whether they meet specific requirements.

A school's administration team decides to install new fibre-optic internet cables underground. The school has eight buildings that need to be connected by these cables. A map of the school is shown below, with the internet access point of each building labelled A–H.



Jonas is planning where to install the underground cables. He begins by determining the distances, in metres, between the underground access points in each of the buildings.

He finds $\mathrm{AD} = 89.2\,\mathrm{m}$, $\mathrm{DF} = 104.9\,\mathrm{m}$ and $\mathrm{A\hat{D}F} = 83^\circ$.

(a) Find AF. [3]

The cost for installing the cable directly between A and F is $\$21\,310$.

(b) Find the cost per metre of installing this cable. [2]

Jonas estimates that it will cost $\$110$ per metre to install the cables between all the other buildings.

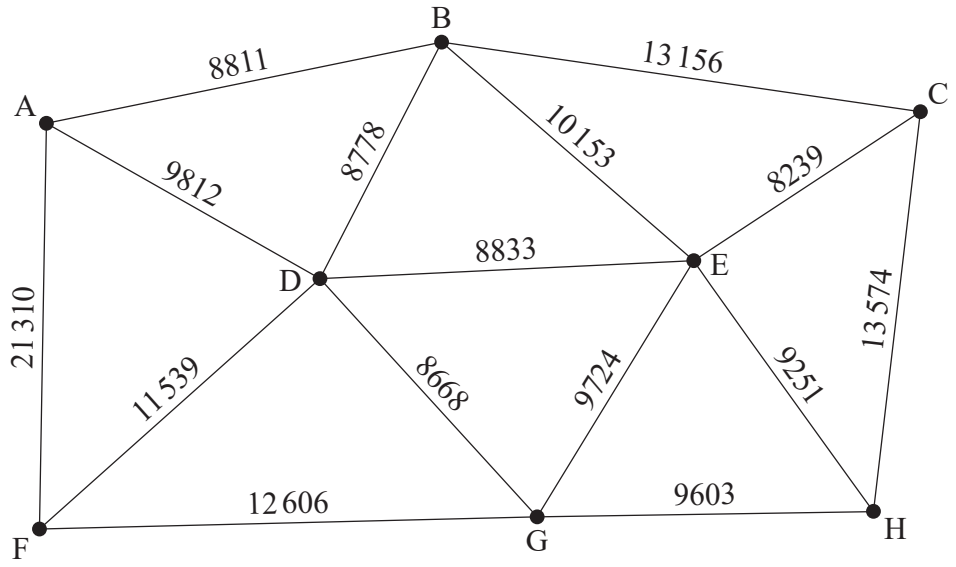
(c) State why the cost for installing the cable between A and F would be higher than between the other buildings. [1]

(This question continues on the following page)

Turn over

(Question 2 continued)

Jonas creates the following graph, S, using the cost of installing the cables between two buildings as the weight of each edge.



The computer network could be designed such that each building is directly connected to at least one other building and hence all buildings are indirectly connected.

(d) (i) By using Kruskal's algorithm, find the minimum spanning tree for S, showing clearly the order in which edges are added. [3]

(ii) Hence find the minimum installation cost for the cables that would allow all the buildings to be part of the computer network. [2]

The computer network fails if any part of it becomes unreachable from any other part. To help protect the network from failing, every building could be connected to at least two other buildings. In this way if one connection breaks, the building is still part of the computer network. Jonas can achieve this by finding a Hamiltonian cycle within the graph.

(e) State why a path that forms a Hamiltonian cycle does not always form an Eulerian circuit. [1]

(f) Starting at D, use the nearest neighbour algorithm to find the upper bound for the installation cost of a computer network in the form of a Hamiltonian cycle.

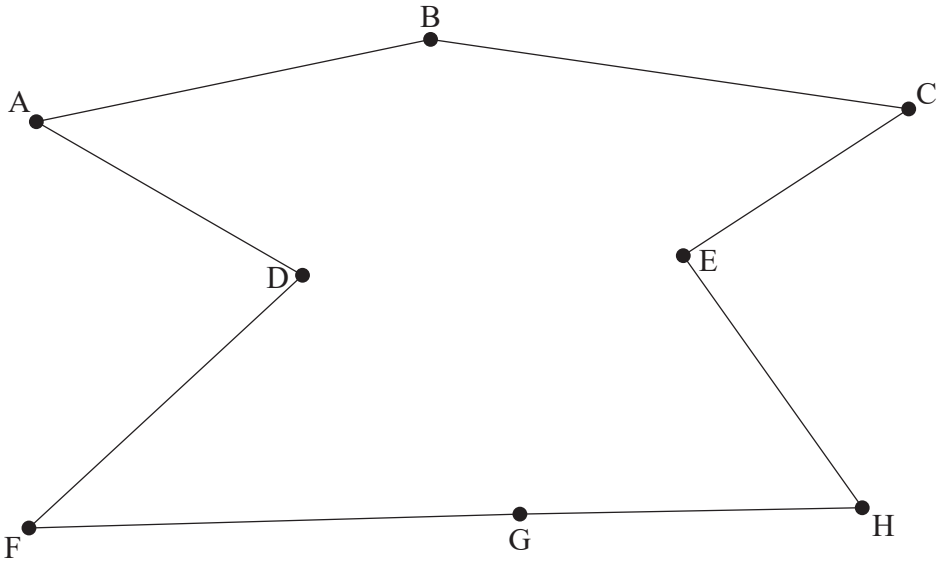
Note: Although the graph is not complete, in this instance it is not necessary to form a table of least distances. [5]

(g) By deleting D, use the deleted vertex algorithm to find the lower bound for the installation cost of the cycle. [6]

(This question continues on the following page)

(Question 2 continued)

After more research, Jonas decides to install the cables as shown in the diagram below.



Each individual cable is installed such that each end of the cable is connected to a building's access point. The connection between each end of a cable and an access point has a 1.4% probability of failing after a power surge.

For the network to be successful, each building in the network must be able to communicate with every other building in the network. In other words, there must be a path that connects any two buildings in the network. Jonas would like the network to have less than a 2% probability of failing to operate after a power surge.

(h) Show that Jonas's network satisfies the requirement of there being less than a 2% probability of the network failing after a power surge. [5]

References:

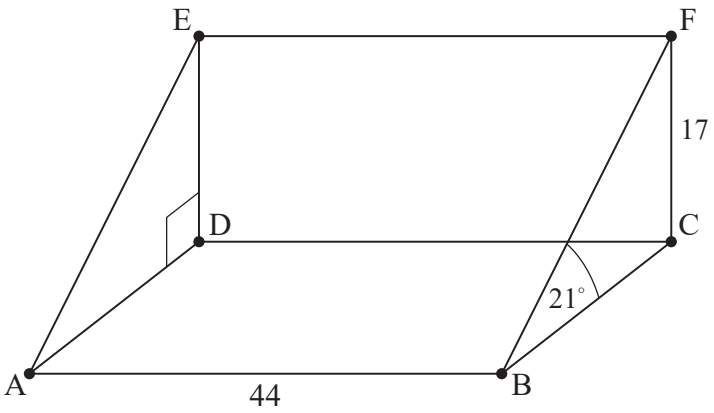
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Question 2 - working space

3. [Maximum mark: 5]

An artificial ski slope can be modelled as a triangular prism, as shown in the diagram. Rectangle $ABCD$ is horizontal, and rectangle $CDEF$ is vertical.

diagram not to scale



The maximum height of the ski slope, CF, is 17 metres and the steepest angle of the ski slope, $F\hat{B}C$, is 21°.

(a) Calculate the length of $[BF]$. [2]

The width of the base of the ski slope, AB, is 44 metres. Mayumi skis in a straight line, starting from point E and finishing at the base of the ski slope.

(b) Find the value of the least steep angle that Mayumi can ski. [3]

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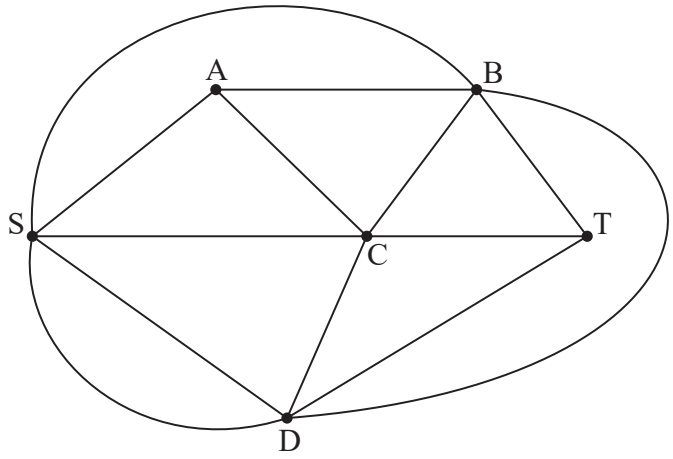
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Question 3 - working space

4. [Maximum mark: 7].

In a competition, a contestant has to move through a maze to find treasure. A graph of the maze is shown below, where each edge represents a corridor in the maze. The contestant starts at S and the treasure is located at T.



(a) Complete the adjacency matrix, $\boldsymbol{M}$, for the graph. [2]

$$\begin{array}{c} \\ \mathrm{S} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathrm{S} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{T} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & \square & 0 \\ 1 & 0 & 1 & 1 & \square & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ \square & \square & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

The competition rules state that the contestant can walk along a maximum of four corridors.

(b) Find the number of walks from S to T with a maximum of 4 edges. [4]

(c) Explain why the number of ways the contestant can reach the treasure is less than the answer to part (b). [1]

(This question continues on the following page)

(Question 4 continued)

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Question 4 - working space

8. [Maximum mark: 7]

Line L_1 has a vector equation $\boldsymbol{r} = \begin{pmatrix} 3p+4 \\ 2p-1 \\ p+9 \end{pmatrix}$, where $p \in \mathbb{R}$.

Line L_2 has a vector equation $\boldsymbol{r} = \begin{pmatrix} q-2 \\ 1-q \\ 2q+1 \end{pmatrix}$, where $q \in \mathbb{R}$.

The two lines intersect at point M.

(a) Find the coordinates of M. [3]

(b) Find the acute angle between the two lines. [4]

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Question 8 - working space

9. [Maximum mark: 8]

The transformation T is represented by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

A pentagon with an area of $12\,\text{cm}^2$ is transformed by T.

(a) Find the area of the image of the pentagon. [2]

Under the transformation T, the image of point X has coordinates $(2t - 3,\ 6 - 5t)$, where $t \in \mathbb{R}$.

(b) Find, in terms of t, the coordinates of X. [6]

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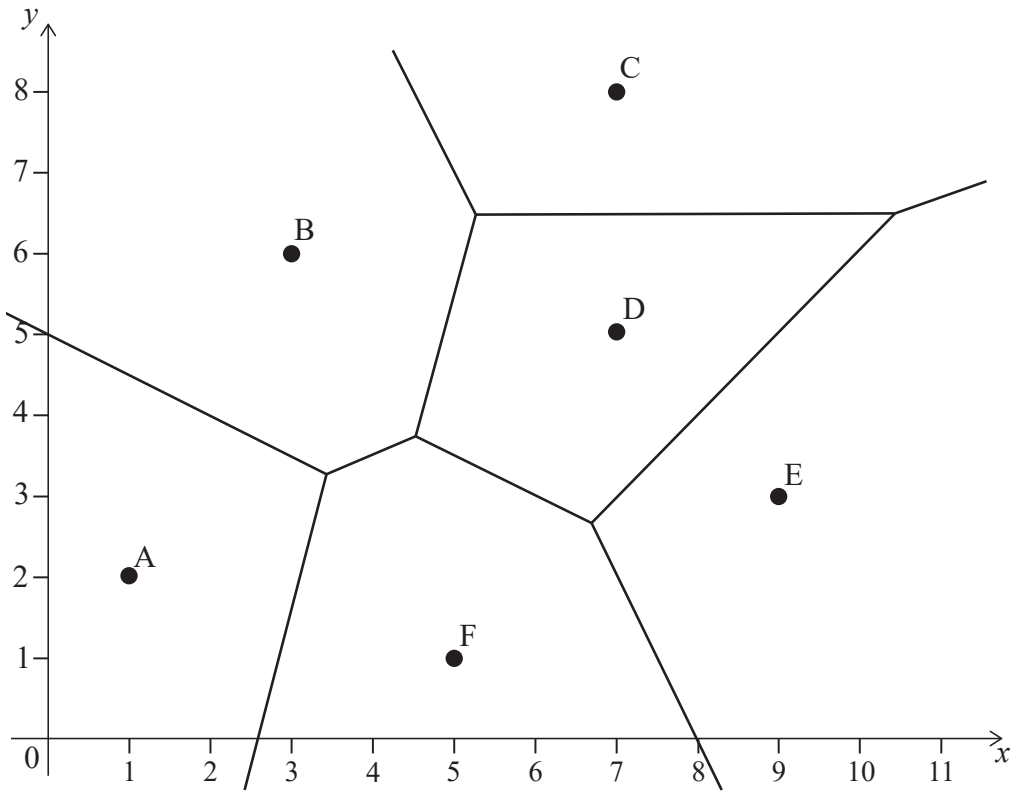
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Question 9 - working space

2. [Maximum mark: 13]

Six restaurant locations (labelled A, B, C, D, E and F) are shown, together with their Voronoi diagram. All distances are measured in kilometres.



(a) Elena wants to eat at the closest restaurant to her. Write down the restaurant she should go to, if she is at

(i) $(2, 7)$.

(ii) $(0, 1)$, when restaurant A is closed. [2]

Restaurant C is at $(7, 8)$ and restaurant D is at $(7, 5)$.

(b) Find the equation of the perpendicular bisector of CD. [2]

Restaurant B is at $(3, 6)$.

(c) Find the equation of the perpendicular bisector of BC. [5]

(d) Hence find

(i) the coordinates of the point which is of equal distance from B, C and D.

(ii) the distance of this point from D. [4]

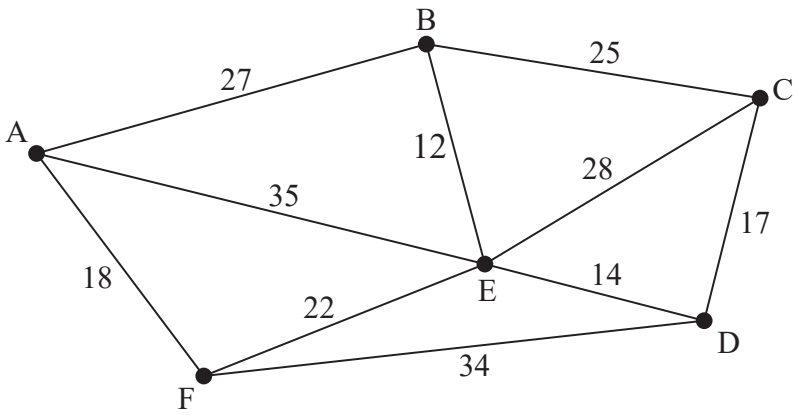
Turn over

Question 2 - working space

4. [Maximum mark: 14]

A company has six offices, A, B, C, D, E and F. One of the company managers, Nanako, needs to visit the offices. She creates the following graph that shows the distances, in kilometres, between some of the offices.

diagram not to scale



(a) Write down a Hamiltonian cycle for this graph. [1]

(b) State, with a reason, whether the graph contains an Eulerian circuit. [1]

Nanako wishes to find the shortest cycle to visit all the offices. She decides to complete a weighted adjacency table, showing the least distance between each pair of offices.

	A	B	C	D	E	F
A		27	52	p	35	18
B			25	26	12	q
C				17	28	r
D					14	34
E						22
F						

(c) Write down the value of

(i) p.

(ii) q.

(iii) r. [3]

(This question continues on the following page)

(Question 4 continued)

(d) Starting at vertex E, use the nearest neighbour algorithm to find an upper bound for Nanako's cycle. [3]

(e) By deleting vertex F, find a lower bound for Nanako's cycle. [4]

(f) Explain, with a reason, why the answer to part (e) might not be the best lower bound. [2]

Question 4 - working space

7. [Maximum mark: 20]

The position vector of a particle at time t is given by $\boldsymbol{r} = 3\cos(3t)\boldsymbol{i} + 4\sin(3t)\boldsymbol{j}$.
Displacement is measured in metres and time is measured in seconds.

(a) (i) Find an expression for the velocity of the particle at time t.

(ii) Hence find the speed when $t = 3$. [4]

(b) (i) Find an expression for the acceleration of the particle at time t.

(ii) Hence show that the acceleration is always directed towards the origin. [4]

The position vector of a second particle is given by $\boldsymbol{r} = -4\sin(4t)\boldsymbol{i} + 3\cos(4t)\boldsymbol{j}$.

(c) For $0 \leq t \leq 10$, find the time when the two particles are closest to each other. [5]

At time k, where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

(d) (i) Find the value of k.

(ii) At time k, show that the two particles are moving in the opposite direction. [7]

References:

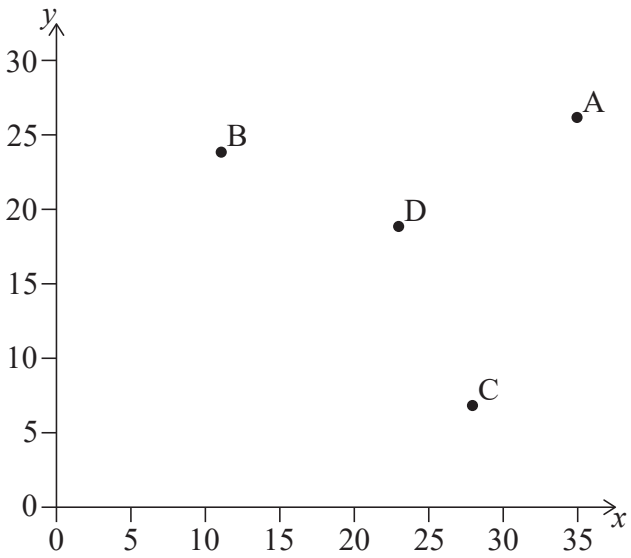
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Question 7 - working space

3. [Maximum mark: 5]

Three towns have positions $A(35, 26)$, $B(11, 24)$, and $C(28, 7)$ according to the coordinate system shown where distances are measured in miles.

Dominique's farm is located at the position $D(24, 19)$.



(a) Find AD. [2]

On a particular day, the mean temperatures recorded in each of towns A, B and C are $34\,°C$, $29\,°C$ and $30\,°C$ respectively.

(b) Use nearest neighbour interpolation to estimate the temperature at Dominique's farm on that particular day. [3]

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Question 3 - working space

5. [Maximum mark: 5]

A boat travels $8\,\text{km}$ on a bearing of $315°$ and then a further $6\,\text{km}$ on a bearing of $045°$. Find the bearing on which the boat should travel to return directly to the starting point.

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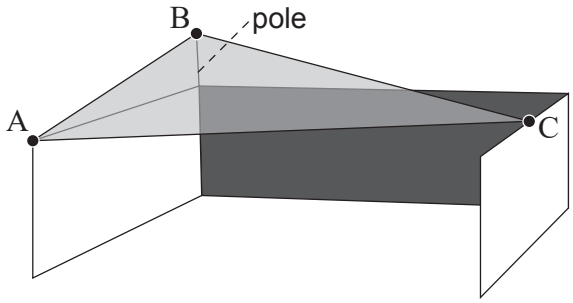
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Question 5 - working space

7. [Maximum mark: 9]

A triangular cover is positioned over a walled garden to provide shade. It is anchored at points A and C, located at the top of a $2\,\mathrm{m}$ wall, and at a point B, located at the top of a $1\,\mathrm{m}$ vertical pole fixed to a top corner of the wall.

The three edges of the cover can be represented by the vectors $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}$, $\overrightarrow{\mathrm{AC}} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$ and $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix}$, where distances are measured in metres.



(a) Calculate the vector product $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$. [2]

(b) Hence find the area of the triangular cover. [2]

The point X on $[\mathrm{AC}]$ is such that $[\mathrm{BX}]$ is perpendicular to $[\mathrm{AC}]$.

(c) Use your answer to part (b) to find the distance BX. [3]

(d) Find the angle the cover makes with the horizontal plane. [2]

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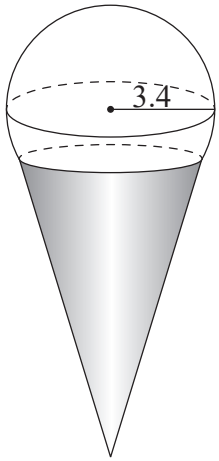
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Question 7 - working space

6. [Maximum mark: 5]

Ruhi buys a scoop of ice cream in the shape of a sphere with a radius of $3.4\,\text{cm}$. The ice cream is served in a cone, and it may be assumed that $\frac{1}{5}$ of the volume of the ice cream is inside the cone. This is shown in the following diagram.

diagram not to scale



(a) Calculate the volume of ice cream that is not inside the cone. [3]

The cone has a slant height of $11\,\text{cm}$ and a radius of $3\,\text{cm}$.

The outside of the cone is covered with chocolate.

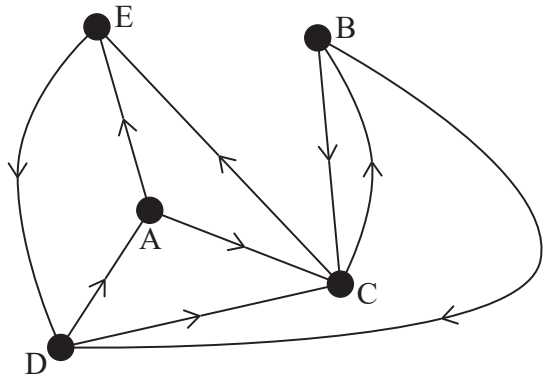
(b) Calculate the surface area of the cone that is covered with chocolate. Give your answer correct to the nearest cm^2. [2]

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Question 6 - working space

8. [Maximum mark: 7]

The following directed, unweighted, graph shows a simplified road network on an island, connecting five small villages marked A to E.



(a) Construct the adjacency matrix $\boldsymbol{M}$ for this network. [3]

Beatriz the bus driver starts at village E and drives to seven villages, such that the seventh village is A.

(b) (i) Determine how many possible routes Beatriz could have taken, to travel from E to A.

(ii) Describe one possible route taken by Beatriz, by listing the villages visited in order. [4]

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Question 8 - working space

13. [Maximum mark: 7]

The matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, T'.

(a) Find the single matrix that represents the transformation $T' \to T$, which will undo the transformation described above. [4]

The area of T' is $273\,\text{cm}^2$.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

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Question 13 - working space

14. [Maximum mark: 8]

In this question, $\boldsymbol{i}$ denotes a unit vector due east, and $\boldsymbol{j}$ denotes a unit vector due north.

Two ships, A and B, are each moving with constant velocities.

The position vector of ship A, at time t hours, is given as $\boldsymbol{r}_A = (1 + 2t)\boldsymbol{i} + (3 - 3t)\boldsymbol{j}$.

The position vector of ship B, at time t hours, is given as $\boldsymbol{r}_B = (-2 + 4t)\boldsymbol{i} + (-4 + t)\boldsymbol{j}$.

(a) Find the bearing on which ship A is sailing. [3]

(b) Find the value of t when ship B is directly south of ship A. [2]

(c) Find the value of t when ship B is directly south-east of ship A. [3]

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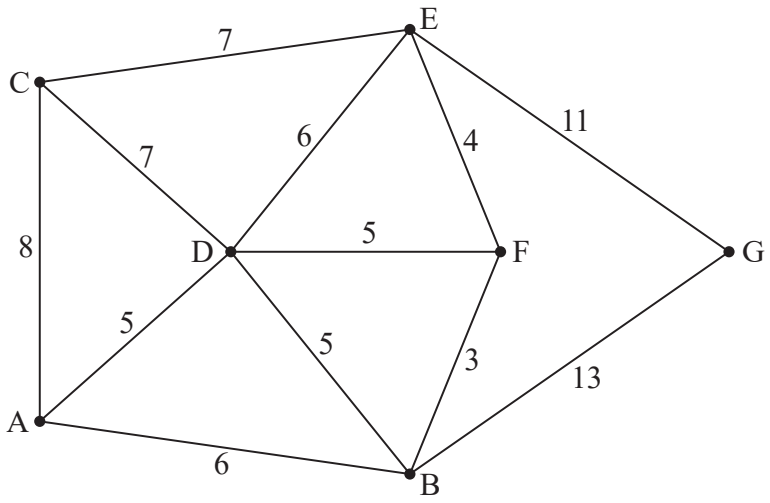
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Question 14 - working space

4. [Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



(a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer. [2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	***a***	***b***
D	5	5	7		6	5	***c***
E	11	7	7	6		4	11
F	9	3	***a***	5	4		***d***
G	19	13	***b***	***c***	11	***d***	

(This question continues on the following page)

(Question 4 continued)

(b) Find the value of

(i) a;

(ii) b;

(iii) c;

(iv) d. [2]

(c) Use the nearest neighbour algorithm, starting at vertex G, to find an upper bound for the travelling salesman problem. [3]

(d) (i) Sketch a minimum spanning tree for the subgraph with vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}$.

(ii) Write down the total weight of the minimum spanning tree. [4]

(e) Hence find a lower bound for the travelling salesman problem. [2]

(f) Explain one way in which an improved lower bound could be found. [1]

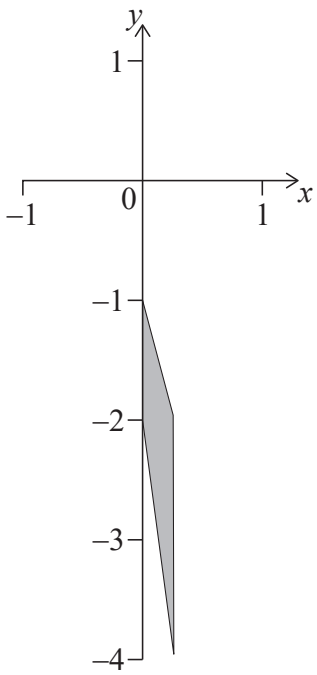
It is found that the optimum solution starting at A is actually $\mathrm{A–C–E–G–B–F–D–A}$.

(g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

Question 4 - working space

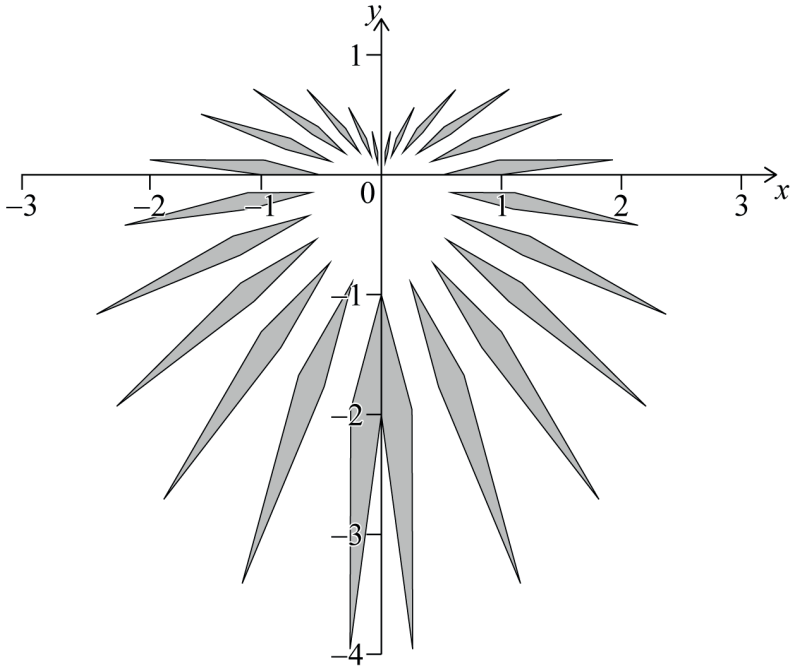
7. [Maximum mark: 18]

A trapezoid, Q, has vertices $(0, -1)$, $(0, -2)$, $(\sin 15°, -3-\cos 15°)$, $(\sin 15°, -1-\cos 15°)$ as shown.



(a) Show that the area of the trapezoid is $\frac{3}{2}\sin 15°$. [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q. These elements are shaded in the following diagram such that the y-axis is a line of symmetry.



(This question continues on the following page)

(Question 7 continued)

The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$\boldsymbol{M}_k = \begin{pmatrix} \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) & -\left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) \\ \left(1-\frac{k}{12}\right)\sin\left(k\times 15^\circ\right) & \left(1-\frac{k}{12}\right)\cos\left(k\times 15^\circ\right) \end{pmatrix}$$

where $k = 0, 1, 2, 3, \ldots, 11$.

(b) (i) Find the matrix $\boldsymbol{M}_6$. Give your answer in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Q}$.

(ii) Hence find the coordinates of the image of the vertex $(0, -1)$ after it is transformed by the matrix $\boldsymbol{M}_6$. [4]

The matrix $\boldsymbol{M}_k$ can be expressed as the product of a rotation matrix and an enlargement matrix.

(c) Write down, in terms of k,

(i) the rotation matrix;

(ii) the enlargement matrix;

(iii) the angle of the rotation;

(iv) the scale factor of the enlargement. [4]

(d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix $\boldsymbol{M}_k$ in terms of k. [2]

(e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix $\boldsymbol{N}_k$, where $k = 0, 1, 2, 3, \ldots, 11$.

(f) Write down the matrix $\boldsymbol{N}_k$ as a product of two matrices. [2]

References:

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Question 7 - working space

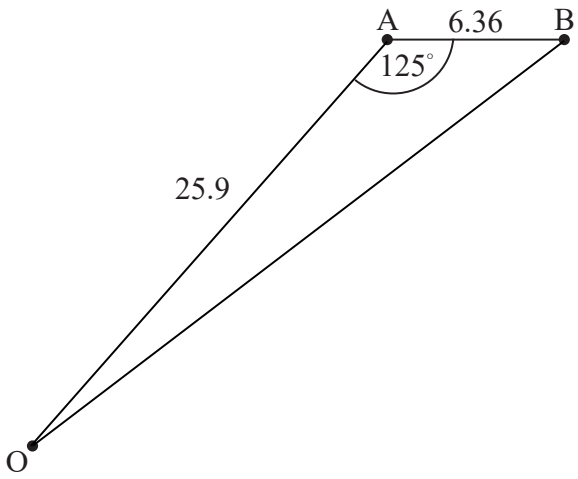
1. [Maximum mark: 13]

The diagram shows points in a park viewed from above, at a specific moment in time.

The distance between two trees, at points A and B, is $6.36\,\text{m}$.

Odette is playing football in the park and is standing at point O, such that $\text{OA} = 25.9\,\text{m}$ and $\text{O}\hat{\text{A}}\text{B} = 125^\circ$.

diagram not to scale



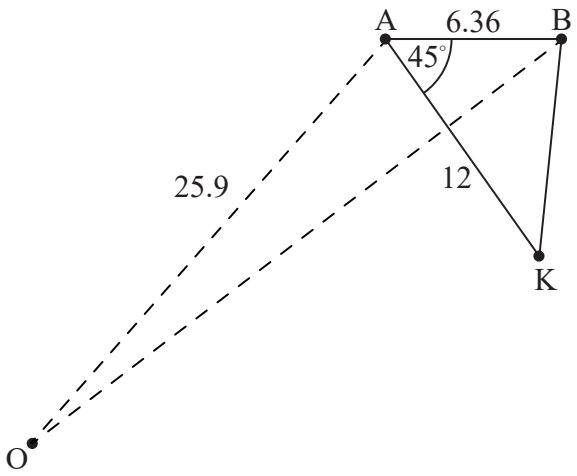
(a) Calculate the area of triangle AOB. [3]

(This question continues on the following page)

(Question 1 continued)

Odette's friend, Khemil, is standing at point K such that he is $12\,\text{m}$ from A and $\text{K}\hat{\text{A}}\text{B} = 45^\circ$.

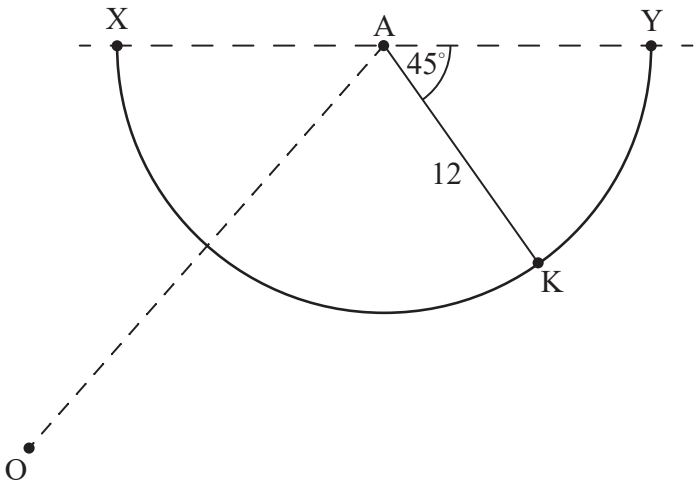
diagram not to scale



(b) Calculate Khemil's distance from B. [3]

XY is a semicircular path in the park with centre A, such that $\text{K}\hat{\text{A}}\text{Y} = 45^\circ$. Khemil is standing on the path and Odette's football is at point X. This is shown in the diagram below.

diagram not to scale



The length $\text{KX} = 22.2\,\text{m}$, $\text{K}\hat{\text{O}}\text{X} = 53.8^\circ$ and $\text{O}\hat{\text{K}}\text{X} = 51.1^\circ$.

(c) Find whether Odette or Khemil is closer to the football. [4]

Khemil runs along the semicircular path to pick up the football.

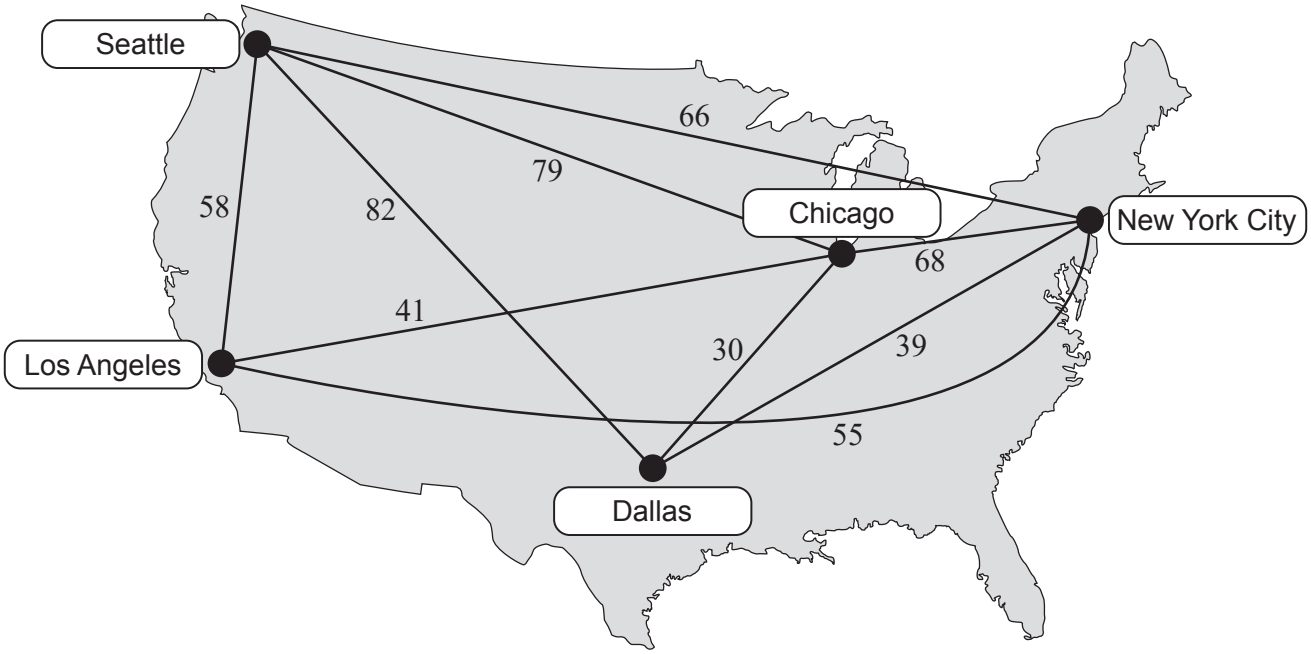
(d) Calculate the distance that Khemil runs. [3]

Turn over

Question 1 - working space

4. [Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($) between cities.



(a) Explain why the graph can be described as "connected", but not "complete". [2]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

(This question continues on the following page)

(Question 4 continued)

(e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem.

(ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

Question 4 - working space

2. [Maximum mark: 29]

In this question you will use vector methods to determine whether aircraft are obeying air traffic regulations.

The base of an air traffic control tower at an airport is taken as the origin of a coordinate system. An aircraft's position is given by the coordinates (x, y, z), where x and y are respectively the aircraft's displacement east and north of the tower, and z is the vertical displacement of the aircraft above the base of the tower. All displacements are measured in kilometres.

At $12{:}00$ two aircraft, A and B, are at the points $\mathrm{P}(100, -82, 10.7)$ and $\mathrm{Q}(215, -197, 10.7)$ respectively.

(a) Find the distance between the two aircraft at $12{:}00$. [2]

The two aircraft are flying along the same straight line (flight path), with B behind A.

They both have the same constant velocity of $\begin{pmatrix} -640 \\ 640 \\ 0 \end{pmatrix}$ kilometres per hour.

(b) Find the speed of both aircraft. [2]

Air traffic regulations state that if two aircraft are on the same flight path then they must always maintain at least a 10 minute gap between them. If at any time two aircraft are too close they are said to be "in conflict".

(c) Find the length of time it takes B to reach point P from point Q, and hence state whether the two aircraft are in conflict. [3]

(d) Write down, $\boldsymbol{r}_A$, the position vector of A, t hours after $12{:}00$. [1]

If two aircraft are not on the same flight path, air traffic regulations state:

When the vertical distance between the two aircraft is less than $300\,\mathrm{m}$ the aircraft must be more than $10\,\mathrm{km}$ apart.

When the vertical distance between the two aircraft is at least $300\,\mathrm{m}$ there are no restrictions.

The air traffic controller notices an aircraft, C, flying on a different flight path but close to A. The position of C, t hours after $12{:}00$, is given by

$$\boldsymbol{r}_C = \begin{pmatrix} -400 \\ -41 \\ 9.1 \end{pmatrix} + t \begin{pmatrix} -140 \\ 604 \\ 2 \end{pmatrix}.$$

(This question continues on the following page)

(Question 2 continued)

(e) (i) Find the two values of t at which the distance between A and C is $10\,\text{km}$. [5]

It is given that the distance between A and C is less than $10\,\text{km}$, only between these two values of t.

(ii) Determine whether the two aircraft, A and C, will break the air traffic regulations if they continue with their current velocities. Justify your answer. [5]

A new coordinate system, (x, y), is defined with an origin R at a point $2\,\text{km}$ directly above the air traffic control tower. When an aircraft is flying in the horizontal plane containing R, the values of x and y represent its displacement, in km, east and north of point R respectively.

A fourth aircraft, D, is flying at a constant height of $2\,\text{km}$ near the airport while waiting for permission to land. Its position at time t is given by

$$\overrightarrow{\text{RD}} = \begin{pmatrix} 6.4\cos(38t) \\ 6.4\sin(38t) \end{pmatrix}.$$

(f) Describe the path followed by D. [3]

Around the same time a small aircraft, E, is flying at the same height as D and along the line with vector equation

$$\overrightarrow{\text{RE}} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(g) Let $\boldsymbol{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(i) Find $\overrightarrow{\text{RE}} \bullet \boldsymbol{b}$ in terms of λ. [2]

(ii) Hence find the value of λ for which the distance from R to the line is minimum. [2]

(iii) Find this minimum distance. [2]

(iv) Show that D and E will not break the air traffic regulations for any value of t. [2]

References:

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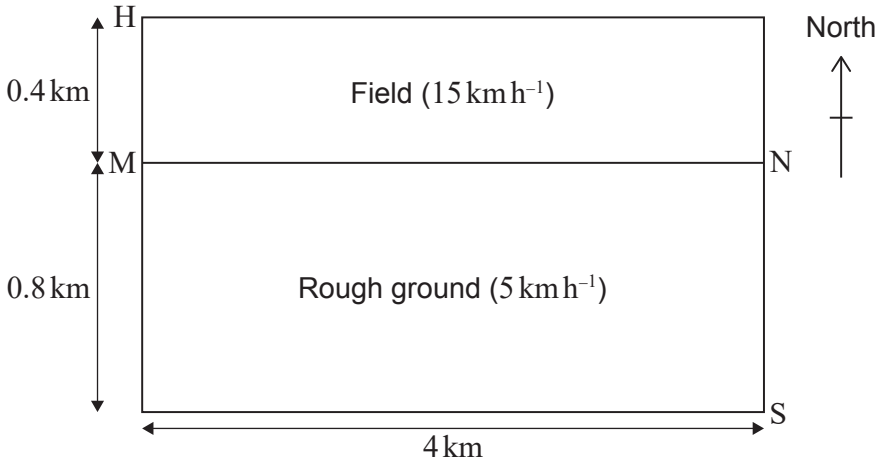
Question 2 - working space

1. [Maximum mark: 26]

This question considers the optimal route between two points, separated by several regions where different speeds are possible.

Huw lives in a house, H, and he attends a school, S, where H and S are marked on the following diagram. The school is situated $1.2\,\text{km}$ south and $4\,\text{km}$ east of Huw's house. There is a boundary [MN], going from west to east, $0.4\,\text{km}$ south of his house. The land north of [MN] is a field over which Huw runs at 15 kilometres per hour ($\text{km}\,\text{h}^{-1}$). The land south of [MN] is rough ground over which Huw walks at $5\,\text{km}\,\text{h}^{-1}$. The two regions are shown in the following diagram.

diagram not to scale



(a) Huw travels in a straight line from H to S. Calculate the time that Huw takes to complete this journey. Give your answer correct to the nearest minute. [6]

(b) Huw realizes that his journey time could be reduced by taking a less direct route. He therefore defines a point P on [MN] that is $x\,\text{km}$ east of M. Huw decides to run from H to P and then walk from P to S. Let $T(x)$ represent the time, in hours, taken by Huw to complete the journey along this route.

(i) Show that $T(x) = \dfrac{\sqrt{0.4^2 + x^2} + 3\sqrt{0.8^2 + (4-x)^2}}{15}$. [3]

(ii) Sketch the graph of $y = T(x)$. [2]

(This question continues on the following page)

(Question 1 continued)

(iii) Hence determine the value of x that minimizes $T(x)$. [1]

(iv) Find by how much Huw's journey time is reduced when he takes this optimal route, compared to travelling in a straight line from H to S. Give your answer correct to the nearest minute. [2]

(c) (i) Determine an expression for the derivative $T'(x)$. [3]

(ii) Hence show that $T(x)$ is minimized when

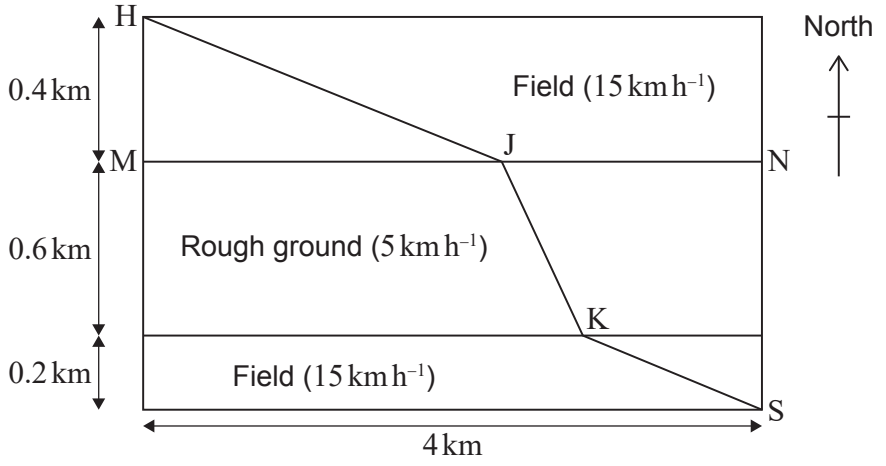
$$\frac{x}{\sqrt{0.16+x^2}} = \frac{3(4-x)}{\sqrt{0.64+(4-x)^2}}.$$ [1]

(iii) For the optimal route, verify that the equation in part (c)(ii) satisfies the following result:

$$\frac{\cos \mathrm{H\hat{P}M}}{\cos \mathrm{S\hat{P}N}} = \frac{\text{speed over field}}{\text{speed over rough ground}}.$$ [2]

(d) The owner of the rough ground converts the southern quarter into a field over which Huw can run at $15\,\mathrm{km\,h^{-1}}$. The following diagram shows the optimal route, HJKS, in this new situation. You are given that [HJ] is parallel to [KS].

diagram not to scale



Using a similar result to that given in part (c)(iii), at the point J, determine MJ. [6]

Question 1 - working space

3. [Maximum mark: 6]

The Great Pyramid of Giza is the oldest of the Seven Wonders of the Ancient World. When it was built, 4500 years ago, the measurements of the pyramid were in Royal Egyptian Cubits (REC).



[Source: Nina Aldin Thune. https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza#/media/File:Kheops-Pyramid.jpg. Licensed under CC BY 2.5 https://creativecommons.org/licenses/by/2.5/#. Image adapted.]

Viktor reads online that $1\,\text{REC}$ is equal to 0.52 metres, rounded to two decimal places.

(a) Write down the upper and lower bounds of $1\,\text{REC}$ in metres. [2]

The Great Pyramid of Giza has a square base with side lengths of $440\,\text{REC}$ and a height of $280\,\text{REC}$. Viktor assumes that these two measurements are exact and that the Great Pyramid can be modelled as a square-based pyramid with smooth faces.

(b) Find the minimum possible volume of the pyramid in cubic metres. [4]

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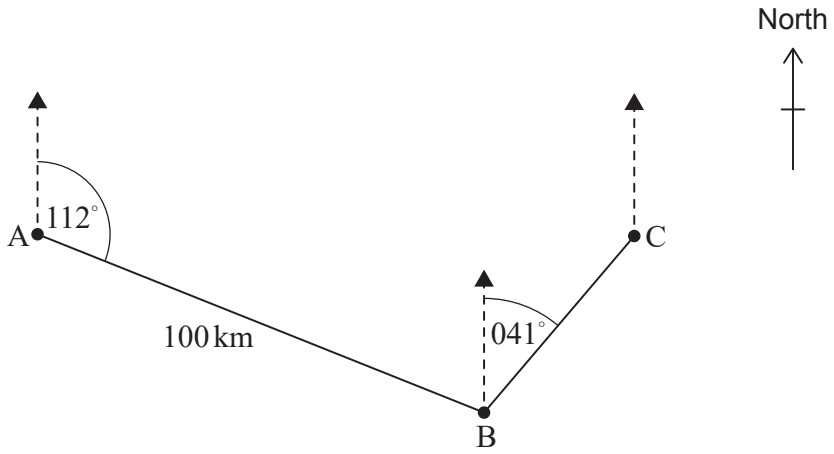
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Question 3 - working space

4. [Maximum mark: 6]

Jason sails his boat from point A for a distance of $100\,\text{km}$, on a bearing of 112°, to arrive at point B. He then sails on a bearing of 041° to point C. Jason's journey is shown in the diagram.

diagram not to scale



(a) Find $A\hat{B}C$. [2]

Point C is directly east of point A.

(b) Calculate the distance that Jason sails to return directly from point C to point A. [4]

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Question 4 - working space

6. [Maximum mark: 8]

On the following Voronoi diagram, the coordinates of three farmhouses are $A(0, 3)$, $B(8, 3)$ and $C(8, 13)$, where distances are measured in kilometres. Each farmhouse owns the land that is closest to it, and their boundaries are defined by the dotted lines on **Diagram 1**.

Diagram 1

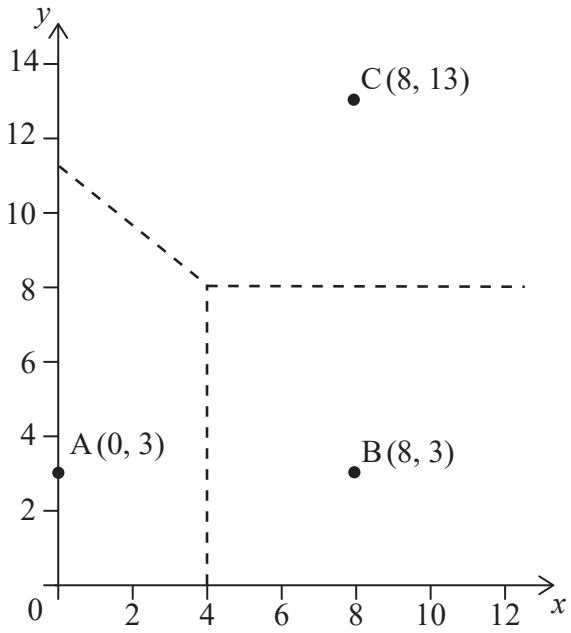
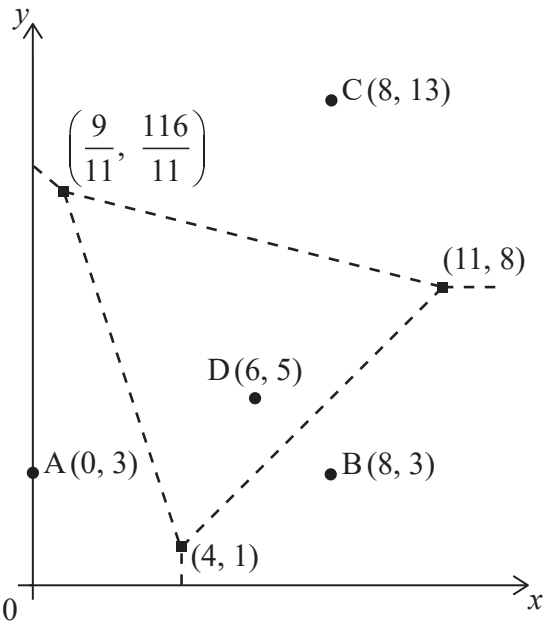


Diagram 2



To provide water to the farms it is decided to construct a well at the point where the boundaries meet on **Diagram 1**.

(a) Write down the coordinates of this point. [1]

(b) Find the equation of the perpendicular bisector of $[AC]$. [3]

An additional farmhouse $D(6, 5)$ is built on the land. The Voronoi diagram has been redrawn to show the new boundaries. The coordinates of the vertices of these boundaries are indicated on **Diagram 2**.

A wind turbine is to be built at one of the vertices.

(c) The wind turbine should be as far from the nearest farmhouses as possible.

(i) By calculating appropriate distances, find the location of the wind turbine.

(ii) Hence, write down the distance of the wind turbine to the nearest farmhouse. [4]

(This question continues on the following page)

(Question 6 continued)

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Question 6 - working space

14. [Maximum mark: 7]

A straight line L has vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and point Q has coordinates $(11, -1, 3)$.

Point P is the point on L closest to Q.

(a) Find the coordinates of P. [4]

(b) Find a vector that is perpendicular to both L and the line passing through points P and Q. [3]

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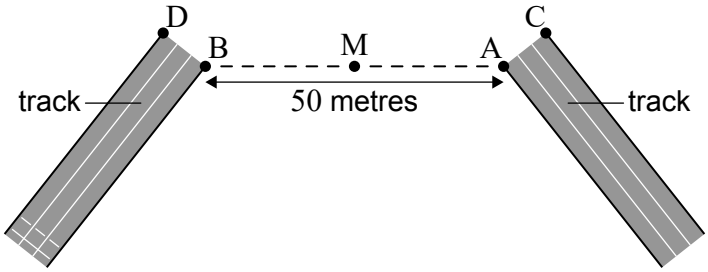
Question 14 - working space

1. [Maximum mark: 15]

Madhu is designing a jogging track for the campus of her school. The following diagram shows an incomplete portion of the track.

Madhu wants to design the track such that the inner edge is a smooth curve from point A to point B, and the other edge is a smooth curve from point C to point D. The distance between points A and B is 50 metres.

diagram not to scale



To create a smooth curve, Madhu first walks to M, the midpoint of $[AB]$.

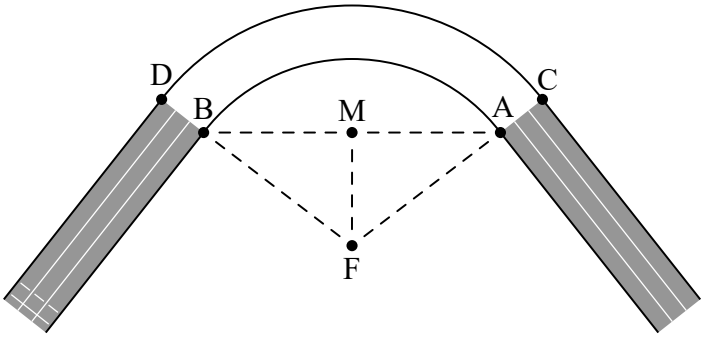
(a) Write down the length of $[BM]$. [1]

(This question continues on the following page)

(Question 1 continued)

Madhu then walks 20 metres in a direction perpendicular to $[AB]$ to get from point M to point F. Point F is the centre of a circle whose arc will form the smooth curve between points A and B on the track, as shown in the following diagram.

diagram not to scale



(b) (i) Find the length of $[BF]$.

(ii) Find $B\hat{F}M$. [4]

(c) Hence, find the length of arc AB. [3]

The outer edge of the track, from C to D, is also a circular arc with centre F, such that the track is 2 metres wide.

(d) Calculate the area of the curved portion of the track, $ABDC$. [4]

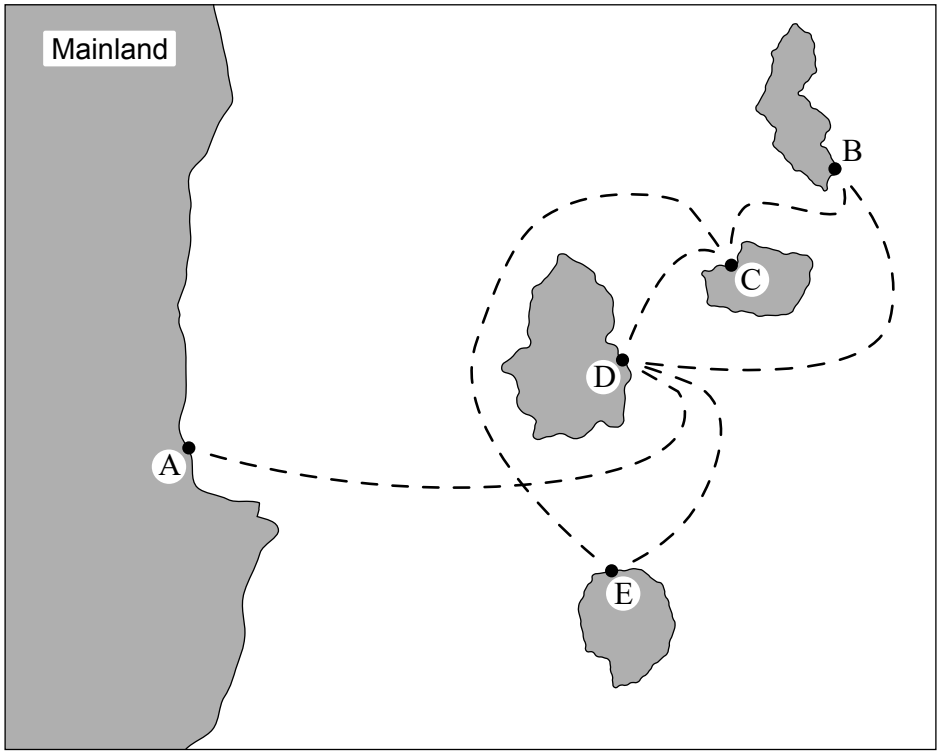
The base of the track will be made of concrete that is $12\,\text{cm}$ deep.

(e) Calculate the volume of concrete needed to create the curved portion of the track. [3]

Question 1 - working space

5. [Maximum mark: 18]

The following diagram is a map of a group of four islands and the closest mainland. Travel from the mainland and between the islands is by boat. The scheduled boat routes between the ports A, B, C, D and E are shown as dotted lines on the map.



Let the undirected graph G represent the boat routes between the ports A, B, C, D and E.

(a) Draw graph G. [1]

(b) Graph G can be represented by an adjacency matrix $\boldsymbol{P}$, where the rows and columns represent the ports in alphabetical order.

(i) Given that $\boldsymbol{P}^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$, find the value of a.

(ii) Hence, write down the number of different ways that someone could start at port B and end at port C, using three boat route journeys. [3]

(c) Find a possible Eulerian trail in G, starting at port A. [2]

(This question continues on the following page)

(Question 5 continued)

The cost of a journey on the different boat routes is given in the following table; all prices are given in USD. The cost of a journey is the same in either direction between two ports.

	A	**B**	**C**	**D**	**E**
A				10	
B			20	25	
C		20		50	45
D	10	25	50		30
E			45	30	

Sofia wants to make a trip where she travels on each of the boat routes at least once, beginning and ending at port A.

(d) Find the minimum cost of Sofia's trip. [3]

The boat company decides to add an additional boat route to make it possible to travel on each boat route **exactly** once, starting and ending at the same port.

(e) (i) Identify between which two ports the additional boat route should be added.

(ii) Determine the cost of the additional boat route such that the overall cost of the trip is the same as your answer to part (d). [2]

The boat company plans to redesign which ports are connected by boat routes. Their aim is to have a single boat trip that visits all the islands and minimizes the total distance travelled, starting and finishing at the mainland, A.

The following table shows the distances in kilometres between the ports A, B, C, D and E.

	A	**B**	**C**	**D**	**E**
A		80	90	50	60
B	80		30	70	120
C	90	30		45	100
D	50	70	45		55
E	60	120	100	55	

(f) (i) Use the nearest neighbour algorithm to find an upper bound for the minimum total distance.

(ii) Use the deleted vertex algorithm on port A to find a lower bound for the minimum total distance. [7]

Turn over

Question 5 - working space

6. [Maximum mark: 14]

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O. In one of his games, an object travels on a straight line L_1 with vector equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where $l,\ m \in \mathbb{Z}$. [1]

François uses the matrix $\boldsymbol{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2. The object will then travel along L_2.

(b) Find the vector equation of L_2. [4]

François knows that the transformation given by matrix $\boldsymbol{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan\alpha,\ 0 \leq \alpha < \pi$

(c) Write down the matrix that represents

(i) the rotation.

(ii) the enlargement. [2]

(d) The matrix $\boldsymbol{R}$ represents the reflection. Write down $\boldsymbol{R}$ in terms of α. [1]

(e) Given that $\boldsymbol{T} = \boldsymbol{RX}$,

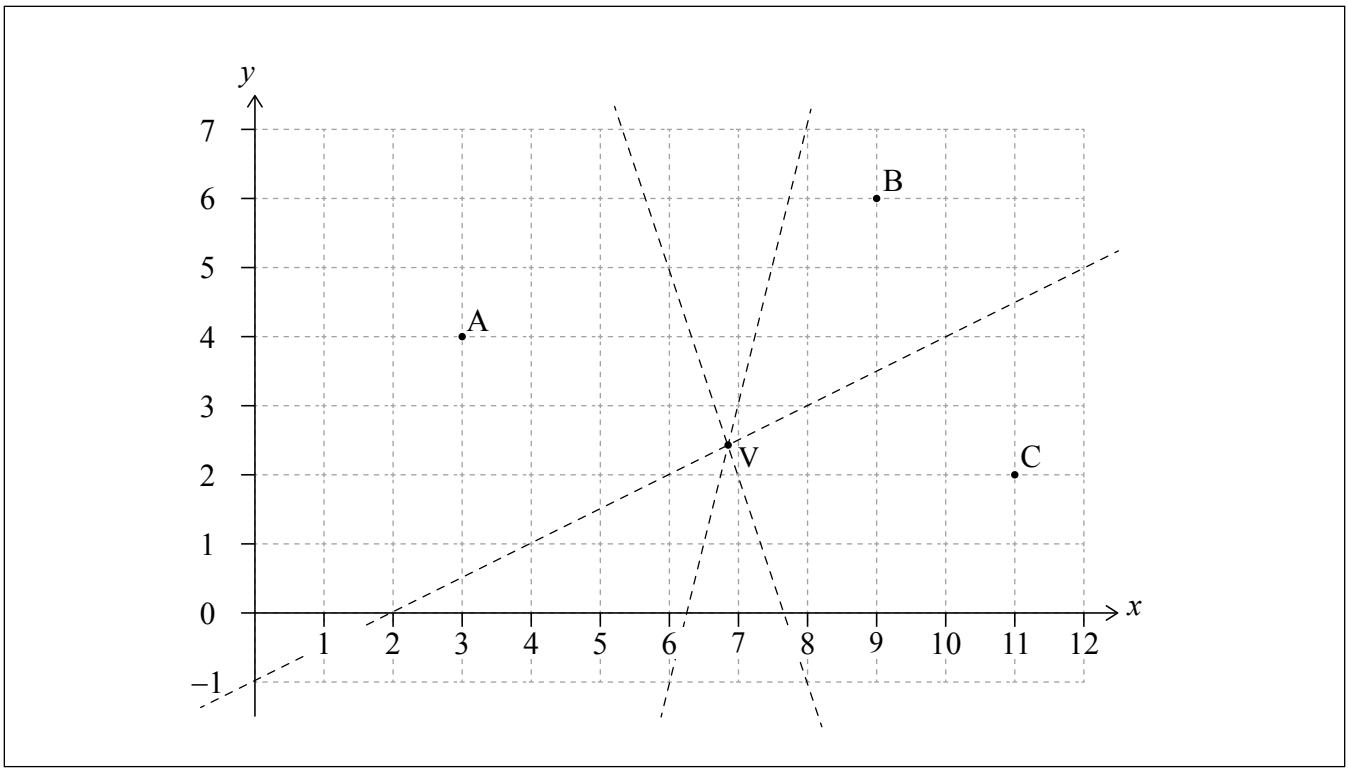
(i) use your answers to part (c) to find matrix $\boldsymbol{X}$.

(ii) hence, find the value of α. [6]

Question 6 - working space

5. [Maximum mark: 6]

Points $\mathrm{A}(3,\ 4)$, $\mathrm{B}(9,\ 6)$ and $\mathrm{C}(11,\ 2)$ are shown on the following diagram, along with the perpendicular bisectors of $[\mathrm{AB}]$, $[\mathrm{AC}]$ and $[\mathrm{BC}]$.



The perpendicular bisector of $[\mathrm{BC}]$ intercepts the axes at coordinates $(0,\ -1)$ and $(2,\ 0)$.

(a) Write down the equation of the perpendicular bisector of $[\mathrm{BC}]$. [2]

The equation of the perpendicular bisector of $[\mathrm{AB}]$ is $y = -3x + 23$.

(b) Find the coordinates of point V where the perpendicular bisectors meet. Give your answer to four significant figures. [2]

A Voronoi diagram is constructed with points A, B and C as the three sites.

(c) Draw, clearly, the edges of the Voronoi diagram on the given diagram. [2]

(This question continues on the following page)

(Question 5 continued)

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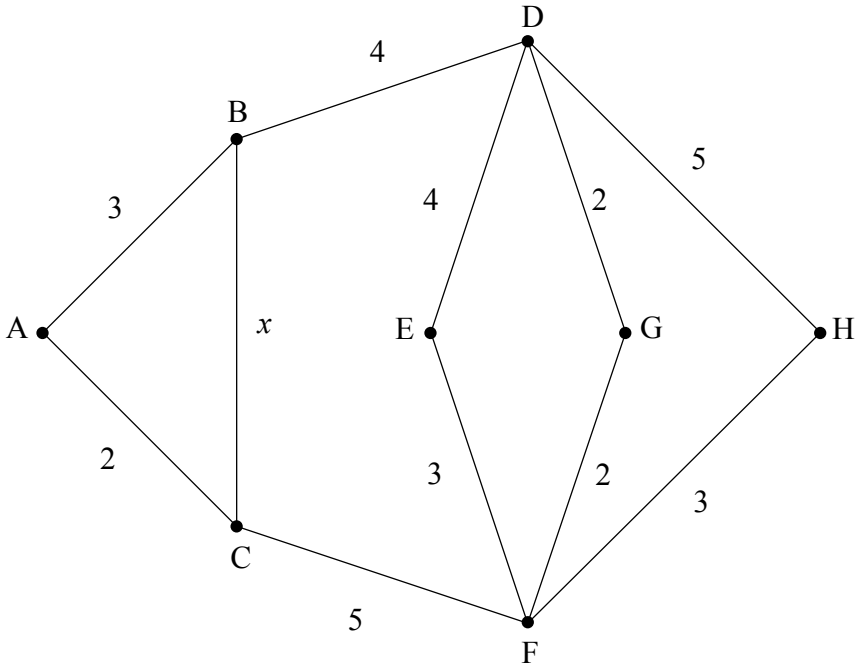
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Question 5 - working space

8. [Maximum mark: 5]

The weights on the following graph represent the lengths of different roads in kilometres.

diagram not to scale



(a) Write down the vertices with odd degree. [1]

The total length of the roads is $33 + x$ km.

(b) Find **two** expressions, in terms of x, for the shortest distance required to walk along all of the paths, beginning and ending at the same vertex.
Include in your answer the interval of values of x for which each expression is valid. [4]

(This question continues on the following page)

(Question 8 continued)

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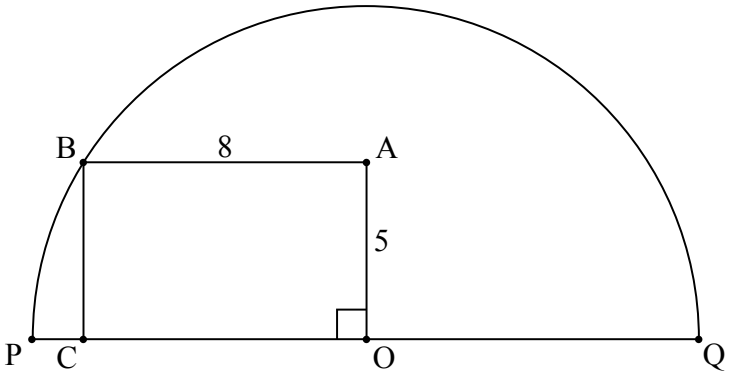


Question 8 - working space

10. [Maximum mark: 5]

The following diagram shows a semicircle with centre O and diameter PQ. A rectangle $OABC$ is also shown, such that $AB = 8$ and $OA = 5$.

diagram not to scale



Find the length of the arc BQ.

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Question 10 - working space

11. [Maximum mark: 8]

Let $\boldsymbol{R}(\alpha)$ be the matrix representing a rotation, counter-clockwise (anticlockwise) about the origin, through an angle of α.

(a) Write down $\boldsymbol{R}(2\alpha)$ as a 2×2 matrix. [2]

(b) Calculate $\boldsymbol{R}(\alpha) \times \boldsymbol{R}(\alpha)$. [2]

(c) Use your answers from part (a) and part (b) to

(i) explain why $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$.

(ii) show that $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$. [4]

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Question 11 - working space

6. [Maximum mark: 6]

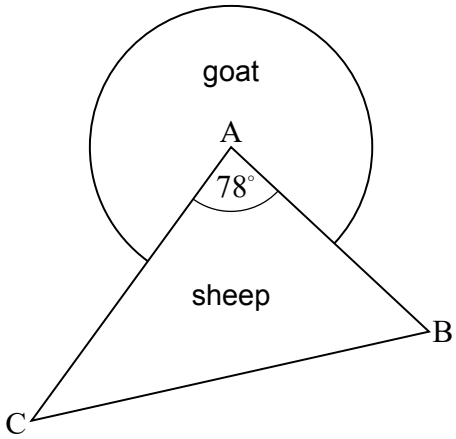
A sheep is in a field in the shape of a triangle, ABC.

$\mathrm{AC} = 21$ metres, $\mathrm{AB} = 15$ metres and $\mathrm{C\hat{A}B} = 78^\circ$.

A goat is in an adjacent field in the shape of a sector of a circle with centre, A, and radius 8 metres.

The fields are shown in the diagram.

diagram not to scale



Determine which animal, the sheep or the goat, is in the field with the larger area, and state how many extra square metres are in this larger field.

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Question 6 - working space

8. [Maximum mark: 10]

The quadrilateral $ABCD$ has coordinates $A(1, 3, 5)$, $B(4, 7, 5)$, $C(5, 8, 7)$ and $D(2, 4, 7)$.

(a) Write down $\vec{AD}$. [1]

(b) Calculate

(i) the size of $B\hat{A}D$.

(ii) the area of triangle BAD. [7]

(c) Show that $ABCD$ is a parallelogram. [2]

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Question 8 - working space

12. [Maximum mark: 7]

A duck is sitting in a duck pond at point $\mathrm{A}(7, 4, 0)$ relative to an origin O, where lengths are measured in metres and time, t, is measured in seconds. It takes off and flies in a straight line with vector equation

$$\boldsymbol{d} = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}.$$

(a) Find the speed of the duck through the air (in $\mathrm{m\,s^{-1}}$). [2]

A hawk hovering at position vector $\begin{pmatrix} -38 \\ 134 \\ 315 \end{pmatrix}$, relative to O, sees the duck take off and immediately dives from its position with constant velocity vector $\begin{pmatrix} 15 \\ -20 \\ -60 \end{pmatrix}$ to intercept the duck.

(b) Write down the vector equation for $\boldsymbol{h}$, that models the flight of the hawk. [1]

(c) Find the position vector at which the hawk intercepts the duck. [4]

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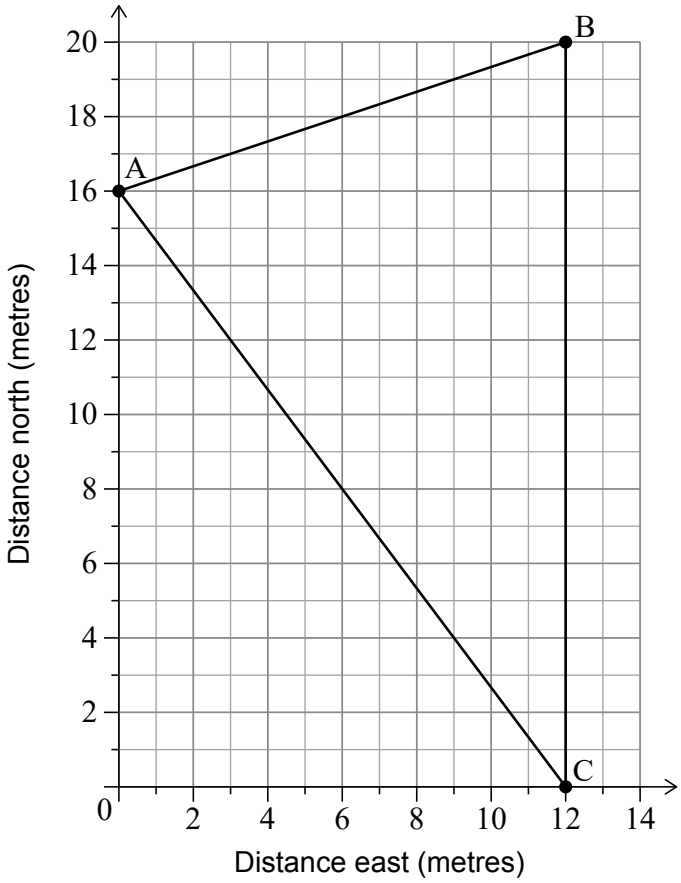
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Question 12 - working space

1. [Maximum mark: 14]

Mai is at an amusement park. A map of part of the amusement park is represented on the following coordinate axes.

Mai's favourite three attractions are positioned at $A(0, 16)$, $B(12, 20)$ and $C(12, 0)$.
All measurements are in metres.



(a) Write down the distance between B and C. [1]

(b) Calculate the distance between A and B. [2]

(This question continues on the following page)

(Question 1 continued)

Mai is standing at the attraction at B and wants to walk directly to the attraction at A.

(c) Calculate the bearing of A from B. [3]

A drinking fountain is to be installed at a point that is an equal distance from each of the attractions at A, B and C.

(d) (i) Write down the gradient of $[\mathrm{AC}]$.

(ii) Write down the mid-point of $[\mathrm{AC}]$.

(iii) Hence calculate the coordinates of the drinking fountain. [8]

Turn over

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Question 1 - working space

3. [Maximum mark: 22]

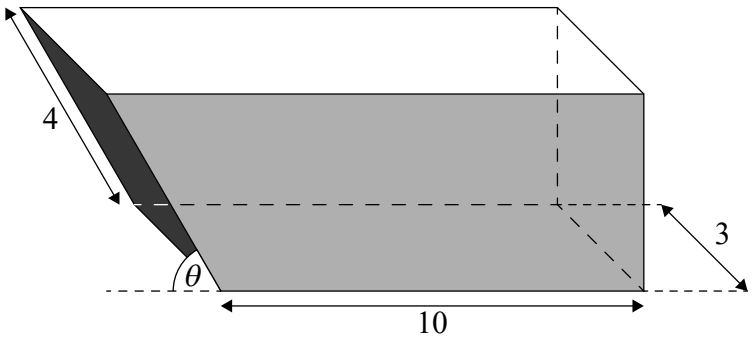
A skip is a container used to carry garbage away from a construction site. For safety reasons the garbage must not extend beyond the top of the skip. The maximum volume of garbage to be removed is therefore equal to the volume of the skip.



A particular design of skip can be modelled as a prism with a trapezoidal cross section. For the skip to be transported, it must have a rectangular base of length $10\,\text{m}$ and width $3\,\text{m}$. The length of the sloping edge is fixed at $4\,\text{m}$, and makes an angle of θ with the horizontal.

The following diagram shows such a skip.

diagram not to scale



(This question continues on the following page)

(Question 3 continued)

(a) Find the volume of this skip,

(i) if the length of the top edge of the skip is $11\,\text{m}$.

(ii) if the height of the skip is $3.2\,\text{m}$.

(iii) if θ is $\frac{\pi}{3}$. [9]

(b) Show that the volume, $V\,\text{m}^3$, of the skip is given by

$$24\sin(\theta)(5+\cos(\theta)).$$ [2]

(c) Explain, in context, why $\theta \neq 0$. [1]

(d) (i) Sketch the graph of $V = 24\sin(\theta)(5+\cos(\theta))$, $0 < \theta < \frac{\pi}{2}$.

(ii) Find the maximum volume of the skip and the value of θ for which this maximum volume occurs. [4]

(e) Show, by differentiation, that the maximum volume occurs at a value of θ that satisfies the equation $2\cos^2\theta + 5\cos\theta - 1 = 0$. [6]

Question 3 - working space

4. [Maximum mark: 12]

A hygiene inspector lives in Town A and must visit restaurants in five towns $(\mathrm{B}-\mathrm{F})$, before returning to A. The inspector must not repeat any of the towns. The distances, in km, between the six towns are shown in the table.

	A	**B**	**C**	**D**	**E**	**F**
A		31	28	26	22	23
B	31		25	20	27	25
C	28	25		19	22	24
D	26	20	19		21	22
E	22	27	22	21		24
F	23	25	24	22	24	

(a) Starting at A, use the nearest neighbour algorithm to find an upper bound for the length of the journey the inspector must take. State the order in which the towns are to be visited. [4]

(b) By deleting A, use Prim's algorithm starting at B to find a lower bound for the length of the inspector's journey. [5]

(c) By considering the minimum spanning tree found in part (b), determine whether the journey given by this lower bound is an achievable solution. [3]

Question 4 - working space

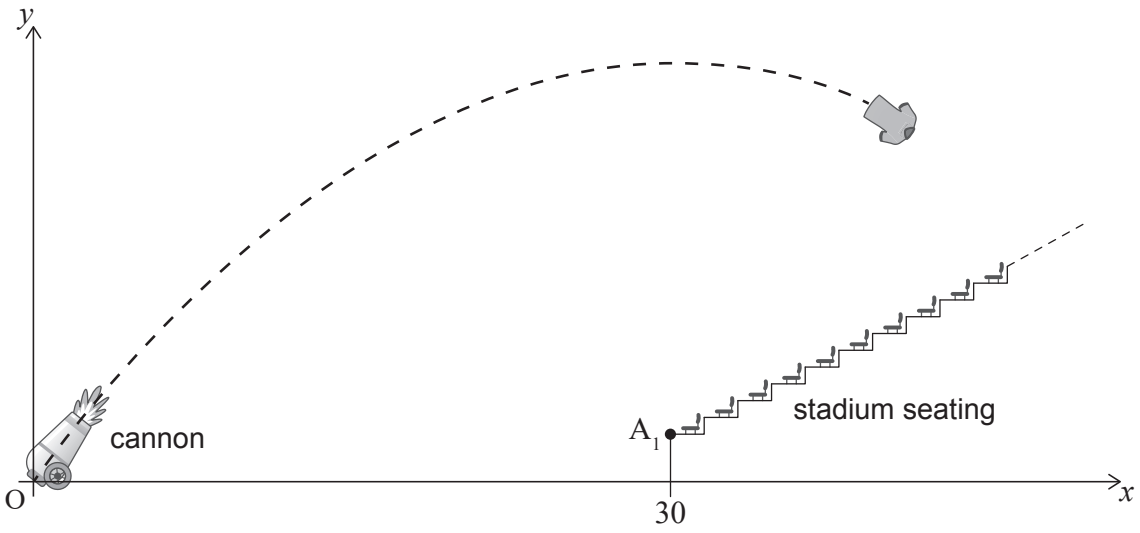
2. [Maximum mark: 29]

A sports stadium has a T-shirt cannon which is used to launch T-shirts into the crowd. The purpose of this question is to determine whether a person sitting in a particular seat will ever receive a T-shirt.

A T-shirt cannon is placed on the horizontal ground of a stadium playing area. A coordinate system is created such that the origin, O, is the point on the ground from where the T-shirts are launched. In this coordinate system, x and y represent the horizontal and vertical displacement from O, and are measured in metres.

Seat A_1 is the nearest seat to the T-shirt cannon. The coordinates of the front of the foot space for seat A_1 are $(30,\ 2.1)$.

diagram not to scale

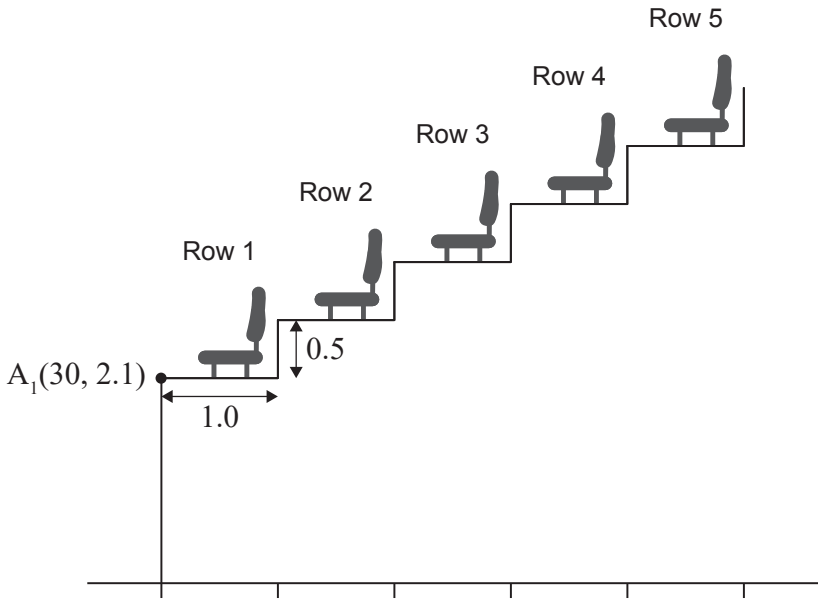


(This question continues on the following page)

(Question 2 continued)

Each seat behind seat A_1 is $1.0\,\text{m}$ further from O horizontally and $0.5\,\text{m}$ higher than the seat in the row below it, as shown on the diagram.

diagram not to scale



Seat A_1 is in row 1. Let seat A_n be the seat directly behind A_1 in row n.

(a) (i) Write down the coordinates of the front of the foot space of seat A_5. [2]

(ii) Find, in terms of n, the coordinates for the front of the foot space of seat A_n. [3]

While in motion, the T-shirt can be treated as a projectile.

Let t be the time, in seconds, after a T-shirt is launched.

At any time $t > 0$, the acceleration of the T-shirt, in m s^{-2}, is given by the vector

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}.$$

The initial velocity, in m s^{-1}, of the T-shirt is given as $\begin{pmatrix} 29.4\cos\theta \\ 29.4\sin\theta \end{pmatrix}$, where θ is the angle to the ground at which the T-shirt is launched and $0^\circ < \theta \leq 90^\circ$.

(b) (i) Find an expression for the velocity, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, at time t. [3]

(ii) Hence show that when the T-shirt is launched vertically, the time for it to reach its maximum height is 3 seconds. [3]

(This question continues on the following page)

(Question 2 continued)

The displacement of the T-shirt, t seconds after it is launched, is given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 29.4(\cos\theta)t \\ 29.4(\sin\theta)t - 4.9t^2 \end{pmatrix}.$$

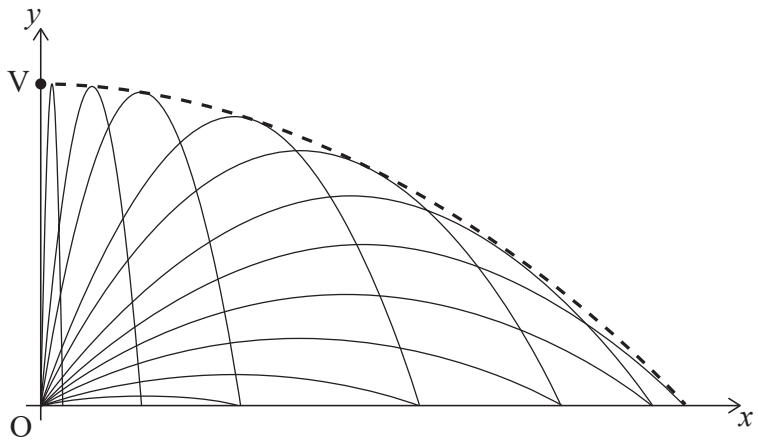
(c) Using the given answer to part (b)(ii) or otherwise, find the maximum height reached by a T-shirt when it is launched vertically. [2]

(d) (i) If there was no seating, and the T-shirt was launched at an angle θ, show that the value of x when it would hit the ground is given by the expression

$$x = 176.4\sin\theta\cos\theta.$$ [3]

(ii) Hence find the maximum possible value for x if there was no seating to block the path of the T-shirt. [2]

In order to calculate the seats in the stadium which can be reached by a T-shirt it is required to find the equation of the curve that forms the boundary of all the points that can be reached. This boundary is represented by the dashed curve in the following diagram, while the solid curves represent some of the possible trajectories for the T-shirts.



It is given that the boundary curve is the parabola $y = ax^2 + bx + c$, with its vertex V on the y-axis.

(e) Using your answers to parts (c) and (d)(ii), or otherwise, find

(i) the value of c. [1]

(ii) the value of b. [2]

(iii) the value of a. [3]

A spectator is sitting in seat A_{40}.

(f) Show that it is not possible for the spectator to ever get a T-shirt. [5]

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References:

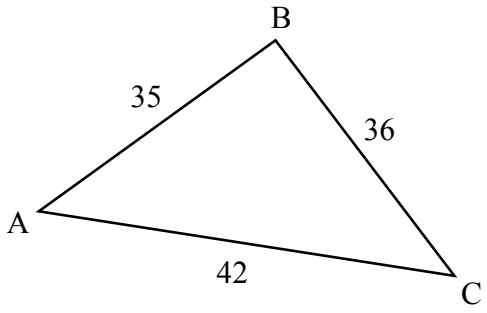
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Question 2 - working space

1. [Maximum mark: 5]

Consider the following triangle, ABC, such that $AB = 35\,\text{cm}$, $BC = 36\,\text{cm}$, and $CA = 42\,\text{cm}$.

diagram not to scale



(a) Find the value of $C\hat{A}B$. [3]

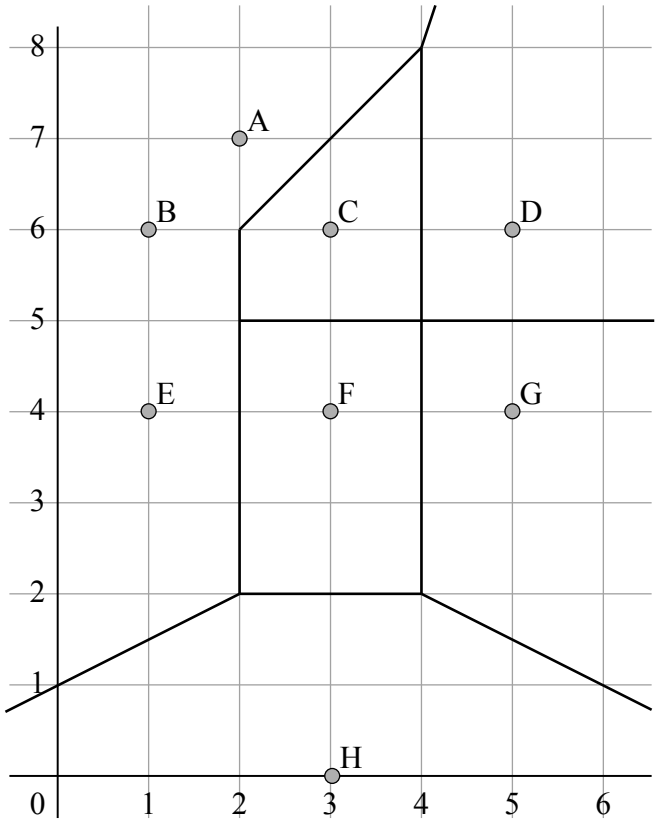
(b) Find the area of the triangle ABC. [2]

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Question 1 - working space

5. [Maximum mark: 6]

The sites in the Voronoi diagram represent eight hospitals in a city.



Each site has integer coordinates. Two edges are missing from the diagram.

(a) Draw the missing edges on the diagram. [2]

One square unit on the diagram represents $4\,\text{km}^2$ in the city.

(b) Find the area, in km^2, of the cell containing

(i) site F

(ii) site C. [3]

The hospitals at sites C and F have the same number of patients each year.

(c) Suggest a reason why the number of patients is not proportional to the area of the cell. [1]

(This question continues on the following page)

(Question 5 continued)

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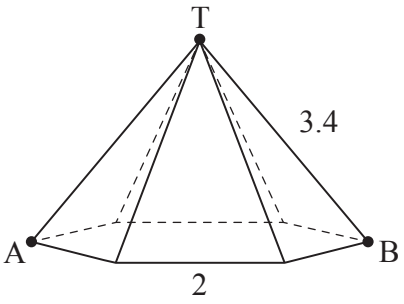
Question 5 - working space

4. [Maximum mark: 17]

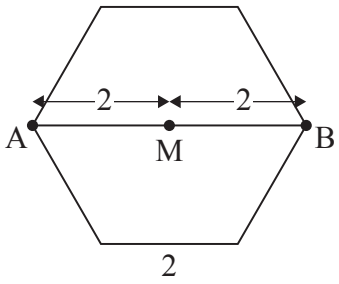
Gaurika is designing a tent in the shape of a right pyramid with a regular hexagonal base, centre M. The length of each side of the base is $2\,\mathrm{m}$, the length of each sloping edge is $3.4\,\mathrm{m}$, and the distance between each vertex on the base and M is $2\,\mathrm{m}$, as shown in the diagrams.

diagrams not to scale

3D view of tent



2D view of base

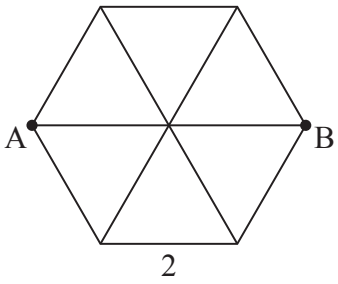


The top of the tent, T, will be supported by a vertical pole from M.

(a) Find the length of the pole, MT. [2]

The hexagonal base can be divided into six equilateral triangles.

diagram not to scale



(b) Find

(i) the area of the base

(ii) the volume of the tent. [5]

(c) Find the value of $\mathrm{M\hat{A}T}$. [2]

(This question continues on the following page)

(Question 4 continued)

For extra support, Gaurika decides to attach a rope, with length $2.6\,\text{m}$, to the tent at a point, X, on the edge AT.

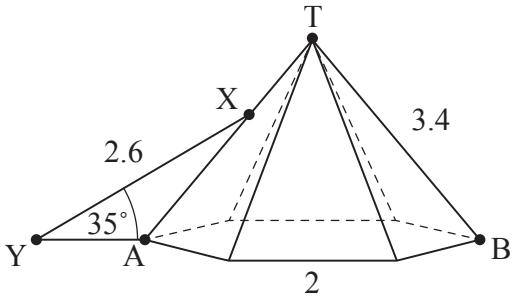
The rope will be fixed to the ground at point Y, such that:

- the rope, $[\text{XY}]$, is straight
- points Y, A and B lie on a straight line
- $\text{A}\hat{\text{Y}}\text{X} = 35°$.

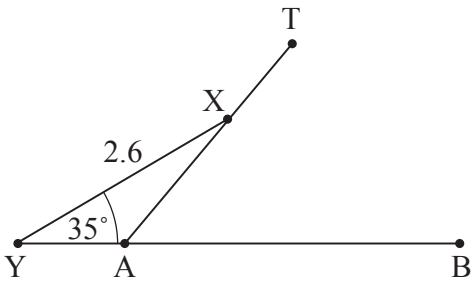
This is shown in the diagrams.

diagrams not to scale

3D view of tent and rope



2D view of triangle AXY



(d) Find AY. [4]

For decoration at night, Gaurika wants to fix a strip of lights from point A to a point, Z, along the rope $[\text{XY}]$.

The strip of lights, $[\text{AZ}]$, is straight and has length $0.9\,\text{m}$.

(e) Find the two possible values of YZ. [4]

Question 4 - working space

5. [Maximum mark: 15]

In this question, all distances are in kilometres and t is in hours.

Let $\begin{pmatrix}1\\0\\0\end{pmatrix}$ be a displacement of $1\,\text{km}$ due east, $\begin{pmatrix}0\\1\\0\end{pmatrix}$ be a displacement of $1\,\text{km}$ due north, and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ be a vertical displacement of $1\,\text{km}$.

Highway 85 in Saudi Arabia is a long, straight, flat road.

Relative to the centre of the town Arar, point O, the position vector of a car, C, travelling along this road is given by:

$$\overrightarrow{\text{OC}} = \begin{pmatrix}10\\-5\\0\end{pmatrix} + t\begin{pmatrix}50\\-33\\0\end{pmatrix}.$$

(a) Find the speed of the car. [2]

The police are testing a long-range drone, D, to monitor cars travelling along this road. The drone is launched at $t = 0$ from the point with position vector $\begin{pmatrix}200\\-100\\0.02\end{pmatrix}$ and flies in a straight line with a constant height of $0.02\,\text{km}$ and a constant velocity of $\begin{pmatrix}-15\\-20\\0\end{pmatrix}$.

(b) Find the angle between the path of the car and the path of the drone. [3]

(c) Write down the position vector, $\overrightarrow{\text{OD}}$, of the drone at time t. [1]

(This question continues on the following page)

(Question 5 continued)

(d) At time t_1, the drone passes through the point with position vector $\begin{pmatrix} 152 \\ p \\ 0.02 \end{pmatrix}$.

Find the value of

(i) t_1

(ii) p. [3]

(e) (i) Find an expression for $\vec{CD}$, the relative position of the drone from the car.

(ii) Hence, find the shortest distance between the car and the drone. [6]

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Question 5 - working space

6. [Maximum mark: 13]

Juan is creating animations for a website. He uses matrices to transform objects relative to the origin, O.

One matrix that he uses is $\boldsymbol{A} = \begin{pmatrix} \cos(15°) & -\sin(15°) \\ \sin(15°) & \cos(15°) \end{pmatrix}$.

(a) Describe fully the transformation represented by matrix $\boldsymbol{A}$. [1]

(b) Find the smallest value of n such that $\boldsymbol{A}^n = I$, $n \in \mathbb{Z}^+$. [2]

Juan also uses matrix $\boldsymbol{B}$, which represents an enlargement with a scale factor of 1.05, centre $(0, 0)$.

(c) (i) Write down matrix $\boldsymbol{B}$.

(ii) Describe fully the transformation represented by $\boldsymbol{B}^n$, where n is the value found in part (b). [3]

Juan creates a new matrix, $\boldsymbol{C} = \boldsymbol{AB}$.

(d) Find matrix $\boldsymbol{C}$. [2]

Juan creates an animation by repeatedly transforming an object by $\boldsymbol{C}$.

A point, P, on the object is initially at $(1, 0)$. Juan sets the speed of the animation to 6 transformations per second.

(e) Sketch the path of P for the first 4 seconds of motion **and** label the coordinates of the start and end points. [2]

Juan uses a different transformation, $\boldsymbol{T}$, defined by

$$\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To create his animation, he repeatedly transforms an object by $\boldsymbol{T}$. After many transformations, he notices that all points, (x, y), on the object tend towards a single point, (p, q), such that

$$\lim_{a \to \infty} T^a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \text{ where } a \in \mathbb{Z}^+.$$

(f) Find $\begin{pmatrix} p \\ q \end{pmatrix}$, where $p, q \in \mathbb{R}$. [3]

Turn over

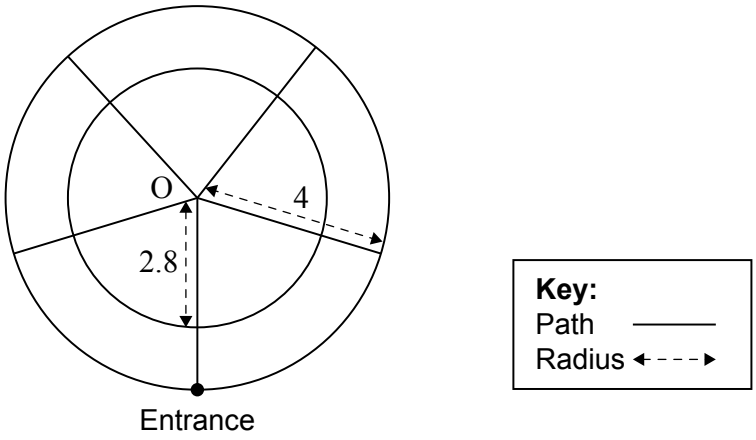
Question 6 - working space

1. [Maximum mark: 6]

A child's game is played by making paths in the snow. First, two circular paths are made using the same centre, O. The radius of the smaller circle is $2.8\,\text{m}$, and the radius of the larger circle is $4\,\text{m}$. Additional paths are then made from O to the outer edge of the larger circle, dividing each circle into 5 equal sectors, as shown in the following diagram.

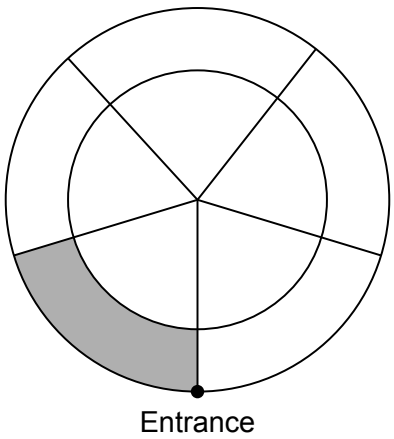
For your calculations, ignore the widths of the paths.

diagram not to scale



Each region **between** paths will be coloured with environmentally friendly dye. The shaded region below will be coloured orange.

diagram not to scale



Maureen has enough orange dye to cover an area of $6\,\text{m}^2$.

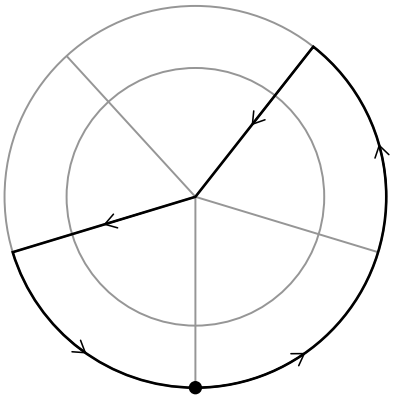
(This question continues on the following page)

(Question 1 continued)

(a) Show that Maureen has enough orange dye to cover the shaded region. [3]

During the game, the players start at the entrance and must travel only along the paths made in the snow. Maureen travels from the entrance along the path shown in the following diagram.

diagram not to scale



(b) Calculate the distance Maureen travels along this path, starting from the entrance and returning to the entrance. [3]

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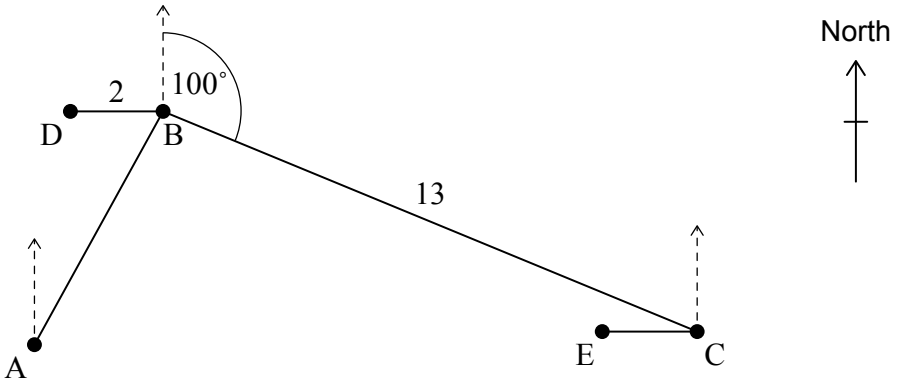
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Question 1 - working space

5. [Maximum mark: 9]

Ticky is sailing a boat in the ocean. For the first part of her trip, she plans to start at point A and travel to point B. She then plans to turn and travel $13\,\text{km}$ on a bearing of $100°$ to point C as shown in the following diagram.

diagram not to scale



Due to navigational error in the first part of the trip, Ticky arrives at point D instead of point B. Point D is $2\,\text{km}$ due west of point B.

(a) Determine

(i) the distance from point D to point C.

(ii) the bearing Ticky must use to travel directly from point D to point C. [7]

Due to navigational error in the second part of the trip, Ticky arrives at point E instead of point C. Point E is $2\,\text{km}$ due west of point C.

(b) Write down the distance between point D and point E. Justify your answer. [2]

(This question continues on the following page)

(Question 5 continued)

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Question 5 - working space

13. [Maximum mark: 7]

The points $\mathrm{R}(-2, 2)$, $\mathrm{S}(3, -5)$, and $\mathrm{T}(5, 1)$ are the vertices of a triangle. The triangle goes through a sequence of transformations in the following order:

I. a transformation using the matrix $\boldsymbol{A} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$

II. a transformation using the matrix $\boldsymbol{B} = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$.

(a) Describe the nature of the transformation represented by $\boldsymbol{A}$. [2]

Instead of using this sequence of transformations, the image of triangle RST can be found using only a reflection of triangle RST in line $y = ax$.

(b) Given that $a > 0$, find the value of a. [5]

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Question 13 - working space

2. [Maximum mark: 9]

The point A has coordinates $(1, 2, 1)$ and the point B has coordinates $(3, 5, 2)$.

(a) Find AB. [2]

Triangle ABC is right-angled with its right angle at B. The point C has coordinates $(2, 8, k)$.

(b) Find the value of k. [4]

(c) Calculate the size of $\mathrm{B\hat{A}C}$. [3]

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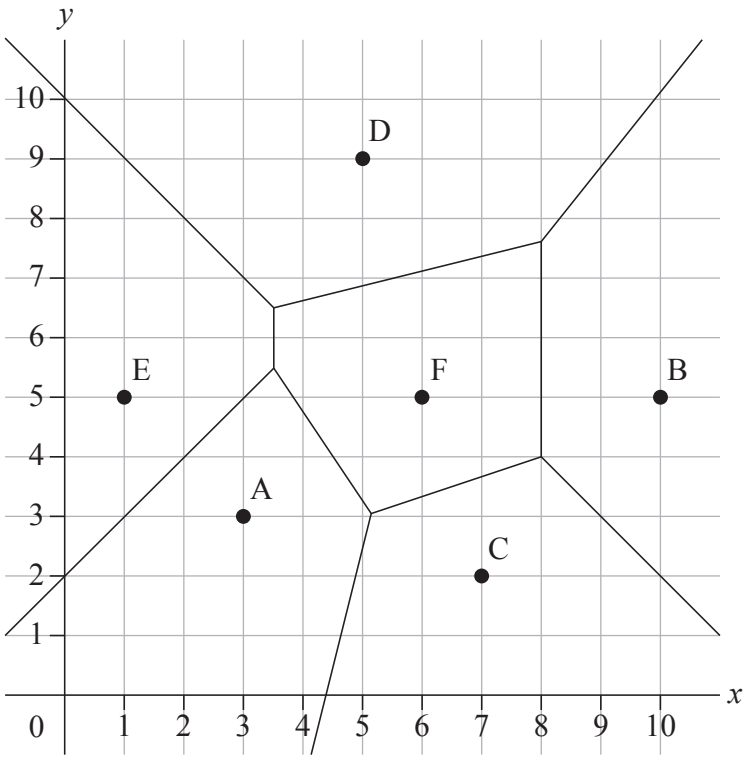
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Question 2 - working space

4. [Maximum mark: 7]

Consider the Voronoi diagram which shows the sites $\mathrm{A}(3, 3)$, $\mathrm{B}(10, 5)$, $\mathrm{C}(7, 2)$, $\mathrm{D}(5, 9)$, $\mathrm{E}(1, 5)$ and $\mathrm{F}(6, 5)$. The diagram also shows the cells formed by each site and their boundaries.



Vertex X is equidistant from sites B, C and F.

(a) (i) Write down the coordinates of X.

(ii) The exact value of BX is $\sqrt{n}$. Write down the value of n. [2]

Vertex $\mathrm{Y}(a, b)$ is equidistant from sites B, D and F.

(b) (i) Write down the value of a.

(ii) Find the exact value of b. [5]

(This question continues on the following page)

(Question 4 continued)

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Question 4 - working space

8. [Maximum mark: 6]

A mapping system stores the connections between 5 towns, labelled A, B, C, D and E, in an adjacency matrix. The adjacency matrix, with rows and columns in alphabetical order, is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Draw and label a graph to represent the adjacency matrix. [2]

(b) Determine the number of walks of length 4 which start and end at the same town. [4]

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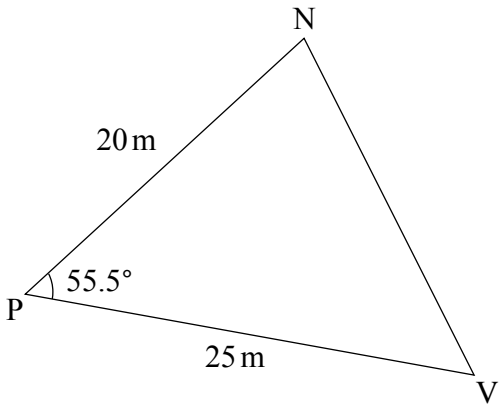
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Question 8 - working space

1. [Maximum mark: 9]

Three points N, P, and V are shown on the following diagram. NP is 20 metres, PV is 25 metres and $V\hat{P}N$ is $55.5°$.

diagram not to scale



(a) Find NV. [3]

(b) Find $P\hat{N}V$. [3]

(c) Hence or otherwise, find the shortest distance between P and $[NV]$. [3]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

8. [Maximum mark: 5]

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ transforms a point A to a point B.

The matrix $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ transforms the point B to a point C with coordinates $(4, 7)$.

Find the coordinates of A.

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Question 8 - working space

14. [Maximum mark: 8]

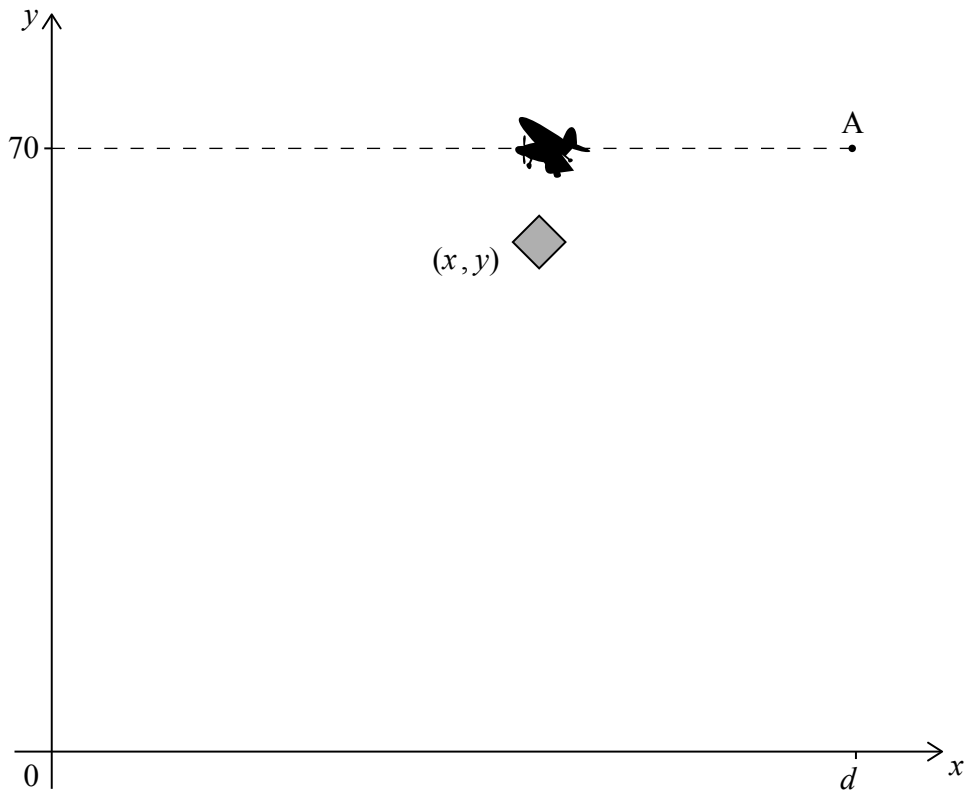
A plane needs to drop a relief package for a remote village. The villagers have marked out a point O in the middle of a field where the package should land.

The package is released from the plane at point A when it is a horizontal distance of d metres from O.

The position vector of the package t seconds after release is given by the vector $\begin{pmatrix} x \\ y \end{pmatrix}$,

where x is the horizontal displacement of the package from O and y is the vertical height of the package above the ground. This is shown in the following diagram.

diagram not to scale



(This question continues on the following page)

(Question 14 continued)

The acceleration of the package is $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \text{ms}^{-2}$ and its velocity at the point of release is $\begin{pmatrix} -20 \\ 0 \end{pmatrix} \text{ms}^{-1}$.

(a) Find the velocity vector of the package $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at time t. [2]

The vertical height of the plane above the ground at the moment the package is released is $70\,\text{m}$.

(b) Find the value of d, given that the package lands at O. [6]

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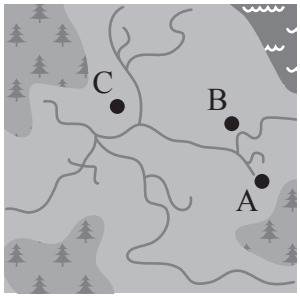
Question 14 - working space

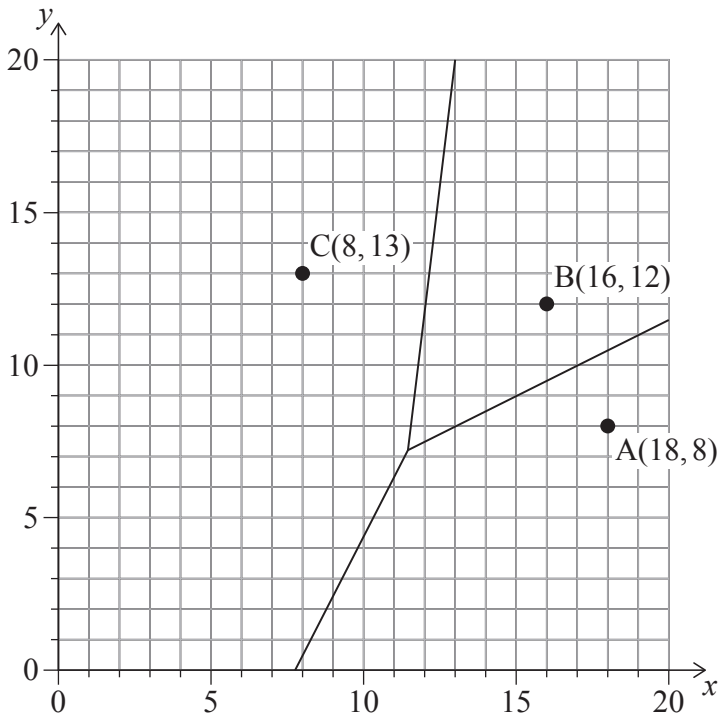
2. [Maximum mark: 19]

The locations of three fire stations within a 50 by 50 kilometre rural area of Japan are shown on the following map.

A Voronoi diagram can be used to determine the part of the rural area for which each fire station is responsible. The Voronoi diagram for the three fire stations is shown on the coordinate grid beside the map, where 1 unit represents $2.5\,\text{km}$.

Map





(a) (i) Find the midpoint of $[\text{BC}]$.

(ii) Find the gradient of the perpendicular bisector of $[\text{BC}]$.

(iii) Hence find the equation of the edge between sites B and C. [6]

(b) (i) Identify the fire station that is expected to respond, based on the Voronoi diagram, if a fire is reported at a location with coordinates $(14, 10)$. Justify your response.

(ii) Suggest a reason why a different fire station might respond to this fire. [2]

(This question continues on the following page)

(Question 2 continued)

A fire is reported at a location D with coordinates $\left(11\frac{1}{3}, 7\frac{1}{6}\right)$. The distance of D from A on the Voronoi diagram, to six significant figures, is 6.71855 units ($16.7964\,\mathrm{km}$ on the map).

(c) (i) Show that the distance of D from B, to six significant figures, is also 6.71855 units.

(ii) Show that any of the three fire stations would be expected to respond to the fire reported at D. [4]

The equation of the edge between fire stations A and B is $y = 0.5x + 1.5$. The edge between fire stations A and C has an x-intercept of 7.75.

(d) (i) Determine the area, to the nearest square unit, of the region within the given Voronoi diagram for which fire station A is responsible.

(ii) Hence, calculate the actual area, in square kilometres, of this region. [7]

Question 2 - working space

4. [Maximum mark: 17]

Lizjerika works in an air traffic control tower. At 10:00 am, she detects a passenger airplane and a flock of birds whose flight paths are given by the following equations:

$$\text{Airplane: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + t_1 \begin{pmatrix} -1.4 \\ 1.65 \\ 0 \end{pmatrix} \qquad \text{Birds: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \\ 6 \end{pmatrix} + t_1 \begin{pmatrix} -0.40 \\ -0.35 \\ 0.1 \end{pmatrix}$$

where the x-coordinate represents displacement east of the control tower, the y-coordinate represents displacement north of the control tower, and the z-coordinate represents height (above sea level).

Distances are measured in kilometres and t_1 is the time, in minutes, since 10:00am.

(a) Determine the speed, in kilometres per minute, at which the flock of birds is travelling. [2]

(b) By making two statements, describe the path of the airplane in context. [2]

(c) (i) Find the time at which the birds and the airplane will have the same height.

(ii) Using your answer to part (c)(i), explain why the birds will not collide with the airplane. [6]

At 10:20 am, Lizjerika detects a jet whose movement is given by the following equation,

$$\text{Jet: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -95 \\ 32 \\ 1.5 \end{pmatrix} + t_2 \begin{pmatrix} 1.3 \\ 1.45 \\ 0.2 \end{pmatrix}$$

where t_2 is the time, in minutes, since 10:20am.

(d) Find the vector equation of the motion of **the passenger airplane** in terms of t_2. [2]

An air traffic law requires that the jet and the airplane must be more than 5 kilometres from each other at all times.

(e) Determine whether the jet and the airplane will break this law if they follow their current paths. [5]

Turn over

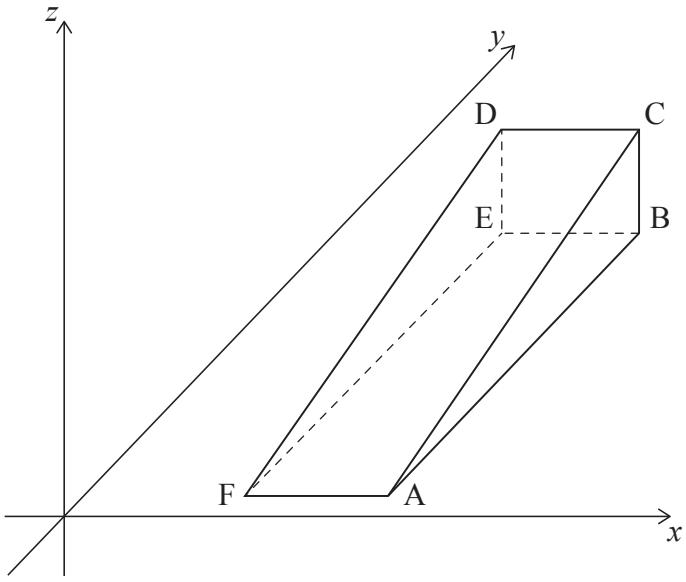
Question 4 - working space

5. [Maximum mark: 16]

AirFlow Industries is designing a doorstop in the shape of a triangular prism, $ABCDEF$.

The points $A(13, 1, 0)$, $B(13, 25, 0)$ and $C(13, 25, 7)$ are the coordinates of the vertices of one of the triangular faces of the prism. Point $D(4, 25, 7)$ is another vertex as shown in the following diagram. All the measurements are in centimetres.

diagram not to scale



(a) Find AC. [2]

(b) Show that triangle ABC is a right-angled triangle. [2]

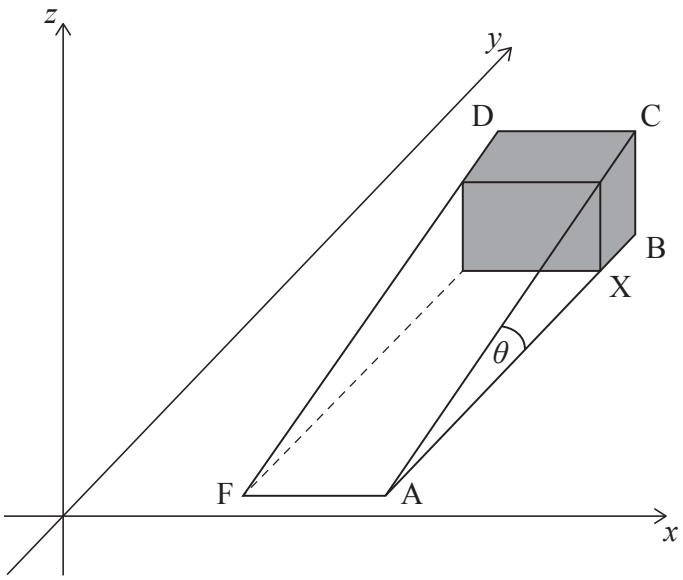
(c) Find the volume of material needed to make the doorstop. [3]

(This question continues on the following page)

(Question 5 continued)

To lower the cost of the doorstop, AirFlow decides to reduce its volume to $625\,\text{cm}^3$ by removing the shaded section shown in the following diagram. The remaining triangular faces are still right-angled triangles.

diagram not to scale



(d) (i) Find the value of θ.

(ii) Hence, or otherwise, find AX. [6]

The material needed to make the doorstop costs 0.025 US dollars (USD) per cubic centimetre. The production of each doorstop requires $10\,\%$ more material than its final volume, due to wastage in the production process.

To make a profit, the company will sell the doorstop for $20\,\%$ more than the cost of the material.

(e) Determine the price, to two decimal places, the company will charge for the **new** doorstop. [3]

Turn over

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Question 5 - working space

4. [Maximum mark: 13]

A wind farm consists of five wind turbines, located at points A to E.

The table below shows the distances, in kilometres, between each pair of turbines.

	A	B	C	D	E
A		0.90	0.88	1.56	0.86
B	0.90		0.74	0.94	1.28
C	0.88	0.74		0.78	0.62
D	1.56	0.94	0.78		1.36
E	0.86	1.28	0.62	1.36	

The turbines must all be connected by cables. However, there does not need to be a direct connection between every pair.

(a) Use Prim's algorithm, starting with vertex A, to find the minimum total length of cable required to connect the turbines. Show the order in which you added the vertices. [4]

The supervisor of the wind farm has a monitoring cabin located at point F. The distances from F to each turbine are shown in the table.

Turbine	A	B	C	D	E
Distance from F (km)	0.96	1.82	1.57	2.24	1.14

The supervisor wants to visit every turbine exactly once for inspection, starting and finishing at the cabin, and using the route of shortest possible length.

(b) By deleting vertex F, find a lower bound for the length of the shortest route. [2]

(c) Use the nearest neighbour algorithm starting at F to find an upper bound for the length of the shortest route. [3]

(This question continues on the following page)

(Question 4 continued)

The table below shows the lower bounds found by deleting each of the other five vertices, and the upper bounds found by starting at each of the other five vertices.

Vertex	A	B	C	D	E
Lower bound	5.02	4.86	5.02	4.90	4.84
Upper bound	6.36	6.36	7.13	7.22	6.82

The supervisor travels between the turbines at a constant speed of $28\,\text{km/h}$ and spends 12 minutes inspecting each turbine.

(d) Based on the information given in the table, find the best possible upper and lower bounds for the shortest amount of time, T hours, required for the inspection. Write your answer as an inequality. [4]

Question 4 - working space

4. [Maximum marks: 16]

The ferry routes between five ports, A to E, can be represented by a graph, G.

The adjacency matrix $\boldsymbol{M}$ for the graph G is shown below.

$$\boldsymbol{M} = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(a) State which direct route between ports can only be travelled in one direction. [1]

(b) By finding an appropriate matrix

(i) show that it is not possible to reach all of the ports from any other port in **exactly** three trips.

(ii) state whether it is possible to reach all of the ports from any other port in three trips or fewer. Justify your answer. [5]

In the summer there is a new timetable for ferry routes. The least cost, in euros (EUR), to travel between any two of the ports is shown in the table below.

Port	A	B	C	D	E
A	0	3	7	8	5
B	3	0	4	9	6
C	7	4	0	a	10
D	8	9	a	0	3
E	5	6	10	3	0

(c) Given there is no direct route between D and C or C and D, show that the value of a is $13\,\mathrm{EUR}$. In your answer ensure you have considered all possible routes. [3]

(d) Starting at A, use the nearest neighbour algorithm to find an upper bound for the cost of visiting all the ports and returning to your starting position. [3]

(e) By deleting the vertex A, use the deleted vertex algorithm to find a lower bound for the cost of visiting all the ports. [4]

Question 4 - working space

5. [Maximum mark: 15]

The position of a particle, P_1, t seconds after leaving a point A, is given by

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 4 \\ 7 \end{pmatrix},\ t \geq 0.$$

The units of distance are metres.

(a) Write down the coordinates of A. [1]

Five seconds after leaving A, P_1 is at point B.

(b) Find

(i) $\vec{\mathrm{AB}}$.

(ii) $\left|\vec{\mathrm{AB}}\right|$. [4]

A second particle P_2 leaves A at the same time as P_1. P_2 is moving in the direction of the vector $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, and passes through a point C.

(c) Find $\mathrm{C\hat{A}B}$. [5]

The particle P_2 has a speed of $12\,\mathrm{m\,s^{-1}}$.

(d) Find the distance from P_1 to P_2 when $t = 5$. [5]

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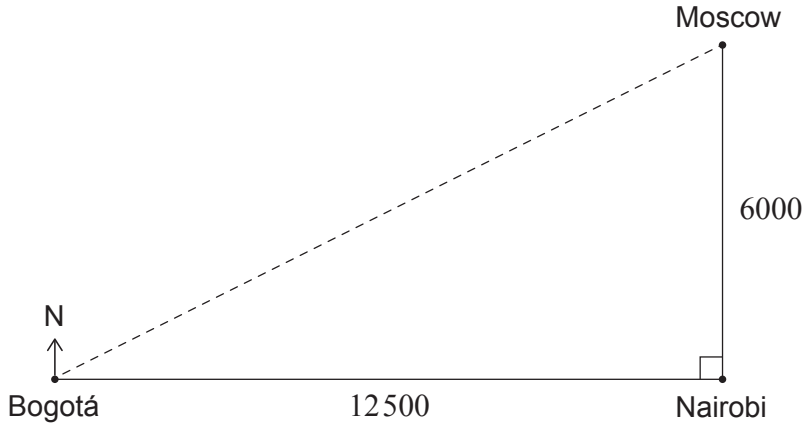
Question 5 - working space

2. [Maximum mark: 27]

The following question compares the distance and direction between cities on a flat surface to the distance and direction between cities on a sphere.

Consider a model where the cities of Bogotá, Moscow, and Nairobi lie on a flat surface. In this model, Nairobi is $6000\,\text{km}$ due south of Moscow and Bogotá is $12\,500\,\text{km}$ due west of Nairobi, as shown in the following diagram.

diagram not to scale



(a) (i) Find the distance from Bogotá to Moscow. [2]

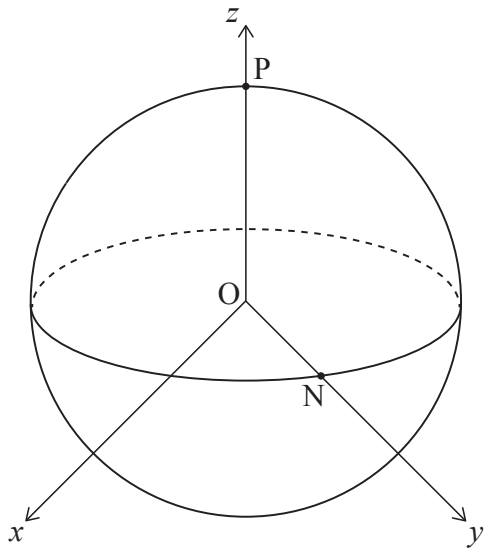
(ii) Find the bearing of Moscow from Bogotá. Give your answer in degrees. [3]

(This question continues on the following page)

(Question 2 continued)

In reality, these three cities lie on the curved surface of the Earth which will change the distances and directions found in part (a).

Now consider a curved model using a coordinate system (x, y, z) with its origin, O, at the centre of the Earth. The units of this system are thousands of kilometres and the Earth is modelled as a sphere with radius $6000\,\mathrm{km}$. The North Pole, P, lies on the z-axis, and Nairobi, N, is modelled as being on the equator and lying on the y-axis.



P has position vector $\overrightarrow{\mathrm{OP}} = \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ and N has position vector $\overrightarrow{\mathrm{ON}} = \boldsymbol{n} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$.

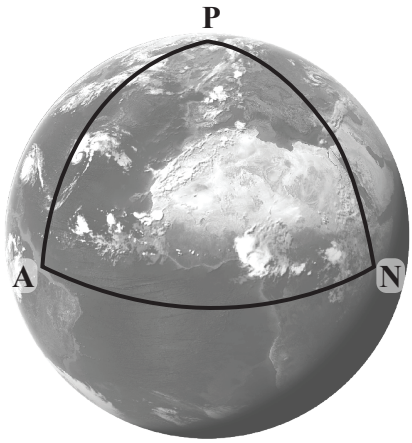
(b) (i) Use the scalar product to find the angle between $\boldsymbol{p}$ and $\boldsymbol{n}$. [2]

(ii) Show that the distance between P and N along the arc from P to N is $3000\pi\,\mathrm{km}$. [2]

(This question continues on the following page)

(Question 2 continued)

Point A, which is also on the equator, has position vector $\boldsymbol{a} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ as shown in the following diagram.



P, N and A, and the arcs connecting them, form a spherical triangle.

The angle at vertex A is defined as the angle between the vectors $\boldsymbol{a} \times \boldsymbol{p}$ and $\boldsymbol{a} \times \boldsymbol{n}$.

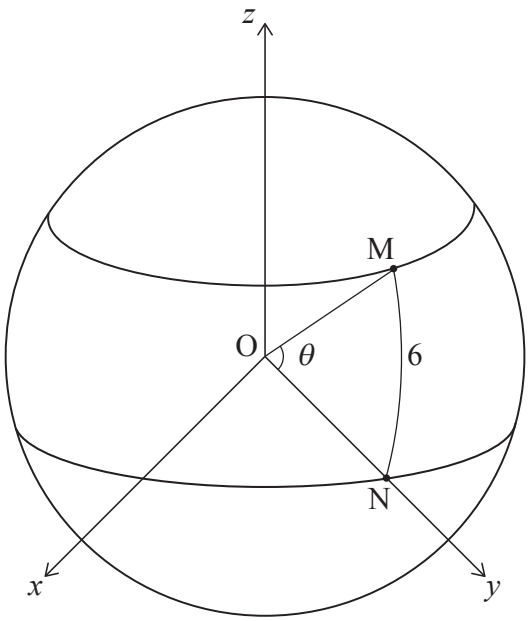
(c) (i) Find the vector $\boldsymbol{a} \times \boldsymbol{p}$. [2]

(ii) Show that the angle at vertex A in the spherical triangle is $90°$. [3]

(This question continues on the following page)

(Question 2 continued)

Moscow, M, has position vector $\overrightarrow{\mathrm{OM}} = \boldsymbol{m} = \begin{pmatrix} 0 \\ 6\cos\theta \\ 6\sin\theta \end{pmatrix}$, as shown in the following diagram.



The shortest distance between two points on the sphere lies along an arc of a circle on the sphere with centre O. In this model the shortest distance from Moscow to Nairobi is $6000\,\mathrm{km}$.

(d) Show that $\theta = 57.3°$, correct to three significant figures. [2]

Bogotá, B, is west of Nairobi and has position vector $\overrightarrow{\mathrm{OB}} = \boldsymbol{b} = \begin{pmatrix} 6\sin 120^\circ \\ 6\cos 120^\circ \\ 0 \end{pmatrix}$.

(e) Find the shortest distance from Bogotá to Moscow on the sphere. [5]

The bearing from B to M is defined as the angle at vertex B in the spherical triangle containing B, M and P. It is given that $\boldsymbol{b} \times \boldsymbol{p} = \begin{pmatrix} 36\cos 120^\circ \\ -36\sin 120^\circ \\ 0 \end{pmatrix}$.

(f) Using the method from part (c), find the bearing from Bogotá to Moscow. [6]

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References:

2(c) Google Maps/Google Earth. Data from Data SIO, NOAA, U.S. Navy, NGA, GEBCOLandsat / Copernicus, IBCAO, U.S. Geological Survey, PGC/NASA. Imagery from 14/12/2015. Image available at: https://earth.google.com/web/@12.01529518,-18.56070747,-158.39383184a,23597813.93249989d,30.00008083y,359.99981502h,0t,0r/data=CgRCAggBOgMKATBCAggASg0I_____ARAA.

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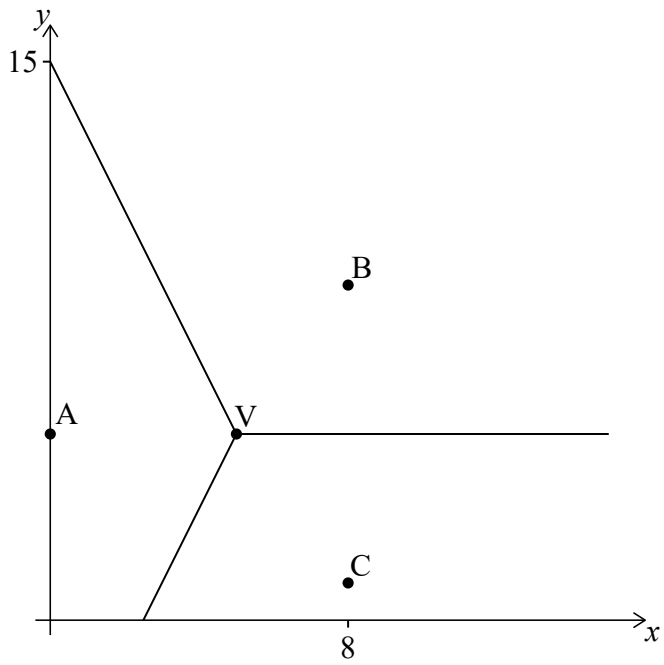
Question 2 - working space

1. [Maximum mark: 28]

This question considers how the assessment of the Air Quality Index (AQI) for a school depends on the method chosen by the person doing the assessing.

Air quality for a district is measured at three monitoring stations. The positions of these stations on a coordinate system with units in kilometres are $\mathrm{A}(0, 5)$, $\mathrm{B}(8, 9)$ and $\mathrm{C}(8, 1)$.

A Voronoi diagram is constructed with the three stations as sites.



(a) (i) Find the equation of the perpendicular bisector of $[\mathrm{BC}]$. [2]

(ii) Given that the equation of the perpendicular bisector of $[\mathrm{AB}]$ is $y = 15 - 2x$, find the coordinates of vertex V. [2]

(This question continues on the following page)

(Question 1 continued)

A school, S, is situated in the district at the point with coordinates $(5, 6)$.

(b) State which station is closest to the school. [1]

The principal of the school is concerned about the air quality in the area. Air quality is measured by the Air Quality Index (AQI). In this district, values less than 50 are taken to indicate good air quality.

The principal contacts the local environmental agency requesting an AQI value for her school. They tell her the mean AQI reading from the closest station to the school.

(c) Write down the type of interpolation being used by the environmental agency. [1]

The principal obtains the mean AQI value from each of the three stations.

Station	Mean AQI value
A	132
B	49
C	125

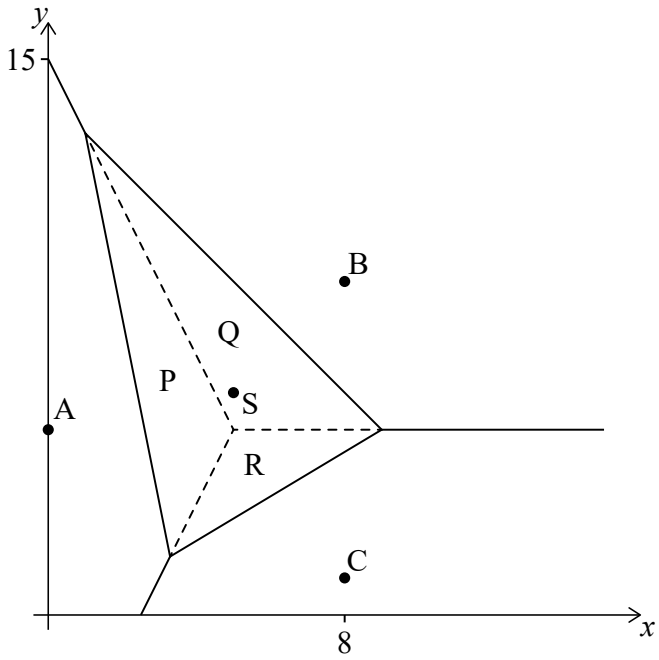
(d) Explain why the principal might not accept that the air quality around the school can be classed as “good”. [1]

(This question continues on the following page)

(Question 1 continued)

The principal decides to obtain an expected value for the AQI at the school that uses all the available information. To do this, she uses an alternative method: the natural neighbour algorithm. This algorithm has two stages.

The first stage is to create a new Voronoi diagram with the school as an extra site. This is shown in the following diagram with the edges of the previous diagram shown by dashed lines.



The second stage is to estimate the AQI value at the school, W, by using the formula

$$W = \frac{w_A a_A + w_B a_B + w_C a_C}{T}.$$

In the formula, a_A is the area within the new cell that has been taken from the cell surrounding site A, shown as region P on the diagram, and w_A is the mean AQI value from site A. This is given as 132 in the table of mean AQI values above. Similarly for sites B and C. T is the total area of the new cell around S.

(e) (i) Find the mid-point of $[\mathrm{SB}]$. [2]

(ii) Show that the equation of the perpendicular bisector of $[\mathrm{SB}]$ is $y = 14 - x$. [3]

(iii) Hence find a_B (the area of region Q on the diagram). [5]

(This question continues on the following page)

(Question 1 continued)

The areas of regions P and R are $a_A = 13.7\,\text{km}^2$ and $a_C = 6.9\,\text{km}^2$ respectively.

(f) Use the natural neighbour algorithm to show that an estimate for the expected AQI value at the school, W, is 94.4. [2]

The principal is still concerned that this method is underestimating the AQI value at the school, as the school is situated close to a busy traffic intersection. She decides to take her own readings (x) over a period of 60 days. Her results are summarized as

$$\overline{x} = 97.8,\ s_{n-1} = 17.2,\ n = 60.$$

The principal assumes that the daily AQI values at the school can be modelled by a normal distribution and that each value is independent of any other value.

(g) Test at the 10% significance level the hypothesis that the mean AQI value at the school is greater than 94.4. State clearly the null and alternative hypotheses and the conclusion of the test in context. [6]

The principal now assumes that the distribution of AQI values at the school follows a normal distribution with mean 97.8 and standard deviation 17.2.

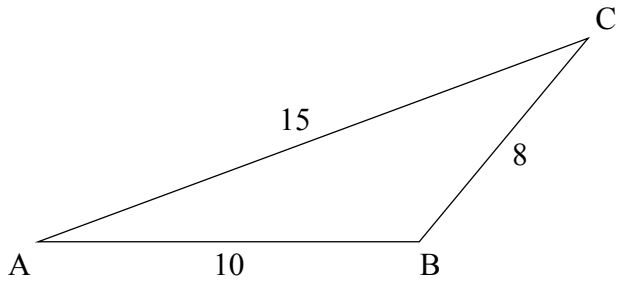
(h) Use this model to find the expected number of days per year (correct to the nearest day) on which the AQI value can be classed as “good”. [3]

Question 1 - working space

4. [Maximum mark: 8]

Triangle ABC has sides of length $AB = 10\,\text{cm}$, $BC = 8\,\text{cm}$ and $AC = 15\,\text{cm}$, as shown in the diagram.

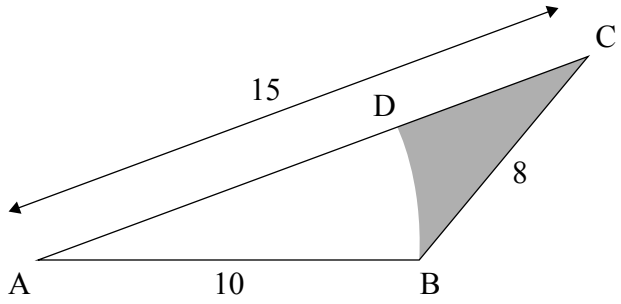
diagram not to scale



(a) Find the size of $B\hat{A}C$. [3]

The circular arc BD, with centre A, is drawn inside the triangle, such that D lies on AC.

diagram not to scale



(b) Find the area of the shaded region. [5]

(This question continues on the following page)

(Question 4 continued)

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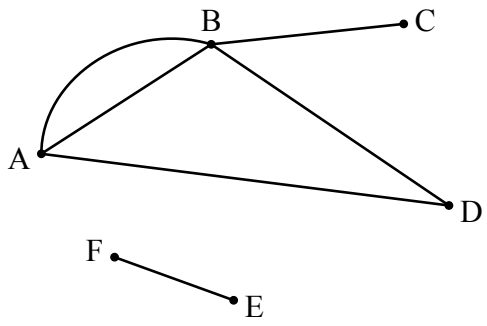
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Question 4 - working space

12. [Maximum mark: 8]

The graph G has six vertices and is shown in the diagram.



(a) State, giving a reason, whether G is

(i) simple;

(ii) connected. [2]

(b) Complete the adjacency matrix for G. [2]

$$\begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \\ \mathrm{F} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} \end{array} \\ \begin{pmatrix} 0 & \square & 0 & 1 & 0 & 0 \\ \square & 0 & \square & 1 & 0 & 0 \\ 0 & \square & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(c) Hence, determine the number of walks from A to B with fewer than four edges. [4]

(This question continues on the following page)

(Question 12 continued)

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Question 12 - working space

13. [Maximum mark: 8]

Air traffic control begins to monitor two airplanes.

The position of the first airplane is given by

$$\boldsymbol{r}_1 = \begin{pmatrix} 29 \\ -32 \\ 5.8 \end{pmatrix} + t \begin{pmatrix} -320 \\ 600 \\ 0.5 \end{pmatrix},$$

where t is the time, in hours, after air traffic control begins to monitor the airplane.

Distances are in kilometres and are relative to an origin at air traffic control.

The second airplane is initially at $\begin{pmatrix} -13 \\ 50 \\ 7.2 \end{pmatrix}$ and is travelling at a speed of $800\,\text{km}\,\text{h}^{-1}$ in the direction $\begin{pmatrix} 7 \\ -24 \\ 0 \end{pmatrix}$.

(a) Find an expression for $\boldsymbol{r}_2$, the position of the second airplane at time t. [3]

(b) Find the least distance between the two airplanes. [5]

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Question 13 - working space

1. [Maximum mark: 26]

This question considers whether it is reasonable to go on all the rides in a theme park and get back to the entrance in two and a half hours.

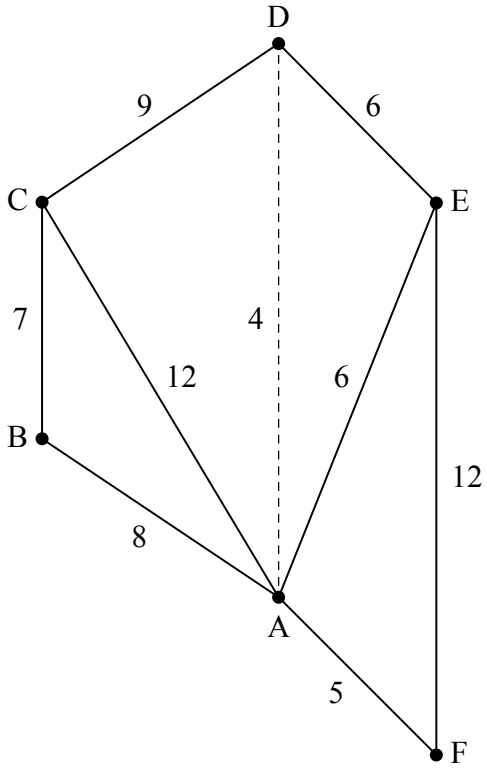
Martin is visiting a theme park. He will enter the park at $09{:}00$ and must leave the park by $11{:}30$. He uses information available on the internet to calculate whether he will be able to go on all of the rides in the two and a half hours.

He begins by constructing a graph which shows the main paths between the rides and the route of the cable car between the entrance/exit A and ride D.

His graph and the names of the rides are shown in the following diagram.

diagram not to scale

A	Entrance/Exit
B	Bullet
C	Canyon
D	Deep Dive
E	Energy Pulse
F	Flash



(This question continues on the following page)

(Question 1 continued)

The weights on the edges of the graph represent the times, in minutes, to walk between the rides and the time to travel by cable car between A and D.

Let T be the shortest possible time, in minutes, taken to visit all the rides, beginning and ending at A.

Martin notices that the graph contains a Hamiltonian cycle. He decides to use the weight of the Hamiltonian cycle as an upper bound for T.

(a) Find the weight of this Hamiltonian cycle. [2]

Martin constructs **Table 1** to show the shortest possible time it takes to travel between any two rides and between the entrance and any ride.

Table 1

	A	B	C	D	E	F
A	0	8	12	4	6	5
B	8	0	7	a	14	13
C	12	7	0	9	15	b
D	4	a	9	0	6	9
E	6	14	15	6	0	11
F	5	13	b	9	11	0

(b) Write down the value of

(i) a; [1]

(ii) b. [1]

(c) Use the nearest neighbour algorithm on **Table 1** to find an upper bound for T. [3]

(This question continues on the following page)

(Question 1 continued)

Martin decides to use the deleted vertex algorithm to find a lower bound for T by first deleting vertex A. The shortest possible time to travel between each ride, with vertex A deleted, is given in **Table 2**.

Table 2

	B	C	D	E	F
B	0	7	a	14	13
C	7	0	9	15	b
D	a	9	0	6	9
E	14	15	6	0	11
F	13	b	9	11	0

(d) (i) Use Prim's algorithm on **Table 2** to find the weight of the minimum spanning tree for the graph with vertices B, C, D, E and F.
Start at vertex B and write down the order in which the edges are selected. [3]

(ii) Hence find a lower bound for T. [2]

Martin finds more lower bounds for T, by deleting each vertex in turn. The results are shown in the following table.

Deleted vertex	Lower bound
B	39
C	39
D	36
E	36
F	39

Martin finds the smallest possible interval within which T lies, based on his calculated values for upper and lower bounds. He writes his answer in the form $p \leq T \leq q$.

(e) Write down the value of

(i) p; [1]

(ii) q. [1]

(This question continues on the following page)

(Question 1 continued)

Martin's favourite ride is Energy Pulse (E), so he decides to go there first. He plans to begin at A and visit the rides in the order $\mathrm{E}, \mathrm{D}, \mathrm{C}, \mathrm{B}, \mathrm{F}$ before returning to A. For the rest of the question, assume that Martin is taking this route.

(f) (i) Find the shortest possible time it would take to complete this route. [1]

(ii) State the edge which would need to be repeated. [1]

Each of the rides takes 2 minutes to complete.

Let the time spent waiting in the queue for ride B be written as the random variable B_t and similarly for the other rides.

The following distributions model the times spent waiting in the queues. Each waiting time is independent of all other waiting times and the time of day.

$$B_t \sim \mathrm{N}(13, 3),\ C_t \sim \mathrm{N}(21, 15),\ D_t \sim \mathrm{N}(16, 8),\ E_t \sim \mathrm{N}(10, 6),\ F_t \sim \mathrm{N}(20, 15)$$

(g) (i) Find the distribution for the total time spent queuing for all five rides. [3]

(ii) Find the probability that Martin manages to go on all five rides and return to the entrance in two and a half hours. [4]

Martin enters the park at $09{:}00$ and decides to follow his planned route, but has two consecutive rides on Energy Pulse.

(h) Find the expected time he will leave the park. [3]

Question 1 - working space

APOLLO

Statistics and probability

117 questions

1. [Maximum mark: 4]

George goes fishing. From experience he knows that the mean number of fish he catches per hour is 1.1. It is assumed that the number of fish he catches can be modelled by a Poisson distribution.

On a day in which George spends 8 hours fishing, find the probability that he will catch more than 9 fish.

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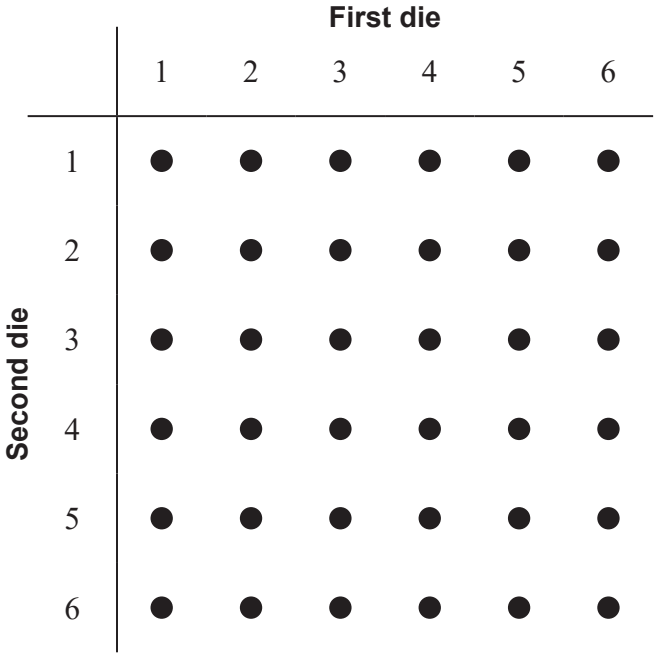
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Question 1 - working space

8. [Maximum mark: 7]

A game is played where two unbiased dice are rolled and the score in the game is the greater of the two numbers shown. If the two numbers are the same, then the score in the game is the number shown on one of the dice. A diagram showing the possible outcomes is given below.



Let T be the random variable "the score in a game".

(a) Complete the table to show the probability distribution of T. [2]

t	1	2	3	4	5	6
$\mathbf{P}(T = t)$						

(b) Find the probability that

(i) a player scores at least 3 in a game.

(ii) a player scores 6, given that they scored at least 3. [3]

(c) Find the expected score of a game. [2]

(This question continues on the following page).

(Question 8 continued)

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Question 8 - working space

11. [Maximum mark: 7]

A factory, producing plastic gifts for a fast food restaurant's Jolly meals, claims that just 1% of the toys produced are faulty.

A restaurant manager wants to test this claim. A box of 200 toys is delivered to the restaurant. The manager checks all the toys in this box and four toys are found to be faulty.

(a) Identify the type of sampling used by the restaurant manager. [1]

The restaurant manager performs a one-tailed hypothesis test, at the 10% significance level, to determine whether the factory's claim is reasonable. It is known that faults in the toys occur independently.

(b) Write down the null and alternative hypotheses. [2]

(c) Find the p-value for the test. [2]

(d) State the conclusion of the test. Give a reason for your answer. [2]

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Question 11 - working space

14. [Maximum mark: 6]

The weights of apples from Tony's farm follow a normal distribution with mean $158\,\text{g}$ and standard deviation $13\,\text{g}$. The apples are sold in bags that contain six apples.

(a) Find the mean weight of a bag of apples. [2]

(b) Find the standard deviation of the weights of these bags of apples. [2]

(c) Find the probability that a bag selected at random weighs more than $1\,\text{kg}$. [2]

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Question 14 - working space

3. [Maximum mark: 6]

At Springfield University, the weights, in kg, of 10 chinchilla rabbits and 10 sable rabbits were recorded. The aim was to find out whether chinchilla rabbits are generally heavier than sable rabbits. The results obtained are summarized in the following table.

Weight of chinchilla rabbits, kg	4.9	4.2	4.1	4.4	4.3	4.6	4.0	4.7	4.5	4.4
Weight of sable rabbits, kg	4.2	4.1	4.1	4.2	4.5	4.4	4.5	3.9	4.2	4.0

A t-test is to be performed at the $5\,\%$ significance level.

(a) Write down the null and alternative hypotheses. [2]

(b) Find the p-value for this test. [2]

(c) Write down the conclusion to the test. Give a reason for your answer. [2]

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Question 3 - working space

9. [Maximum mark: 8]

A newspaper vendor in Singapore is trying to predict how many copies of *The Straits Times* they will sell. The vendor forms a model to predict the number of copies sold each weekday. According to this model, they expect the same number of copies will be sold each day.

To test the model, they record the number of copies sold each weekday during a particular week. This data is shown in the table.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Number of copies sold	74	97	91	86	112

A goodness of fit test at the 5% significance level is used on this data to determine whether the vendor's model is suitable. The critical value for the test is 9.49.

(a) Find an estimate for how many copies the vendor expects to sell each day. [1]

(b) (i) State the null and alternative hypotheses for this test.

(ii) Write down the degrees of freedom for this test.

(iii) Write down the conclusion to the test. Give a reason for your answer. [7]

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Question 9 - working space

10. [Maximum mark: 6]

A manufacturer of chocolates produces them in individual packets, claiming to have an average of 85 chocolates per packet.

Talha bought 30 of these packets in order to check the manufacturer's claim.

Given that the number of individual chocolates is x, Talha found that, from his 30 packets, $\sum x = 2506$ and $\sum x^2 = 209\,738$.

(a) Find an unbiased estimate for the mean number (μ) of chocolates per packet. [1]

(b) Use the formula $s_{n-1}^2 = \frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}$ to determine an unbiased estimate for the variance of the number of chocolates per packet. [2]

(c) Find a $95\,\%$ confidence interval for μ. You may assume that all conditions for a confidence interval have been met. [2]

(d) Suggest, with justification, a valid conclusion that Talha could make. [1]

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Question 10 - working space

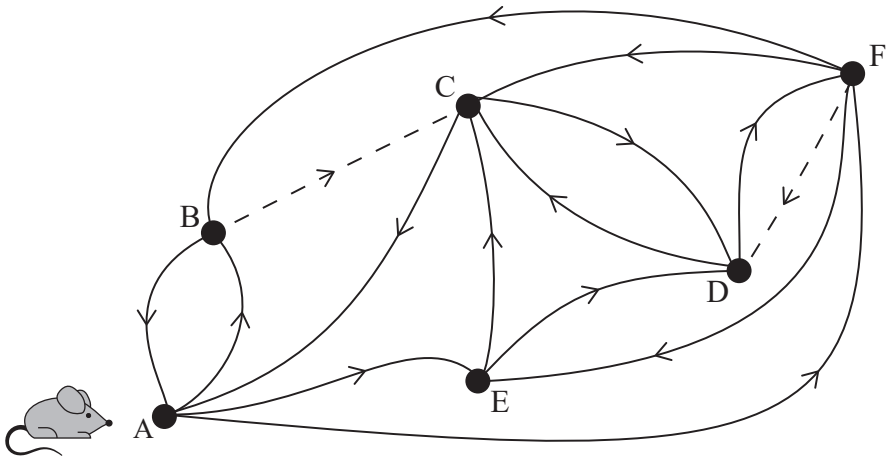
13. [Maximum mark: 8]

The graph below shows a small maze, in the form of a network of directed routes. The vertices A to F show junctions in the maze and the edges show the possible paths available from one vertex to another.

A mouse is placed at vertex A and left to wander the maze freely. The routes shown by dashed lines indicate paths sprinkled with sugar.

When the mouse reaches any junction, she rests for a constant time before continuing.

At any junction, it may also be assumed that
- the mouse chooses any available normal path with equal probability
- if the junction includes a path sprinkled with sugar, the probability of choosing this path is twice that of a normal path.



(a) Determine the transition matrix for this graph. [3]

(b) If the mouse was left to wander indefinitely, use your graphic display calculator to estimate the percentage of time that the mouse would spend at point F. [3]

(c) Comment on your answer to part (b), referring to at least one limitation of the model. [2]

(This question continues on the following page)

(Question 13 continued)

Question 13 - working space

15. [Maximum mark: 7]

The number of coffees sold per hour at an independent coffee shop is modelled by a Poisson distribution with a mean of 22 coffees per hour.

Sheila, the shop's owner wants to increase the number of coffees sold in the shop. She decides to offer a discount to customers who buy more than one coffee.

To test how successful this strategy is, Sheila records the number of coffees sold over a single 5-hour period. Sheila decides to use a 5% level of significance in her test.

(a) State the null and alternative hypotheses for the test. [1]

(b) Find the probability that Sheila will make a type I error in her test conclusion. [4]

Sheila finds 126 coffees were sold during the 5-hour period.

(c) State Sheila's conclusion to the test. Justify your answer. [2]

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Question 15 - working space

3. [Maximum mark: 13]

The stopping distances for bicycles travelling at $20\,\text{km}\,\text{h}^{-1}$ are assumed to follow a normal distribution with mean $6.76\,\text{m}$ and standard deviation $0.12\,\text{m}$.

(a) Under this assumption, find, correct to four decimal places, the probability that a bicycle chosen at random travelling at $20\,\text{km}\,\text{h}^{-1}$ manages to stop

(i) in less than $6.5\,\text{m}$.

(ii) in more than $7\,\text{m}$. [3]

1000 randomly selected bicycles are tested and their stopping distances when travelling at $20\,\text{km}\,\text{h}^{-1}$ are measured.

(b) Find, correct to four significant figures, the expected number of bicycles tested that stop between

(i) $6.5\,\text{m}$ and $6.75\,\text{m}$.

(ii) $6.75\,\text{m}$ and $7\,\text{m}$. [3]

The measured stopping distances of the 1000 bicycles are given in the table.

Measured stopping distance	**Number of bicycles**
Less than $6.5\,\text{m}$	12
Between $6.5\,\text{m}$ and $6.75\,\text{m}$	428
Between $6.75\,\text{m}$ and $7\,\text{m}$	527
More than $7\,\text{m}$	33

It is decided to perform a χ^2 goodness of fit test at the 5% level of significance to decide whether the stopping distances of bicycles travelling at $20\,\text{km}\,\text{h}^{-1}$ can be modelled by a normal distribution with mean $6.76\,\text{m}$ and standard deviation $0.12\,\text{m}$.

(c) State the null and alternative hypotheses. [2]

(d) Find the p-value for the test. [3]

(e) State the conclusion of the test. Give a reason for your answer. [2]

Turn over

Question 3 - working space

5. [Maximum mark: 13]

Long term experience shows that if it is sunny on a particular day in Vokram, then the probability that it will be sunny the following day is 0.8. If it is not sunny, then the probability that it will be sunny the following day is 0.3.

The transition matrix $\boldsymbol{T}$ is used to model this information, where $\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

(a) It is sunny today. Find the probability that it will be sunny in three days' time. [2]

(b) Find the eigenvalues and eigenvectors of $\boldsymbol{T}$. [5]

The matrix $\boldsymbol{T}$ can be written as a product of three matrices, $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c) (i) Write down the matrix $\boldsymbol{P}$.

(ii) Write down the matrix $\boldsymbol{D}$. [2]

(d) Hence find the long-term percentage of sunny days in Vokram. [4]

Question 5 - working space

2. [Maximum mark: 16]

It is known that the weights of male Persian cats are normally distributed with mean $6.1\,\text{kg}$ and variance $0.5^2\,\text{kg}^2$.

(a) Sketch a diagram showing the above information. [2]

(b) Find the proportion of male Persian cats weighing between $5.5\,\text{kg}$ and $6.5\,\text{kg}$. [2]

A group of 80 male Persian cats are drawn from this population.

(c) Determine the expected number of cats in this group that have a weight of less than $5.3\,\text{kg}$. [3]

The male cats are now joined by 80 female Persian cats. The female cats are drawn from a population whose weights are normally distributed with mean $4.5\,\text{kg}$ and standard deviation $0.45\,\text{kg}$.

(d) Ten female cats are chosen at random.

(i) Find the probability that exactly one of them weighs over $4.62\,\text{kg}$.

(ii) Let N be the number of cats weighing over $4.62\,\text{kg}$.

Find the variance of N. [5]

A cat is selected at random from all 160 cats.

(e) Find the probability that the cat was female, given that its weight was over $4.7\,\text{kg}$. [4]

Turn over

Question 2 - working space

4. [Maximum mark: 18]

In a small village there are two doctors' clinics, one owned by Doctor Black and the other owned by Doctor Green. It was noted after each year that 3.5% of Doctor Black's patients moved to Doctor Green's clinic and 5% of Doctor Green's patients moved to Doctor Black's clinic. All additional losses and gains of patients by the clinics may be ignored.

At the start of a particular year, it was noted that Doctor Black had 2100 patients on their register, compared to Doctor Green's 3500 patients.

(a) Write down a transition matrix $\boldsymbol{T}$ indicating the annual population movement between clinics. [2]

(b) Find a prediction for the ratio of the number of patients Doctor Black will have, compared to Doctor Green, after two years. [2]

(c) Find a matrix $\boldsymbol{P}$, with integer elements, such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [6]

(d) Hence, show that the long-term transition matrix $\boldsymbol{T}^{\infty}$ is given by $\boldsymbol{T}^{\infty} = \begin{pmatrix} \frac{10}{17} & \frac{10}{17} \\ \frac{7}{17} & \frac{7}{17} \end{pmatrix}$. [6]

(e) Hence, or otherwise, determine the expected ratio of the number of patients Doctor Black would have compared to Doctor Green in the long term. [2]

Turn over

Question 4 - working space

5. [Maximum mark: 14]

Hank sets up a bird table in his garden to provide the local birds with some food. Hank notices that a specific bird, a large magpie, visits several times per month and he names him Bill. Hank models the number of times per month that Bill visits his garden as a Poisson distribution with mean 3.1.

(a) Using Hank's model, find the probability that Bill visits the garden on exactly four occasions during one particular month. [1]

(b) Over the course of 3 consecutive months, find the probability that Bill visits the garden:

(i) on exactly 12 occasions.

(ii) during the first and third month only. [5]

(c) Find the probability that over a 12-month period, there will be exactly 3 months when Bill does not visit the garden. [4]

After the first year, a number of baby magpies start to visit Hank's garden. It may be assumed that each of these baby magpies visits the garden randomly and independently, and that the number of times each baby magpie visits the garden per month is modelled by a Poisson distribution with mean 2.1.

(d) Determine the least number of magpies required, including Bill, in order that the probability of Hank's garden having at least 30 magpie visits per month is greater than 0.2. [4]

Question 5 - working space

2. [Maximum mark: 28]

A firm wishes to review its recruitment processes. This question considers the validity and reliability of the methods used.

Every year an accountancy firm recruits new employees for a trial period of one year from a large group of applicants.

At the start, all applicants are interviewed and given a rating. Those with a rating of either *Excellent*, *Very good* or *Good* are recruited for the trial period. At the end of this period, some of the new employees will stay with the firm.

It is decided to test how valid the interview rating is as a way of predicting which of the new employees will stay with the firm.

Data is collected and recorded in a contingency table.

	Interview rating		
	Excellent	**Very good**	**Good**
Stay	12	20	19
Leave	10	15	24

(a) Use an appropriate test, at the 5% significance level, to determine whether a new employee staying with the firm is independent of their interview rating. State the null and alternative hypotheses, the p-value and the conclusion of the test. [6]

The next year's group of applicants are asked to complete a written assessment which is then analysed. From those recruited as new employees, a random sample of size 18 is selected.

The sample is stratified by department. Of the 91 new employees recruited that year, 55 were placed in the national department and 36 in the international department.

(b) Show that 11 employees are selected for the sample from the national department. [2]

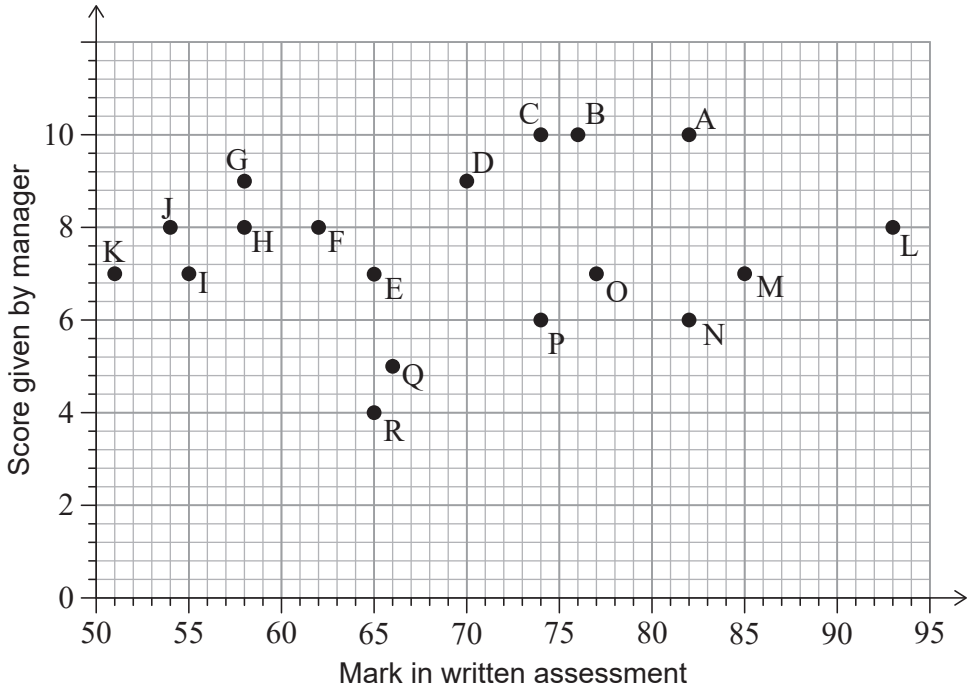
(This question continues on the following page)

(Question 2 continued)

At the end of their first year, the level of performance of each of the 18 employees in the sample is assessed by their department manager. They are awarded a score between 1 (low performance) and 10 (high performance).

The marks in the written assessment and the scores given by the managers are shown in both the table and the scatter diagram.

	National department											International department						
Employee	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
Mark in written assessment	82	76	74	70	65	62	58	58	55	54	51	93	85	82	77	74	66	65
Score given by manager	10	10	10	9	7	8	9	8	7	8	7	8	7	6	7	6	5	4



The firm decides to find a Spearman's rank correlation coefficient, r_s, for this data.

(c) (i) Without calculation, explain why it might not be appropriate to calculate a correlation coefficient for the whole sample of 18 employees. [2]

(ii) Find r_s for the seven employees working in the **international** department. [4]

(iii) Hence comment on the validity of the written assessment as a measure of the level of performance of employees in this department. Justify your answer. [2]

(This question continues on the following page)

Turn over

(Question 2 continued)

The same seven employees are given the written assessment a second time, at the end of the first year, to measure its reliability. Their marks are shown in the table below.

	International department						
	L	M	N	O	P	Q	R
First mark	93	85	82	77	74	66	65
Second mark	90	92	85	73	79	71	65

(d) (i) State the name of this type of test for reliability. [1]

(ii) For the data in this table, test the null hypothesis, $H_0 : \rho = 0$, against the alternative hypothesis, $H_1 : \rho > 0$, at the 5% significance level. You may assume that all the requirements for carrying out the test have been met. [4]

(iii) Hence comment on the reliability of the written assessment. [1]

(e) The written assessment is in five sections, numbered 1 to 5. At the end of the year, the employees are also given a score for each of five professional attributes: V, W, X, Y and Z.

The firm decides to test the hypothesis that there is a correlation between the mark in a section and the score for an attribute.

They compare marks in **each** of the sections with scores for **each** of the attributes.

(i) Write down the number of tests they carry out. [1]

(ii) The tests are performed at the 5% significance level.

Assuming that:
- there is no correlation between the marks in any of the sections and scores in any of the attributes,
- the outcome of each hypothesis test is independent of the outcome of the other hypothesis tests,

find the probability that at least one of the tests will be significant. [4]

(iii) The firm obtains a significant result when comparing section 2 of the written assessment and attribute X. Interpret this result. [1]

References:

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Question 2 - working space

1. [Maximum mark: 24]

Juliet is a sociologist who wants to investigate if income affects happiness amongst doctors. This question asks you to review Juliet's methods and conclusions.

Juliet obtained a list of email addresses of doctors who work in her city. She contacted them and asked them to fill in an anonymous questionnaire. Participants were asked to state their annual income and to respond to a set of questions. The responses were used to determine a *happiness score* out of 100. Of the 415 doctors on the list, 11 replied.

(a) (i) Describe **one** way in which Juliet could improve the reliability of her investigation. [1]

(ii) Describe **one** criticism that can be made about the validity of Juliet's investigation. [1]

Juliet's results are summarized in the following table.

Response	Annual income ($)	Happiness score
A	65 000	60
B	63 000	52
C	40 000	31
D	125 000	81
E	100 000	48
F	245 000	61
G	48 000	42
H	39 000	40
I	85 000	57
J	92 000	53
K	123 456 789	56

(b) Juliet classifies response K as an outlier and removes it from the data. Suggest **one** possible justification for her decision to remove it. [1]

(This question continues on the following page)

(Question 1 continued)

(c) For the remaining ten responses in the table, Juliet calculates the mean happiness score to be 52.5.

(i) Calculate the mean **annual income** for these remaining responses. [2]

(ii) Determine the value of r, Pearson's product-moment correlation coefficient, for these remaining responses. [2]

Juliet decides to carry out a hypothesis test on the correlation coefficient to investigate whether increased annual income is associated with greater happiness.

(d) (i) State why the hypothesis test should be one-tailed. [1]

(ii) State the null and alternative hypotheses for this test. [2]

The critical value for this test, at the 5% significance level, is 0.549. Juliet assumes that the population is bivariate normal.

(iii) Determine whether there is significant evidence of a positive correlation between annual income and happiness. Justify your answer. [2]

(This question continues on page 5)

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(Question 1 continued)

(e) Juliet wants to create a model to predict how changing annual income might affect happiness scores. To do this, she assumes that annual income in dollars, X, is the independent variable and the happiness score, Y, is the dependent variable.

She first considers a linear model of the form

$$Y = aX + b.$$

(i) Use Juliet's data to find the value of a and of b. [1]

(ii) Interpret, referring to income and happiness, what the value of a represents. [1]

Juliet then considers a quadratic model of the form

$$Y = cX^2 + dX + e.$$

(iii) Find the value of c, of d and of e. [1]

(iv) Find the coefficient of determination for each of the two models she considers. [2]

(v) Hence compare the two models. [1]

Juliet decides to use the coefficient of determination to choose between these two models.

(vi) Comment on the validity of her decision. [1]

After presenting the results of her investigation, a colleague questions whether Juliet's sample is representative of all doctors in the city.

A report states that the mean annual income of doctors in the city is $\$80\,000$. Juliet decides to carry out a test to determine whether her sample could realistically be taken from a population with a mean of $\$80\,000$.

(f) (i) State the name of the test which Juliet should use. [1]

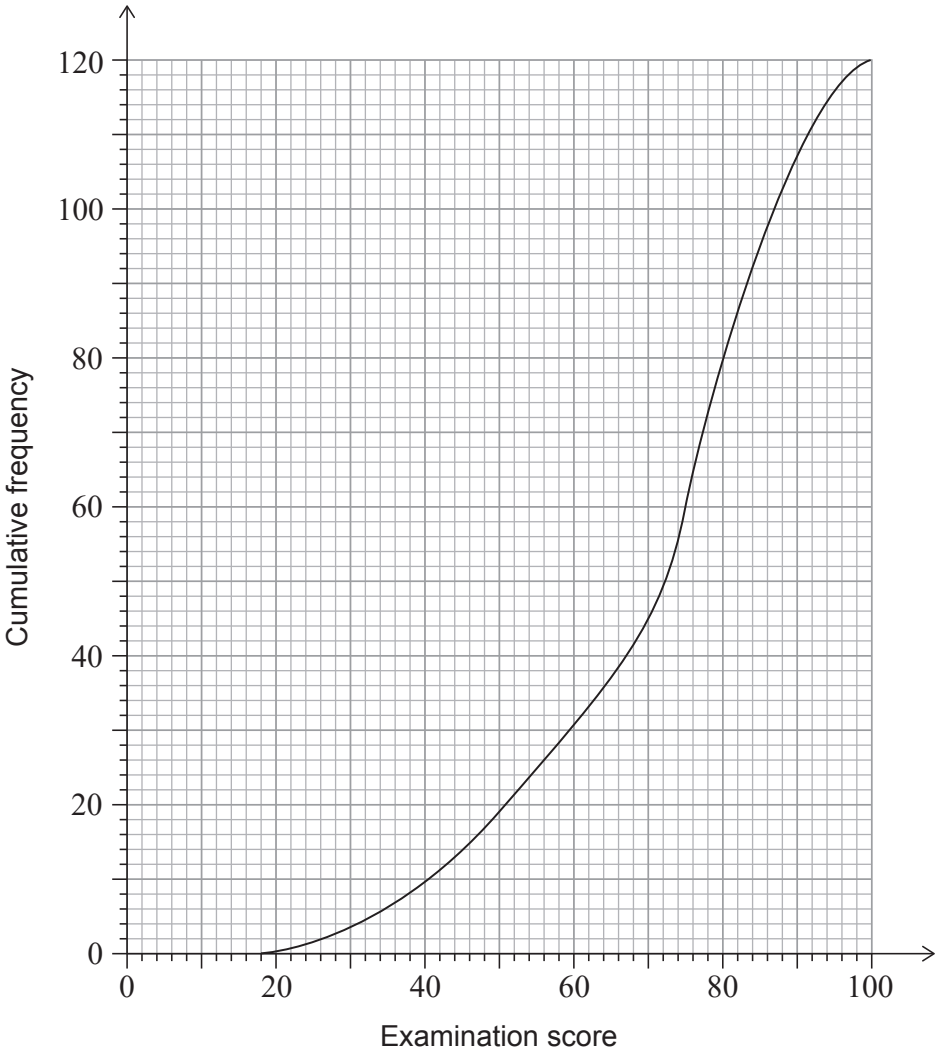
(ii) State the null and alternative hypotheses for this test. [1]

(iii) Perform the test, using a 5% significance level, and state your conclusion in context. [3]

Question 1 - working space

7. [Maximum mark: 8]

A group of 120 students sat a history exam. The cumulative frequency graph shows the scores obtained by the students.



(a) Find the median of the scores obtained. [1]

The students were awarded a grade from 1 to 5, depending on the score obtained in the exam. The number of students receiving each grade is shown in the following table.

Grade	1	2	3	4	5
Number of students	6	13	26	a	b

(b) Find an expression for a in terms of b. [2]

(This question continues on the following page)

(Question 7 continued)

(c) The mean grade for these students is 3.65.

(i) Find the number of students who obtained a grade 5.

(ii) Find the minimum score needed to obtain a grade 5. [5]

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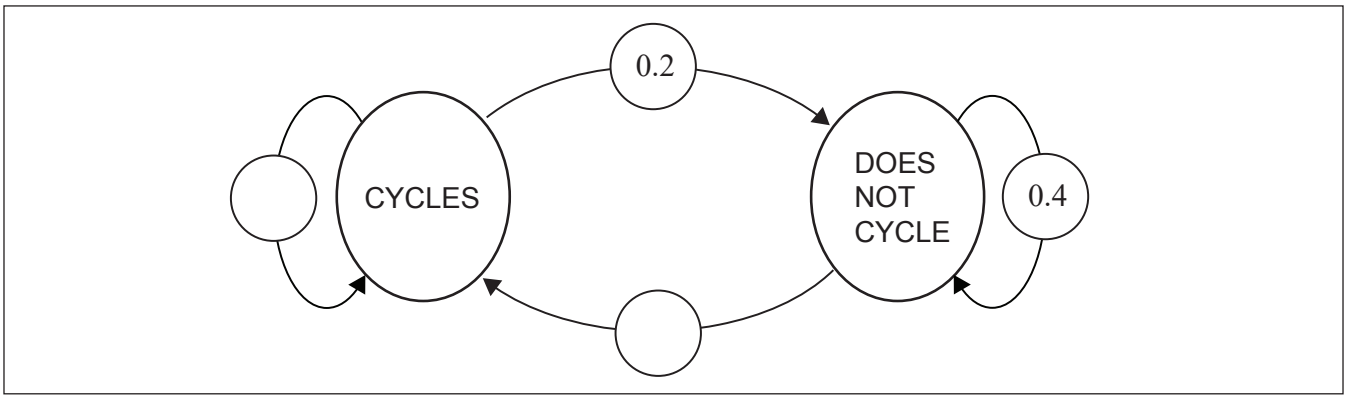
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Question 7 - working space

9. [Maximum mark: 5]

Katie likes to cycle to work as much as possible. If Katie cycles to work one day then she has a probability of 0.2 of not cycling to work on the next work day. If she does not cycle to work one day then she has a probability of 0.4 of not cycling to work on the next work day.

(a) Complete the following transition diagram to represent this information. [2]



Katie works for 180 days in a year.

(b) Find the probability that Katie cycles to work on her final working day of the year. [3]

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Question 9 - working space

12. [Maximum mark: 5]

The following table shows the time, in days, from December 1st and the percentage of Christmas trees in stock at a shop on the beginning of that day.

Days since December 1st (d)	1	3	6	9	12	15	18
Percentage of Christmas trees left in stock (x)	100	51	29	21	18	16	14

The following table shows the natural logarithm of both d and x on these days to 2 decimal places.

ln (d)	0	1.10	1.79	2.20	2.48	2.71	2.89
ln (x)	4.61	3.93	3.37	3.04	2.89	2.77	2.64

(a) Use the data in the second table to find the value of m and the value of b for the regression line, $\ln x = m(\ln d) + b$. [2]

(b) Assuming that the model found in part (a) remains valid, estimate the percentage of trees in stock when $d = 25$. [3]

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Question 12 - working space

14. [Maximum mark: 7]

On Paul's farm, potatoes are packed in sacks labelled $50\,\text{kg}$. The weights of the sacks of potatoes can be modelled by a normal distribution with mean weight $49.8\,\text{kg}$ and standard deviation $0.9\,\text{kg}$.

(a) Find the probability that a sack is under its labelled weight. [2]

(b) Find the lower quartile of the weights of the sacks of potatoes. [2]

The sacks of potatoes are transported in crates. There are 10 sacks in each crate and the weights of the sacks of potatoes are independent of each other.

(c) Find the probability that the total weight of the sacks of potatoes in a crate exceeds $500\,\text{kg}$. [3]

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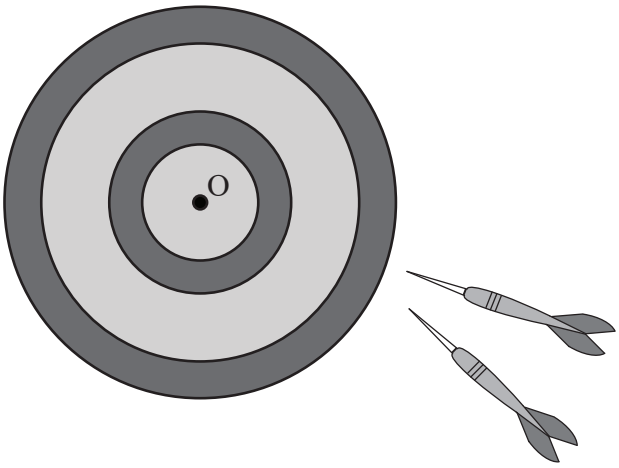
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Question 14 - working space

3. [Maximum mark: 16]

Arianne plays a game of darts.



The distance that her darts land from the centre, O, of the board can be modelled by a normal distribution with mean $10\,\text{cm}$ and standard deviation $3\,\text{cm}$.

(a) Find the probability that

(i) a dart lands less than $13\,\text{cm}$ from O.

(ii) a dart lands more than $15\,\text{cm}$ from O. [3]

Each of Arianne's throws is independent of her previous throws.

(b) Find the probability that Arianne throws two consecutive darts that land more than $15\,\text{cm}$ from O. [2]

In a competition a player has three darts to throw on each turn. A point is scored if a player throws **all** three darts to land within a central area around O. When Arianne throws a dart the probability that it lands within this area is 0.8143.

(c) Find the probability that Arianne does **not** score a point on a turn of three darts. [2]

In the competition Arianne has ten turns, each with three darts.

(d) (i) Find Arianne's expected score in the competition.

(ii) Find the probability that Arianne scores at least 5 points in the competition.

(iii) Find the probability that Arianne scores at least 5 points and less than 8 points.

(iv) Given that Arianne scores at least 5 points, find the probability that Arianne scores less than 8 points. [9]

Question 3 - working space

7. [Maximum mark: 14]

Loreto is a manager at the Da Vinci health centre. If the mean rate of patients arriving at the health centre exceeds 1.5 per minute then Loreto will employ extra staff. It is assumed that the number of patients arriving in any given time period follows a Poisson distribution.

Loreto performs a hypothesis test to determine whether she should employ extra staff. She finds that 320 patients arrived during a randomly selected 3-hour clinic.

(a) (i) Write down null and alternative hypotheses for Loreto's test.

(ii) Using the data from Loreto's sample, perform the hypothesis test at a 5% significance level to determine if Loreto should employ extra staff. [7]

Loreto is also concerned about the average waiting time for patients to see a nurse. The health centre aims for at least 95% of patients to see a nurse in under 20 minutes.

Loreto assumes that the waiting times for patients are independent of each other and decides to perform a hypothesis test at a 10% significance level to determine whether the health centre is meeting its target.

Loreto surveys 150 patients and finds that 11 of them waited more than 20 minutes.

(b) (i) Write down null and alternative hypotheses for this test.

(ii) Perform the test, clearly stating the conclusion in context. [7]

References:

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Question 7 - working space

1. [Maximum mark: 6]

At the end of a school day, the Headmaster conducted a survey asking students in how many classes they had used the internet.

The data is shown in the following table.

Number of classes in which the students used the internet	0	1	2	3	4	5	6
Number of students	20	24	30	k	10	3	1

(a) State whether the data is discrete or continuous. [1]

The mean number of classes in which a student used the internet is 2.

(b) Find the value of k. [4]

It was not possible to ask every person in the school, so the Headmaster arranged the student names in alphabetical order and then asked every 10th person on the list.

(c) Identify the sampling technique used in the survey. [1]

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Question 1 - working space

6. [Maximum mark: 6]

Jae Hee plays a game involving a biased six-sided die.
The faces of the die are labelled -3, -1, 0, 1, 2 and 5.
The score for the game, X, is the number which lands face up after the die is rolled.
The following table shows the probability distribution for X.

Score x	-3	-1	0	1	2	5
$\mathbf{P}(X = x)$	$\frac{1}{18}$	p	$\frac{3}{18}$	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{7}{18}$

(a) Find the exact value of p. [1]

Jae Hee plays the game once.

(b) Calculate the expected score. [2]

Jae Hee plays the game twice and adds the two scores together.

(c) Find the probability Jae Hee has a **total** score of -3. [3]

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Question 6 - working space

9. [Maximum mark: 5]

A manager wishes to check the mean weight of flour put into bags in his factory. He randomly samples 10 bags and finds the mean weight is $1.478\,\text{kg}$ and the standard deviation of the sample is $0.0196\,\text{kg}$.

(a) Find s_{n-1} for this sample. [2]

(b) Find a $95\,\%$ confidence interval for the population mean, giving your answer to 4 significant figures. [2]

(c) The bags are labelled as being $1.5\,\text{kg}$ weight. Comment on this claim with reference to your answer in part (b). [1]

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Question 9 - working space

10. [Maximum mark: 6]

In a coffee shop, the time it takes to serve a customer can be modelled by a normal distribution with a mean of 1.5 minutes and a standard deviation of 0.4 minutes.

Two customers enter the shop together. They are served one at a time.

Find the probability that the total time taken to serve both customers will be less than 4 minutes.

Clearly state any assumptions you have made.

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Question 10 - working space

12. [Maximum mark: 7]

Product research leads a company to believe that the revenue (R) made by selling its goods at a price (p) can be modelled by the equation.

$$R(p) = cpe^{dp},\ c,\ d \in \mathbb{R}$$

There are two competing models, A and B with different values for the parameters c and d.

Model A has $c = 3$, $d = -0.5$ and model B has $c = 2.5$, $d = -0.6$.

The company experiments by selling the goods at three different prices in three similar areas and the results are shown in the following table.

Area	Price (p)	Revenue (R)
1	1	1.5
2	2	1.8
3	3	1.5

The company will choose the model with the smallest value for the sum of square residuals.

Determine which model the company chose.

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Question 12 - working space

16. [Maximum mark: 6]

The number of fish that can be caught in one hour from a particular lake can be modelled by a Poisson distribution.

The owner of the lake, Emily, states in her advertising that the average number of fish caught in an hour is three.

Tom, a keen fisherman, is not convinced and thinks it is less than three. He decides to set up the following test. Tom will fish for one hour and if he catches fewer than two fish he will reject Emily's claim.

(a) State a suitable null and alternative hypotheses for Tom's test. [1]

(b) Find the probability of a Type I error. [2]

The average number of fish caught in an hour is actually 2.5.

(c) Find the probability of a Type II error. [3]

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Question 16 - working space

17. [Maximum mark: 6]

Mr Burke teaches a mathematics class with 15 students. In this class there are 6 female students and 9 male students.

Each day Mr Burke randomly chooses one student to answer a homework question.

In the first month, Mr Burke will teach his class 20 times.

(a) Find the probability he will choose a female student 8 times. [2]

The Head of Year, Mrs Smith, decides to select a student at random from the year group to read the notices in assembly. There are 80 students in total in the year group. Mrs Smith calculates the probability of picking a male student 8 times in the first 20 assemblies is 0.153357 correct to 6 decimal places.

(b) Find the number of male students in the year group. [4]

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Question 17 - working space

2. [Maximum mark: 12]

Slugworth Candy Company sell a variety pack of colourful, shaped sweets.

The sweets are produced such that 80% are star shaped and 20% are shaped like a crescent moon. It is known that 10% of the stars and 30% of the crescent moons are coloured yellow.

(a) A sweet is selected at random.

(i) Find the probability that the sweet is yellow.

(ii) Given that the sweet is yellow, find the probability it is star shaped. [4]

According to manufacturer specifications, the colours in each variety pack should be distributed as follows.

Colour	Brown	Red	Green	Orange	Yellow	Purple
Percentage (%)	15	25	20	20	10	10

Mr Slugworth opens a pack of 80 sweets and records the frequency of each colour.

Colour	Brown	Red	Green	Orange	Yellow	Purple
Observed Frequency	10	20	16	18	12	4

To investigate if the sample is consistent with manufacturer specifications, Mr Slugworth conducts a χ^2 goodness of fit test. The test is carried out at a 5% significance level.

(b) Write down the null hypothesis for this test. [1]

(c) **Copy** and complete the following table in your answer booklet. [2]

Colour	Brown	Red	Green	Orange	Yellow	Purple
Expected Frequency						

(d) Write down the number of degrees of freedom. [1]

(e) Find the p-value for the test. [2]

(f) State the conclusion of the test. Give a reason for your answer. [2]

Turn over

Question 2 - working space

6. [Maximum mark: 14]

A city has two cable companies, X and Y. Each year $20\,\%$ of the customers using company X move to company Y and $10\,\%$ of the customers using company Y move to company X.
All additional losses and gains of customers by the companies may be ignored.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movements between the two companies in a particular year. [2]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$. [4]

(c) Hence write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$. [2]

Initially company X and company Y both have 1200 customers.

(d) Find an expression for the number of customers company X has after n years, where $n \in \mathbb{N}$. [5]

(e) Hence write down the number of customers that company X can expect to have in the long term. [1]

Turn over

Question 6 - working space

1. [Maximum mark: 27]

Two IB schools, A and B, follow the IB Diploma Programme but have different teaching methods. A research group tested whether the different teaching methods lead to a similar final result.

For the test, a group of eight students were randomly selected from each school. Both samples were given a standardized test at the start of the course and a prediction for total IB points was made based on that test; this was then compared to their points total at the end of the course.

Previous results indicate that both the predictions from the standardized tests and the final IB points can be modelled by a normal distribution.

It can be assumed that:
- the standardized test is a valid method for predicting the final IB points
- that variations from the prediction can be explained through the circumstances of the student or school.

(a) Identify a test that might have been used to verify the null hypothesis that the predictions from the standardized test can be modelled by a normal distribution. [1]

(b) State why comparing only the final IB points of the students from the two schools would not be a valid test for the effectiveness of the two different teaching methods. [1]

(This question continues on the following page)

(Question 1 continued)

The data for school A is shown in the following table.

School A

Student number	Gender	Predicted IB points (p)	Final IB points (f)
1	male	43.2	44
2	male	36.5	34
3	female	37.1	38
4	male	30.9	28
5	male	41.1	39
6	female	35.1	39
7	male	36.4	40
8	male	38.2	38
	Mean	37.31	37.5

(c) For each student, the change from the predicted points to the final points ($f - p$) was calculated.

(i) Find the mean change.

(ii) Find the standard deviation of the changes. [3]

(d) Use a paired t-test to determine whether there is significant evidence that the students in school A have improved their IB points since the start of the course. [4]

(This question continues on the following page)

(Question 1 continued)

The data for school B is shown in the following table.

School B

Student number	Gender	Final IB points– Predicted IB points ($f-p$)
1	male	8.7
2	female	−1.1
3	female	4.8
4	female	−1.5
5	male	2.5
6	female	3.2
7	female	−1.3
8	female	3.1
	Mean	2.3

(e) (i) Use an appropriate test to determine whether there is evidence, at the 5% significance level, that the students in school B have improved more than those in school A.

(ii) State why it was important to test that both sets of points were normally distributed. [6]

(This question continues on the following page)

(Question 1 continued)

School A also gives each student a score for effort in each subject. This effort score is based on a scale of 1 to 5 where 5 is regarded as outstanding effort.

Student number	Gender	Predicted IB points	Final IB points	Average effort score
1	male	43.2	44	4.4
2	male	36.5	34	4.2
3	female	37.1	38	4.7
4	male	30.9	28	4.3
5	male	41.1	39	3.9
6	female	35.1	39	4.9
7	male	36.4	40	4.9
8	male	38.2	38	4.3
	Mean	37.31	37.5	4.45

It is claimed that the effort put in by a student is an important factor in improving upon their predicted IB points.

(f) (i) Perform a test on the data from school A to show it is reasonable to assume a linear relationship between effort scores and improvements in IB points. You may assume effort scores follow a normal distribution.

(ii) Hence, find the expected improvement between predicted and final points for an increase of one unit in effort grades, giving your answer to one decimal place. [4]

A mathematics teacher in school A claims that the comparison between the two schools is not valid because the sample for school B contained mainly girls and that for school A, mainly boys. She believes that girls are likely to show a greater improvement from their predicted points to their final points.

She collects more data from other schools, asking them to class their results into four categories as shown in the following table.

	$(f-p)<-2$	$-2\leq(f-p)<0$	$0\leq(f-p)<2$	$(f-p)\geq 2$
Male	6	8	10	9
Female	3	8	14	8

(g) Use an appropriate test to determine whether showing an improvement is independent of gender. [6]

(h) If you were to repeat the test performed in part (e) intending to compare the quality of the teaching between the two schools, suggest **two** ways in which you might choose your sample to improve the validity of the test. [2]

Turn over

Question 1 - working space

3. [Maximum mark: 7]

Leo is investigating whether a six-sided die is fair. He rolls the die 60 times and records the observed frequencies in the following table:

Number on die	1	2	3	4	5	6
Observed frequency	8	7	6	15	12	12

Leo carries out a χ^2 goodness of fit test at a 5% significance level.

(a) Write down the null and alternative hypotheses. [1]

(b) Write down the degrees of freedom. [1]

(c) Write down the expected frequency of rolling a 1. [1]

(d) Find the p-value for the test. [2]

(e) State the conclusion of the test. Give a reason for your answer. [2]

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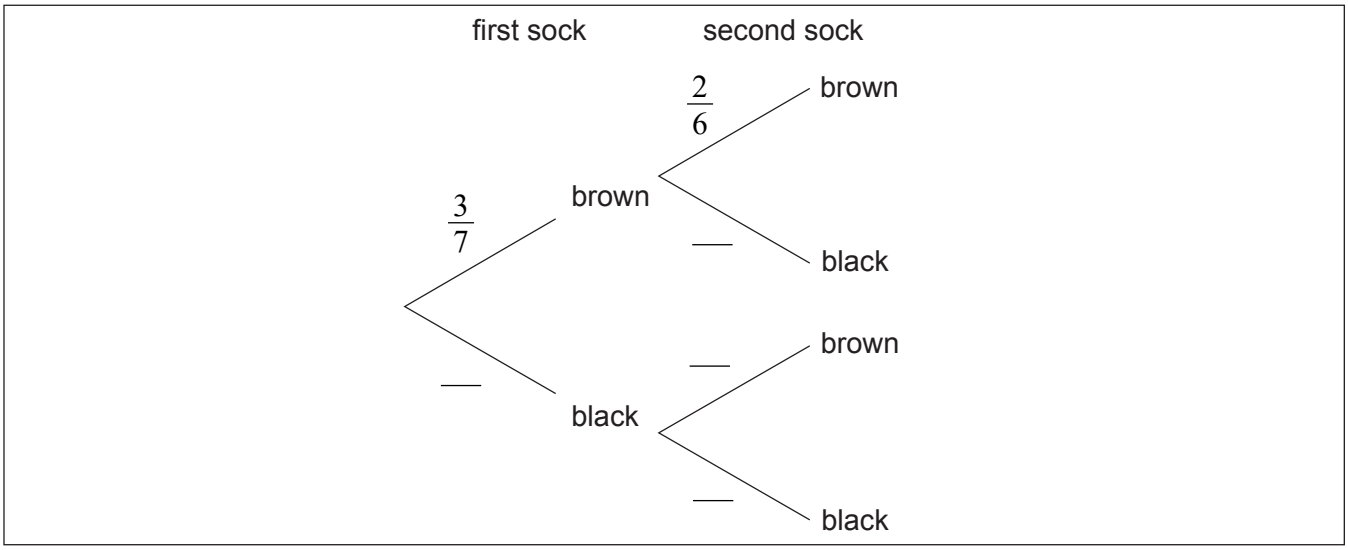
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Question 3 - working space

4. [Maximum mark: 7]

Karl has three brown socks and four black socks in his drawer. He takes two socks at random from the drawer.

(a) Complete the tree diagram. [2]



(b) Find the probability that Karl takes two socks of the same colour. [2]

(c) Given that Karl has two socks of the same colour find the probability that he has two brown socks. [3]

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Question 4 - working space

9. [Maximum mark: 6]

A company produces bags of sugar with a labelled weight of $1\,\text{kg}$. The weights of the bags are normally distributed with a mean of $1\,\text{kg}$ and a standard deviation of $100\,\text{g}$. In an inspection, if the weight of a randomly chosen bag is less than $950\,\text{g}$ then the company fails the inspection.

(a) Find the probability that the company fails the inspection. [2]

A statistician in the company suggests it would be fairer if the company passes the inspection when the mean weight of five randomly chosen bags is greater than $950\,\text{g}$.

(b) Find the probability of passing the inspection if the statistician's suggestion is followed. [4]

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Question 9 - working space

15. [Maximum mark: 7]

The number of cars arriving at a junction in a particular town in any given minute between $9{:}00\ \text{am}$ and $10{:}00\ \text{am}$ is historically known to follow a Poisson distribution with a mean of 5.4 cars per minute.

A new road is built near the town. It is claimed that the new road has decreased the number of cars arriving at the junction.

To test the claim, the number of cars, X, arriving at the junction between $9{:}00\ \text{am}$ and $10{:}00\ \text{am}$ on a particular day will be recorded. The test will have the following hypotheses:

H_0: the mean number of cars arriving at the junction has not changed,
H_1: the mean number of cars arriving at the junction has decreased.

The alternative hypothesis will be accepted if $X \leq 300$.

(a) Assuming the null hypothesis to be true, state the distribution of X. [1]

(b) Find the probability of a Type I error. [2]

(c) Find the probability of a Type II error, if the number of cars now follows a Poisson distribution with a mean of 4.5 cars per minute. [4]

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Question 15 - working space

1. [Maximum mark: 6]

A group of 130 applicants applied for admission into either the Arts programme or the Sciences programme at a university. The outcomes of their applications are shown in the following table.

	Accepted	**Rejected**
Arts programme	17	24
Sciences programme	25	64

(a) Find the probability that a randomly chosen applicant from this group was accepted by the university. [1]

An applicant is chosen at random from this group. It is found that they were accepted into the programme of their choice.

(b) Find the probability that the applicant applied for the Arts programme. [2]

Two different applicants are chosen at random from the original group.

(c) Find the probability that both applicants applied to the Arts programme. [3]

(This question continues on the following page)

(Question 1 continued)

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Question 1 - working space

3. [Maximum mark: 7]

A polygraph test is used to determine whether people are telling the truth or not, but it is not completely accurate. When a person tells the truth, they have a 20% chance of failing the test. Each test outcome is independent of any previous test outcome.

10 people take a polygraph test and all 10 tell the truth.

(a) Calculate the expected number of people who will pass this polygraph test. [2]

(b) Calculate the probability that exactly 4 people will fail this polygraph test. [2]

(c) Determine the probability that fewer than 7 people will pass this polygraph test. [3]

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Question 3 - working space

9. [Maximum mark: 8]

A psychologist records the number of digits (d) of π that a sample of IB Mathematics higher level candidates could recall.

d	2	3	4	5	6	7
Frequency	2	6	24	21	11	3

(a) Find an unbiased estimate of the population mean of d. [1]

(b) Find an unbiased estimate of the population variance of d. [2]

The psychologist has read that in the general population people can remember an average of 4.4 digits of π. The psychologist wants to perform a statistical test to see if IB Mathematics higher level candidates can remember more digits than the general population.

(c) $H_0: \mu = 4.4$ is the null hypothesis for this test.

(i) State the alternative hypothesis.

(ii) Given that all assumptions for this test are satisfied, carry out an appropriate hypothesis test. State and justify your conclusion. Use a 5% significance level. [5]

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Question 9 - working space

12. [Maximum mark: 6]

The sex of cuttlefish is difficult to determine visually, so it is often found by weighing the cuttlefish.

The weights of adult male cuttlefish are known to be normally distributed with mean $10\,\text{kg}$ and standard deviation $0.5\,\text{kg}$.

The weights of adult female cuttlefish are known to be normally distributed with mean $12\,\text{kg}$ and standard deviation $1\,\text{kg}$.

A zoologist uses the null hypothesis that in the absence of information a cuttlefish is male.

If the weight is found to be above $11.5\,\text{kg}$ the cuttlefish is classified as female.

(a) Find the probability of making a Type I error when weighing a male cuttlefish. [2]

(b) Find the probability of making a Type II error when weighing a female cuttlefish. [2]

$90\,\%$ of adult cuttlefish are male.

(c) Find the probability of making an error using the zoologist's method. [2]

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Question 12 - working space

1. [Maximum mark: 16]

The scores of the eight highest scoring countries in the 2019 Eurovision song contest are shown in the following table.

	Eurovision score
Netherlands	498
Italy	472
Russia	370
Switzerland	364
Sweden	334
Norway	331
North Macedonia	305
Azerbaijan	302

(a) For this data, find

(i) the upper quartile.

(ii) the interquartile range. [4]

(b) Determine if the Netherlands' score is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 1 continued)

Chester is investigating the relationship between the highest-scoring countries' Eurovision score and their population size to determine whether population size can reasonably be used to predict a country's score.

The populations of the countries, to the nearest million, are shown in the table.

	Population (x) (millions)	**Eurovision score (y)**
Netherlands	17	498
Italy	60	472
Russia	145	370
Switzerland	9	364
Sweden	10	334
Norway	5	331
North Macedonia	2	305
Azerbaijan	10	302

Chester finds that, for this data, the Pearson's product moment correlation coefficient is $r = 0.249$.

(c) State whether it would be appropriate for Chester to use the equation of a regression line for y on x to predict a country's Eurovision score. Justify your answer. [2]

Chester then decides to find the Spearman's rank correlation coefficient for this data, and creates a table of ranks.

	Population rank (to the nearest million)	**Eurovision score rank**
Netherlands	3	1
Italy	2	2
Russia	1	3
Switzerland	a	4
Sweden	b	5
Norway	7	6
North Macedonia	8	7
Azerbaijan	c	8

(This question continues on the following page)

(Question 1 continued)

(d) Write down the value of:

(i) a,

(ii) b,

(iii) c. [3]

(e) (i) Find the value of the Spearman's rank correlation coefficient r_s.

(ii) Interpret the value obtained for r_s. [3]

(f) When calculating the ranks, Chester incorrectly read the Netherlands' score as 478. Explain why the value of the Spearman's rank correlation r_s does not change despite this error. [1]

Question 1 - working space

3. [Maximum mark: 13]

A Principal would like to compare the students in his school with a national standard. He decides to give a test to eight students made up of four boys and four girls. One of the teachers offers to find the volunteers from his class.

(a) Name the type of sampling that best describes the method used by the Principal. [1]

The marks out of 40, for the students who took the test, are:

$$25,\ 29,\ 38,\ 37,\ 12,\ 18,\ 27,\ 31.$$

(b) For the eight students find

(i) the mean mark.

(ii) the standard deviation of the marks. [3]

The national standard mark is 25.2 out of 40.

(c) Perform an appropriate test at the 5% significance level to see if the mean marks achieved by the students in the school are higher than the national standard. It can be assumed that the marks come from a normal population. [5]

(d) State one reason why the test might not be valid. [1]

Two additional students take the test at a later date and the mean mark for all ten students is 28.1 and the standard deviation is 8.4.

For further analysis, a standardized score out of 100 for the ten students is obtained by multiplying the scores by 2 and adding 20.

(e) For the ten students, find

(i) their mean standardized score.

(ii) the standard deviation of their standardized score. [3]

Turn over

Question 3 - working space

5. [Maximum mark: 15]

The aircraft for a particular flight has 72 seats. The airline's records show that historically for this flight only 90% of the people who purchase a ticket arrive to board the flight. They assume this trend will continue and decide to sell extra tickets and hope that no more than 72 passengers will arrive.

The number of passengers that arrive to board this flight is assumed to follow a binomial distribution with a probability of 0.9.

(a) The airline sells 74 tickets for this flight. Find the probability that more than 72 passengers arrive to board the flight. [3]

(b) (i) Write down the expected number of passengers who will arrive to board the flight if 72 tickets are sold. [2]

(ii) Find the maximum number of tickets that could be sold if the expected number of passengers who arrive to board the flight must be less than or equal to 72. [2]

Each passenger pays $\$150$ for a ticket. If too many passengers arrive, then the airline will give $\$300$ in compensation to each passenger that cannot board.

(c) Find, to the nearest integer, the expected increase or decrease in the money made by the airline if they decide to sell 74 tickets rather than 72. [8]

Turn over

Question 5 - working space

5. [Maximum mark: 12]

A geneticist uses a Markov chain model to investigate changes in a specific gene in a cell as it divides. Every time the cell divides, the gene may mutate between its normal state and other states.

The model is of the form

$$\begin{pmatrix} X_{n+1} \\ Z_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} X_n \\ Z_n \end{pmatrix}$$

where X_n is the probability of the gene being in its normal state after dividing for the nth time, and Z_n is the probability of it being in another state after dividing for the nth time, where $n \in \mathbb{N}$.

Matrix $\boldsymbol{M}$ is found to be $\begin{pmatrix} 0.94 & b \\ 0.06 & 0.98 \end{pmatrix}$.

(a) (i) Write down the value of b.

(ii) What does b represent in this context? [2]

(b) Find the eigenvalues of $\boldsymbol{M}$. [3]

(c) Find the eigenvectors of $\boldsymbol{M}$. [3]

(d) The gene is in its normal state when $n = 0$. Calculate the probability of it being in its normal state

(i) when $n = 5$;

(ii) in the long term. [4]

Question 5 - working space

1. [Maximum mark: 27]

This question uses statistical tests to investigate whether advertising leads to increased profits for a grocery store.

Aimmika is the manager of a grocery store in Nong Khai. She is carrying out a statistical analysis on the number of bags of rice that are sold in the store each day. She collects the following sample data by recording how many bags of rice the store sells each day over a period of 90 days.

Number of bags of rice sold	0	1	2	3	4	5	6	7	8	9	10
Number of days	1	8	12	11	19	14	13	8	2	0	2

She believes that her data follows a Poisson distribution.

(a) (i) Find the mean and variance for the sample data given in the table. [2]

(ii) Hence state why Aimmika believes her data follows a Poisson distribution. [1]

(b) State one assumption that Aimmika needs to make about the sales of bags of rice to support her belief that it follows a Poisson distribution. [1]

Aimmika knows from her historic sales records that the store sells an average of 4.2 bags of rice each day. The following table shows the expected frequency of bags of rice sold each day during the 90 day period, assuming a Poisson distribution with mean 4.2.

Number of bags of rice sold	≤ 1	2	3	4	5	6	7	≥ 8
Expected frequency	a	11.903	16.665	b	14.698	10.289	6.173	c

(c) Find the value of a, of b, and of c. Give your answers to 3 decimal places. [5]

(This question continues on the following page)

(Question 1 continued)

Aimmika decides to carry out a χ^2 goodness of fit test at the 5% significance level to see whether the data follows a Poisson distribution with mean 4.2.

(d) (i) Write down the number of degrees of freedom for her test. [1]

(ii) Perform the χ^2 goodness of fit test and state, with reason, a conclusion. [7]

Aimmika claims that advertising in a local newspaper for 300 Thai Baht (THB) per day will increase the number of bags of rice sold. However, Nichakarn, the owner of the store, claims that the advertising will **not** increase the store's overall profit.

Nichakarn agrees to advertise in the newspaper for the next 60 days. During that time, Aimmika records that the store sells 282 bags of rice with a profit of 495 THB on each bag sold.

(e) Aimmika wants to carry out an appropriate hypothesis test to determine whether the number of bags of rice sold during the 60 days increased when compared with the historic sales records.

(i) By finding a critical value, perform this test at a 5% significance level. [6]

(ii) Hence state the probability of a Type I error for this test. [1]

(f) By considering the claims of both Aimmika and Nichakarn, explain whether the advertising was beneficial to the store. [3]

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Question 1 - working space

1. [Maximum mark: 5]

Sergio is interested in whether an adult's favourite breakfast berry depends on their income level. He obtains the following data for 341 adults and decides to carry out a χ^2 test for independence, at the 10% significance level.

		Income level		
		Low	**Medium**	**High**
Favourite berry	**Strawberry**	21	39	30
	Blueberry	39	67	42
	Other berry	32	45	26

(a) Write down the null hypothesis. [1]

(b) Find the value of the χ^2 statistic. [2]

The critical value of this χ^2 test is 7.78.

(c) Write down Sergio's conclusion to the test in context. Justify your answer. [2]

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Question 1 - working space

5. [Maximum mark: 7]

Taizo plays a game where he throws one ball at two bottles that are sitting on a table. The probability of knocking over bottles, in any given game, is shown in the following table.

Number of bottles knocked over	0	1	2
Probability	0.5	0.4	0.1

(a) Taizo plays two games that are independent of each other. Find the probability that Taizo knocks over a **total** of two bottles. [4]

In any given game, Taizo will win k points if he knocks over two bottles, win 4 points if he knocks over one bottle and lose 8 points if no bottles are knocked over.

(b) Find the value of k such that the game is fair. [3]

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Question 5 - working space

16. [Maximum mark: 8]

The principal of a school is concerned that only 30% of her students are choosing healthy options from the school canteen. She organizes a campaign to promote healthy eating and decides to test if the campaign has increased the number of students choosing healthy options. She assumes that a student's choice is independent of other students' choices.

(a) Write down suitable hypotheses for this test. [2]

The principal decides to take a random sample of 80 students. She will reject the null hypothesis if at least 31 students choose a healthy option.

(b) Find the probability that she makes a Type I error. [3]

In fact, the campaign led to 40% of her students choosing a healthy option.

(c) Find the probability that she makes a Type II error. [3]

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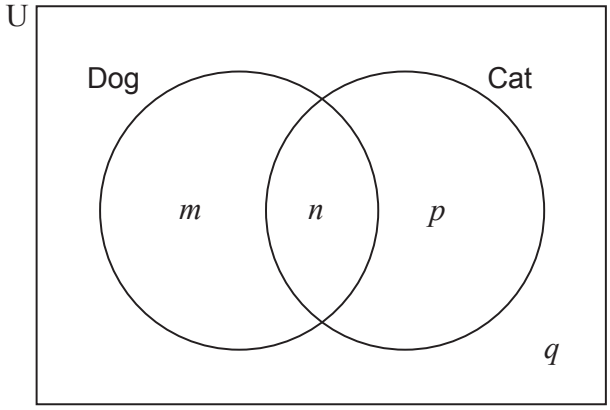
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Question 16 - working space

1. [Maximum mark: 15]

At Mirabooka Primary School, a survey found that 68% of students have a dog and 36% of students have a cat. 14% of students have both a dog and a cat.

This information can be represented in the following Venn diagram, where m, n, p and q represent the percentage of students within each region.



(a) Find the value of

(i) m.

(ii) n.

(iii) p.

(iv) q. [4]

(b) Find the probability that a randomly chosen student

(i) has a dog but does not have a cat.

(ii) has a dog given that they do not have a cat. [3]

(This question continues on the following page)

(Question 1 continued)

Each year, one student is chosen randomly to be the school captain of Mirabooka Primary School.

Tim is using a binomial distribution to make predictions about how many of the next 10 school captains will own a dog. He assumes that the percentages found in the survey will remain constant for future years and that the events “being a school captain” and “having a dog” are independent.

Use Tim’s model to find the probability that in the next 10 years

(c) (i) 5 school captains have a dog.

(ii) more than 3 school captains have a dog.

(iii) exactly 9 school captains in succession have a dog. [7]

John randomly chooses 10 students from the survey.

(d) State why John should not use the binomial distribution to find the probability that 5 of these students have a dog. [1]

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Question 1 - working space

6. [Maximum mark: 18]

A company makes doors for kitchen cupboards from two layers. The inside layer is wood, and its thickness is normally distributed with mean $7\,\text{mm}$ and standard deviation $0.3\,\text{mm}$. The outside layer is plastic, and its thickness is normally distributed with mean $3\,\text{mm}$ and standard deviation $0.16\,\text{mm}$. The thickness of the plastic is independent of the thickness of the wood.

(a) Find the probability that a randomly chosen door has a total thickness of less than $9.5\,\text{mm}$. [5]

Eight doors are to be packed into a box to send to a customer. The width of the box is $82\,\text{mm}$. The thickness of each door is independent.

(b) Find the probability that the total thickness of the eight doors is greater than the width of the box. [4]

The company buys two new machines, A and B, to make the wooden layers. An employee claims that the layers from machine B are thinner than the layers from machine A. In order to test this claim, a random sample is taken from each machine.

The seven layers in the sample from machine A have a thickness, in mm, of

$$6.23,\ 7.04,\ 7.31,\ 6.79,\ 6.91,\ 6.79,\ 7.47.$$

(c) Find the

(i) mean.

(ii) unbiased estimate of the population variance. [3]

The eight layers in the sample from machine B have a mean thickness of $6.89\,\text{mm}$ and $s_{n-1} = 0.31$.

(d) Perform a suitable test, at the 5% significance level, to test the employee's claim. You may assume the thickness of the wooden layers from each machine are normally distributed with equal population variance. [6]

Turn over

Question 6 - working space

1. [Maximum mark: 29]

In this question, you will explore possible approaches to using historical sports results for making predictions about future sports matches.

Two friends, Peter and Helen, are discussing ways of predicting the outcomes of international football matches involving Argentina.

Peter suggests analysing historical data to help make predictions. He lists the results of the most recent 240 matches in which Argentina played, in chronological order, then considers blocks of four matches at a time. He counts how many times Argentina has won in each block. The following table shows his results for the 60 blocks of four matches.

Number of wins for Argentina (per block of four matches)	Frequency
0	0
1	11
2	21
3	21
4	7

(a) Determine the mean number of wins per block of four matches for Argentina. [2]

Peter thinks that this data can be modelled by a binomial distribution with $n = 4$ and decides to carry out a χ^2 goodness of fit test.

(b) Use Peter's data to write down an estimate for the probability p for this binomial model. [1]

(c) (i) Use the binomial model to find the probability that Argentina win zero matches in a block of four matches. [1]

(ii) Find the expected frequency for zero wins. [2]

(This question continues on the following page)

(Question 1 continued)

As some expected frequencies are less than 5, Peter combines rows in his table to produce the following observed frequencies. He then uses his binomial model to find appropriate expected frequencies, correct to one decimal place.

Number of wins for Argentina (per block of four matches)	Observed frequency	Expected frequency
0 or 1	11	10.8
2	21	20.7
3	21	20.7
4	7	7.8

(d) Peter uses this table to carry out a χ^2 goodness of fit test, to test the hypothesis that the data follows a binomial distribution with $n = 4$, at the 5% significance level.

For this test, state

(i) the null hypothesis; [1]

(ii) the number of degrees of freedom; [1]

(iii) the p-value; [2]

(iv) the conclusion, justifying your answer. [2]

(e) Using Peter's binomial model, find the probability that Argentina will win at least one of their next four international football matches. [2]

(This question continues on the following page)

(Question 1 continued)

Helen thinks that a better prediction might be made by considering the transition between matches. To keep the model simple, she decides to use only two states: Argentina won (A) or Argentina did not win (B). Helen looks at Peter's list of results and counts the number of times that:

- Argentina won, twice in succession (AA),
- Argentina won, then did not win (AB),
- Argentina did not win, then won (BA),
- Argentina did not win, twice in succession (BB).

She recorded the following results.

Transition	Frequency
AA	85
AB	60
BA	62
BB	32

Helen uses the relative frequencies to estimate the probabilities in a transition matrix.

(f) (i) Given that Argentina won the previous match, show that Helen's estimate for the probability of Argentina winning the next match is $\frac{17}{29}$. [2]

(ii) Write down the transition matrix, $\boldsymbol{T}$, for Helen's model. [2]

(g) (i) Show that the characteristic polynomial of $\boldsymbol{T}$ is $1363\lambda^2 - 1263\lambda - 100 = 0$. [3]

(ii) Hence or otherwise, find the eigenvalues of $\boldsymbol{T}$. [1]

(iii) Find the corresponding eigenvectors. [3]

(h) In her retirement, many years from now, Helen is planning to travel to three consecutive international football matches involving Argentina. Use Helen's model to find the probability that Argentina will win all three matches. [4]

Question 1 - working space

2. [Maximum mark: 6]

Carys believes that, on a memory retention test, the mean score of bilingual people (μ_b) will be higher than the mean score of monolingual people (μ_m). Carys gave a memory retention test to a random sample of students in her class. The results are shown in the two tables.

	Scores									
Bilingual	100	94	100	90	100	94	98	98	98	98

	Scores							
Monolingual	97	92	88	98	88	94	100	100

Carys performs a one-tailed t-test at a 5% level of significance. It is assumed that the scores are normally distributed and the samples have equal variances.

(a) State the null and alternative hypotheses. [2]

(b) Calculate the p-value for this test. [2]

(c) State the conclusion of the test in the context of the question. Justify your answer. [2]

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Question 2 - working space

8. [Maximum mark: 7]

The random variables (X, Y) follow a bivariate normal distribution with product-moment correlation coefficient ρ. The values of six random observations of (X, Y) are shown in the table.

x	6.3	4.1	5.6	9.2	7.8	8.2
y	9.2	4.9	8.9	10.3	8.9	9.8

(a) State null and alternative hypotheses which could be used to test whether there is a linear correlation between X and Y. [2]

(b) Determine the value of

(i) the product-moment correlation coefficient, r, of the sample.

(ii) the corresponding p-value. [3]

(c) State whether your result from part (b)(ii) indicates there is sufficient evidence to claim that, at the 5% significance level, X and Y are not linearly correlated.

Give a reason for your answer. [2]

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Question 8 - working space

9. [Maximum mark: 5]

On a specific day, the speed of cars as they pass a speed camera can be modelled by a normal distribution with a mean of $67.3\,\text{km h}^{-1}$.

A speed of $75.7\,\text{km h}^{-1}$ is two standard deviations from the mean.

(a) Find the standard deviation for the speed of the cars. [2]

It is found that 82% of cars on this road travel at speeds between $p\,\text{km h}^{-1}$ and $q\,\text{km h}^{-1}$, where $p < q$. This interval includes cars travelling at a speed of $74\,\text{km h}^{-1}$.

(b) Show that the region of the normal distribution between p and q is **not** symmetrical about the mean. [3]

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Question 9 - working space

11. [Maximum mark: 6]

A shop sells oranges and lemons. The weights of the oranges are assumed to be normally distributed with mean 205 grams and standard deviation 5 grams. The weights of the lemons are assumed to be normally distributed with mean 105 grams and standard deviation 3 grams.

Nelia selects 1 orange and 2 lemons at random and independent of each other. Calculate the probability that the weight of her orange exceeds the combined weight of her lemons.

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Question 11 - working space

16. [Maximum mark: 6]

The relationship between the intensity, I, of a light source and the distance, d, from the light source can be modelled by $I = \frac{k}{d^2}$.

Pablo measures the intensity of a light source at different distances. The data collected is shown in the table.

d (m)	1	2	5
I (lm)	42	11	1.5

Pablo finds the sum of square residuals in the form $1.0641k^2 - 89.62k + c$.

(a) Find the exact value of c. [4]

(b) Hence find the least squares regression curve of the form $I = \frac{k}{d^2}$. [2]

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Question 16 - working space

2. [Maximum mark: 4]

A company that owns many restaurants wants to determine if there are differences in the quality of the food cooked for three different meals: breakfast, lunch and dinner.

Their quality assurance team randomly selects 500 items of food to inspect. The quality of this food is classified as perfect, satisfactory, or poor. The data is summarized in the following table.

		Quality			
		Perfect	**Satisfactory**	**Poor**	**Total**
Meal	**Breakfast**	101	124	7	**232**
	Lunch	68	81	5	**154**
	Dinner	35	69	10	**114**
	Total	**204**	**274**	**22**	**500**

A χ^2 test at the 5% significance level is carried out to determine if there is significant evidence of a difference in the quality of the food cooked for the three meals.

The critical value for this test is 9.488.

The hypotheses for this test are:

H_0: The quality of the food and the type of meal are independent.

H_1: The quality of the food and the type of meal are not independent.

(a) Find the χ^2 statistic. [2]

(b) State, with justification, the conclusion for this test. [2]

(This question continues on the following page)

(Question 2 continued)

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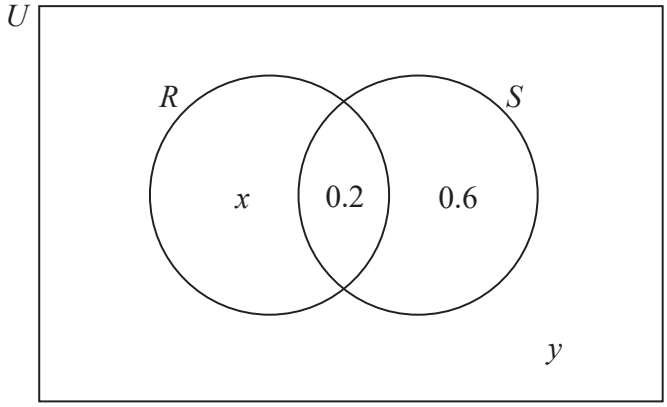
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Question 2 - working space

3. [Maximum mark: 7]

The following Venn diagram shows two independent events, R and S. The values in the diagram represent probabilities.



(a) Find the value of x. [3]

(b) Find the value of y. [2]

(c) Find $\mathrm{P}(R' \mid S')$. [2]

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Question 3 - working space

5. [Maximum mark: 6]

The lengths of the seeds from a particular mango tree are approximated by a normal distribution with a mean of $4\,\text{cm}$ and a standard deviation of $0.25\,\text{cm}$.

A seed from this mango tree is chosen at random.

(a) Calculate the probability that the length of the seed is less than $3.7\,\text{cm}$. [2]

It is known that $30\,\%$ of the seeds have a length greater than $k\,\text{cm}$.

(b) Find the value of k. [2]

For a seed of length $d\,\text{cm}$, chosen at random, $\text{P}(4 - m < d < 4 + m) = 0.6$.

(c) Find the value of m. [2]

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Question 5 - working space

7. [Maximum mark: 6]

Akar starts a new job in Australia and needs to travel daily from Wollongong to Sydney and back. He travels to work for 28 consecutive days and therefore makes 56 single journeys. Akar makes all journeys by bus.

The probability that he is successful in getting a seat on the bus for any single journey is 0.86.

(a) Determine the expected number of these 56 journeys for which Akar gets a seat on the bus. [1]

(b) Find the probability that Akar gets a seat on at least 50 journeys during these 28 days. [3]

The probability that Akar gets a seat on at most n journeys is at least 0.25.

(c) Find the smallest possible value of n. [2]

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Question 7 - working space

9. [Maximum mark: 9]

At a running club, Sung-Jin conducts a test to determine if there is any association between an athlete's age and their best time taken to run $100\,\text{m}$. Eight athletes are chosen at random, and their details are shown below.

Athlete	A	B	C	D	E	F	G	H
Age (years)	13	17	22	18	19	25	11	36
Time (seconds)	13.4	14.6	13.4	12.9	12.0	11.8	17.0	13.1

Sung-Jin decides to calculate the Spearman's rank correlation coefficient for his set of data.

(a) Complete the table of ranks. [2]

Athlete	A	B	C	D	E	F	G	H
Age rank			3					
Time rank							1	

(b) Calculate the Spearman's rank correlation coefficient, r_s. [2]

(c) Interpret this value of r_s in the context of the question. [1]

(d) Suggest a mathematical reason why Sung-Jin may have decided not to use Pearson's product-moment correlation coefficient with his data from the original table. [1]

(e) (i) Find the coefficient of determination for the data from the original table.

(ii) Interpret this value in the context of the question. [3]

(This question continues on the following page)

(Question 9 continued)

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Question 9 - working space

10. [Maximum mark: 6]

A chocolate company plans to produce chocolate bars with special flavours. They survey 246 people to determine if there is any particular preference for one of the flavours.

The table below shows the information collected.

Hot chilli	Almond crunch	Spiced Chai	Ginger'n'lime
75	59	46	66

A χ^2 goodness of fit test at the 5% significance level is carried out on the data.

The critical value for the test is 7.82.

(a) State the null and alternative hypotheses for this test. [2]

(b) Perform the test and give your conclusion in context. [4]

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Question 10 - working space

15. [Maximum mark: 6]

A random sample of eight packets of Apollo coffee granules are selected from a supermarket shelf.

The weights of the coffee granules present in each packet are as follows:

$$222\,\text{g} \quad 226\,\text{g} \quad 221\,\text{g} \quad 228\,\text{g} \quad 227\,\text{g} \quad 225\,\text{g} \quad 222\,\text{g} \quad 223\,\text{g}$$

(a) (i) Find an unbiased estimate for the mean weight of coffee granules in a packet of Apollo coffee.

(ii) Calculate a $95\,\%$ confidence interval for the population mean. Give your answer to four significant figures. [3]

(b) State one assumption you have made in order for your interval to be valid. [1]

(c) The label of each packet has a description which includes the phrase: "contains $226\,\text{g}$ of coffee granules".

Using your answer to part (a)(ii), briefly comment on the claim on the label. [2]

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Question 15 - working space

1. [Maximum mark: 13]

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958	2008
Temperature °C (y)	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates $(1708, 8.73)$ and $(1958, 9.45)$.

(a) Calculate the gradient of the straight line that passes through these two points. [2]

(b) (i) Interpret the meaning of the gradient in the context of the question.

(ii) State appropriate units for the gradient. [2]

(c) Find the equation of this line giving your answer in the form $y = mx + c$. [2]

(d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Thandizo uses linear regression to obtain a model for the data.

(e) (i) Find the equation of the regression line y on x.

(ii) Find the value of r, the Pearson's product-moment correlation coefficient. [3]

(f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

Turn over

Question 1 - working space

3. [Maximum mark: 17]

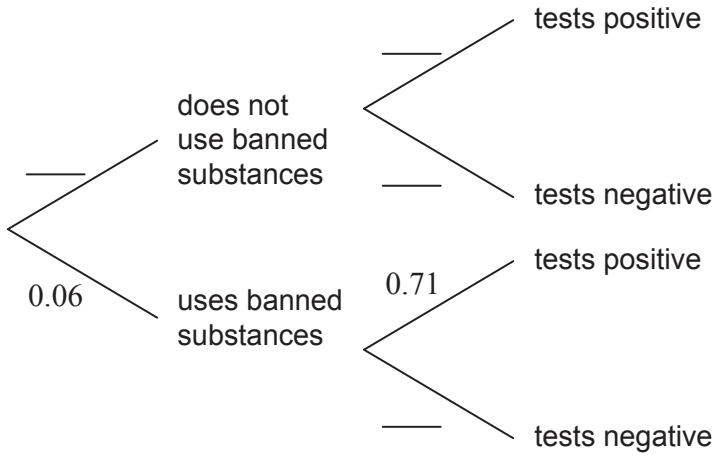
A large international sports tournament tests their athletes for banned substances.
They interpret a positive test result as meaning that the athlete uses banned substances.
A negative result means that they do not.

The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

(a) Using the information given, **copy** (into your answer booklet) and complete the following tree diagram. [2]



(b) (i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative.

(ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [4]

(c) (i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result.

(ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [5]

Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use banned substances.

(d) Calculate the probability that none of the athletes in Team X will test positive. [4]

(e) Determine the probability that more than 2 athletes in Team X will test positive. [2]

Turn over

Question 3 - working space

5. [Maximum mark: 15]

Goran is interested in the number of sightings of a particular bird each week in the 50 weeks following the first day of September. He collects some data which is shown in the table.

Number of sightings	0	1	2	3	4	5	More than 5
Number of weeks	8	16	13	8	3	2	0

The sample mean number of sightings per week for this data is 1.76.

(a) Calculate the unbiased estimate of the population variance of sightings per week. [3]

Goran believes that the data follows a Poisson distribution.

(b) State why your answer to part (a) supports Goran's belief. [1]

Goran decides to test at the 5% significance level to see if his belief is correct.

His null hypothesis is $X \sim \text{Po}(m)$, where the random variable, X, is defined as the number of sightings per week.

Goran estimates parameter m to be the mean of the sample, 1.76. He calculates the expected frequencies for sightings per week in the 50 weeks after the first day of September. These are shown to two decimal places in the following table.

Number of sightings	0	1	2	3	4	5 or more
Expected frequencies	8.60	15.14	13.32	7.82	j	k

(c) Find the value of

(i) j;

(ii) k. [5]

(d) State a reason why Goran should combine groups to conduct his significance test. [1]

(e) Write down the degrees of freedom for the test. [1]

(f) Find the p-value for the test. [2]

(g) State the conclusion of the test. Justify your answer. [2]

Question 5 - working space

5. [Maximum mark: 14]

The three countries of Belgium, Germany and The Netherlands meet at a single point called Vaalserberg.

To support future transport planning, a $10\,\text{km}$ circle was drawn around Vaalserberg on a map. A study was carried out over five years to determine what percentage of people living in each of these countries (within the $10\,\text{km}$ circular region) either stayed in their own country or moved to another country within the circle.

From this study, the following movements were observed during the five years.
- From Belgium, $5\,\%$ moved to Germany, and $0.5\,\%$ moved to The Netherlands.
- From Germany, $2\,\%$ moved to The Netherlands, and $1.5\,\%$ moved to Belgium.
- From The Netherlands, $3\,\%$ moved to Germany, and $2\,\%$ moved to Belgium.

All additional population changes within the circular region may be ignored.

(a) Represent the above information in a transition matrix $\boldsymbol{T}$. [3]

At the end of the study, the population of the Belgian side was $26\,000$, the population of the German side was $240\,000$, and the population of The Netherlands side was $50\,000$.

(b) By using $\boldsymbol{T}$, find the expected population of the German side of Vaalserberg 30 years after the end of the study. [4]

For matrix $\boldsymbol{T}$ there exists a steady state vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where u_1, u_2 and u_3 are the proportions of the total population on the Belgian side, the German side and The Netherlands side respectively.

The steady state vector $\boldsymbol{u}$ may be found by solving a system of equations.

(c) (i) Determine these equations that are to be solved.

(ii) By solving your system of equations, find $\boldsymbol{u}$. [3]

(d) Use your answer to part (c)(ii) to determine the long-term expected population of the German side. [2]

(e) Suggest two reasons why your answer to part (d) is not likely to be accurate. You may comment on both the model and the situation in context. [2]

Question 5 - working space

6. [Maximum mark: 18]

The gardener in a local park suggested that the number of snails found in the park can be modelled by a Poisson distribution.



(a) Suggest two observations that the gardener may have made that led him to suggest this model. [2]

Now assume that the model is valid and that the mean number of snails per m^2 is 0.2.
The gardener inspects, at random, a $12\,\mathrm{m}^2$ area of the park.

(b) Find the probability that the gardener finds exactly four snails. [3]

(c) Find the probability that the gardener finds fewer than three snails. [2]

(d) Find the probability that, in three consecutive inspections, the gardener finds at least one snail per inspection. [3]

Following heavy rain overnight, the gardener wished to determine whether the number of snails found in a random $12\,\mathrm{m}^2$ area of the park had increased.

(e) State the hypotheses for the test. [2]

(f) Find the critical region for the test at the $1\,\%$ significance level. [3]

(g) Given that the mean number of snails per m^2 has actually risen to 0.75, find the probability that the gardener makes a Type II error. [3]

Turn over

Question 6 - working space

2. [Maximum mark: 29]

This question considers the analysis of several datasets of examination marks using a variety of standard procedures and also an unfamiliar statistical test.

A class of eight students sits two examinations, one in French and one in German.

The marks in these examinations are given in **Table 1**.

Table 1

Student	French mark	German mark
S_1	42	39
S_2	65	66
S_3	82	71
S_4	50	53
S_5	48	32
S_6	73	59
S_7	34	40
S_8	59	56

The maximum mark in both examinations is the same.

You may assume that these data are a random sample from a bivariate normal distribution with mean μ_F for the French examination, mean μ_G for the German examination and Pearson's product-moment correlation coefficient ρ.

Before the examinations were sat, the Head of Languages, Pierre, decided to investigate whether there would be significant evidence of a difference between μ_F and μ_G. He decided to analyse the data using a two-tailed paired t-test with significance level 5%.

(a) Explain briefly

(i) why he chose to use a t-test and not a z-test; [1]

(ii) why he chose to use a two-tailed test and not a one-tailed test. [1]

(b) (i) State suitable hypotheses for the t-test. [1]

(ii) Find the p-value for this test. [2]

(iii) The p-value is a probability. State the event for which it gives the probability. [1]

(iv) State, giving a reason, what conclusion Pierre should reach. [2]

(This question continues on the following page)

(Question 2 continued)

(c) Pierre believes that students who score well in one language examination tend to score well in the other language examination. He therefore decides to carry out a test at the 5% significance level to investigate whether there is a positive correlation between the French examination marks and the German examination marks.

(i) State appropriate hypotheses in terms of ρ. [1]

(ii) Perform a suitable test and state the p-value. State, in context, the conclusion that Pierre should reach, giving a reason. [4]

(d) There are actually two more students in this particular class, Paul and Sue. Paul sat the French examination, but he was unable to sit the German examination. Sue sat the German examination, but she was unable to sit the French examination.

(i) Paul's mark in the French examination was 58. Use the data in **Table 1** to predict the mark that Paul would have obtained in his German examination. [3]

(ii) Based on her mark in the German examination, Sue's mark in the French examination was predicted to be 71. Find the mark she obtained in the German examination. [2]

Six students sit examinations in mathematics and history and their marks are shown in **Table 2**. The Vice Principal, Angela, decides to investigate whether there is any association between the marks obtained in these two subjects.

Table 2

Student	Mathematics mark (x)	History mark (y)
P_1	53	41
P_2	76	70
P_3	50	62
P_4	65	47
P_5	61	66
P_6	84	50

(This question continues on the following page)

(Question 2 continued)

Angela is informed that the maximum mark in each subject is 100.

Angela believes that the data might not be normally distributed, so she investigates what suitable tests are available which do not assume the data is normally distributed. She decides to use an unfamiliar test based on a statistic called Kendall's τ.

Consider n bivariate observations (x_i, y_i), $i = 1, 2, \ldots n$, such that there are no equal x-values and no equal y-values. Any pair of distinct bivariate observations (x_i, y_i) and (x_j, y_j) is said to be concordant if $(x_i - x_j)(y_i - y_j) > 0$ and discordant if $(x_i - x_j)(y_i - y_j) < 0$. For n bivariate observations, there are $\frac{n(n-1)}{2}$ distinct pairs.

Kendall's τ is defined as $\frac{2(C-D)}{n(n-1)}$ where C and D denote respectively the number of concordant pairs and discordant pairs.

(e) (i) Show that the value of Kendall's τ always lies in the interval $[-1, +1]$. [1]

(ii) For students P_1 and P_2, show that their pair is concordant. [1]

(iii) Show that the value of Kendall's τ for the mathematics and history data is 0.2. [4]

(f) Angela decided to use this statistic in a two-tailed test at the 10% significance level. The critical region for her test is $|\tau| \geq 0.733$.

(i) State, in words, her null and alternative hypotheses. [1]

(ii) State the conclusion that Angela should reach. Give a reason for your answer. [2]

Angela now finds that the history marks are actually out of 120. The history teacher advises Angela to scale the history marks so that they are out of 100 and then redo the calculations for the value of τ.

(g) State, with a reason, whether you agree with the advice given by the history teacher. [2]

References:

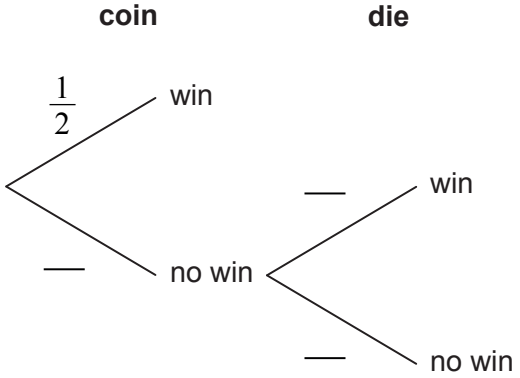
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Question 2 - working space

7. [Maximum mark: 7]

Michèle is playing a game. In the game, she must first flip a fair coin which will result in the coin landing on heads or tails. If the coin lands on heads, then she wins a prize. If it lands on tails, then she has another chance but this time she must roll a fair six-sided dice and get a five or six in order to win a prize.

(a) Complete the tree diagram by writing in the three missing probabilities. [2]



(b) Find the probability that Michèle does **not** win a prize. [2]

(c) Given that Michèle won a prize, find the probability that the coin landed on heads. [3]

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Question 7 - working space

10. [Maximum mark: 7]

The decay of a chemical isotope over five years is recorded in **Table 1**. The mass of the chemical M is measured to the nearest gram at the beginning of each year t of the experiment.

Table 1

Time t (years)	1	2	3	4	5
Mass M (grams)	1000	660	517	435	381

It is believed that the decay of the isotope can be modelled by an equation of the form $M = a \times t^b$.

(a) Use power regression on your graphic display calculator to find the value of a and the value of b. [2]

The values of t and M can be transformed such that $x = \ln t$ and $y = \ln M$. **Table 2** shows data for x and y to three decimal places.

Table 2

x	0	0.693	1.099	1.386	1.609
y	6.908	6.492	6.248	6.075	5.943

(b) Find the linear regression equation of y on x, in the form $y = cx + d$. Give the values of c and d to three decimal places. [2]

(c) Hence, show that this linear regression is equivalent to the power regression found in part (a). [3]

(This question continues on the following page)

(Question 10 continued)

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Question 10 - working space

15. [Maximum mark: 7]

The eating habits of students in a school are studied over a number of months. The focus of the study is whether non-vegetarians become vegetarians, and whether vegetarians remain vegetarians.

Each month, students choose between the vegetarian or non-vegetarian lunch options. Once they have chosen for the month, they cannot change the option until the next month.

In any month during the study, it is noticed that the probability of a non-vegetarian becoming vegetarian the following month is 0.1, and that the probability of a vegetarian remaining a vegetarian the following month is 0.8.

This situation can be represented by the transition matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}.$$

(a) Interpret the value 0.9 in $\boldsymbol{T}$ in terms of the changes in the eating habits of the students in the school. [1]

(b) Find the eigenvalues of matrix $\boldsymbol{T}$. [3]

One of the eigenvectors of $\boldsymbol{T}$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Find another, non-parallel, eigenvector and interpret it in context. [3]

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Question 15 - working space

16. [Maximum mark: 6]

When Jef plays basketball, the number of shots he takes during any 6 minutes of play can be modelled by a Poisson distribution with mean 2.5.

(a) Find the probability that Jef takes less than 7 shots during any 12 minutes of play. [2]

It can be assumed that the outcomes of the shots are independent of each other, and the probability of success of a shot is constant. The probability that Jef is successful with a shot is 0.4.

It can be assumed that the probability of Jef's success with a shot is independent of the number of shots that he takes.

(b) Find the probability that during any 6 minutes of play Jef takes fewer than 4 shots and is successful at least once. [4]

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References:

3. Nina Aldin Thune. https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza#/media/File:Kheops-Pyramid.jpg. Licensed under CC BY 2.5 https://creativecommons.org/licenses/by/2.5/#. Image adapted.

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will not be marked.

Question 16 - working space

2. [Maximum mark: 18]

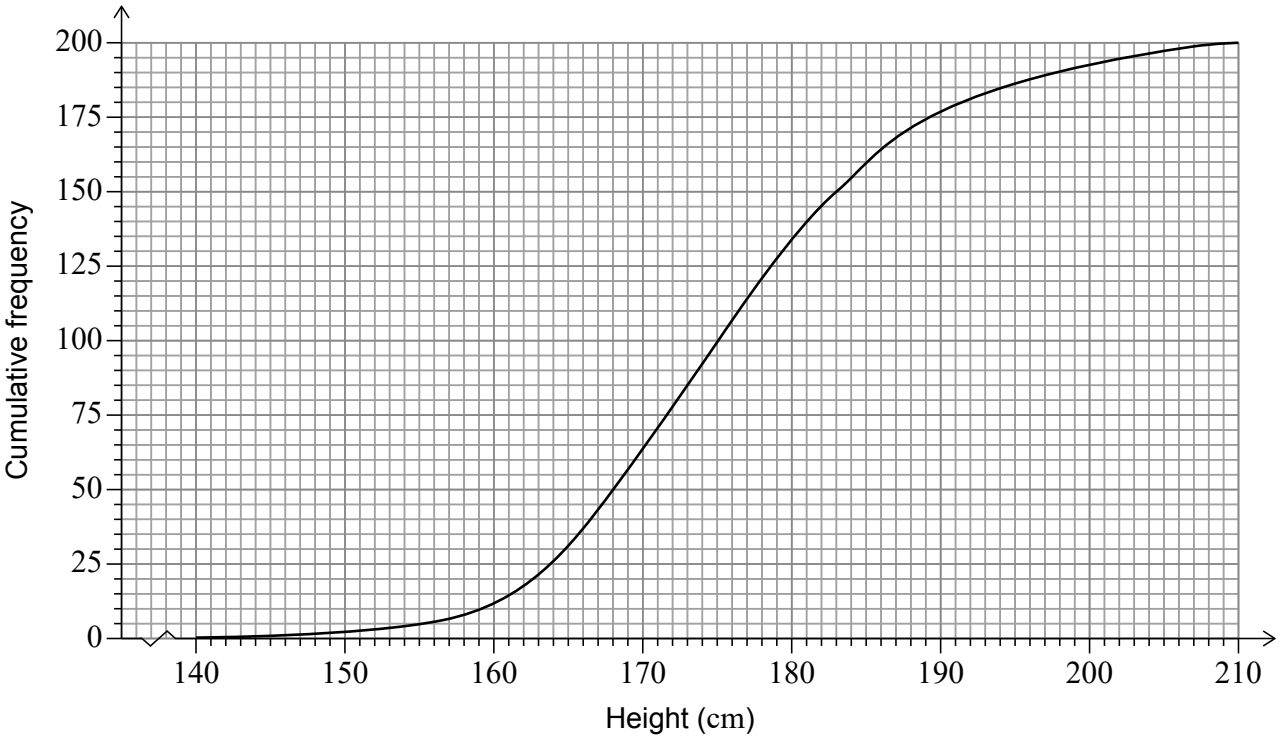
The heights, h, of 200 university students are recorded in the following table.

Height (cm)	Frequency
$140 \leq h < 160$	11
$160 \leq h < 170$	51
$170 \leq h < 180$	68
$180 \leq h < 190$	47
$190 \leq h < 210$	23

(a) (i) Write down the mid-interval value of $140 \leq h < 160$.

(ii) Calculate an estimate of the mean height of the 200 students. [3]

This table is used to create the following cumulative frequency graph.



(b) Use the cumulative frequency curve to estimate the interquartile range. [2]

Laszlo is a student in the data set and his height is $204\,\text{cm}$.

(c) Use your answer to part (b) to estimate whether Laszlo's height is an outlier for this data. Justify your answer. [3]

(This question continues on the following page)

(Question 2 continued)

It is believed that the heights of university students follow a normal distribution with mean $176\,\text{cm}$ and standard deviation $13.5\,\text{cm}$.

It is decided to perform a χ^2 goodness of fit test on the data to determine whether this sample of 200 students could have plausibly been drawn from an underlying distribution $\text{N}(176, 13.5^2)$.

(d) Write down the null and the alternative hypotheses for the test. [2]

As part of the test, the following table is created.

Height of student (cm)	Observed frequency	Expected frequency
$h < 160$	11	23.6
$160 \leq h < 170$	51	42.1
$170 \leq h < 180$	68	a
$180 \leq h < 190$	47	46.7
$190 \leq h$	23	b

(e) (i) Find the value of a and the value of b.

(ii) Hence, perform the test to a $5\,\%$ significance level, clearly stating the conclusion in context. [8]

Question 2 - working space

7. [Maximum mark: 17]

The city of Melba has an adult population of four million people. It is assumed that the weights of adults in Melba can be modelled by a normal distribution with mean $72\,\text{kg}$ and standard deviation $10\,\text{kg}$.

(a) If 10 adults in Melba are chosen independently and at random, find the probability that more than 3 of them have a weight greater than $85\,\text{kg}$. [4]

Laetitia runs a travel agency in Melba. The elevator to her office is designed to hold a maximum of 8 people.

(b) Write down a probability distribution that models the total weight of 8 adults chosen independently and at random from Melba. [3]

The total weight of 8 adults exceeds w on less than $1\,\%$ of all occasions that 8 adults enter the elevator.

(c) Find the value of w. [2]

A newspaper claims that $42\,\%$ of the adults in Melba who go on holiday choose to go abroad. Laetitia believes that this is an overestimation of the true number. During the past month, Laetitia found that 67 of her clients chose a holiday abroad, and 133 chose a holiday that was not abroad.

(d) Laetitia decides to perform a test using the binomial distribution on her data for the population proportion, p, that go on holiday abroad.

(i) State **two** assumptions that Laetitia makes in order to conduct the test.

(ii) Write down the null and the alternative hypotheses for Laetitia's test, in terms of p.

(iii) Using the data from Laetitia's sample, perform the test at a $5\,\%$ significance level to determine whether Laetitia's belief is reasonable. [8]

Question 7 - working space

2. [Maximum mark: 27]

This question is about applying ideas from logarithms, calculus and probability to an unfamiliar mathematical theory called information theory.

Claude Shannon developed a mathematical theory called information theory to measure the information gained when random events occur. He defined the information, I, that is gained when an event with probability p occurs as

$$I = -\ln p$$

where $0 < p \leq 1$. For example, no information is gained ($I = 0$) when an event is certain to occur ($p = 1$).

(a) (i) Sketch the graph of $I = -\ln p$, for $0 < p \leq 1$, labelling all axes intercepts and asymptotes. [3]

(ii) Show, using calculus, that I is a decreasing function of p. [3]

(iii) Interpret what "I is a decreasing function of p" means in the given context. [1]

(b) A computer selects at random an integer x from 1 to 10, inclusive. Each outcome is equally likely.

Alessia is trying to determine the value of x and asks if x is odd.

(i) Write down the probability that x is odd. [1]

(ii) Alessia is told that x is odd. Find how much information Alessia gains. [2]

The computer then selects at random an integer y from 1 to 10, inclusive.
Each outcome is equally likely.

Daniel is trying to determine the value of y and asks if y is 7. He is told that it is not 7.

(iii) Find how much information Daniel gains. [2]

(This question continues on the following page)

(Question 2 continued)

If a random variable has n possible outcomes with probabilities $p_1, p_2 \ldots p_n$, then the expected information gained, $\mathrm{E}(I)$, is defined as

$$\mathrm{E}(I) = \sum_{r=1}^{n} -p_r \ln p_r .$$

(c) For the integer guessing game described in part (b), when Daniel asks if y is 7, there are two possible outcomes: "y is 7" or "y is not 7".

(i) Show that the expected information gained by Daniel is 0.325, correct to three significant figures. [2]

(ii) Alessia asks if x is odd. Show that her expected information gained is greater than Daniel's expected information gained. [2]

Information theory can be applied to a variety of situations.

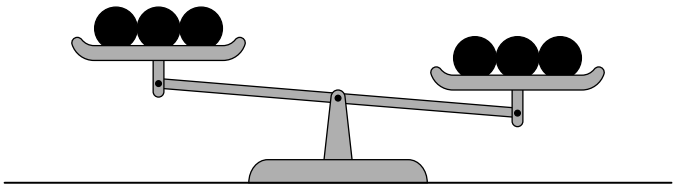
(d) When a coin is flipped, the outcome is either heads or tails. The coin may be biased. Let p be the probability of the outcome being heads.

(i) Find, in terms of p, the information gained when the outcome is tails. [1]

(ii) Find, in terms of p, the expected information gained when the coin is flipped once. [1]

(iii) Hence, find the value of p when the expected information gained is maximized. [2]

A famous puzzle uses 12 balls which appear identical. 11 have the same weight, but one is either lighter or heavier than the others. A pair of scales can be repeatedly used to compare the weights of different combinations of the balls.



The outcome of each weighing can be "balanced", "left-hand side heavier" or "right-hand side heavier". The aim of the puzzle is to identify the ball which is the different weight, and whether it is heavier or lighter than the others, in as few weighings as possible.

(This question continues on the following page)

(Question 2 continued)

(e) Angela wants to decide how many balls should be compared to each other in the **first weighing**. She produces the following table to help plan her strategy.

Number of balls on each side	**Probability of balanced**	**Probability of left-hand side heavier**	**Probability of right-hand side heavier**	**Expected information, $\mathrm{E}(I)$ (3 decimal places)**
1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	0.566
2	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\boldsymbol{z}$
3	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{y}$	1.040
4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1.099
5	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{5}{12}$	1.028
6	0	$\frac{1}{2}$	$\frac{1}{2}$	0.693

(i) Find the value of x. Justify your answer. [2]

(ii) Find the value of y. [2]

(iii) Find the value of z. [2]

(iv) Use the table to suggest the best choice for Angela's first weighing. Justify your answer. [1]

Question 2 - working space

2. [Maximum marks: 7]

After taking a mathematics test, Fatima wonders how many more marks she would have achieved if she had spent an extra 1.5 hours studying.

To find out, she randomly selects five students from her class who took the same test and asks them how many hours (t) they spent studying for the test and the marks (m) they achieved. Their responses are shown in the following table.

Hours, *t*	0	1.2	1.6	2.5	4
Marks, *m*	45	54	61	72	86

Fatima believes there might be a linear relationship between the time spent studying and the results obtained.

(a) (i) Find the Pearson's product moment correlation coefficient, r, for this data.

(ii) Find the least squares regression line of m on t for this data. [4]

(b) According to her model, find how many more marks Fatima would have achieved if she spent an extra 1.5 hours studying. [2]

(c) State one reason why the value obtained in part (b) might not be valid. [1]

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Question 2 - working space

3. [Maximum mark: 6]

The formula $F = 1.8C + 32$ is used to convert a temperature in degrees Celsius, C, to degrees Fahrenheit, F.

(a) (i) Find a formula for converting a temperature in degrees Fahrenheit to degrees Celsius.

(ii) Find the temperature in degrees Celsius that is recorded as 77 degrees Fahrenheit. [3]

Over one year, the mean daily temperature in Mexico City was calculated to be 17 degrees Celsius with a standard deviation of 9 degrees Celsius.

(b) For the same year, find in degrees Fahrenheit

(i) the mean daily temperature in Mexico City.

(ii) the standard deviation of the daily temperature in Mexico City. [3]

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Question 3 - working space

12. [Maximum mark: 8]

Roma is told by the manufacturers of their phone that, when charging from 0%, the percentage P of charge can be modelled by $P = 100 - 100 \times 2^{-t}$, where t is the number of hours since the phone started charging.

(a) Find the time taken for the amount of charge to go from 0% to 40%, according to this model. [2]

Roma decides to test the model. They charge the phone from 0% and record the charge of the phone for different values of t. The results are shown in the following table.

t **hours**	1	2	3	4
P **percent**	48	74	86	91

(b) Find the sum of the square residuals (SS_{res}) when using $P = 100 - 100 \times 2^{-t}$ to fit this data. [4]

Roma now uses a quadratic function to model the charge in the phone. They calculate that, when using this model, the sum of the square residuals for these four points is 2.45.

(c) State one reason why Roma might prefer to use

(i) the quadratic model.

(ii) $P = 100 - 100 \times 2^{-t}$. [2]

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Question 12 - working space

17. [Maximum mark: 9]

Phoebe opens a coffee shop, near to a well-established Apollo coffee shop.

After being open for a few months, Phoebe notices that

- 10% of customers who preferred the Apollo coffee shop in one month preferred her coffee shop the following month.
- 25% of customers who preferred her coffee shop in one month preferred the Apollo coffee shop the following month.

She decides to show these changes in the following transition matrix.

$$\begin{pmatrix} 0.9 & 0.25 \\ 0.1 & 0.75 \end{pmatrix}$$

The two eigenvalues for this matrix are 1 and 0.65. An eigenvector corresponding to the eigenvalue of 1 is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

(a) Find an eigenvector corresponding to the eigenvalue of 0.65. [2]

A diagonal matrix of eigenvalues is $\boldsymbol{D} = \begin{pmatrix} 0.65 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Write down an expression for $\boldsymbol{D}^n$, giving your answer as a 2×2 matrix in terms of n. [1]

When Phoebe's coffee shop first opened, the Apollo shop had 7000 customers the previous month.

(c) Assuming all 7000 customers continue to go to one of these coffee shops, find an expression for the number that will favour Phoebe's coffee shop after n months. [6]

(This question continues on the following page)

(Question 17 continued)

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7. Saddako, n.d. *Cheetah (Acinonyx jubatus) Running - stock photo.* [image online] Available at: https://www.gettyimages.co.uk/detail/photo/cheetah-running-royalty-free-image/523244194?phrase=cheetah+speed&adppopup=true [Accessed 2 May 2023]. Source adapted.

GlobalP, n.d. *Lion, Panthera leo, 8 years old, standing - stock photo.* [image online] Available at: https://www.gettyimages.co.uk/detail/photo/lion-panthera-leo-8-years-old-standing-royalty-free-image/134976936?phrase=Lion+standing&adppopup=true [Accessed 2 May 2023]. Source adapted.

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Question 17 - working space

2. [Maximum mark: 6]

Jerry makes handcrafted chocolates. On average, 1 in 25 of the chocolates that Jerry makes is flawed. Whether or not a chocolate is flawed is independent of all other chocolates.

(a) In a batch of 20 chocolates, chosen at random, find the probability that

(i) two are flawed.

(ii) more than two are flawed. [4]

Jerry sells the perfect chocolates for 50 pesos each and the flawed ones for 15 pesos each.

(b) Calculate the expected number of pesos Jerry makes from selling a batch of 20 randomly selected chocolates. [2]

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Question 2 - working space

4. [Maximum mark: 7]

Gustav plays a game in which he first tosses an unbiased coin and then rolls an unbiased six-sided die.

If the coin shows tails, the score on the die is Gustav's final number of points.

If the coin shows heads, one is added to the score on the die for Gustav's final number of points.

(a) Find the probability that Gustav's final number of points is 7. [2]

(b) Complete the following table. [3]

Final number of points	1	2	3	4	5	6	7
Probability							

(c) Calculate the expected value of Gustav's final number of points. [2]

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Question 4 - working space

9. [Maximum mark: 6]

The number of traffic accidents at a road junction is modelled by a Poisson distribution with a mean of 0.76 accidents per week.

(a) Under this model, calculate the probability that

(i) there are at least 2 accidents in a particular week.

(ii) there will be exactly 3 accidents in a particular 4-week period. [4]

The local traffic authority wishes to determine the probability that, in an 8-week period, fewer than 2 accidents occur in a week on exactly 5 occasions. It assumes that the weekly occurrence of accidents is independent of the week in which these occur.

(b) State the appropriate model that the traffic authority should use to determine this probability. [2]

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Question 9 - working space

14. [Maximum mark: 6]

A cup of hot water is placed in a room and is left to cool for half an hour. Its temperature, measured in $^{\circ}\mathrm{C}$, is recorded every 5 minutes. The results are shown in the table.

Time (mins)	5	10	15	20	25	30
Temperature ($^{\circ}\mathrm{C}$)	59	52	46	40	35	29

Akira uses the power function $T(t) = at^b + 25$ to model the temperature, T, of the water t minutes after it was placed in the room.

(a) State what the value of 25 represents in this context. [1]

(b) Use your graphic display calculator to find the value of a and of b. [3]

Soo Min models the temperature, T, of the water t minutes after it was placed in the room as $T(t) = kc^t + 25$.

(c) Find the value of k and of c. [1]

(d) State a reason why Soo Min's model of the temperature is a better fit for the data than Akira's model. [1]

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Question 14 - working space

5. [Maximum mark: 16]

The drivers of a delivery company can park their vans overnight either at its headquarters or at home.

Urvashi is a driver for the company. If Urvashi has parked her van overnight at headquarters on a given day, the probability that she parks her van at headquarters on the following day is 0.88. If Urvashi has parked her van overnight at her home on a given day, the probability that she parks her van at home on the following day is 0.92.

(a) Write down a transition matrix, $\boldsymbol{T}$, that shows the movement of Urvashi's van between headquarters and home. [2]

On Monday **morning** she collected her van from headquarters where it was parked overnight.

(b) Find the probability that Urvashi's van will be parked at home at the end of the week on Friday **evening**. [3]

(c) Write down the characteristic polynomial for the matrix $\boldsymbol{T}$. Give your answer in the form $\lambda^2 + b\lambda + c$. [2]

(d) Calculate eigenvectors for the matrix $\boldsymbol{T}$. [4]

(e) Write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix. [2]

(f) Hence find the long-term probability that Urvashi's van is parked at home. [3]

Turn over

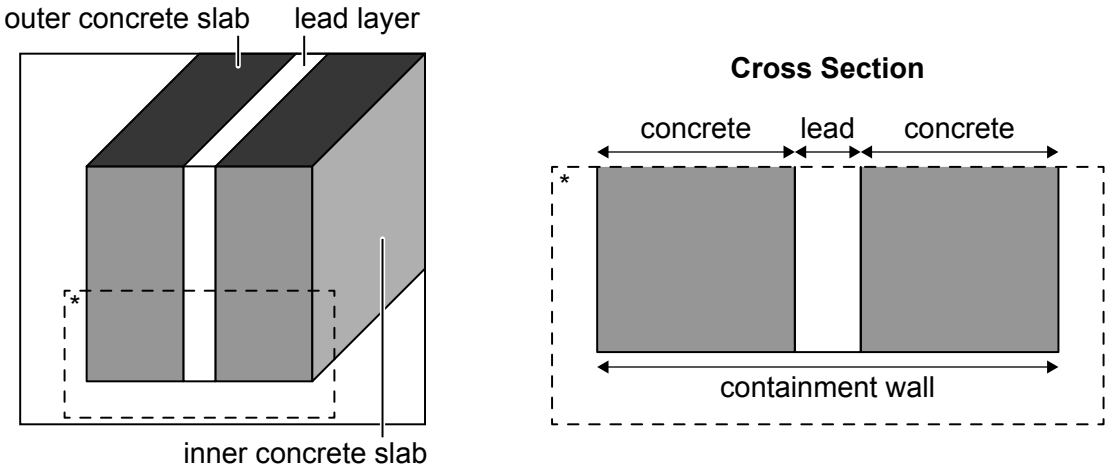
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Question 5 - working space

7. [Maximum mark: 17]

Containment walls to protect against radiation are constructed from two parallel concrete slabs that have a layer of lead between them as shown in the diagram.

diagram not to scale



The width of a concrete slab is modelled by a normal distribution with mean $350\,\text{mm}$ and standard deviation $10\,\text{mm}$.

(a) Find the probability that a randomly chosen concrete slab is less than $340\,\text{mm}$ in width. [2]

(b) Find the endpoints of the interval, symmetric about the mean, such that $95\,\%$ of the slabs have a width that lies in this interval. [3]

Stephen assumes the lead layer is also modelled by a normal distribution, but with mean $100\,\text{mm}$ and standard deviation $5\,\text{mm}$ and is independent of the width of the slabs.

Let W be the random variable that represents the total width of the wall, measured in mm.

(c) (i) Given that the widths of any two concrete slabs are independent, calculate Stephen's value for the mean and standard deviation of W.

(ii) Hence find $\text{P}(780 < W < 810)$. [7]

(This question continues on the following page)

(Question 7 continued)

There are concerns that the mean and standard deviation for Stephen's model of the lead layer are incorrect. However, his assumption that the model is normal and the width of the lead is independent of the width of the concrete slabs still holds.

On investigation it is found that the total width of the containment wall is normally distributed with mean $810\,\text{mm}$ and standard deviation $16\,\text{mm}$. The model for the width of a concrete slab does not change.

(d) Use the results for the **sum** of independent random variables to find a revised value for

(i) the mean of the width of the lead layer.

(ii) the standard deviation of the width of the lead layer. [4]

Under this revised model, $80\,\%$ of the lead layers have a width less than $k\,\text{mm}$.

(e) Calculate the value of k. [1]

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References:

3. Andyqwe, n.d. *Dumpster truck* [image online] Available at: https://www.gettyimages.co.uk/detail/photo/dumpster-truck-royalty-free-image/157611454 [Accessed 18 April 2023] Source adapted.

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Question 7 - working space

1. [Maximum mark: 26]

The purpose of this question is to help a company decide whether or not they should use a new technique to make a component.

A factory produces components for a tractor. They have designed a new technique to produce one of their components that they hope will increase its useful lifespan.

They test 120 components made with the new technique and 240 with the technique they currently use. At the end of 250 hours of use, they check the components, and record whether they have no cracks, minor cracks or major cracks.

The data from the trial are given in the table.

	New technique	**Current technique**
No cracks	a	88
Minor cracks	54	96
Major cracks	b	56

In total 141 components had no cracks.

(a) (i) Show that the value of a is 53. [1]

(ii) Find the value of b. [1]

(This question continues on the following page)

(Question 1 continued)

One of the components from the trial is selected at random.

(b) Given that this component had minor cracks find the probability that it was produced by the new technique. [2]

(c) A χ^2 test for independence is performed at the 5% significance level to determine whether a component having no cracks, minor cracks or major cracks is independent of the production technique used.

(i) State the null and alternative hypotheses. [1]

(ii) Find the p-value. [2]

(iii) State the conclusion of the test in context, justifying your answer. [2]

(d) For the components in the trial that were made with the current technique, show that the proportion which developed cracks is $\frac{19}{30}$. [1]

As an alternative measure, the researchers decide to let p be the probability that a component, made with the new technique, develops cracks. They then test the following hypotheses

$$\mathrm{H}_0\colon\ p = \frac{19}{30},$$

$$\mathrm{H}_1\colon\ p < \frac{19}{30}.$$

In a randomly selected sample of 120 components made with the **new technique** let X be the number which developed cracks. The researchers assume that, under the null hypothesis, $X \sim \mathrm{B}\left(120, \frac{19}{30}\right)$.

(e) State one additional assumption that the researchers are making in choosing this distribution. [1]

(f) Use appropriate data from the trial to perform the test proposed by the researchers, at the 5% significance level. State the conclusion of the test, justifying your answer. [5]

(g) In comparison with the test in part (c), state one mathematical reason why

(i) the test in part (f) might be preferred. [1]

(ii) the test in part (f) might **not** be preferred. [1]

(This question continues on page 5)

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(Question 1 continued)

For these components, the researchers also consider the mean time taken until cracks develop. It is hoped that using the new technique will increase this value. A second trial is carried out and the times, in hours, taken for cracks to appear is recorded.

The mean time taken for cracks to appear ($\bar{t}$) and the value of s_{n-1} for each technique are given in the following table.

	Number of components tested	$\bar{t}$ **(hours)**	s_{n-1} **(hours)**
New technique	100	250.1	3.8
Current technique	200	249.1	3.8

(h) Perform an appropriate test at the 5% significance level to determine whether the new technique increases the mean time taken for cracks to appear. [7]

The company decides to go ahead with the new technique and publishes the following statement: “statistical tests show the new technique will significantly increase the time before components crack and need to be replaced”.

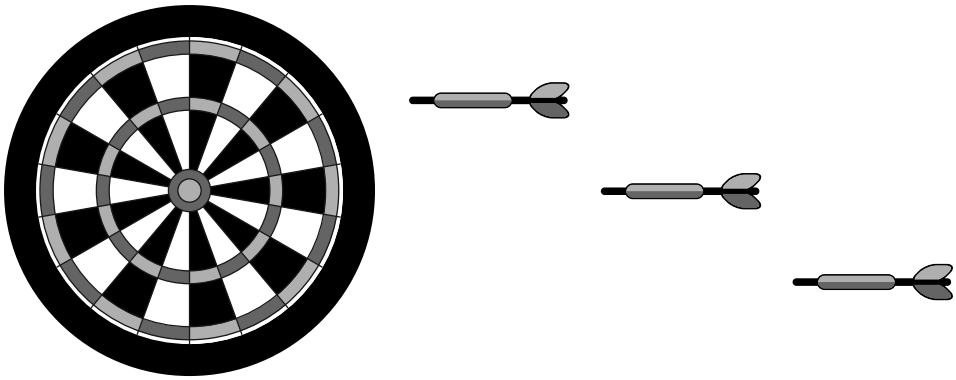
(i) Comment on this statement. [1]

Question 1 - working space

1. [Maximum mark: 24]

The following question examines the changes in darts players' scores using two statistical tests.

In the sport of darts, players take turns throwing darts at a board in order to score points.



A player's "three dart average" refers to the mean score achieved when throwing three darts.

Valia aimed to find out whether amateur darts players in her local area improved over a 12-month period. An increase in their "three dart average" would indicate an improvement.

She selected a random sample of eight darts players and recorded their mean "three dart average" from Year 1.

(This question continues on the following page)

(Question 1 continued)

She then recorded their mean "three dart average" from Year 2.

Valia's results were as follows:

Table 1

Player	Year 1 mean	Year 2 mean
Justin	68.1	70.1
Fallon	72.2	72.0
Rob	65.8	65.0
Michael	73.7	74.9
Mensur	69.8	68.0
Deta	64.5	68.0
Meaghan	69.2	70.6
Peter	64.9	92.9

Valia calculated the median, quartiles and inter-quartile range for each year. The results are shown in **Table 2**.

Table 2

	Year 1 mean	Year 2 mean
Lower quartile	65.35	68
Median	68.65	a
Upper quartile	71	b
Inter-quartile range	5.65	c

(a) Determine the values of a, b and c. [3]

(b) By comparing the results for both years summarized in **Table 2**, state one conclusion, in context, that Valia might be justified in making. [1]

(This question continues on the following page)

(Question 1 continued)

Valia then decided to analyse the data from **Table 1** using a one-tailed paired t-test at the 10% significance level to determine whether the players' averages have increased.

(c) State an assumption about the differences in means that is necessary in order for the test to be valid. [1]

(d) (i) State the null and alternative hypotheses for the test. [2]

(ii) Find the p-value. [2]

(iii) State the conclusion of the test in context, justifying your answer. [2]

(e) State one way Valia could have reduced the chance of her making

(i) a Type I error. [2]

(ii) a Type II error. [2]

Valia was not sure the assumption made in part (c) was correct and hence thought the results obtained from her paired t-test may not be valid.

Following further research, Valia decided to use the Wilcoxon signed-rank test, which does not require the assumption she made in part (c).

For this test, the magnitudes of the differences between the Year 2 and Year 1 means are ranked from 1 to 8, with the ranks of the positive differences (P) and the ranks of the negative differences (N) separated in columns.

This is partially shown in the following table, which Valia constructs to perform the test.

Year 1 mean	Year 2 mean	Difference	P	N
68.1	70.1	2.0	A	
72.2	72.0	–0.2		1
65.8	65.0	–0.8		2
73.7	74.9	1.2	3	
69.8	68.0	–1.8		B
64.5	68.0	3.5	C	
69.2	70.6	1.4	D	
64.9	92.9	28	8	
		Total	$\sum P = 28$	$\sum N$

(This question continues on the following page)

(Question 1 continued)

(f) Determine the values of

(i) A, B, C and D. [3]

(ii) $\sum N$. [1]

For this test:

- the Wilcoxon signed-rank test statistic is $T =$ the smaller value from a choice of $\sum P$ or $\sum N$.
- the null hypothesis is that the population's median for "three dart average" is the same in both years.

Valia chooses to carry out the test at the 5% level of significance. From statistical tables, she determines that the critical region is $T \leq 5$.

(g) (i) State the alternative hypothesis H_1 for this test. [1]

(ii) Write down the value of the test statistic, T. [1]

(iii) Determine the conclusion of the test in context. [2]

(h) Suggest briefly how Valia could assess the reliability of her results for either test. [1]

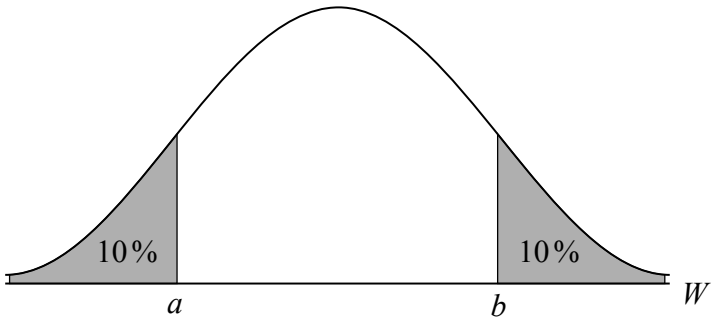
Question 1 - working space

3. [Maximum mark: 7]

The mass, W, of Manx cats is normally distributed with a mean of $4.5\,\text{kg}$ and a standard deviation of $0.4\,\text{kg}$.

(a) A Manx cat is selected at random. Calculate the probability this cat's mass is more than $3.5\,\text{kg}$. [2]

The following curve represents this distribution. It is known that $\text{P}(W < a) = 0.1$ and $\text{P}(W > b) = 0.1$.



(b) Find the value of

(i) a

(ii) b. [3]

(c) Two Manx cats are selected at random from a large population. Find the probability that they both have a mass less than $3.5\,\text{kg}$. [2]

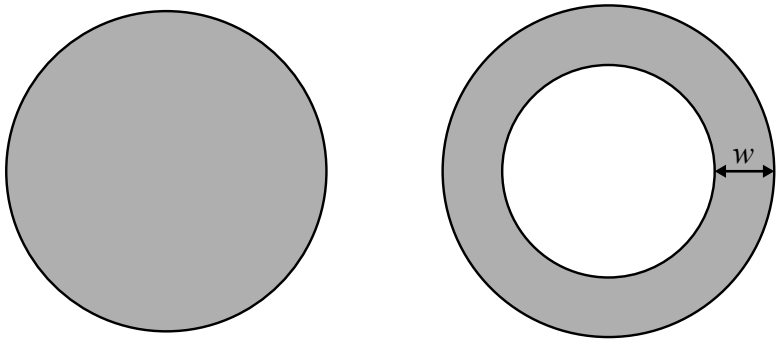
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Question 3 - working space

15. [Maximum mark: 8]

A metal ring, of width w, is made by first cutting a circular disc and then cutting a circular hole exactly in the centre of this disc. This is shown in the diagram.

diagram not to scale



disc ring

A machine produces many of these rings each day.

The diameters of the discs are normally distributed with mean $30\,\text{cm}$ and standard deviation $0.8\,\text{cm}$.

(a) Find the distribution of the radii of the discs. [3]

The radii of the **holes** are normally distributed with mean $12\,\text{cm}$ and standard deviation $0.25\,\text{cm}$.

(b) Calculate the probability that the width of a randomly chosen ring is less than $2.5\,\text{cm}$. [5]

(This question continues on the following page)

(Question 15 continued)

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Question 15 - working space

1. [Maximum mark: 14]

A survey was answered by $20\,000$ expatriates (people living in a country that is not their own). The data ranked countries in order of the country they felt was best for expatriates. The highest-ranked country was Switzerland.

These results were compared to happiness scores taken from *The World Happiness Report 2022*. The following table shows this data for the top 10 expatriate countries.

Country	Switzerland	New Zealand	Spain	Australia	Cyprus	Portugal	Ireland	United Arab Emirates	France	Netherlands
Expatriate country rank	1	2	3	4	5	6	7	8	9	10
Happiness score	7.5	7.2	6.5	7.2	6.2	6.0	7.0	6.6	6.7	7.4

(a) For the **happiness score**, find

(i) the upper quartile

(ii) the interquartile range. [4]

(b) Show that Switzerland's happiness score is not an outlier for this data. [3]

(This question continues on the following page)

(Question 1 continued)

The happiness scores were ranked to calculate Spearman's rank correlation coefficient, r_s. These ranks are shown in the following table.

Country	Switzerland	New Zealand	Spain	Australia	Cyprus	Portugal	Ireland	United Arab Emirates	France	Netherlands
Happiness score	7.5	7.2	6.5	7.2	6.2	6.0	7.0	6.6	6.7	7.4
Happiness rank	1	a	b	c	9	10	5	7	6	2

(c) Write down the value of

(i) a

(ii) b

(iii) c. [3]

(d) (i) Find r_s.

(ii) If France's happiness score is upgraded to 6.9, explain why the value of r_s does not change. [3]

Jose concludes from this data that countries with high happiness scores are likely to be favourite expatriate countries.

(e) State, with a reason, whether Jose's conclusion is appropriate. [1]

Question 1 - working space

1. [Maximum mark: 24]

In this question you will use graph theory and transition matrices to solve problems about a manager visiting five factories.

Audrey is the quality control manager for a manufacturing company that has five factories, A, B, C, D and E.

She is planning a route to visit each factory once, starting and finishing from her home, H.

She determines the distance between each location, in kilometres, as shown in the table.

	A	B	C	D	E	H
A		100	150	70	50	40
B	100		80	140	130	85
C	150	80		90	120	160
D	70	140	90		60	100
E	50	130	120	60		70
H	40	85	160	100	70	

Audrey wants to find an upper and lower bound for the shortest total distance travelled on her route.

(a) Starting at H, use the nearest-neighbour algorithm to find an upper bound. [3]

To find a lower bound, Audrey uses the deleted vertex algorithm and deletes vertex H.

(b) (i) Use Prim's algorithm, starting at E, to find the weight of the minimum spanning tree for A, B, C, D and E. You should clearly state the order in which the edges are selected by the algorithm. [3]

(ii) Hence, find a lower bound. [2]

(This question continues on the following page)

(Question 1 continued)

After her initial visit to all factories, Audrey now decides she will visit one factory each day. She decides which factory to visit according to the following transition matrix, $\boldsymbol{T}$.

$$\begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \end{array} \\ \begin{pmatrix} 0 & 0.3 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.2 & 0.4 \\ 0 & 0.2 & 0.2 & p & 0.2 \\ 0.7 & 0.25 & q & r & 0 \\ 0.3 & 0.25 & 0.2 & 0.2 & 0.3 \end{pmatrix} \end{array}$$

After visiting factory D, there is a probability of 0.4 that Audrey will visit factory C next.

(c) Write down the value of

(i) p [1]

(ii) q [1]

(iii) r. [1]

(d) Audrey first visits factory A.

(i) Write down the initial state matrix, $\boldsymbol{S}_0$. [1]

(ii) Find the probability that the fifth factory that Audrey visits is C. [2]

(iii) Find the probability that the fifth factory that Audrey visits is the same as the second factory she visits. [5]

(e) Over a long period of time,

(i) find the proportion of Audrey's visits that are to factory A [2]

(ii) find the expected distance Audrey travels in a day, given that she always travels directly from home to a factory and then back home. [3]

Turn over

Question 1 - working space

2. [Maximum mark: 31]

In this question, you will first consider a statistical model for the number of fish caught in a lake and then consider a differential equation to model the growth of fish in the lake.

Althea enjoys fishing in a lake near her home. She usually fishes for 4 hours each day, and she records the length of each fish before releasing it back in the lake.

Althea decides to perform a χ^2 goodness of fit test, at the 5% significance level, to determine whether the number of fish caught can be modelled by a Poisson distribution.

She uses her records from the last 50 days to produce this table.

Number of fish caught per day	0	1	2	3	4	5	>5
Frequency	2	7	12	11	10	8	0

(a) Calculate the mean number of fish caught per day. [2]

(b) Hence, calculate the expected number of days, in a 50-day period, that Althea would catch no fish according to the Poisson model. [3]

Althea calculates the expected frequencies and records them in this table.

Number of fish caught per day	≤1	2	3	4	≥5
Expected frequency	10.9	11.6	11.2	8.05	8.25

(c) Explain why Althea has combined the columns for 0 and 1. [1]

(d) For this χ^2 goodness of fit test,

(i) write down a suitable null hypothesis [1]

(ii) write down the number of degrees of freedom [1]

(iii) find the p-value [3]

(iv) write down the conclusion, justifying your answer. [2]

(This question continues on the following page)

(Question 2 continued)

A factory is built on the edge of the lake, and Althea is concerned about the impact of pollution on the length of the fish. Based on her records, before the factory was built, the length of the fish could be modelled by a normal distribution with a mean of $50\,\text{cm}$.

After the factory is built, Althea collects a random sample of 15 fish and calculates that $\bar{x} = 48.6\,\text{cm}$ and $s_{n-1} = 2.56\,\text{cm}$.

Althea decides to test if there is sufficient evidence of a decrease in the population's mean length. The hypotheses for her test are H_0: $\mu = 50$ and H_1: $\mu < 50$.

(e) Perform the test, at the 5% significance level, to show that there is evidence of a decrease in the population's mean length. [3]

Althea now decides to consider the length of an individual fish in the lake as it grows. She researches the von Bertalanffy growth model, which states that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = k(L_\infty - L),$$

where:

- L is the length of the fish (in centimetres) at time t (in months since birth)
- $L_\infty = \lim_{t\to\infty} L$, the length the fish approaches as it grows
- k is a constant.

(f) Justifying your answer,

(i) state whether k is positive, negative, or could be either [2]

(ii) state the value of $\frac{\mathrm{d}L}{\mathrm{d}t}$ as $t \to \infty$ [2]

(iii) state a parameter in the model that could have been affected by the factory pollution in the lake. [2]

At $t = 0$, a fish has length $L_0\,\text{cm}$.

(g) Sketch a graph of $\frac{\mathrm{d}L}{\mathrm{d}t}$ against t. You should label the coordinates of any axes intercepts and the equation of any asymptotes on your sketch. [3]

(h) Solve the differential equation $\frac{\mathrm{d}L}{\mathrm{d}t} = k(L_\infty - L)$, using the initial condition. Write your answer in the form $L = f(t)$. [6]

Question 2 - working space

9. [Maximum mark: 7]

In a year, Louis attends school for a total of 180 school days. Louis leaves his house at exactly $7{:}45\,\text{am}$ every day and takes public transportation from his house to school.

His travel times to school are normally distributed with a mean of 35.5 minutes and a standard deviation of 5.2 minutes. The travel times are independent of each other.

He is late if he arrives at school after $8{:}30\,\text{am}$.

(a) Find the probability that Louis is late on at least 10 days in the year. [4]

Mrs. Beale is responsible for recording students who arrive late. However, Mrs. Beale does not always accurately mark lateness.

On days when Louis is late, there is a $35\,\%$ chance that Mrs. Beale will still mark him as being on time. Mrs. Beale's accuracy for one school day is independent of her accuracy for any other school day.

Louis will receive a detention if he is marked late to school at least 4 times during the year.

(b) Find the probability of Louis receiving a detention in the year. [3]

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Question 9 - working space

3. [Maximum mark: 6]

Two judges, Brett and Clarence, rank the skill levels of eight sheepdogs in a competition. The sheepdogs are labelled A to H and the judges rank the dogs as shown in the table.

Rank	1	2	3	4	5	6	7	8
Brett	A	C	D	B	E	F	G	H
Clarence	A	B	D	C	E	G	F	H

(a) Write down the rank that Brett awards sheepdog B. [1]

(b) Calculate Spearman's rank correlation coefficient for these data. [4]

(c) Comment on your answer to part (b) in terms of the ranks awarded by Brett and Clarence. [1]

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Question 3 - working space

6. [Maximum mark: 5]

The temperatures at 5 different locations on a coral reef were measured. The mean of this sample was $20.1°\text{C}$ and the standard deviation of this sample, s_n, was $3.2°\text{C}$.

(a) Find an unbiased estimate of the population variance. [2]

(b) Assuming that the temperatures are normally distributed, find a 95% confidence interval for the mean temperature on the coral reef. [2]

(c) Using your answer to part (b), determine if it is plausible that the mean temperature on the coral reef could be $17°\text{C}$. [1]

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Question 6 - working space

13. [Maximum mark: 9]

A biologist uses a wire frame to count the number of worms in a $1\,\text{m}^2$ section.

She models the number of worms found in each $1\,\text{m}^2$ section as following a Poisson distribution with mean 1.2.

(a) Find the probability of observing exactly one worm in one $1\,\text{m}^2$ section. [1]

(b) Find the probability of observing at least one worm in one $1\,\text{m}^2$ section. [1]

The biologist looks at 5 independent $1\,\text{m}^2$ sections.

(c) Find the probability of observing a total of five worms in 5 sections. [2]

(d) Find the probability of observing exactly one worm in all 5 sections. [2]

(e) Find the probability of observing at least one worm in exactly 3 of the 5 sections. [3]

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Question 13 - working space

14. [Maximum mark: 6]

A vet wants to find a relationship between the age in days of a breed of puppy (d) and its weight in $\text{kg}(w)$.

To do this he collects a large quantity of data and plots two graphs.

He finds the regression line for each graph. His results are summarized in the table.

	Horizontal axis	Vertical axis	Gradient	Intercept on vertical axis	R^2
Graph 1	d	$\ln w$	0.00571	1.54	0.72
Graph 2	$\ln d$	$\ln w$	0.302	0.693	0.95

Based on these results, find the best of the two possible relationships between w and d.

Express your relationship in the form $w = f(d)$ where f is a simplified expression.

Justify your choice of expression.

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Question 14 - working space

3. [Maximum mark: 7]

Prakash is the leader of a customer service team and is interested in determining whether there is a relationship between a customer's satisfaction level and the type of service interaction they have experienced.

He collects data from a random sample of 250 customers and tracks their satisfaction level after three types of service interactions: in-person, online chat bots and website contact forms.

He categorizes the satisfaction levels as satisfied, neutral and dissatisfied.

He records the data in the following table.

		Satisfaction level		
		Satisfied	Neutral	Dissatisfied
Type of service interaction	In-person	35	30	23
	Online chat bots	31	39	23
	Website contact forms	19	28	22

Prakash performs a χ^2 test for independence at the 5% significance level.

The critical value is 9.488.

The null hypothesis, H_0, is the satisfaction level and the type of service interaction are independent.

(a) State the alternative hypothesis for this test. [1]

(b) Find the degrees of freedom for this test. [1]

(c) Find χ^2_{calc}, the chi-squared test statistic. [2]

Prakash concludes that there is sufficient evidence to reject the null hypothesis.

(d) (i) State whether Prakash is correct. Justify your answer.

(ii) Write down the conclusion for this test in context. [3]

(This question continues on the following page)

(Question 3 continued)

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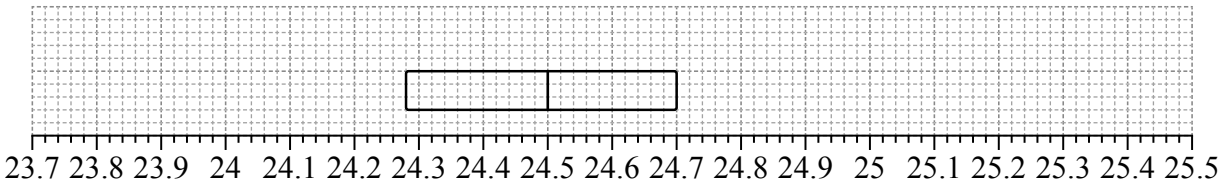
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Question 3 - working space

4. [Maximum mark: 7]

The times, in seconds, for the fastest 16 women in a $50\,\text{m}$ freestyle swimming championship event were recorded. All swimmers recorded different times.

Part of a box and whisker diagram for these times is shown in the following diagram.



Swimming times in seconds

(a) Write down the number of swimmers who took more than 24.70 seconds to complete the race. [1]

(b) Find the interquartile range (IQR) for the data. [2]

An outlier is defined as a value that satisfies one of the following:

- more than $1.5 \times \text{IQR}$ below the lower quartile
- more than $1.5 \times \text{IQR}$ above the upper quartile.

Of the 16 women, the two fastest swimmers took 23.96 and 24.12 seconds and the two slowest women took 25.12 and 25.40 seconds to complete the race.

(c) (i) Show that only one of these times is an outlier.

(ii) Complete the box and whisker diagram above. [4]

(This question continues on the following page)

(Question 4 continued)

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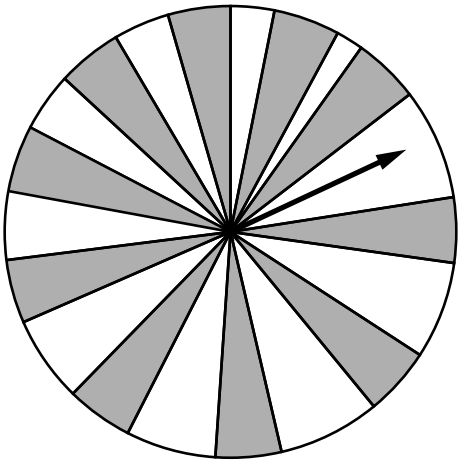
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Question 4 - working space

6. [Maximum mark: 6]

A circular disk has 10 white sectors and 10 shaded sectors. The size of the angles, made at the centre of the circle by the white sectors, form an arithmetic sequence. The common difference of this sequence is $3°$ and the first term is $6°$.

An arrow is attached to the centre of the circle as shown in the following diagram.



The arrow can be spun and has an equal chance of stopping at any point.

(a) Find the probability that the arrow will stop in a white sector. [3]

In a game, the arrow is spun 4 times and the player wins if the arrow stops in a white sector at least 3 times.

(b) Find the probability that the player wins the game. [3]

(This question continues on the following page)

(Question 6 continued)

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Question 6 - working space

7. [Maximum mark: 7]

A team of researchers is using a model to predict the relative happiness of different countries. To do this, a value x is calculated based on easily measured parameters, for example, life expectancy, or available social support. It is assumed that higher values of x indicate greater happiness.

To test the model a survey is conducted in six countries, A, B, C, D, E and F. In these countries the level of happiness is assessed directly using questionnaires and given a score y, out of 10, with higher scores indicating greater happiness.

To select the countries for the survey, all countries are divided into three equal groups based on wealth and two countries are chosen randomly from each group.

(a) Write down the name of this type of sampling. [1]

The results of the survey, along with the value obtained from the model, are given in the following table.

Country	A	B	C	D	E	F
Value from the model (x)	12.3	15.2	14.1	18.5	20.1	19.2
Happiness score (y)	5.2	7.3	6.2	6.9	8.0	7.2

The researchers will accept the model is a valid predictor of happiness score if the Pearson's product-moment correlation coefficient, r, is greater than 0.8.

(b) (i) Find the value of r.

(ii) Hence state whether the model can be regarded as a valid predictor of happiness score. [3]

(c) Find the equation of the regression line y on x. [1]

For a particular country $x = 17.2$.

(d) Use the regression line to predict the happiness score for this country. [2]

(This question continues on the following page)

(Question 7 continued)

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Question 7 - working space

12. [Maximum mark: 5]

Tajinder believes that the number of cars arriving at an intersection every minute follows a Poisson distribution. Under this assumption he decides to test the null hypothesis that the mean is 12.4 cars per minute against the alternative hypothesis that the mean is greater than 12.4.

The test consists of observing the intersection for 10 minutes and then rejecting the null hypothesis if more than 130 cars arrive.

(a) (i) State the meaning of a Type I error.

(ii) Find the probability of a Type I error for Tajinder's test. [4]

(b) Write down one reason why a Poisson distribution might not be a suitable model for the number of cars arriving at the intersection. [1]

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Question 12 - working space

15. [Maximum mark: 6]

A Markov chain has two states, A and B. The transition matrix for the Markov chain is

$$\begin{matrix} & \mathrm{A} & \mathrm{B} \\ \mathrm{A} & p & 1-q \\ \mathrm{B} & 1-p & q \end{matrix}.$$

One of the eigenvalues for this matrix is equal to 1.

(a) Find the eigenvector corresponding to the eigenvalue of 1. [3]

(b) Hence, find an expression, in terms of p and q, for the proportion of time spent in state A. [2]

(c) Hence or otherwise, find the proportion of time spent in state A when $p = 0.3$ and $q = 0.6$. [1]

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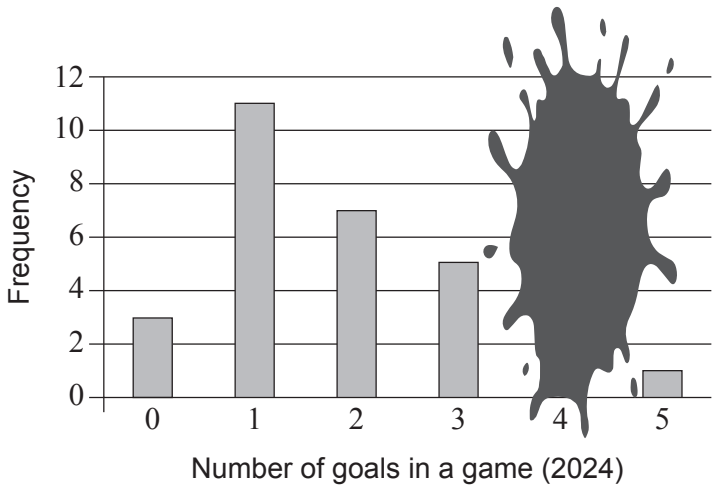
Question 15 - working space

1. [Maximum mark: 15]

Paul has a bar graph for the total number of goals scored in each game of a soccer tournament in 2024. The bar graph is shown below, however the frequency of 4 goals in a game is unreadable.

Paul uses this bar graph to create a frequency table.

Bar graph



Frequency table

Number of goals in a game (2024)	Frequency
0	3
1	11
2	7
3	k
4	p
5	1

(a) Write down the value of k. [1]

Paul knows that the mean number of goals per game scored during the tournament was 2.2.

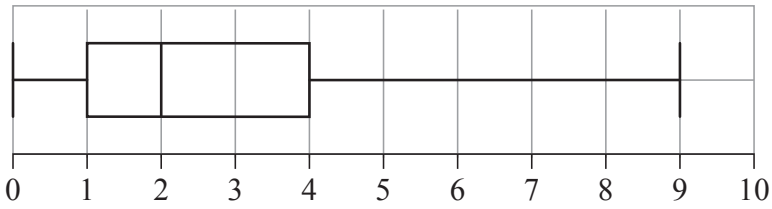
(b) (i) Write down an equation for the mean in terms of p.

(ii) Determine the value of p. [3]

(This question continues on the following page)

(Question 1 continued)

Data for the number of goals per game in the **2025 soccer tournament** are shown in the following box and whisker diagram.



Number of goals in a game (2025)

After comparing the box and whisker diagram from the 2025 tournament with the frequency table from the 2024 tournament, Paul concludes that the distribution of goals is consistent between the two tournaments.

(c) State two observations that support Paul's conclusion using values from the data to compare any **two** of:

range, symmetry, median, and interquartile range. [3]

Paul plans to watch all the games from the **2024** tournament in a random order.
He will watch each game once.

For the first game he watches, he defines event F as:

"scoring either 0 goals or exactly 1 goal".

(d) Write down the event(s) from the table that are equivalent to F'. There may be more than one correct event. [2]

Event	Description
A	Scoring exactly 2 goals in this game
B	Scoring more than 1 goal in this game
C	Scoring at least 2 goals in this game
D	Scoring either 0 goals or exactly 1 goal in all games except this game
E	Not scoring 0 goals or exactly 1 goal in any game

(e) If exactly 1 goal was scored in the first game Paul watches, write down the probability that exactly 1 goal was scored in the second game he watches. Give your answer as a fraction. [2]

(f) Calculate the probability that 5 goals were scored in the first game that Paul watches **and** 0 goals were scored in the second game he watches. [4]

Question 1 - working space

5. [Maximum mark: 15]

A zoologist collects a sample of cane beetles. He measures their length and categorizes them as “small” meaning from 10 to $12\,\text{mm}$ long, “medium” meaning from 12 to $16\,\text{mm}$ long and “large” meaning from 16 to $18\,\text{mm}$ long. He also notes their sex and records the frequencies in the following table.

		Length, x mm		
		Small $10 < x \leq 12$	Medium $12 < x \leq 16$	Large $16 < x \leq 18$
Sex	Female	42	25	19
	Male	61	27	12

(a) Find how many cane beetles are in the zoologist’s sample. [1]

(b) Based on this data set, estimate the mean length of a cane beetle. [2]

Two female beetles are chosen at random with replacement from the sample.

(c) Find the probability that they are both categorized as small. [3]

(d) Test, at the 5% significance level, the null hypothesis that length category and sex are independent. State the p-value of your test and write your conclusion in context. Justify your answer. [4]

Let ϕ be the population proportion of cane beetles that are male.

(e) Test, at the 5% significance level, the hypothesis that more than 45% of cane beetles are male. Write the null and alternative hypotheses. State the p-value of your test and write your conclusion in context. [5]

Question 5 - working space

7. [Maximum mark: 17]

On any given day, the probability that Emlyn charges his phone depends only on whether he charged it the previous day.

If he charged his phone the previous day, the probability he charges it today is 0.4.

If he did not charge his phone the previous day, the probability he charges it today is p.

On day n this can be represented using the vector $\boldsymbol{v}_n$ where

$$\boldsymbol{v}_n = \begin{pmatrix} \text{probability that Emlyn charges his phone on day } n \\ \text{probability that Emlyn does not charge his phone on day } n \end{pmatrix}$$

A Markov chain model is formed where

$$\boldsymbol{v}_{n+1} = \boldsymbol{M}\boldsymbol{v}_n.$$

Matrix $\boldsymbol{M}$ is of the form $\begin{pmatrix} a & p \\ b & 1-p \end{pmatrix}$.

(a) Write down the value of

(i) a.

(ii) b. [2]

(b) On day zero Emlyn charges his phone. Find the probability

(i) that Emlyn charges his phone on all days from $n = 1$ to $n = 4$.

(ii) that Emlyn charges his phone on day 4, when $p = 0.7$. [5]

(c) Demonstrate that, for all values of p, one eigenvector of $\boldsymbol{M}$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and hence state the associated eigenvalue. [4]

(d) Find, in terms of p, the steady state probability that Emlyn charges his phone on a given day. [4]

In the long term, Emlyn wants to charge his phone on at least 60% of days.

(e) Find the minimum value of p required for this to occur. [2]

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References:

1. naisupakit, 2016. *Triangle Pillow tradition native Thai style pillow*. [image online] Available at: https://www.gettyimages.co.uk/detail/photo/triangle-pillow-tradition-native-thai-style-pillow-royalty-free-image/623127206 [Accessed 9 April 2024]. Source adapted.

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Question 7 - working space

6. [Maximum mark: 15]

Daphne runs a coffee shop and needs to decide whether she should hire more staff.

Over a long period of time, the staff collect data on the number of customers, C, who are queuing when a new customer enters the shop. The probability distribution of C is shown in the following table.

Number of customers in queue, c	0	1	2	3	≥ 4
$\mathrm{P}(C = c)$	0.21	0.34	0.28	0.17	0

(a) (i) Find the probability that there are at least two customers in the queue when a new customer arrives.

(ii) Find $\mathrm{E}(C)$. [3]

The time in seconds, T, taken to serve a single customer can be modelled by the normal distribution $T \sim \mathrm{N}(115, 28^2)$.

Daphne is told that an estimate for the expected time a customer will need to wait before being served can be found by calculating $\mathrm{E}(C) \times \mathrm{E}(T)$.

(b) Find the value of $\mathrm{E}(C) \times \mathrm{E}(T)$. [2]

Daphne now decides that she will employ a new member of staff if the probability that a customer has to wait more than three minutes before being served is greater than 0.4.

(c) Using the distribution of T given above, find the probability that it takes more than three minutes to serve a randomly selected customer. [2]

(d) Find the probability it takes more than three minutes in total to serve two randomly selected customers. You may assume all serving times are independent of all other serving times. [4]

Daphne assumes that when a new customer arrives the person at the front of the queue has only just reached the service counter. She also assumes that the probability that it takes more than three minutes to serve three customers is equal to 1.

(e) Using Daphne's assumptions and the probabilities for C given in the table above

(i) find the probability a customer just arriving at the coffee shop will wait more than three minutes before being served.

(ii) Hence state whether Daphne will decide to employ more staff. [4]

Turn over

Question 6 - working space

1. [Maximum mark: 29]

The following question explores how sequences, series and Markov chains may be used in modelling the number of customers in a commercial setting.

In a town, there are three stores: Aroma, Bodega and Clover.

Ashley is the manager of Aroma. She gathers data to determine whether there is significant movement of customers between the three stores over the course of one year.

She found that:

- 91% of Aroma customers stayed with Aroma, 5% moved to Bodega, and 4% moved to Clover.
- 95% of Bodega customers stayed with Bodega, 4% moved to Aroma, and 1% moved to Clover.
- 92% of Clover customers stayed with Clover, 6% moved to Aroma, and 2% moved to Bodega.

This information is used to form a transition matrix, $\boldsymbol{T}$.

(a) Write down the transition matrix $\boldsymbol{T}$. [3]

It is assumed that the movement of customers between stores remains constant from year to year.

(b) Determine the percentage of Clover customers expected to move to Aroma over a 5-year period. [4]

(c) Find the steady state vector for $\boldsymbol{T}$. [2]

(d) Hence, state the percentage of Clover customers expected to move to Aroma in the long term. [1]

(This question continues on the following page)

(Question 1 continued)

Ashley's initial findings suggested that 6% of Clover's customers moved to Aroma over the course of one year. Ashley is instructed to increase this figure so that at least 40% of Clover's customers move to Aroma **in the long term**.

It may be assumed that all other annual percentage changes remain the same, other than the percentage that stay with Clover.

(e) Determine the minimum integer percentage to which the 6% figure will need to be raised to achieve this objective, justifying your answer. [3]

(f) State, with an explanation, whether Bodega's manager would benefit from Ashley attracting more of Clover's customers. [2]

(g) Suggest a contextual reason why the annual percentage changes are unlikely to be constant from year to year. [1]

Ashley moves to a new store, Dusk, in a different town.

She notes that in her first week, Dusk had 600 customers and that every week, the number of customers increases by 30. Therefore, Ashley models the number of customers as an arithmetic sequence (**Model 1**).

(h) Use **Model 1** to find the number of customers Ashley can expect Dusk to have in her 10th week. [2]

Ashley wants to increase the number of customers at Dusk, so she introduces a loyalty scheme after her 5th week. She anticipates that the number of **new customers** she can expect each week should be 12% greater than the number of **new customers** the previous week (**Model 2**).

(i) Show that the number of customers in Ashley's 6th week, according to **Model 2**, is 753.6. [2]

(j) Use **Model 2** to find how many customers Ashley should expect in her 10th week. [3]

(k) By comparing the two models, determine the first week in which there would be an expected difference of at least 500 customers. [6]

Question 1 - working space

2. [Maximum mark: 26]

The following question uses statistical tests to compare the weights of eggs in different situations.

Farmer Giles owns a chicken farm and sells eggs at the local market. He assumes that the weights of his eggs follow a normal distribution, with mean $52.0\,\text{g}$ and standard deviation $3.7\,\text{g}$.

Giles selects 25 eggs at random.

(a) Find the probability that at least 5 of these eggs weigh at least $55.0\,\text{g}$. [4]

(b) Calculate the probability that the mean weight of these 25 eggs lies within 1 gram of the population mean. [3]

In an effort to increase the mean weight of his eggs, Farmer Giles gives the chickens a new food, *Chick Crackle*, for one month. He then randomly selects 10 eggs and weighs them. His results, in grams, are shown in **Table 1**.

Table 1

54.5	52.4	50.4	54.3	54.2	53.1	50.2	51.8	52.2	53.5

Farmer Giles assumes the weights of all his eggs are still normally distributed but believes the mean and standard deviation of the weights may have changed.

(c) Perform a suitable test at the 5% significance level to investigate whether the mean weight of his eggs has increased from $52.0\,\text{g}$. [6]

(d) Find a 99% confidence interval for the mean weight of Giles's eggs, after feeding his chickens *Chick Crackle*. [2]

To analyse further the effect of *Chick Crackle*, Giles decides to choose 8 specific chickens. He weighs a randomly selected egg from each chicken before feeding them *Chick Crackle*. He then weighs another randomly selected egg from each chicken after feeding them *Chick Crackle* for one month, to determine whether the new feed has the required effect.

His results, in grams, are shown in **Table 2**.

Table 2

Chicken	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**
Egg weight before	53.8	51.5	49.3	52.3	46.4	55.0	51.2	52.4
Egg weight after	55.1	51.9	50.0	54.8	47.3	53.5	52.2	54.4

(e) (i) State a suitable test that Giles should use. [1]

(ii) State one assumption that Giles needs to make before carrying out the test. [1]

(This question continues on the following page)

(Question 2 continued)

(f) Perform the test at the 5% significance level, stating your conclusion in context. [6]

From the next village, Farmer Ray offers Farmer Giles a random sample of 10 eggs from his own chickens and requests that Giles carries out a t-test to see whose farm produces the heavier eggs.

Farmer Giles recalls from his statistics degree that a t-test is only valid if both population variances are the same.

(g) Given that Giles uses his data from **Table 1**, calculate the value of S_G^2, the unbiased estimator for the population variance of his eggs. [1]

He sets out to determine the maximum value of S_R^2, the unbiased estimator for the population variance of Ray's eggs, for which the t-test is considered valid. To do this, he uses a test called the F-test.

The test statistic for the F-test in this case is $F = \frac{s_R^2}{s_G^2}$.

The critical value at the 5% level for comparing two samples of 10 eggs is $F_{crit} = 3.1789$.

(h) Determine the maximum value of S_R^2 for the t-test to be valid. [2]

Question 2 - working space

1. [Maximum mark: 28]

This question is about comparing the academic performance of two schools.

At age 18, all students in school A and school B take the same final exam. Augustin is studying the results in these schools.

Augustin chooses to take a representative sample of size six from each school. For each student in the sample, he will conduct an interview.

(a) (i) State one advantage of increasing the sample size. [1]

(ii) State one disadvantage of increasing the sample size. [1]

The data in **Table 1** shows the results of the final exam as a percentage for each of the six students, sampled from school A.

Table 1

Candidate number	Final exam
10001	63.5
10003	52.5
10012	50.7
10045	42.8
10073	44.7
10090	56.1

The mean result for the sample from school A is 51.7 to three significant figures.

(b) Find the value of s_{n-1} for the sample from school A. [2]

(This question continues on the following page)

(Question 1 continued)

The value of s_{n-1} for the sample from school B is 7.66 to three significant figures.

Augustin makes the following claim:

"The spread of results in school A must be less than the spread of results in school B."

(c) Make one criticism of Augustin's claim. [1]

The examination board claims that the final exam results in each school are approximately normally distributed. You may assume that this claim is correct.

Augustin decides to use a pooled t-test to compare the mean results of school A and of school B.

(d) (i) State the condition regarding population variances required to use a pooled t-test. [1]

(ii) State whether Augustin should use a pooled t-test in this case. Justify your answer. [2]

Prior to collecting data, Augustin believed that the mean result of school B was higher than that of school A. From his data he finds that the mean final exam result for the sample from school B is exactly 60.

(e) (i) State appropriate null and alternative hypotheses for the pooled t-test. [2]

(ii) Find the *p*-value. [2]

(iii) Given that the test is carried out at the 5% significance level, state the appropriate conclusion in context. Justify your answer. [2]

All students in the two schools took the same entry exam at age 11.

Augustin wants to determine if there is evidence of any correlation between the entry exam result and final exam result. For the 12 students in the sample, Augustin collects their entry exam results.

The Pearson's product moment correlation coefficient between the results for the entry exam and the final exam is $r = 0.876$ to three significant figures. The critical value of r at the 5% significance level is 0.576.

(f) (i) Assuming all requirements are met, perform a test at the 5% significance level.

State the hypotheses and justify your conclusion. [4]

(ii) If the requirements are not met, state an alternative test. [1]

(This question continues on page 5)

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(Question 1 continued)

The examination board uses a model to make a prediction of a student's final exam result $(\hat{y})$ based on their entry exam result (x).

The model used is:

$$\hat{y} = 0.37x + 37.6$$

(g) Give, in context, an interpretation of the gradient 0.37 in the model. [1]

For each student, Augustin uses the value $y - \hat{y}$ rounded to one decimal place to measure the extent to which a school helped to improve a student's results. He calls this "school value added". This is shown in **Table 2**.

Table 2

Candidate number	School	Entry exam (x)	Final exam (y)	School value added $(y - \hat{y})$
10001	A	53.2	63.5	6.2
10003	A	38.7	52.5	q
10012	A	32.0	50.7	1.3
10045	A	20.0	42.8	–2.2
10073	A	23.2	44.7	–1.5
10090	A	42.4	56.1	2.8
20004	B	72.0	60.8	–3.4
20008	B	69.5	60.3	–3.0
20023	B	52.3	47.1	–9.9
20044	B	84.1	69.9	1.2
20061	B	74.7	64.6	–0.6
20083	B	66.9	57.3	–5.1

(h) (i) Show that the value of q in **Table 2** is 0.6. [2]

(ii) Assuming that the appropriate requirements are met, use a pooled t-test at a 5% significance level to determine if the mean "school value added" is higher in school A than in school B. Write down your null and alternative hypotheses and justify your conclusion. [4]

(i) Using Augustin's results, explain how each school could claim they are performing better than the other school. [2]

Turn over

Question 1 - working space

1. [Maximum mark: 7]

Under controlled driving conditions, Jacob investigated the fuel efficiency of his car when using premium fuel compared to standard fuel.

Jacob recorded the distance travelled per litre ($\mathrm{km\,L^{-1}}$) using standard fuel for six days and then using premium fuel for seven days. This information is shown in the following tables.

	Standard fuel						
Distance travelled per litre ($\mathbf{km\,L^{-1}}$)	8.1	8.2	7.9	8.2	7.9	8.0	

	Premium fuel						
Distance travelled per litre ($\mathbf{km\,L^{-1}}$)	8.3	8.4	8.1	8.3	8.2	8.0	8.2

At the 5% significance level, Jacob performs a t-test to determine whether there is sufficient evidence that his car travels further using premium fuel compared to standard fuel.

(a) State one mathematical assumption made for this test to be valid. [1]

(b) Write down the

(i) null hypothesis.

(ii) alternative hypothesis. [2]

(c) (i) Find the p-value.

(ii) State your conclusion to the test in context. Justify your answer. [4]

(This question continues on the following page)

(Question 1 continued)

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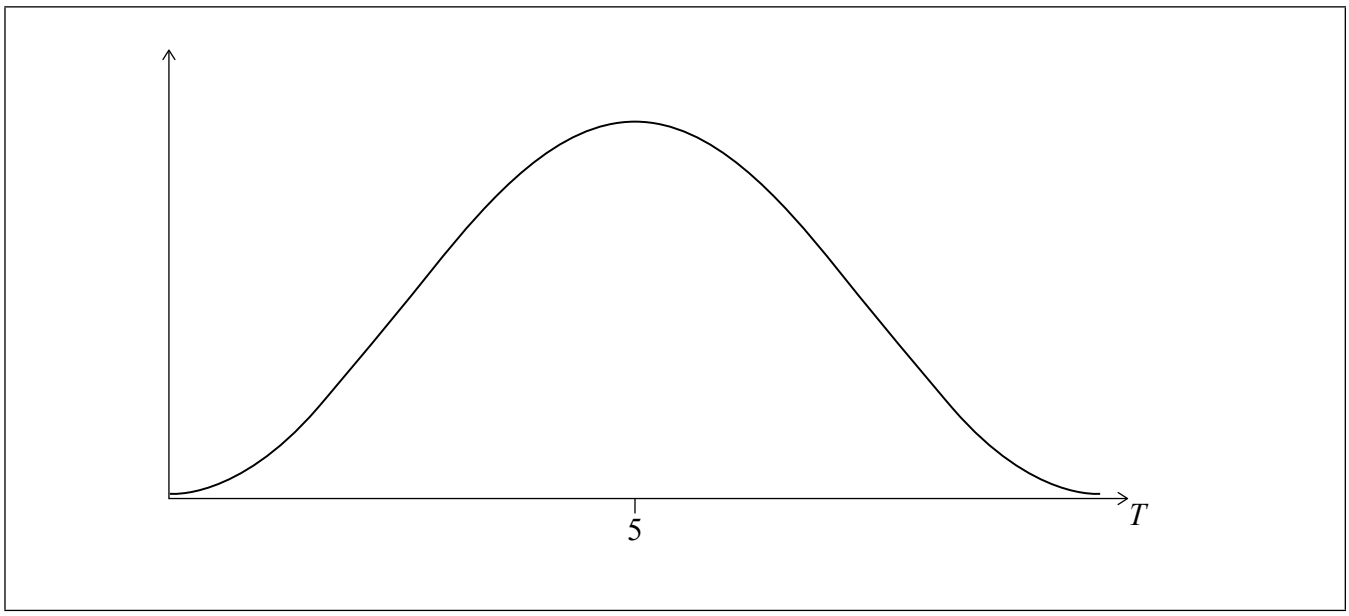
Question 1 - working space

3. [Maximum mark: 5]

In a reaction test, the time, T seconds, taken to complete the test can be modelled by a normal distribution with a mean of 5 seconds and a standard deviation of 1 second.

(a) Find the probability that a randomly selected person completes the reaction test in less than 4 seconds. [2]

The graph of the distribution of T is shown in the following diagram.



The slowest 15% of participants will be given training to reduce their time to complete the test.

(b) (i) Shade the region on the given diagram that corresponds to these participants.

(ii) Find the least time taken to complete the test by a person who will be given training. [3]

(This question continues on the following page)

(Question 3 continued)

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Question 3 - working space

9. [Maximum mark: 6]

A help desk receives an average of 34 calls each hour. The number of calls received can be modelled by a Poisson distribution.

The help desk staff can answer a maximum of 22 calls in any 30-minute period.

(a) Find the probability that some calls are not answered in a 30-minute period. [3]

A supervisor is deciding whether to employ extra staff for the help desk. They monitor the phone calls for a random 30-minute period each day for 10 days.

(b) Find the probability that all the monitored calls are answered on at least 9 of these 10 days. [3]

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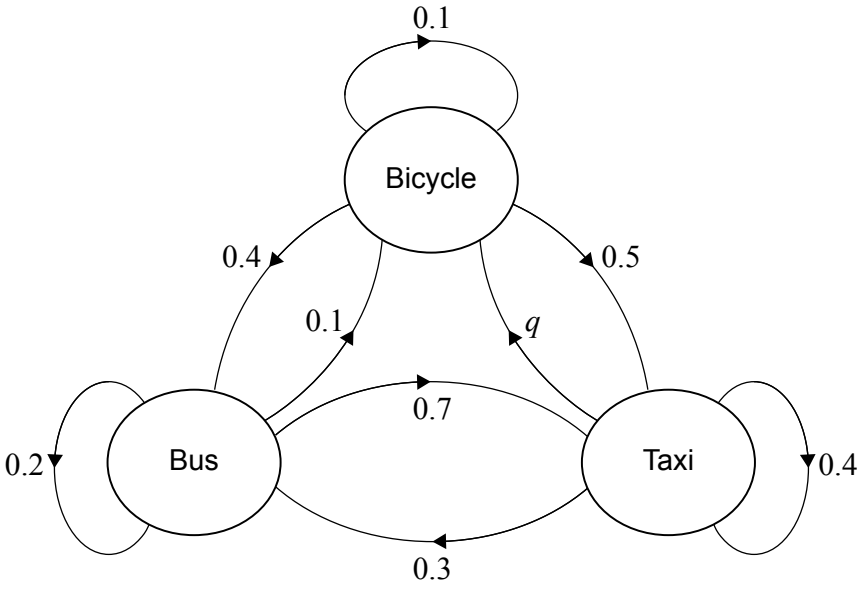
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Question 9 - working space

16. [Maximum mark: 8]

Each day Anirudh goes to work by bus (B) or bicycle (C) or taxi (T). The probability of him using a mode of transport depends on which mode of transport he used the previous day, and is shown in the following transition diagram.



(a) Write down the value of q. [1]

On day 1, Anirudh goes to work by bus.

(b) Write down the probability that he goes to work by taxi on day 2. [1]

(c) Find the probability that the mode of transport Anirudh uses on day 3 is the same as the one he uses on day 2. [3]

(d) Complete the following transition matrix. [1]

$$\begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.1 & 0.1 & \square \\ 0.7 & \square & \square \end{pmatrix}$$

(e) Hence, find the long-term proportion of days that he goes to work by bicycle. [2]

(This question continues on the following page)

(Question 16 continued)

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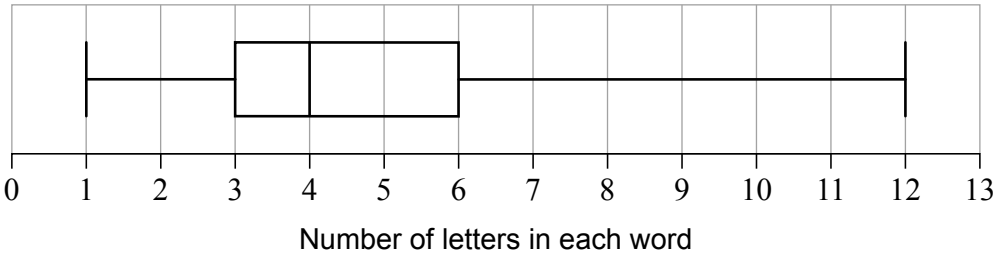
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Question 16 - working space

1. [Maximum mark: 15]

Dr Petrillo wrote a short scientific essay. He analysed the readability of his essay by counting the number of letters in each word.

Dr Petrillo constructs a box and whisker diagram for his data.



Number of letters in each word

(a) Write down

(i) the median;

(ii) the upper quartile, Q_3;

(iii) the interquartile range, IQR. [3]

Dr Petrillo now wants to modify his diagram to show any outliers. He considers the longer words in his data and uses the following formula:

$$\text{outliers} > (1.5 \times \mathrm{IQR}) + \mathrm{Q}_3.$$

Words with at least k letters are considered outliers.

(b) Find the value of k. [2]

(This question continues on the following page)

(Question 1 continued)

Dr Petrillo further considers the outliers and sees no reason to exclude them from his analysis.

The length of each word in the essay, n, and its associated frequency are given in the following table.

n	$1 \leq n \leq 2$	$3 \leq n \leq 4$	$5 \leq n \leq 6$	$7 \leq n \leq 8$	$9 \leq n \leq 10$	$11 \leq n \leq 12$	$n \geq 13$
Frequency	289	480	220	136	90	45	0

(c) Use the mid-interval values to calculate an estimate of the mean number of letters in each word. [3]

Dr Petrillo conducts a χ^2 goodness of fit test at the 1% significance level, to test the following null hypothesis:

H_0: The frequency of the number of letters in each word in his essay is consistent with the English language.

(d) Write down the alternative hypothesis for this test. [1]

The observed and expected frequencies of the number of letters in each word in his essay are listed in the following table.

n	$1 \leq n \leq 2$	$3 \leq n \leq 4$	$5 \leq n \leq 6$	$7 \leq n \leq 8$	$9 \leq n \leq 10$	$11 \leq n \leq 12$	$n \geq 13$
Observed frequency	289	480	220	136	90	45	0
Expected frequency	252.0	491.4	239.4	151.2	88.2	31.8	6.0

(e) (i) Write down the number of degrees of freedom.

(ii) Find the χ^2 statistic for this test.

(iii) Find the p-value for this test. [4]

The critical value for this test, at the 1% significance level, is 16.812.

(f) State whether Dr Petrillo should reject the null hypothesis. Justify your answer. [2]

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Question 1 - working space

5. [Maximum mark: 17]

A company sells spicy snacks. Over a long period of time, the company finds that 60% of people who taste the snacks will buy them. The company wants to test a new recipe to see if a greater percentage of people who taste these new snacks will buy them.

The company surveys the first 80 customers at a local supermarket who agree to taste the new snacks.

(a) State the name of the sampling method used. [1]

Out of the 80 people who taste the new snacks, 54 buy them. The company performs a hypothesis test to determine whether the new recipe is an improvement.

It assumes the responses of the 80 people are independent of each other. The null hypothesis is

$$\mathrm{H}_0 : p = 0.6,$$

where p is the proportion of people who buy the new snacks after tasting them.

(b) (i) Write down H_1, the alternative hypothesis.

(ii) Perform a suitable test, at the 5% significance level, stating the conclusion in context. [6]

(c) (i) Find the critical region for this test.

(ii) Given that $p = 0.65$, find the probability that the company makes a Type II error. [4]

During the survey, the company also recorded the age of each person. They want to find out if a person's age is independent of whether they will buy these snacks. The data for the 80 people is shown in the following table.

Age, x	**Will buy**	**Will not buy**
$x < 20$	22	8
$20 \leq x < 30$	18	5
$30 \leq x < 40$	6	8
$40 \leq x$	8	5

(d) Perform a suitable test, at the 5% significance level, to determine if age is independent of whether someone will buy these snacks. [6]

Turn over

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Question 5 - working space

APOLLO

Calculus

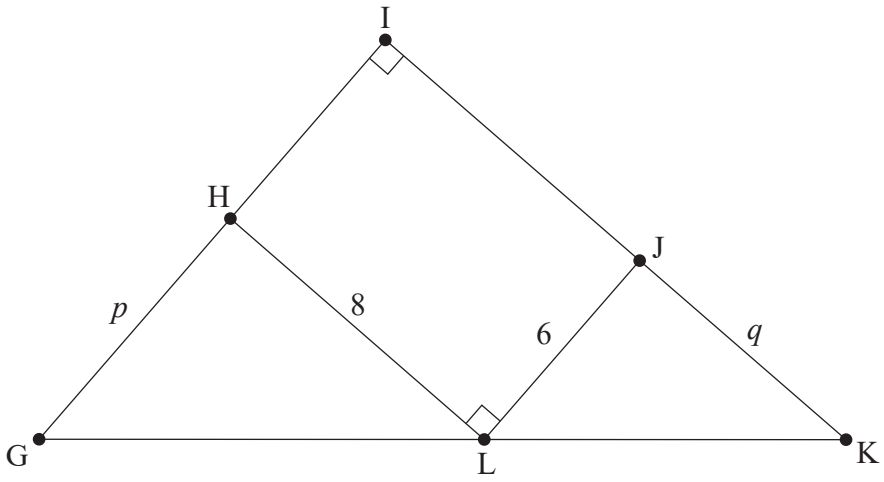
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7. [Maximum mark: 8]

Ellis designs a gift box. The top of the gift box is in the shape of a right-angled triangle GIK.

A rectangular section HIJL is inscribed inside this triangle. The lengths of GH, JK, HL, and LJ are p cm, q cm, 8 cm and 6 cm respectively.

diagram not to scale



The area of the top of the gift box is $A\,\text{cm}^2$.

(a) (i) Find A in terms of p and q.

(ii) Show that $A = \frac{192}{q} + 3q + 48$. [4]

(b) Find $\frac{\mathrm{d}A}{\mathrm{d}q}$. [2]

Ellis wishes to find the value of q that will minimize the area of the top of the gift box.

(c) (i) Write down an equation Ellis could solve to find this value of q.

(ii) Hence, or otherwise, find this value of q. [2]

(This question continues on the following page)

(Question 7 continued)

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Question 7 - working space

12. [Maximum mark: 8]

A tank of water initially contains 400 litres. Water is leaking from the tank such that after 10 minutes there are 324 litres remaining in the tank.

The volume of water, V litres, remaining in the tank after t minutes, can be modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{V}, \text{ where } k \text{ is a constant.}$$

(a) Show that $V = \left(20 - \frac{t}{5}\right)^2$. [6]

(b) Find the time taken for the tank to empty. [2]

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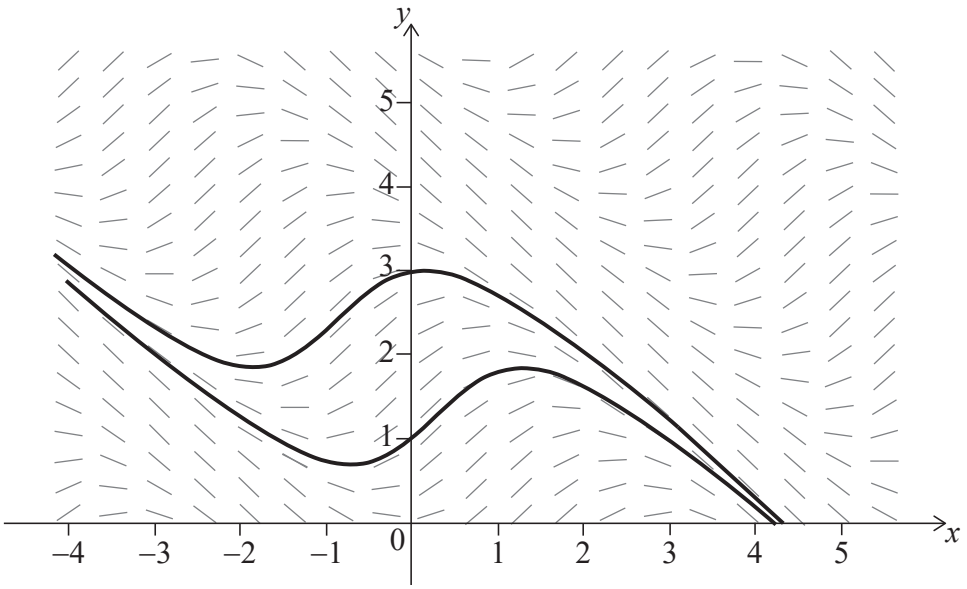
Question 12 - working space

15. [Maximum mark: 5]

The diagram shows the slope field for the differential equation

$$\frac{dy}{dx} = \sin(x+y),\ -4 \le x \le 5,\ 0 \le y \le 5.$$

The graphs of the two solutions to the differential equation that pass through points $(0, 1)$ and $(0, 3)$ are shown.



For the two solutions given, the local minimum points lie on the straight line L_1.

(a) Find the equation of L_1, giving your answer in the form $y = mx + c$. [3]

For the two solutions given, the local maximum points lie on the straight line L_2.

(b) Find the equation of L_2. [2]

(This question continues on the following page)

(Question 15 continued)

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Question 15 - working space

16. [Maximum mark: 8]

A particle P moves in a straight line, such that its displacement x at time $t\,(t \geq 0)$ is defined by the differential equation $\dot{x} = x\cos t\left(\mathrm{e}^{-\sin t}\right)$. At time $t = 0$, $x = \frac{1}{\mathrm{e}}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x when $t = 0.3$. [3]

(b) By solving the differential equation, find the percentage error in your approximation for x when $t = 0.3$. [5]

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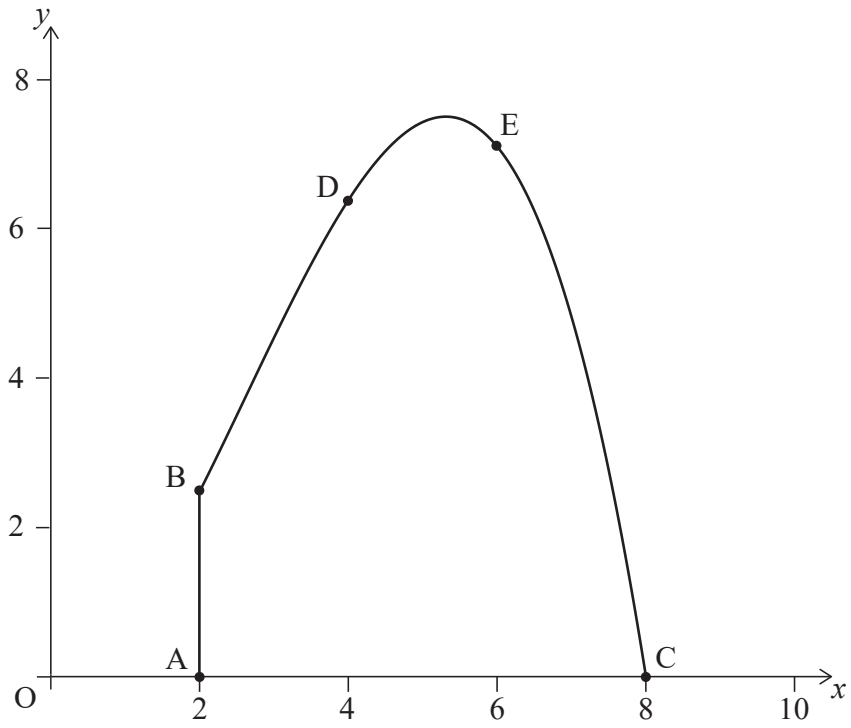
References:

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Question 16 - working space

2. [Maximum mark: 16]

The cross-sectional view of a tunnel is shown on the axes below. The line [AB] represents a vertical wall located at the left side of the tunnel. The height, in metres, of the tunnel above the horizontal ground is modelled by $y = -0.1x^3 + 0.8x^2,\ 2 \le x \le 8$, relative to an origin O.



Point A has coordinates $(2, 0)$, point B has coordinates $(2, 2.4)$, and point C has coordinates $(8, 0)$.

(a) (i) Find $\frac{dy}{dx}$.

(ii) Hence find the maximum height of the tunnel. [6]

(b) Find the height of the tunnel when

(i) $x = 4$.

(ii) $x = 6$. [3]

(c) Use the trapezoidal rule, with three intervals, to estimate the cross-sectional area of the tunnel. [3]

(d) (i) Write down the integral which can be used to find the cross-sectional area of the tunnel.

(ii) Hence find the cross-sectional area of the tunnel. [4]

Question 2 - working space

6. [Maximum mark: 18]

An ice-skater is skating such that her position vector when viewed from above at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\,\mathrm{e}^{bt}\cos t \\ a\,\mathrm{e}^{bt}\sin t \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are in metres.

(a) Find the velocity vector at time t. [3]

(b) Show that the magnitude of the velocity of the ice-skater at time t is given by

$$a\,\mathrm{e}^{bt}\sqrt{(1+b^2)}\,.$$ [4]

At time $t = 0$, the displacement of the ice-skater is given by $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and the velocity of the ice-skater is given by $\begin{pmatrix} -3.5 \\ 5 \end{pmatrix}$.

(c) Find the value of a and the value of b. [3]

(d) Find the magnitude of the velocity of the ice-skater when $t = 2$. [2]

At a point P, the ice-skater is skating parallel to the y-axis for the first time.

(e) Find OP. [6]

Turn over

Question 6 - working space

7. [Maximum mark: 17]

A biologist introduces 100 rabbits to an island and records the size of their population (x) over a period of time. The population growth of the rabbits can be approximately modelled by the following differential equation, where t is time measured in years.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$$

(a) Find the population of rabbits 1 year after they were introduced. [5]

A population of 100 foxes is introduced to the island when the population of rabbits has reached 1000. The subsequent population growth of rabbits and foxes, where y is the population of foxes at time t, can be approximately modelled by the coupled equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(2 - 0.01y)$$

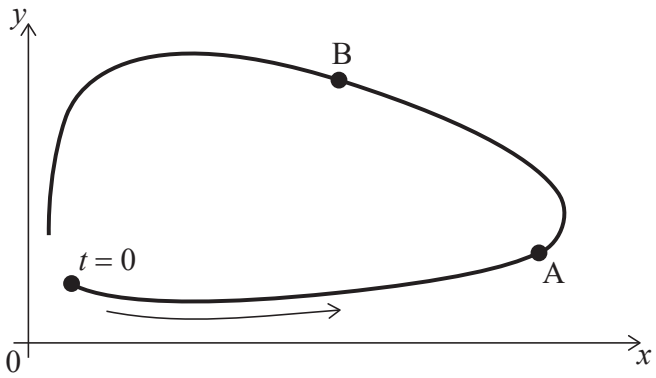
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(0.0002x - 0.8)$$

(b) Use Euler's method with a step size of 0.25, to find

(i) the population of rabbits 1 year after the foxes were introduced.

(ii) the population of foxes 1 year after the foxes were introduced. [6]

(c) The graph of the population sizes, according to this model, for the first 4 years after the foxes were introduced is shown below.



Describe the changes in the populations of rabbits and foxes for these 4 years at

(i) point A.

(ii) point B. [3]

(d) Find the non-zero equilibrium point for the populations of rabbits and foxes. [3]

References:

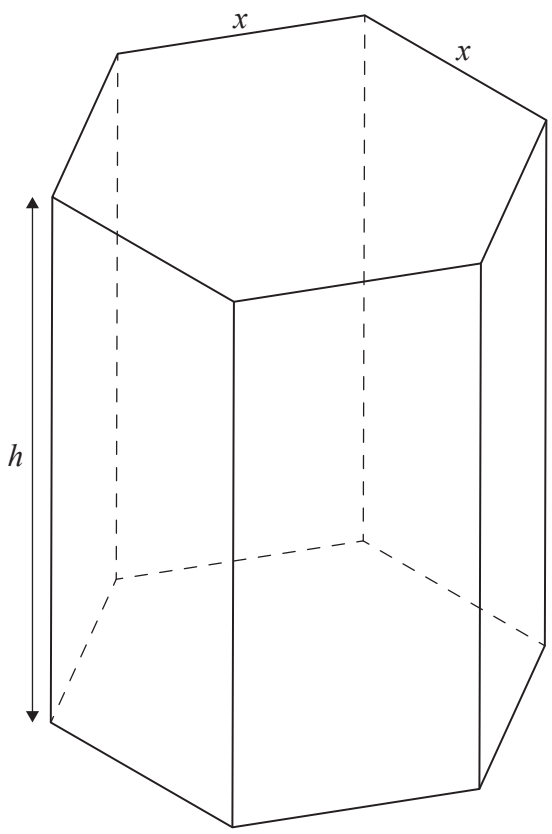
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Question 7 - working space

3. [Maximum mark: 15]

A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is $h\,\text{cm}$, and the top and base of the prism have sides of length $x\,\text{cm}$.

diagram not to scale



(a) Given that $\sin 60^\circ = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]

(b) Given that the total external surface area of the box is $1200\,\text{cm}^2$, show that the volume of the box may be expressed as $V = 300\sqrt{3}\,x - \frac{9}{4}x^3$. [5]

(c) Sketch the graph of $V = 300\sqrt{3}\,x - \frac{9}{4}x^3$, for $0 \le x \le 16$. [2]

(d) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}x}$. [2]

(e) Find the value of x which maximizes the volume of the box. [2]

(f) Hence, or otherwise, find the maximum possible volume of the box. [2]

Question 3 - working space

6. [Maximum mark: 15]

A particle P moves along the x-axis. The velocity of P is $v\,\text{m s}^{-1}$ at time t seconds, where $v = -2t^2 + 16t - 24$ for $t \geq 0$.

(a) Find the times when P is at instantaneous rest. [2]

(b) Find the magnitude of the particle's acceleration at 6 seconds. [4]

(c) Find the greatest speed of P in the interval $0 \leq t \leq 6$. [2]

(d) The particle starts from the origin O. Find an expression for the displacement of P from O at time t seconds. [4]

(e) Find the total distance travelled by P in the interval $0 \leq t \leq 4$. [3]

Turn over

Question 6 - working space

7. [Maximum mark: 17]

Consider the following system of coupled differential equations.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 2y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix}$. [6]

(b) Hence, write down the general solution of the system. [2]

(c) Determine, with justification, whether the equilibrium point $(0, 0)$ is stable or unstable. [2]

(d) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$

(i) at $(4, 0)$.

(ii) at $(-4, 0)$. [3]

(e) Sketch a phase portrait for the general solution to the system of coupled differential equations for $-6 \leq x \leq 6$, $-6 \leq y \leq 6$. [4]

References:

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Question 7 - working space

2. [Maximum mark: 31]

Alessia is an ecologist working for Mediterranean fishing authorities. She is interested in whether the mackerel population density is likely to fall below 5000 mackerel per km^3, as this is the minimum value required for sustainable fishing. She believes that the primary factor affecting the mackerel population is the interaction of mackerel with sharks, their main predator.

The population densities of mackerel (M thousands per km^3) and sharks (S per km^3) in the Mediterranean Sea are modelled by the coupled differential equations:

$$\frac{\text{d}M}{\text{d}t} = \alpha M - \beta MS$$

$$\frac{\text{d}S}{\text{d}t} = \gamma MS - \delta S$$

where t is measured in years, and α, β, γ and δ are parameters.

This model assumes that no other factors affect the mackerel or shark population densities.

The term αM models the population growth rate of the mackerel in the absence of sharks.
The term βMS models the death rate of the mackerel due to being eaten by sharks.

(a) Suggest similar interpretations for the following terms.

(i) γMS [1]

(ii) δS [1]

(b) An equilibrium point is a set of values of M and S, such that $\frac{\text{d}M}{\text{d}t} = 0$ and $\frac{\text{d}S}{\text{d}t} = 0$.

Given that both species are present at the equilibrium point,

(i) show that, at the equilibrium point, the value of the mackerel population density is $\frac{\delta}{\gamma}$; [3]

(ii) find the value of the shark population density at the equilibrium point. [2]

(c) The equilibrium point found in part (b) gives the average values of M and S over time.

Use the model to predict how the following events would affect the average value of M. Justify your answers.

(i) Toxic sewage is added to the Mediterranean Sea. Alessia claims this reduces the shark population growth rate and hence the value of γ is halved. No other parameter changes. [2]

(ii) Global warming increases the temperature of the Mediterranean Sea. Alessia claims that this promotes the mackerel population growth rate and hence the value of α is doubled. No other parameter changes. [2]

(This question continues on the following page)

(Question 2 continued)

(d) To estimate the value of α, Alessia considers a situation where there are no sharks and the initial mackerel population density is M_0.

(i) Write down the differential equation for M that models this situation. [1]

(ii) Show that the expression for the mackerel population density after t years is $M = M_0 e^{\alpha t}$. [4]

(iii) Alessia estimates that the mackerel population density increases by a factor of three every two years. Show that $\alpha = 0.549$ to three significant figures. [3]

Based on additional observations, it is believed that

$$\alpha = 0.549,$$

$$\beta = 0.236,$$

$$\gamma = 0.244,$$

$$\delta = 1.39.$$

Alessia decides to use Euler's method to estimate future mackerel and shark population densities. The initial population densities are estimated to be $M_0 = 5.7$ and $S_0 = 2$.
She uses a step length of 0.1 years.

(e) (i) Write down expressions for M_{n+1} and S_{n+1} in terms of M_n and S_n. [3]

(ii) Use Euler's method to find an estimate for the mackerel population density after one year. [2]

(f) Alessia will use her model to estimate whether the mackerel population density is likely to fall below the minimum value required for sustainable fishing, 5000 per km^3, during the first nine years.

(i) Use Euler's method to sketch the trajectory of the phase portrait, for $4 \leq M \leq 7$ and $1.5 \leq S \leq 3$, over the first nine years. [3]

(ii) Using your phase portrait, or otherwise, determine whether the mackerel population density would be sufficient to support sustainable fishing during the first nine years. [2]

(iii) State **two** reasons why Alessia's conclusion, found in part (f)(ii), might not be valid. [2]

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Question 2 - working space

2. [Maximum mark: 5]

Inspectors are investigating the carbon dioxide emissions of a power plant. Let R be the rate, in tonnes per hour, at which carbon dioxide is being emitted and t be the time in hours since the inspection began.

When R is plotted against t, the total amount of carbon dioxide produced is represented by the area between the graph and the horizontal t-axis.

The rate, R, is measured over the course of two hours. The results are shown in the following table.

t	0	0.4	0.8	1.2	1.6	2
R	30	50	60	40	20	50

(a) Use the trapezoidal rule with an interval width of 0.4 to estimate the total amount of carbon dioxide emitted during these two hours. [3]

The real amount of carbon dioxide emitted during these two hours was 72 tonnes.

(b) Find the percentage error of the estimate found in part (a). [2]

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Question 2 - working space

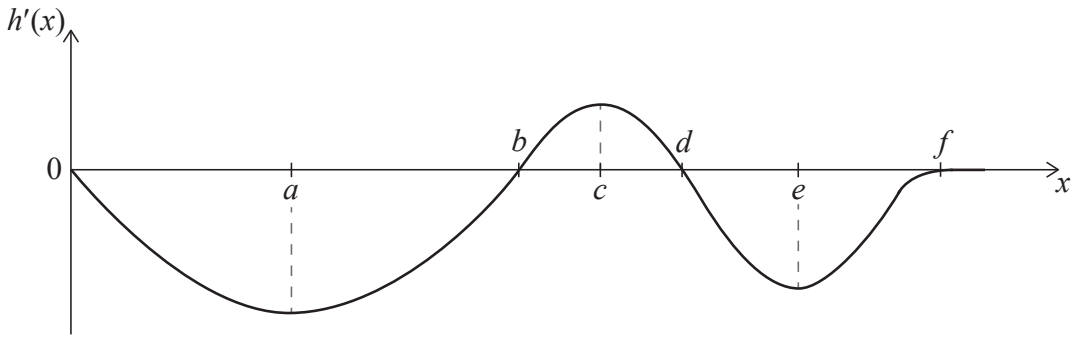
8. [Maximum mark: 5]

Juri skis from the top of a hill to a finishing point at the bottom of the hill. She takes the shortest route, heading directly to the finishing point (F).



Let $h(x)$ define the height of the hill above F at a horizontal distance x from the starting point at the top of the hill.

The graph of the **derivative** of $h(x)$ is shown below. The graph of $h'(x)$ has local minima and maxima when x is equal to a, c and e. The graph of $h'(x)$ intersects the x-axis when x is equal to b, d, and f.



(a) (i) Identify the x value of the point where $|h'(x)|$ has its maximum value.

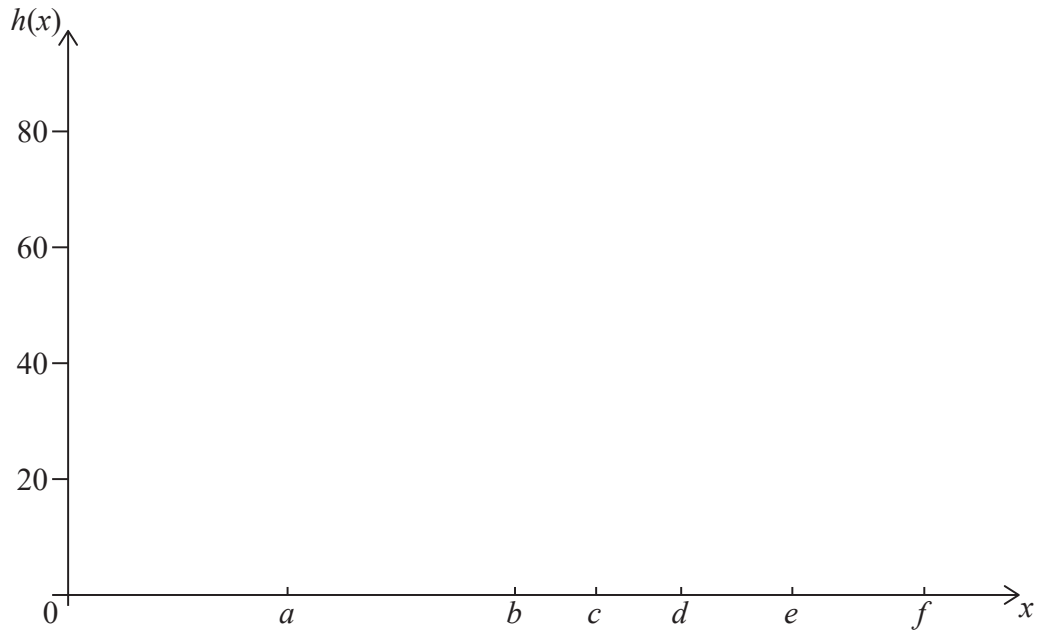
(ii) Interpret this point in the given context. [2]

(This question continues on the following page)

(Question 8 continued)

Juri starts at a height of 60 metres and finishes at F, where $x = f$.

(b) Sketch a possible diagram of the hill on the following pair of coordinate axes. [3]



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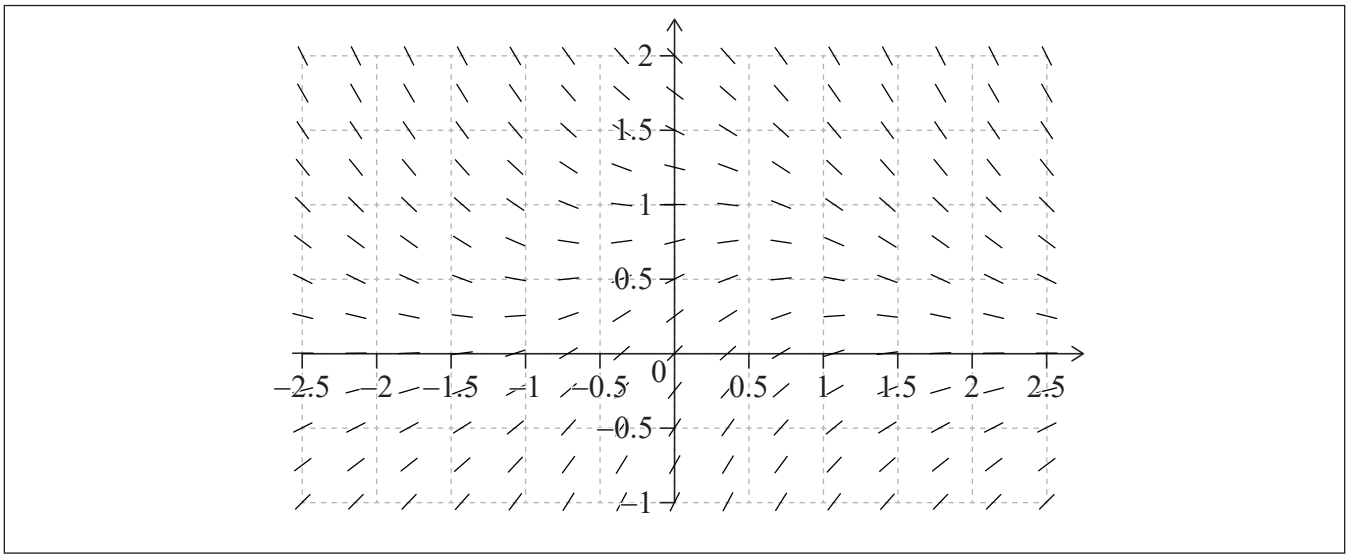
Question 8 - working space

13. [Maximum mark: 7]

The slope field for the differential equation $\frac{dy}{dx} = e^{-x^2} - y$ is shown in the following two graphs.

(a) Calculate the value of $\frac{dy}{dx}$ at the point $(0, 1)$. [1]

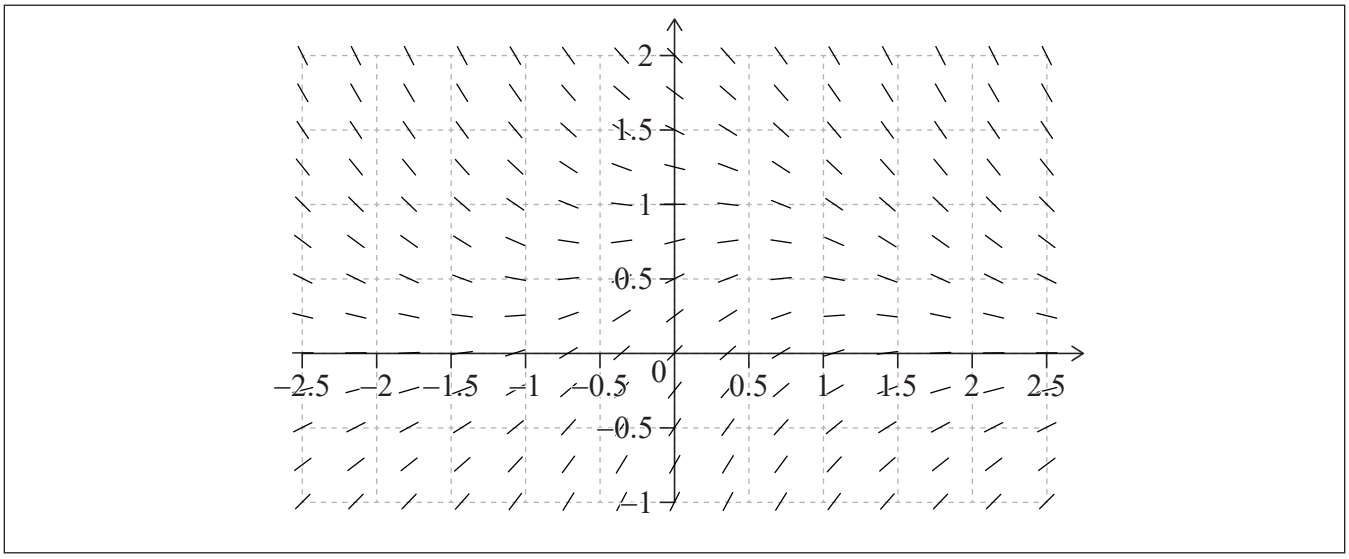
(b) Sketch, on the first graph, a curve that represents the points where $\frac{dy}{dx} = 0$. [2]



(c) On the second graph,

(i) sketch the solution curve that passes through the point $(0, 0)$.

(ii) sketch the solution curve that passes through the point $(0, 0.75)$. [4]



(This question continues on the following page)

(Question 13 continued)

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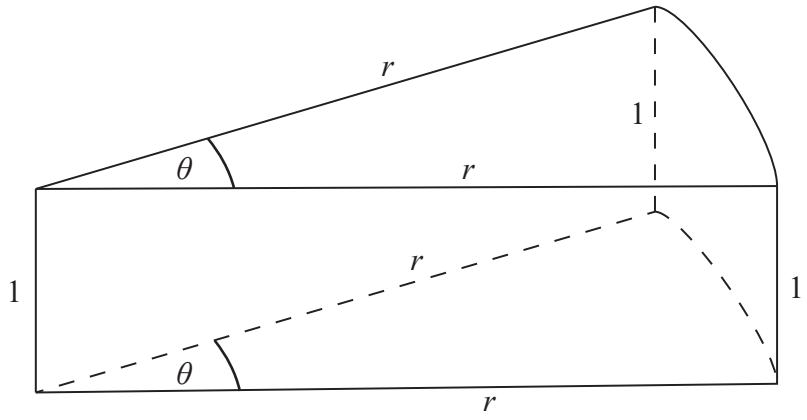
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Question 13 - working space

15. [Maximum mark: 9]

The following diagram shows a frame that is made from wire. The total length of wire is equal to $15\,\text{cm}$. The frame is made up of two identical sectors of a circle that are parallel to each other. The sectors have angle θ radians and radius $r\,\text{cm}$. They are connected by $1\,\text{cm}$ lengths of wire perpendicular to the sectors. This is shown in the diagram below.



(a) Show that $r = \frac{6}{2+\theta}$. [2]

The faces of the frame are covered by paper to enclose a volume, V.

(b) (i) Find an expression for V in terms of θ.

(ii) Find the expression $\frac{\mathrm{d}V}{\mathrm{d}\theta}$.

(iii) Solve algebraically $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ to find the value of θ that will maximize the volume, V. [7]

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Question 15 - working space

17. [Maximum mark: 7]

The sides of a bowl are formed by rotating the curve $y = 6\ln x$, $0 \le y \le 9$, about the y-axis, where x and y are measured in centimetres. The bowl contains water to a height of $h\,\text{cm}$.

(a) Show that the volume of water, V, in terms of h is $V = 3\pi\left(e^{\frac{h}{3}} - 1\right)$. [5]

(b) Hence find the maximum capacity of the bowl in cm^3. [2]

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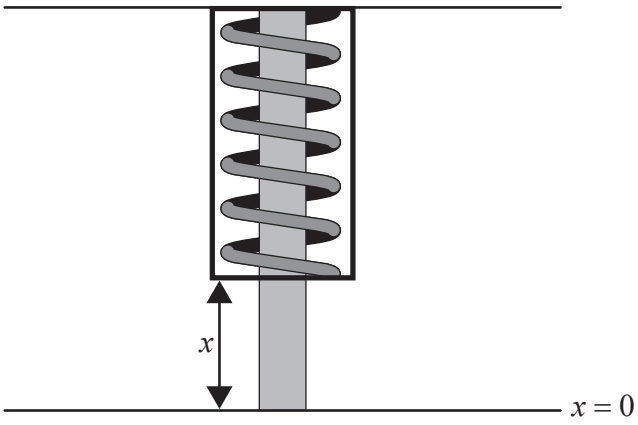
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Question 17 - working space

6. [Maximum mark: 15]

A shock absorber on a car contains a spring surrounded by a fluid. When the car travels over uneven ground the spring is compressed and then returns to an equilibrium position.



The displacement, x, of the spring is measured, in centimetres, from the equilibrium position of $x = 0$. The value of x can be modelled by the following second order differential equation, where t is the time, measured in seconds, after the initial displacement.

$$\ddot{x} + 3\dot{x} + 1.25x = 0$$

(a) Given that $y = \dot{x}$, show that $\dot{y} = -1.25x - 3y$. [2]

The differential equation can be expressed in the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\boldsymbol{A}$ is a 2×2 matrix.

(b) Write down the matrix $\boldsymbol{A}$. [1]

(c) (i) Find the eigenvalues of matrix $\boldsymbol{A}$.

(ii) Find the eigenvectors of matrix $\boldsymbol{A}$. [6]

(d) Given that when $t = 0$ the shock absorber is displaced $8\,\text{cm}$ and its velocity is zero, find an expression for x in terms of t. [6]

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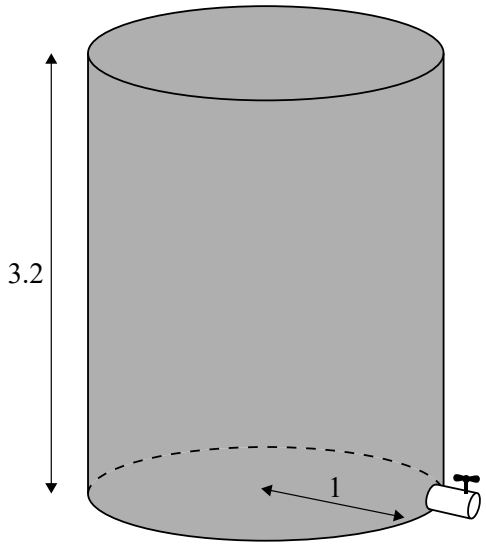
Question 6 - working space

2. [Maximum mark: 30]

This question explores models for the height of water in a cylindrical container as water drains out.

The diagram shows a cylindrical water container of height 3.2 metres and base radius 1 metre. At the base of the container is a small circular valve, which enables water to drain out.

diagram not to scale



Eva closes the valve and fills the container with water.

At time $t = 0$, Eva opens the valve. She records the height, h metres, of water remaining in the container every 5 minutes.

Time, t (minutes)	Height, h (metres)
0	3.2
5	2.4
10	1.6
15	1.1
20	0.5

Eva first tries to model the height using a linear function, $h(t) = at + b$, where a, $b \in \mathbb{R}$.

(a) (i) Find the equation of the regression line of h on t. [2]

(ii) Interpret the meaning of parameter a in the context of the model. [1]

(This question continues on the following page)

(Question 2 continued)

Eva uses the equation of the regression line of h on t, to predict the time it will take for all the water to drain out of the container.

(iii) Suggest why Eva's use of the linear regression equation in this way could be unreliable. [1]

Eva thinks she can improve her model by using a quadratic function, $h(t) = pt^2 + qt + r$, where p, q, $r \in \mathbb{R}$.

(b) (i) Find the equation of the least squares quadratic regression curve. [1]

Eva uses this equation to predict the time it will take for all the water to drain out of the container and obtains an answer of k minutes.

(ii) Find the value of k. [2]

(iii) Hence, write down a suitable domain for Eva's function $h(t) = pt^2 + qt + r$. [1]

Let V be the volume, in cubic metres, of water in the container at time t minutes.
Let R be the radius, in metres, of the circular valve.

Eva does some research and discovers a formula for the rate of change of V.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi R^2 \sqrt{70\,560h}$$

(c) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2\sqrt{70\,560h}$. [3]

(d) By solving the differential equation $\frac{\mathrm{d}h}{\mathrm{d}t} = -R^2\sqrt{70\,560h}$, show that the general solution is given by $h = 17\,640(c - R^2t)^2$, where $c \in \mathbb{R}$. [5]

Eva measures the radius of the valve to be 0.023 metres. Let T be the time, in minutes, it takes for all the water to drain out of the container.

(e) Use the general solution from part (d) and the initial condition $h(0) = 3.2$ to predict the value of T. [4]

Eva wants to use the container as a timer. She adjusts the initial height of water in the container so that all the water will drain out of the container in 15 minutes.

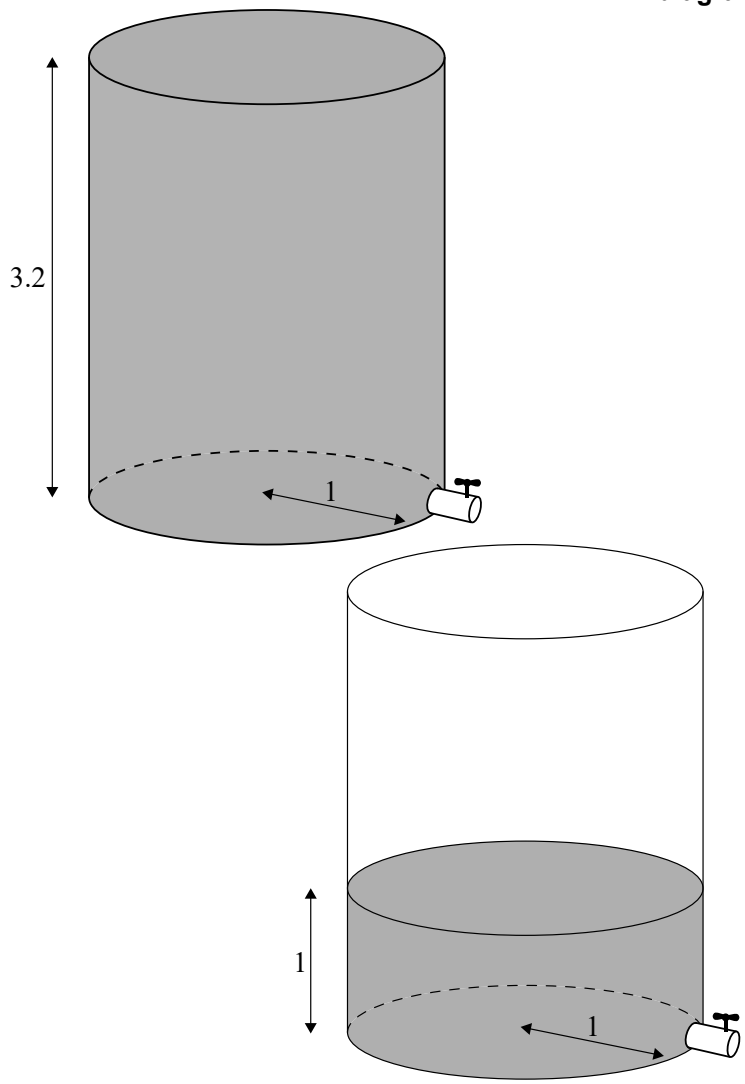
(f) Find this new height. [3]

(This question continues on the following page)

(Question 2 continued)

Eva has another water container that is identical to the first one. She places one water container above the other one, so that all the water from the highest container will drain into the lowest container. Eva completely fills the highest container, but only fills the lowest container to a height of 1 metre, as shown in the diagram.

diagram not to scale



At time $t = 0$ Eva opens both valves. Let H be the height of water, in metres, in the lowest container at time t.

(g) (i) Show that $\frac{\mathrm{d}H}{\mathrm{d}t} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H}$, where $0 \leq t \leq T$. [4]

(ii) Use Euler's method with a step length of 0.5 minutes to estimate the maximum value of H. [3]

References:

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Question 2 - working space

7. [Maximum mark: 5]

A particle, A, moves so that its velocity ($v\,\text{ms}^{-1}$) at time t is given by $v = 2\sin t,\ t \geq 0$.

The kinetic energy (E) of the particle A is measured in joules (J) and is given by $E = 5v^2$.

(a) Write down an expression for E as a function of time. [1]

(b) Hence find $\frac{\mathrm{d}E}{\mathrm{d}t}$. [2]

(c) Hence or otherwise find the first time at which the kinetic energy is changing at a rate of $5\,\text{J}\,\text{s}^{-1}$. [2]

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Question 7 - working space

13. [Maximum mark: 6]

The rates of change of the area covered by two types of fungi, X and Y,on a particular tree are given by the following equations, where x is the area covered by X and y is the area covered by Y.

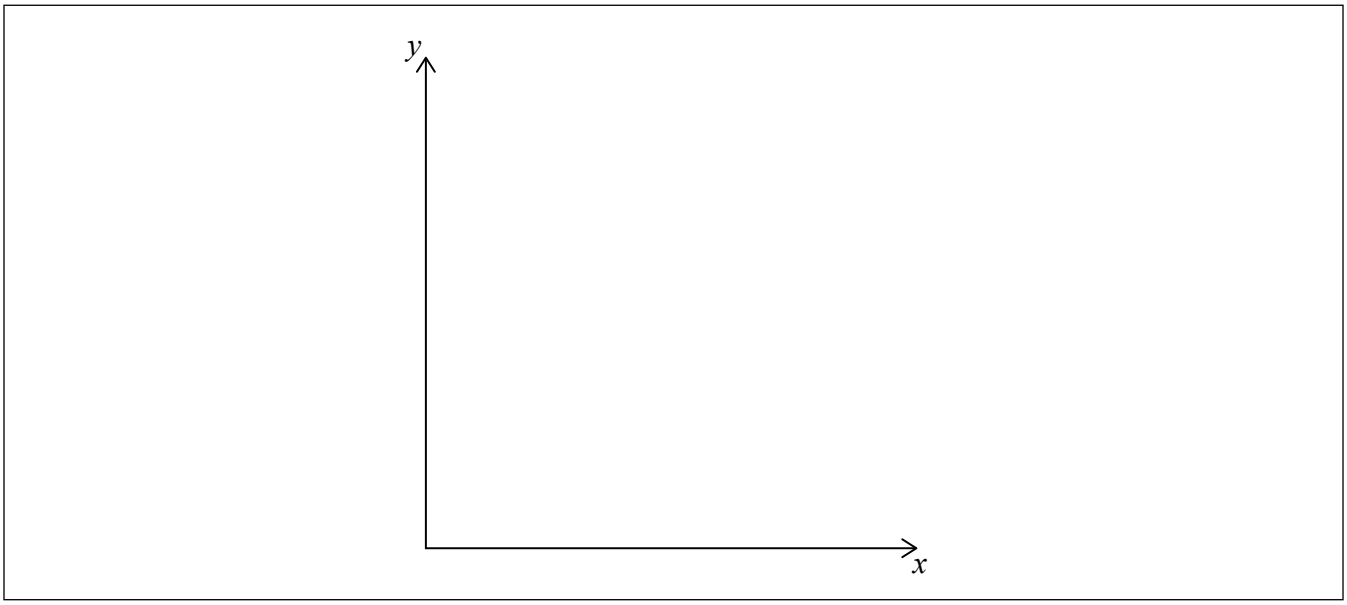
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 2y$$

The matrix $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ has eigenvalues of 2 and -1 with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Initially $x = 8\,\mathrm{cm}^2$ and $y = 10\,\mathrm{cm}^2$.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $t = 0$. [2]

(b) On the following axes, sketch a possible trajectory for the growth of the two fungi, making clear any asymptotic behaviour. [4]



(This question continues on the following page)

(Question 13 continued)

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Question 13 - working space

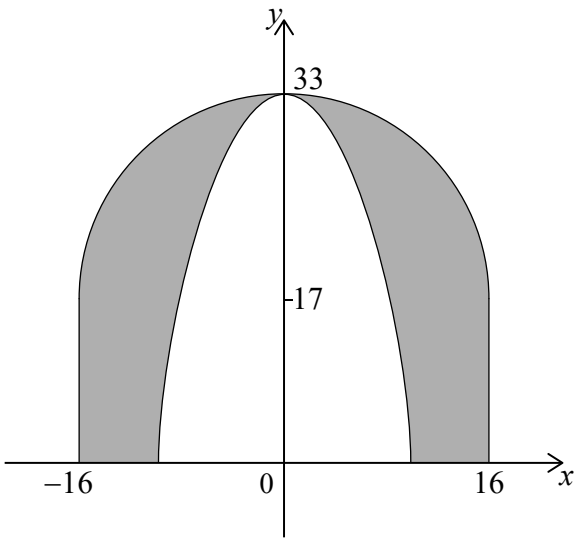
14. [Maximum mark: 8]

(a) The graph of $y = -x^3$ is transformed onto the graph of $y = 33 - 0.08x^3$ by a translation of a units vertically and a stretch parallel to the x-axis of scale factor b.

(i) Write down the value of a.

(ii) Find the value of b. [3]

(b) The outer dome of a large cathedral has the shape of a hemisphere of diameter $32\,\text{m}$, supported by vertical walls of height $17\,\text{m}$. It is also supported by an inner dome which can be modelled by rotating the curve $y = 33 - 0.08x^3$ through 360° about the y-axis between $y = 0$ and $y = 33$, as indicated in the diagram.



Find the volume of the space between the two domes. [5]

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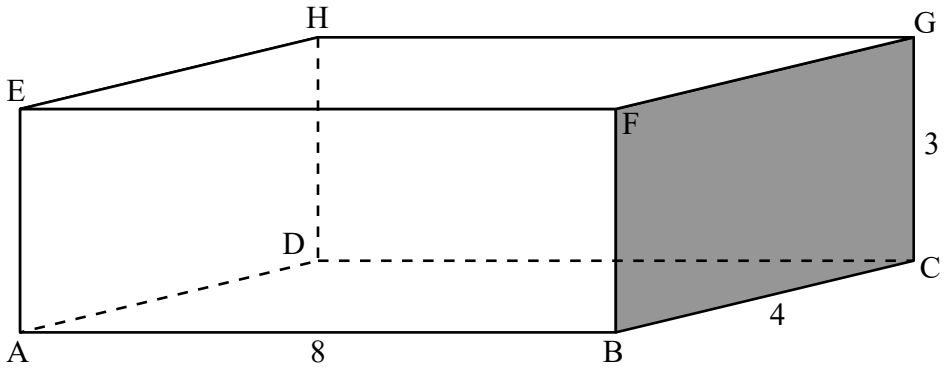
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Question 14 - working space

1. [Maximum mark: 15]

The Happy Straw Company manufactures drinking straws.

The straws are packaged in small closed rectangular boxes, each with length $8\,\text{cm}$, width $4\,\text{cm}$ and height $3\,\text{cm}$. The information is shown in the diagram.



(a) Calculate the surface area of the box in cm^2. [2]

(b) Calculate the length AG. [2]

Each week, the Happy Straw Company sells x boxes of straws. It is known that $\frac{\text{d}P}{\text{d}x} = -2x + 220$, $x \geq 0$, where P is the weekly profit, in dollars, from the sale of x thousand boxes.

(c) Find the number of boxes that should be sold each week to maximize the profit. [3]

The profit from the sale of $20\,000$ boxes is $\$1700$.

(d) Find $P(x)$. [5]

(e) Find the least number of boxes which must be sold each week in order to make a profit. [3]

Question 1 - working space

7. [Maximum mark: 20]

An object is placed into the top of a long vertical tube, filled with a thick viscous fluid, at time $t = 0$.

Initially it is thought that the resistance of the fluid would be proportional to the velocity of the object. The following model was proposed, where the object's displacement, x, from the top of the tube, measured in metres, is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9.81 - 0.9\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right).$$

(a) By substituting $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ into the equation, find an expression for the velocity of the particle at time t. Give your answer in the form $v = f(t)$. [7]

The maximum velocity approached by the object as it falls is known as the terminal velocity.

(b) From your solution to part (a), or otherwise, find the terminal velocity of the object predicted by this model. [2]

An experiment is performed in which the object is placed in the fluid on a number of occasions and its terminal velocity recorded. It is found that the terminal velocity was consistently smaller than that predicted by the model used. It was suggested that the resistance to motion is actually proportional to the velocity squared and so the following model was set up.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9.81 - 0.9\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$$

(c) Write down the differential equation as a system of first order differential equations. [2]

(d) Use Euler's method, with a step length of 0.2, to find the displacement and velocity of the object when $t = 0.6$. [4]

(e) By repeated application of Euler's method, find an approximation for the terminal velocity, to five significant figures. [1]

At terminal velocity the acceleration of an object is equal to zero.

(f) Use the differential equation to find the terminal velocity for the object. [2]

(g) Use your answers to parts (d), (e) and (f) to comment on the accuracy of the Euler approximation to this model. [2]

Question 7 - working space

2. [Maximum mark: 28]

The number of brown squirrels, x, in an area of woodland can be modelled by the following differential equation.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1000}(2000 - x)\text{, where } x > 0$$

(a) (i) Find the equilibrium population of brown squirrels suggested by this model.

(ii) Explain why the population of squirrels is increasing for values of x less than this value. [3]

One year conservationists notice that some black squirrels are moving into the woodland. The two species of squirrel are in competition for the same food supplies. Let y be the number of black squirrels in the woodland.

Conservationists wish to predict the likely future populations of the two species of squirrels. Research from other areas indicates that when the two populations come into contact the growth can be modelled by the following differential equations, in which t is measured in tens of years.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1000}(2000 - x - 2y),\ x,\ y \geq 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{1000}(3000 - 3x - y),\ x,\ y \geq 0$$

An equilibrium point for the populations occurs when both $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$.

(b) (i) Verify that $x = 800$, $y = 600$ is an equilibrium point.

(ii) Find the other three equilibrium points. [6]

(This question continues on the following page)

(Question 2 continued)

When the two populations are small the model can be reduced to the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3y\ .$$

(c) (i) By using separation of variables, show that the general solution of $\frac{\mathrm{d}x}{\mathrm{d}t} = 2x$ is $x = A\mathrm{e}^{2t}$.

(ii) Write down the general solution of $\frac{\mathrm{d}y}{\mathrm{d}t} = 3y$.

(iii) If both populations contain 10 squirrels at $t = 0$ use the solutions to parts (c) (i) and (ii) to estimate the number of black and brown squirrels when $t = 0.2$. Give your answers to the nearest whole numbers. [7]

For larger populations, the conservationists decide to use Euler's method to find the long-term outcomes for the populations. They will use Euler's method with a step length of 2 years ($t = 0.2$).

(d) (i) Write down the expressions for x_{n+1} and y_{n+1} that the conservationists will use.

(ii) Given that the initial populations are $x = 100$, $y = 100$, find the populations of each species of squirrel when $t = 1$.

(iii) Use further iterations of Euler's method to find the long-term population for each species of squirrel from these initial values.

(iv) Use the same method to find the long-term populations of squirrels when the initial populations are $x = 400$, $y = 100$. [7]

(e) Use Euler's method with step length 0.2 to sketch, on the same axes, the approximate trajectories for the populations with the following initial populations.

(i) $x = 1000$, $y = 1500$

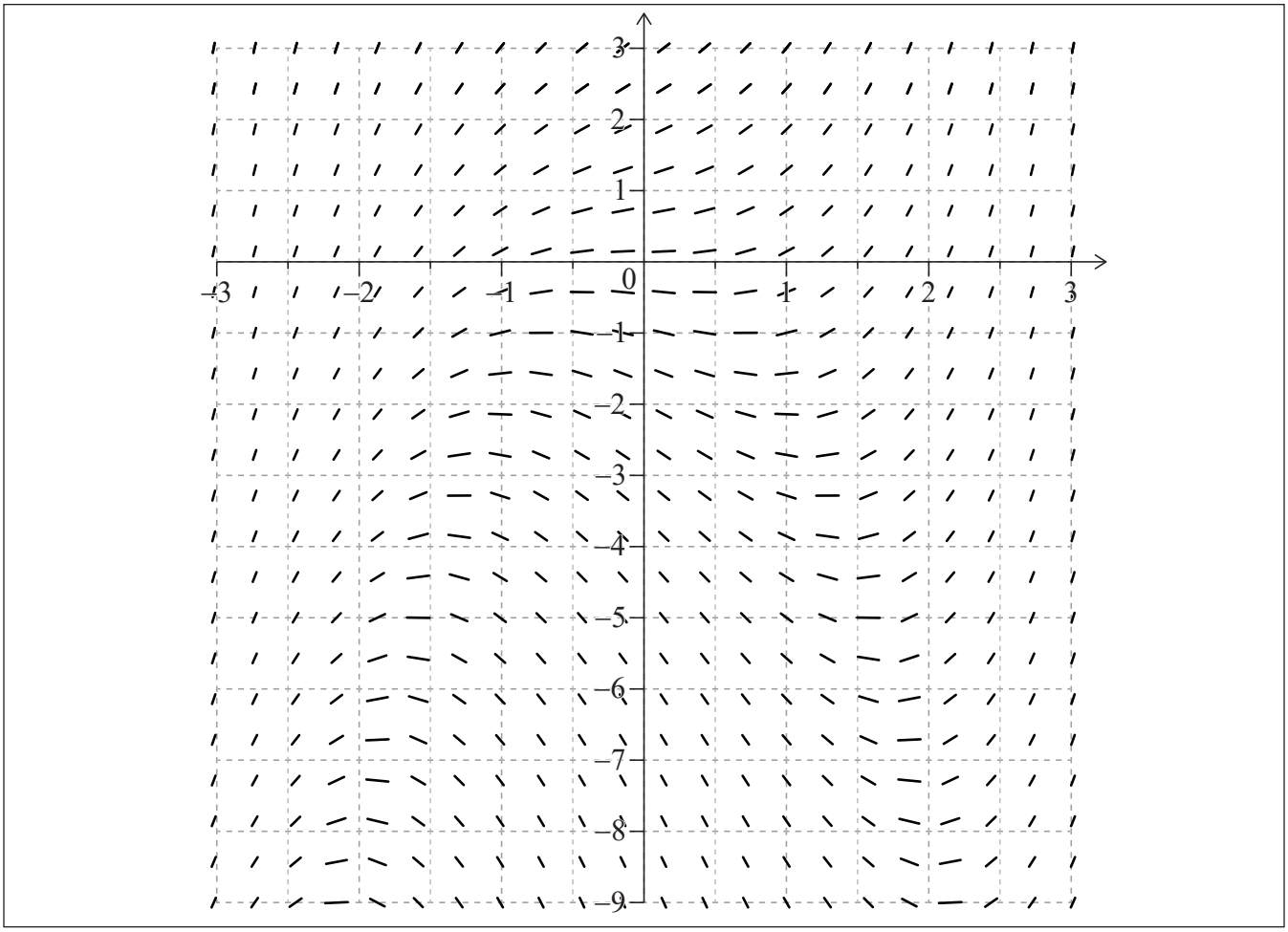
(ii) $x = 1500$, $y = 1000$ [3]

(f) Given that the equilibrium point at $(800, 600)$ is a saddle point, sketch the phase portrait for $x \geq 0$, $y \geq 0$ on the same axes used in part (e). [2]

Question 2 - working space

7. [Maximum mark: 4]

A slope field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2}$ is shown.



Some of the solutions to the differential equation have a local maximum point and a local minimum point.

(a) (i) Write down the equation of the curve on which all these maximum and minimum points lie.

(ii) Sketch this curve on the slope field. [2]

The solution to the differential equation that passes through the point $(0, -2)$ has both a local maximum point and a local minimum point.

(b) On the slope field, sketch the solution to the differential equation that passes through $(0, -2)$. [2]

(This question continues on the following page)

(Question 7 continued)

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Question 7 - working space

8. [Maximum mark: 6]

Consider the curve $y = 2x(4 - e^x)$.

(a) Find

(i) $\frac{dy}{dx}$.

(ii) $\frac{d^2y}{dx^2}$. [4]

The curve has a point of inflexion at (a, b).

(b) Find the value of a. [2]

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Question 8 - working space

14. [Maximum mark: 8]

(a) (i) Expand $\left(\frac{1}{u}+1\right)^2$.

(ii) Find $\int\left(\frac{1}{(x+2)}+1\right)^2 \mathrm{d}x$. [4]

The region bounded by $y=\frac{1}{(x+2)}+1$, $x=0$, $x=2$ and the x-axis is rotated through 2π about the x-axis to form a solid.

(b) Find the volume of the solid formed. Give your answer in the form $\frac{\pi}{4}(a+b\ln(c))$, where $a, b, c \in \mathbb{Z}$. [4]

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Question 14 - working space

16. [Maximum mark: 7]

The wind chill index W is a measure of the temperature, in $^{\circ}\mathrm{C}$, felt when taking into account the effect of the wind.

When Frieda arrives at the top of a hill, the relationship between the wind chill index and the speed of the wind v in kilometres per hour $(\mathrm{km\,h^{-1}})$ is given by the equation

$$W = 19.34 - 7.405v^{0.16}$$

(a) Find an expression for $\frac{\mathrm{d}W}{\mathrm{d}v}$. [2]

When Frieda arrives at the top of a hill, the speed of the wind is 10 kilometres per hour and increasing at a rate of $5\,\mathrm{km\,h^{-1}\,minute^{-1}}$.

(b) Find the rate of change of W at this time. [5]

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Question 16 - working space

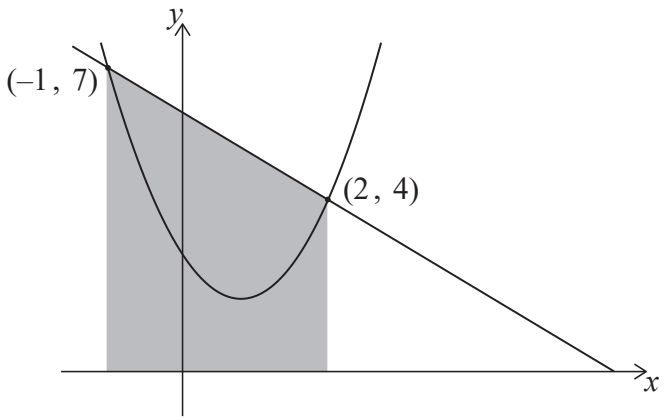
4. [Maximum mark: 7]

The graphs of $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$ intersect at $(2, 4)$ and $(-1, 7)$, as shown in the following diagrams.

In **diagram 1**, the region enclosed by the lines $y = 6 - x$, $x = -1$, $x = 2$ and the x-axis has been shaded.

diagram not to scale

Diagram 1

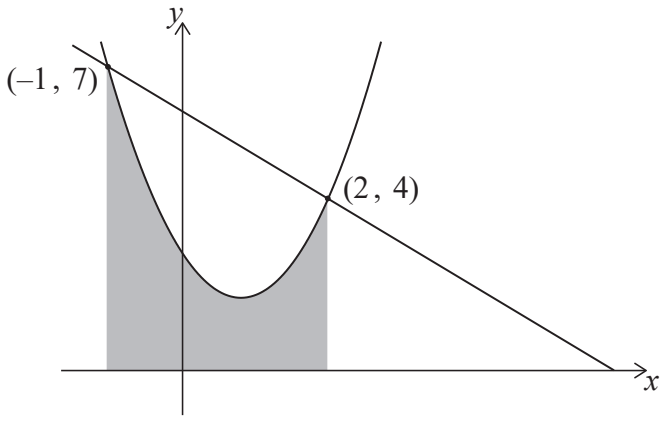


(a) Calculate the area of the shaded region in **diagram 1**. [2]

In **diagram 2**, the region enclosed by the curve $y = 1.5x^2 - 2.5x + 3$, and the lines $x = -1$, $x = 2$ and the x-axis has been shaded.

diagram not to scale

Diagram 2



(b) (i) Write down an integral for the area of the shaded region in **diagram 2**.

(ii) Calculate the area of this region. [3]

(c) Hence, determine the area enclosed between $y = 6 - x$ and $y = 1.5x^2 - 2.5x + 3$. [2]

(This question continues on the following page)

(Question 4 continued)

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Question 4 - working space

14. [Maximum mark: 4]

The shape of a vase is formed by rotating a curve about the y-axis.

The vase is $10\,\text{cm}$ high. The internal radius of the vase is measured at $2\,\text{cm}$ intervals along the height:

Height (cm)	Radius (cm)
0	4
2	6
4	8
6	7
8	3
10	5

Use the trapezoidal rule to estimate the volume of water that the vase can hold.

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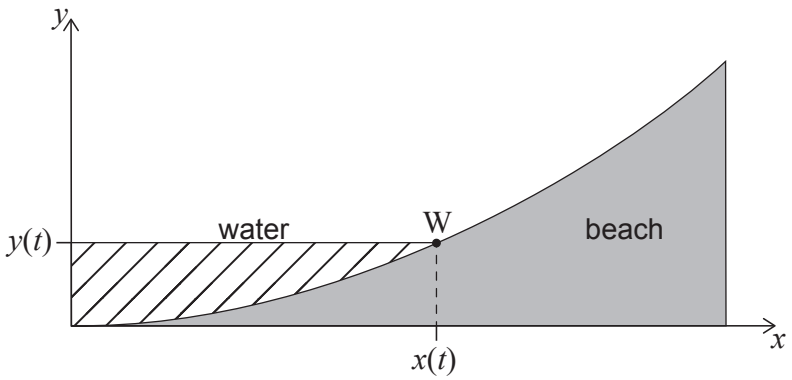
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Question 14 - working space

17. [Maximum mark: 8]

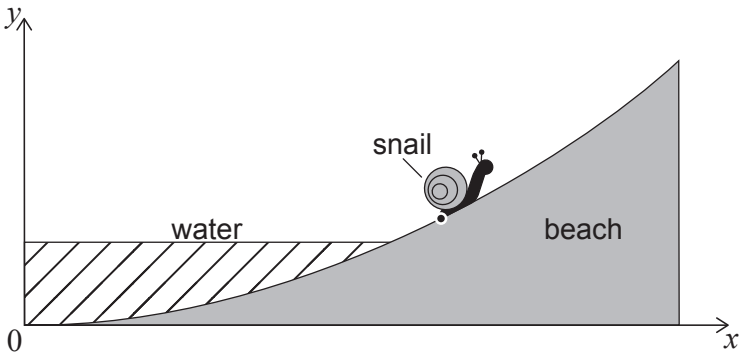
The cross-section of a beach is modelled by the equation $y = 0.02x^2$ for $0 \le x \le 10$ where y is the height of the beach (in metres) at a horizontal distance x metres from an origin. t is the time in hours after low tide.

At $t = 0$ the water is at the point $(0, 0)$. The height of the water rises at a rate of 0.2 metres per hour. The point $\mathrm{W}(x(t), y(t))$ indicates where the water level meets the beach at time t.



(a) When W has an x-coordinate equal to 1, find the horizontal component of the velocity of W. [3]

A snail is modelled as a single point. At $t = 0$ it is positioned at $(1, 0.02)$. The snail travels away from the incoming water at a speed of 1 metre per hour in the direction along the curve of the cross-section of the beach. The following diagram shows this for a value of t, such that $t > 0$.



(b) (i) Find the time taken for the snail to reach the point $(10, 2)$.

(ii) Hence show that the snail reaches the point $(10, 2)$ before the water does. [5]

(This question continues on the following page)

(Question 17 continued)

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References:

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Question 17 - working space

4. [Maximum mark: 13]

A particle moves such that its displacement, x metres, from a point O at time t seconds is given by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0$$

(a) (i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to show that this equation can be written as

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(ii) Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

(iii) Hence state the long-term velocity of the particle. [5]

The equation for the motion of the particle is amended to

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 3t + 4.$$

(b) (i) Use the substitution $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ to write the differential equation as a system of coupled, first order differential equations.

When $t = 0$ the particle is stationary at O.

(ii) Use Euler's method with a step length of 0.1 to find the displacement of the particle when $t = 1$.

(iii) Find the long-term velocity of the particle. [8]

Question 4 - working space

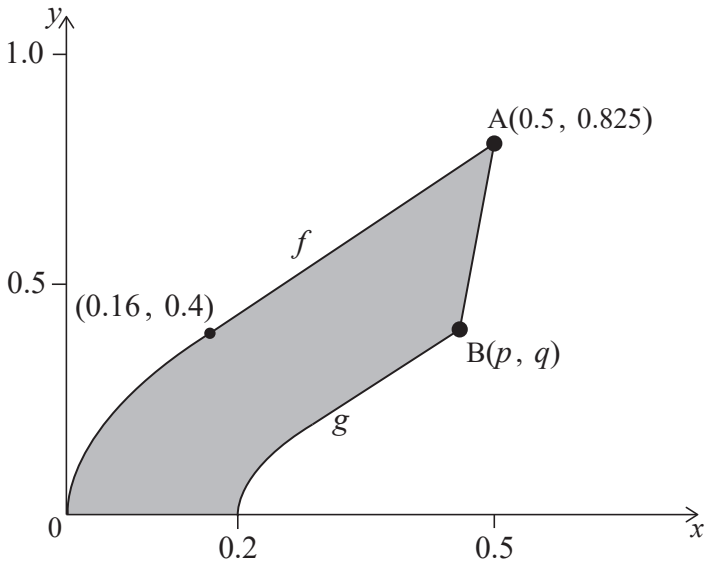
6. [Maximum mark: 18]

Consider the curve $y = \sqrt{x}$.

(a) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Hence show that the equation of the tangent to the curve at the point $(0.16,\ 0.4)$ is $y = 1.25x + 0.2$. [4]

The shape of a piece of metal can be modelled by the region bounded by the functions f, g, the x-axis and the line segment $[\mathrm{AB}]$, as shown in the following diagram. The units on the x and y axes are measured in metres.



The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \le x \le 0.16 \\ 1.25x + 0.2 & 0.16 < x \le 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5,\ 0.825)$. Point B is the image of A under the given transformations and has coordinates $(p,\ q)$.

(b) Find the value of p and the value of q. [2]

(This question continues on the following page)

(Question 6 continued)

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \le x \le a \\ 1.25x + b & a < x \le p \end{cases}$$

(c) Find

(i) an expression for $h(x)$.

(ii) the value of a.

(iii) the value of b. [5]

(d) (i) Find the area enclosed by $y = f(x)$, the x-axis and the line $x = 0.5$.

The area enclosed by $y = g(x)$, the x-axis and the line $x = p$ is $0.0627292\,\text{m}^2$ correct to six significant figures.

(ii) Find the area of the shaded region on the diagram. [7]

Question 6 - working space

2. [Maximum mark: 15]

A cafe makes x litres of coffee each morning. The cafe's profit each morning, C, measured in dollars, is modelled by the following equation

$$C = \frac{x}{10}\left(k^2 - \frac{3}{100}x^2\right)$$

where k is a positive constant.

(a) Find an expression for $\frac{\mathrm{d}C}{\mathrm{d}x}$ in terms of k and x. [3]

(b) Hence find the maximum value of C in terms of k. Give your answer in the form pk^3, where p is a constant. [4]

The cafe's manager knows that the cafe makes a profit of $\$426$ when 20 litres of coffee are made in a morning.

(c) (i) Find the value of k.

(ii) Use the model to find how much coffee the cafe should make each morning to maximize its profit. [3]

(d) Sketch the graph of C against x, labelling the maximum point and the x-intercepts with their coordinates. [3]

The manager of the cafe wishes to serve as many customers as possible.

(e) Determine the maximum amount of coffee the cafe can make that will not result in a loss of money for the morning. [2]

Turn over

Question 2 - working space

7. [Maximum mark: 16]

An environmental scientist is asked by a river authority to model the effect of a leak from a power plant on the mercury levels in a local river. The variable x measures the concentration of mercury in micrograms per litre.

The situation is modelled using the second order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 0$$

where $t \geq 0$ is the time measured in days since the leak started. It is known that when $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$.

(a) Show that the system of coupled first order equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y$$

can be written as the given second order differential equation. [2]

(b) Find the eigenvalues of the system of coupled first order equations given in part (a). [3]

(c) Hence find the exact solution of the second order differential equation. [5]

(d) Sketch the graph of x against t, labelling the maximum point of the graph with its coordinates. [2]

If the mercury levels are greater than 0.1 micrograms per litre, fishing in the river is considered unsafe and is stopped.

(e) Use the model to calculate the total amount of time when fishing should be stopped. [3]

The river authority decides to stop people from fishing in the river for 10% longer than the time found from the model.

(f) Write down one reason, with reference to the context, to support this decision. [1]

References:

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Question 7 - working space

1. [Maximum mark: 28]

This question is about modelling the spread of a computer virus to predict the number of computers in a city which will be infected by the virus.

A systems analyst defines the following variables in a model:

- t is the number of days since the first computer was infected by the virus.
- $Q(t)$ is the total number of computers that have been infected up to and including day t.

The following data were collected:

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146

(a) (i) Find the equation of the regression line of $Q(t)$ on t. [2]

(ii) Write down the value of r, Pearson's product-moment correlation coefficient. [1]

(iii) Explain why it would not be appropriate to conduct a hypothesis test on the value of r found in (a)(ii). [1]

A model for the early stage of the spread of the computer virus suggests that

$$Q'(t) = \beta NQ(t)$$

where N is the total number of computers in a city and β is a measure of how easily the virus is spreading between computers. Both N and β are assumed to be constant.

(b) (i) Find the general solution of the differential equation $Q'(t) = \beta NQ(t)$. [4]

(ii) Using the data in the table write down the equation for an appropriate non-linear regression model. [2]

(iii) Write down the value of R^2 for this model. [1]

(iv) Hence comment on the suitability of the model from (b)(ii) in comparison with the linear model found in part (a). [2]

(v) By considering large values of t write down one criticism of the model found in (b)(ii). [1]

(This question continues on the following page)

(Question 1 continued)

(c) Use your answer from part (b)(ii) to estimate the time taken for the number of infected computers to double. [2]

The data above are taken from city X which is estimated to have 2.6 million computers. The analyst looks at data for another city, Y. These data indicate a value of $\beta = 9.64 \times 10^{-8}$.

(d) Find in which city, X or Y, the computer virus is spreading more easily. Justify your answer using your results from part (b). [3]

An estimate for $Q'(t)$, $t \geq 5$, can be found by using the formula:

$$Q'(t) \approx \frac{Q(t+5) - Q(t-5)}{10}.$$

The following table shows estimates of $Q'(t)$ for city X at different values of t.

t	10	15	20	25	30	35	40
$Q(t)$	20	90	403	1806	8070	32 667	120 146
$Q'(t)$		a	171.6	766.7	b	11 207.6	

(e) Determine the value of a and of b. Give your answers correct to one decimal place. [2]

An improved model for $Q(t)$, which is valid for large values of t, is the logistic differential equation

$$Q'(t) = kQ(t)\left(1 - \frac{Q(t)}{L}\right)$$

where k and L are constants.

Based on this differential equation, the graph of $\frac{Q'(t)}{Q(t)}$ against $Q(t)$ is predicted to be a straight line.

(f) (i) Use linear regression to estimate the value of k and of L. [5]

(ii) The solution to the differential equation is given by

$$Q(t) = \frac{L}{1 + Ce^{-kt}}$$

where C is a constant.

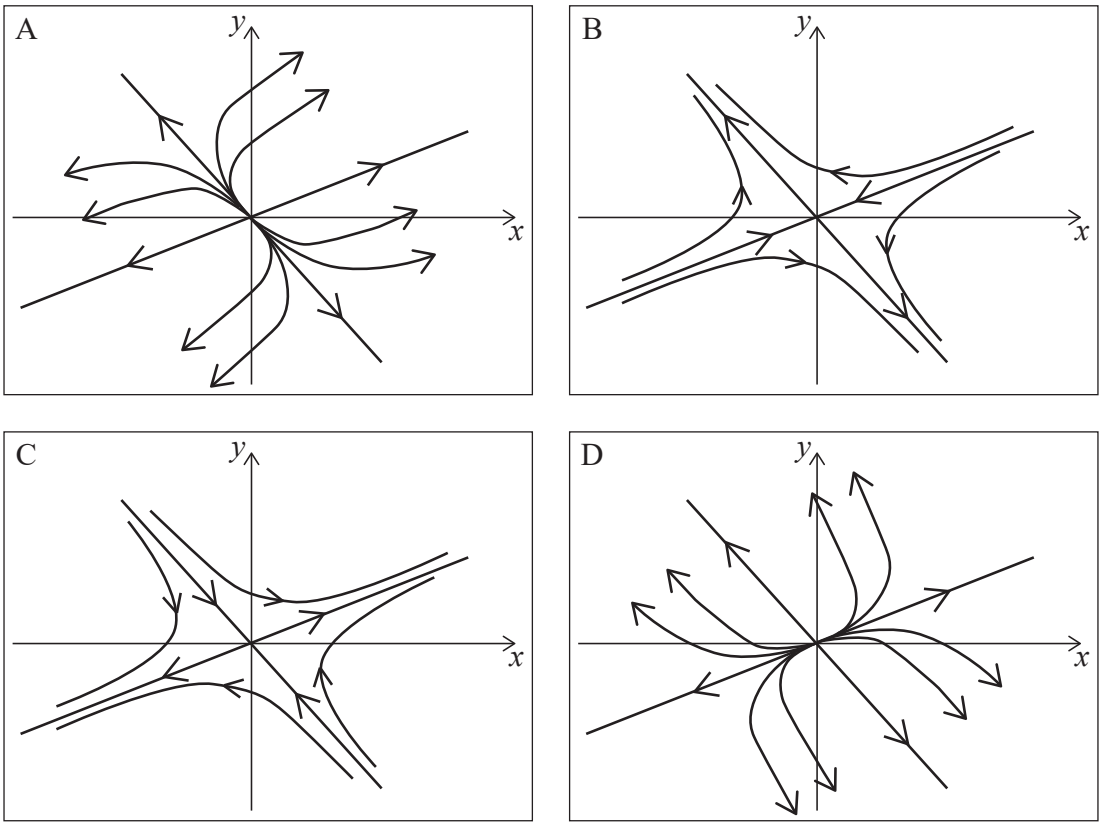
Using your answer to part (f)(i), estimate the percentage of computers in city X that are expected to have been infected by the virus over a long period of time. [2]

Turn over

Question 1 - working space

12. [Maximum mark: 5]

Four possible phase portraits for the coupled differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ are shown, labelled A, B, C and D.



The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2.

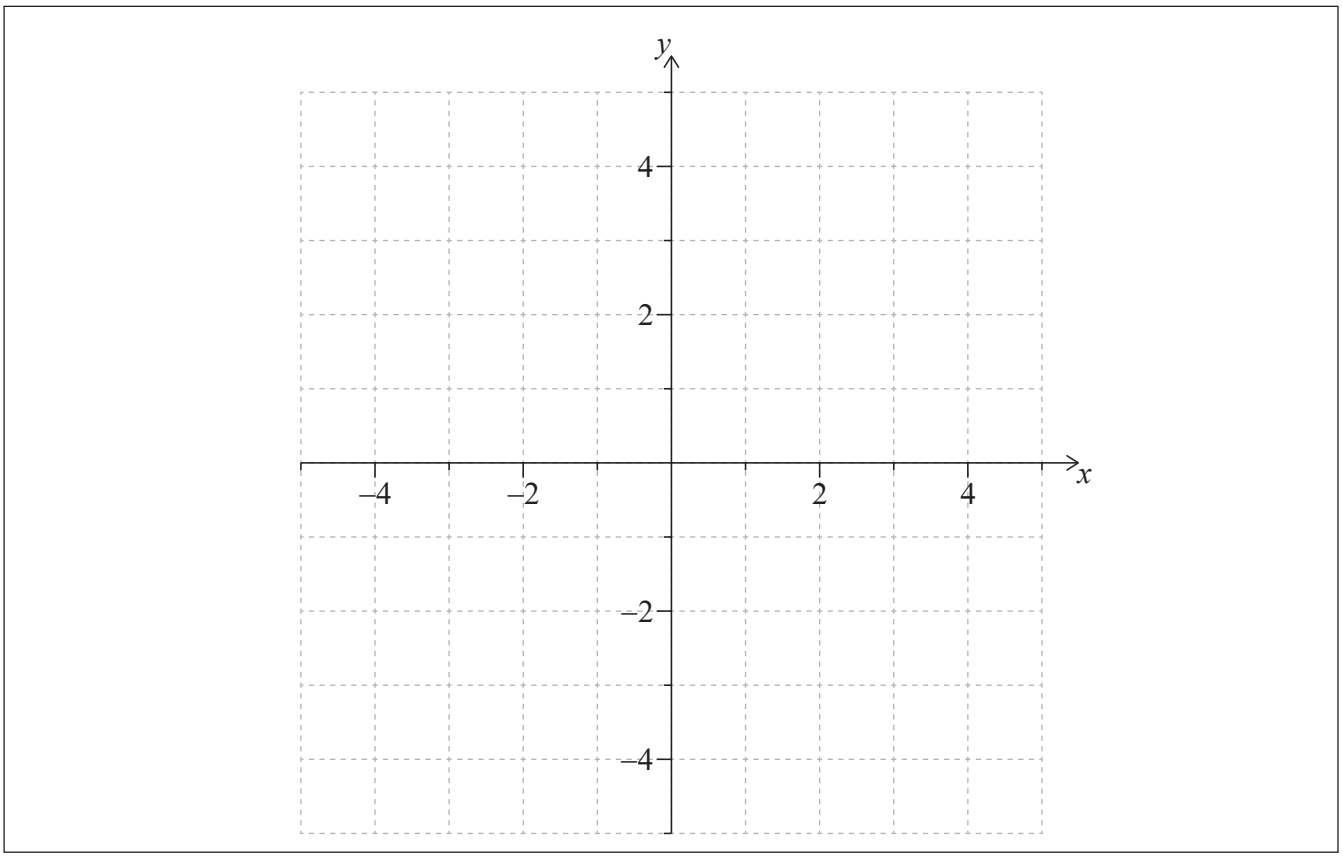
(a) Complete the following table by writing down the letter of the phase portrait that best matches the description. [3]

Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	

(This question continues on the following page)

(Question 12 continued)

(b) On the following axes, sketch the phase portrait that corresponds to $\lambda_1 = -2 + 3\mathrm{i}$ and $\lambda_2 = -2 - 3\mathrm{i}$, given that $\frac{\mathrm{d}y}{\mathrm{d}t} = -12$ at $(3,\ 0)$. [2]



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Question 12 - working space

14. [Maximum mark: 9]

A particle moves such that its velocity, v metres per second, at time t seconds, is given by $v = t\sin(t^2)$.

(a) Find an expression for the acceleration of the particle. [2]

(b) Hence, or otherwise, find its greatest acceleration for $0 \leq t \leq 8$. [2]

The particle starts at the origin.

(c) Find an expression for the displacement of the particle. [3]

(d) Hence show that the particle never has a negative displacement. [2]

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Question 14 - working space

15. [Maximum mark: 5]

An electrical circuit contains a capacitor. The charge on the capacitor, q Coulombs, at time t seconds, satisfies the differential equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 5\frac{\mathrm{d}q}{\mathrm{d}t} + 20q = 200\,.$$

Initially $q = 1$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = 8$.

Use Euler's method with $h = 0.1$ to estimate the maximum charge on the capacitor during the first second.

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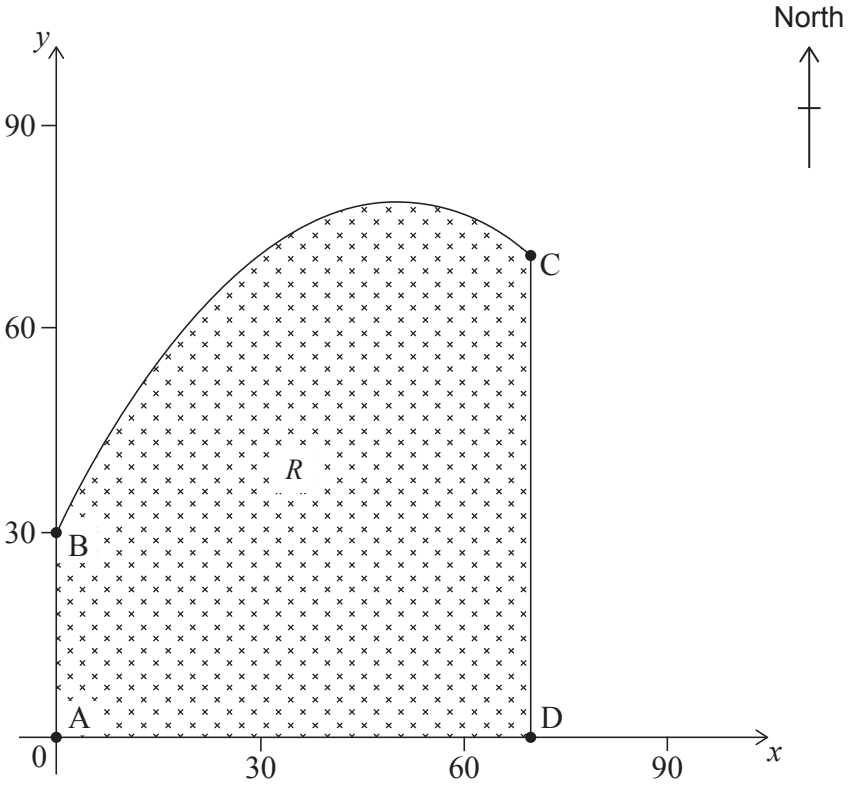
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Question 15 - working space

3. [Maximum mark: 17]

Linda owns a field, represented by the shaded region R. The plan view of the field is shown in the following diagram, where both axes represent distance and are measured in metres.



The segments [AB], [CD] and [AD] respectively represent the western, eastern and southern boundaries of the field. The function, $f(x)$, models the northern boundary of the field between points B and C and is given by

$$f(x) = \frac{-x^2}{50} + 2x + 30, \text{ for } 0 \le x \le 70.$$

(a) (i) Find $f'(x)$.

(ii) Hence find the coordinates of the point on the field that is furthest north. [5]

Point A has coordinates $(0, 0)$, point B has coordinates $(0, 30)$, point C has coordinates $(70, 72)$ and point D has coordinates $(70, 0)$.

(b) (i) Write down the integral which can be used to find the area of the shaded region R.

(ii) Find the area of Linda's field. [4]

(This question continues on the following page)

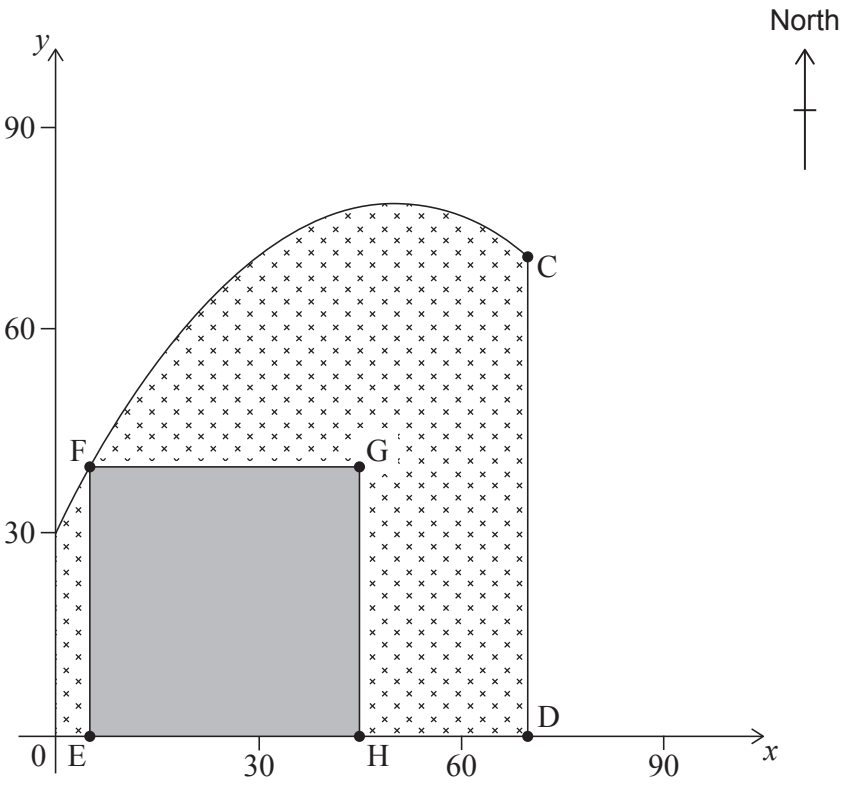
(Question 3 continued)

Linda used the trapezoidal rule with ten intervals to estimate the area. This calculation underestimated the area by $11.4\,\text{m}^2$.

(c) (i) Calculate the percentage error in Linda's estimate.

(ii) Suggest how Linda might be able to reduce the error whilst still using the trapezoidal rule. [3]

Linda would like to construct a building on her field. The **square** foundation of the building, EFGH, will be located such that [EH] is on the southern boundary and point F is on the northern boundary of the property. A possible location of the foundation of the building is shown in the following diagram.



The area of the square foundation will be largest when [GH] lies on [CD].

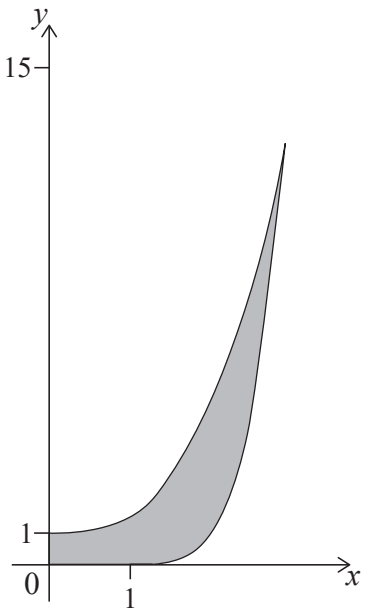
(d) (i) Find the x-coordinate of point E for the largest area of the square foundation of building EFGH.

(ii) Find the largest area of the foundation. [5]

Question 3 - working space

5. [Maximum mark: 13]

Adesh is designing a glass. The glass has an inner surface and an outer surface. Part of the cross section of his design is shown in the following graph, where the shaded region represents the glass. The two surfaces meet at the top of the glass. 1 unit represents $1\,\text{cm}$.



The inner surface is modelled by $f(x) = \frac{1}{2}x^3 + 1$ for $0 \le x \le p$.

The outer surface is modelled by $g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ (x-1)^4 & \text{for } 1 \le x \le p \end{cases}$.

(a) Find the value of p. [2]

The glass design is finished by rotating the shaded region in the diagram through 360° about the y-axis.

(b) Find the volume of liquid that can be contained inside the finished glass. [5]

(c) Find the volume of the region between the two surfaces of the finished glass. [6]

Question 5 - working space

2. [Maximum mark: 26]

Some medical conditions require patients to take medication regularly for long periods of time. In this question, you will explore the concentration of a medicinal drug in the body, when the drug is given repeatedly.

Once a drug enters the body, it is absorbed into the blood. As the body breaks down the drug over time, the concentration of the drug decreases. Let $C(t)$, measured in milligrams per millilitre (mg ml^{-1}), be the concentration of the drug, t hours after the drug is given to the patient. The rate at which the drug is broken down is modelled as directly proportional to its concentration, leading to the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -kC \text{, where } k \in \mathbb{R}^+.$$

The initial concentration is $d \text{ mg ml}^{-1}$, $d > 0$.

(a) By solving the differential equation, show that $C = d\mathrm{e}^{-kt}$. [3]

For the remainder of this question, you will consider a particular drug where it is known that $k = 0.2$. The first dose is given at time $t = 0$ and it is assumed that before this there is no drug present in the blood.

(b) Find the time, in hours, for this drug to reach 5% of its initial concentration. [2]

The drug is to be given every T hours and in constant doses, such that the concentration of the drug is increased by an amount $d \text{ mg ml}^{-1}$. To simplify the model, it is assumed that each time the drug is given the concentration of the drug in the blood increases instantaneously.

(c) Show that the concentration of the drug is $d\left(1 + \mathrm{e}^{-0.2T} + \mathrm{e}^{-0.4T}\right)$ immediately after the third dose is given. [4]

Immediately after the $n\text{th}$ dose is given, the concentration of the drug is

$$d\left(1 + \mathrm{e}^{-0.2T} + \mathrm{e}^{-0.4T} + \ldots + \mathrm{e}^{-0.2(n-1)T}\right).$$

(d) Show that this concentration can be expressed as $d\left(\frac{1 - \mathrm{e}^{-0.2nT}}{1 - \mathrm{e}^{-0.2T}}\right)$. [2]

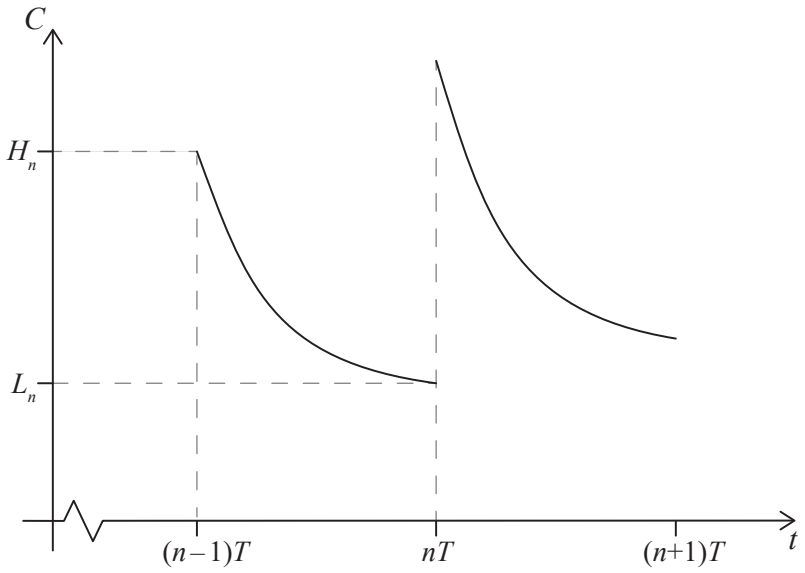
(This question continues on the following page)

(Question 2 continued)

After a patient has been taking this drug for a long time, it is required to keep the concentration within a particular range so that it is both safe and effective.

Let H_n be the highest concentration of the drug in the body for the interval $(n-1)T \leq t < nT$.

Let L_n be the lowest concentration of the drug in the body for the interval $(n-1)T \leq t < nT$. This is shown in the following graph.



H_∞ is defined as $\lim_{n\to\infty} H_n$ and L_∞ is defined as $\lim_{n\to\infty} L_n$.

(e) Find, in terms of d and T, an expression for

(i) H_∞. [2]

(ii) L_∞. [3]

(f) Show that

(i) $H_\infty - L_\infty = d$. [2]

(ii) $5\ln\left(\frac{H_\infty}{L_\infty}\right) = T$. [3]

(This question continues on the following page)

(Question 2 continued)

It is known that this drug is ineffective if the long-term concentration is less than $0.06\,\text{mg}\,\text{ml}^{-1}$ and safe if it never exceeds $0.28\,\text{mg}\,\text{ml}^{-1}$.

(g) Hence, for this drug, find a suitable value for

(i) d. [1]

(ii) T. [1]

(h) For the values of d and T found in part (g), find the proportion of time for which the concentration of the drug is at least $0.06\,\text{mg}\,\text{ml}^{-1}$ between the first and second doses. [2]

(i) Suggest a reason why the instructions on the label of the drug might use a different value for T to that found in part (g)(ii). [1]

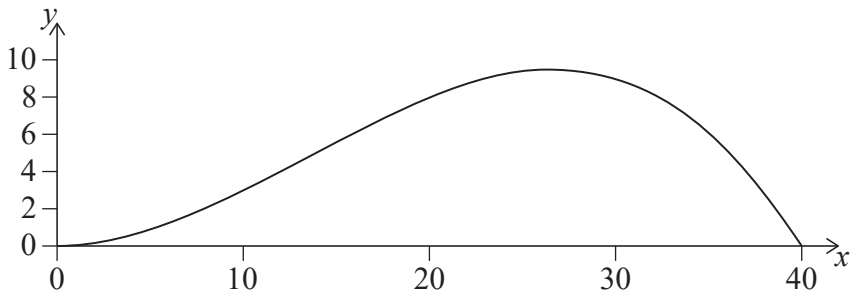
References:

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Question 2 - working space

4. [Maximum mark: 8]

The cross section of a scale model of a hill is modelled by the following graph.



The heights of the model are measured at horizontal intervals and are given in the table.

Horizontal distance, x cm	0	10	20	30	40
Vertical distance, y cm	0	3	8	9	0

(a) Use the trapezoidal rule with $h = 10$ to find an approximation for the cross-sectional area of the model. [2]

It is given that the equation of the curve is $y = 0.04x^2 - 0.001x^3,\ 0 \le x \le 40$.

(b) (i) Write down an integral to find the exact cross-sectional area.

(ii) Calculate the value of the cross-sectional area to two decimal places. [4]

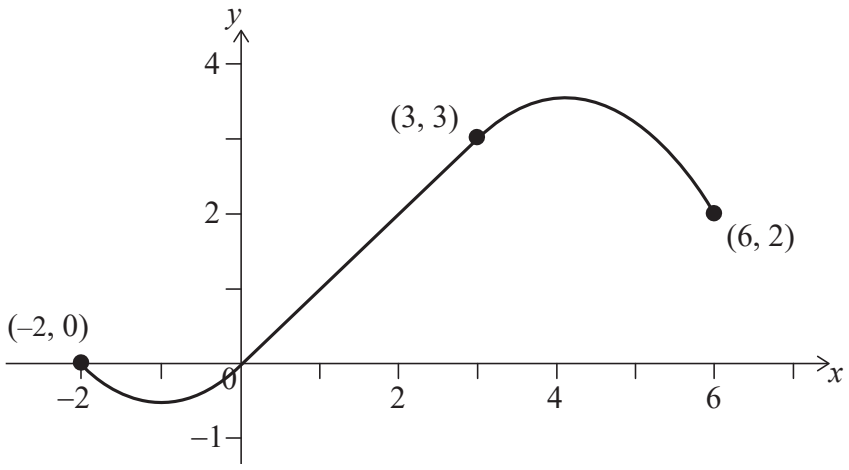
(c) Find the percentage error in the area found using the trapezoidal rule. [2]

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Question 4 - working space

10. [Maximum mark: 9]

A decorative hook can be modelled by the curve with equation $y = f(x)$. The graph of $y = f(x)$ is shown and consists of a line segment from $(0, 0)$ to $(3, 3)$ and two sections formed by quadratic curves.



(a) Write down the equation of the line segment for $0 \leq x \leq 3$. [1]

The quadratic curve, with endpoints $(-2, 0)$ and $(0, 0)$, has the same gradient at $(0, 0)$ as the line segment.

(b) Find the equation of the curve between $(-2, 0)$ and $(0, 0)$. [3]

The second quadratic curve, with endpoints $(3, 3)$ and $(6, 2)$, has the same gradient at $(3, 3)$ as the line segment.

(c) Find the equation of this curve. [4]

(d) Write down f as a piecewise function. [1]

(This question continues on the following page)

(Question 10 continued)

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Question 10 - working space

13. [Maximum mark: 6]

The displacement, $x(\text{cm})$, of the end of a spring, at time t (seconds), is given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}+10x=0\,.$$

At $t=0$, $x=0.75$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=0$.

Use Euler's method, with a step length 0.1 seconds, to estimate the value of x when $t=0.5$.

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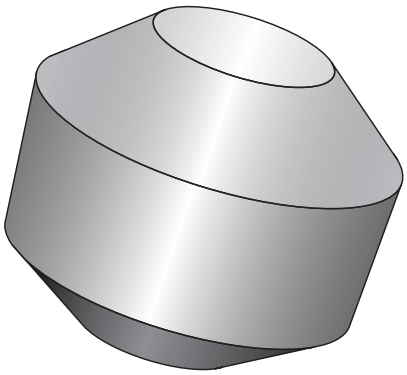
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Question 13 - working space

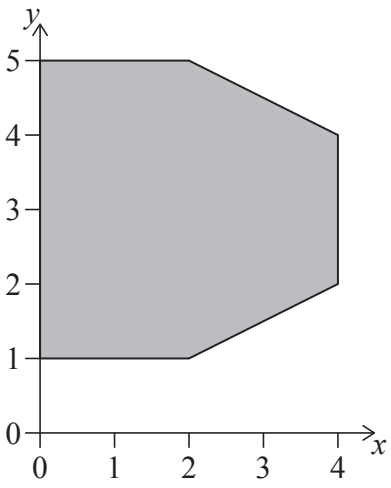
15. [Maximum mark: 6]

The solid shown is formed by rotating the hexagon with vertices $(2, 1)$, $(0, 1)$, $(0, 5)$, $(2, 5)$, $(4, 4)$ and $(4, 2)$ about the y-axis.

Solid



Hexagon



Find the volume of this solid.

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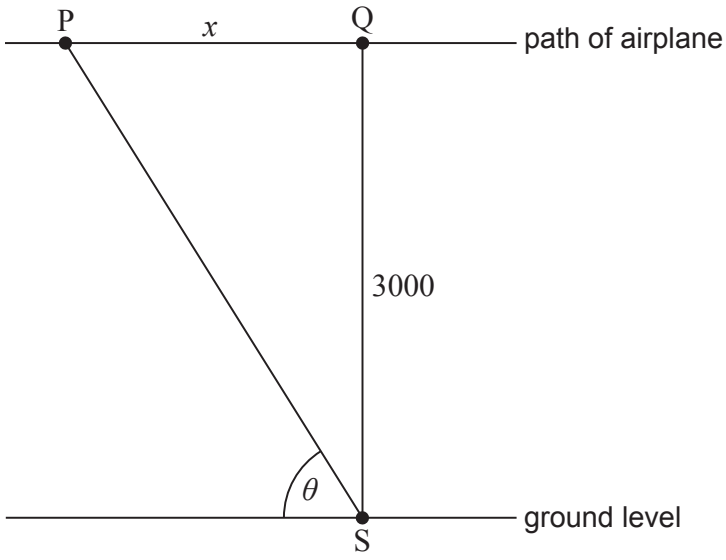
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Question 15 - working space

17. [Maximum mark: 9]

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.



(a) Use related rates to show that, at time T, $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -\dfrac{10\,000}{3}$. [2]

(b) Find $x(\theta)$, x as a function of θ. [1]

(c) Find an expression for $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ in terms of $\sin\theta$. [3]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

(This question continues on the following page)

(Question 17 continued)

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References:

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Question 17 - working space

12. [Maximum mark: 5]

A spherical balloon is being inflated such that its volume is increasing at a rate of $15\,\text{cm}^3\,\text{s}^{-1}$.

(a) Find the radius of the balloon when its volume is $288\pi\,\text{cm}^3$. [2]

(b) Hence or otherwise, find the rate of change of the radius at this instant. [3]

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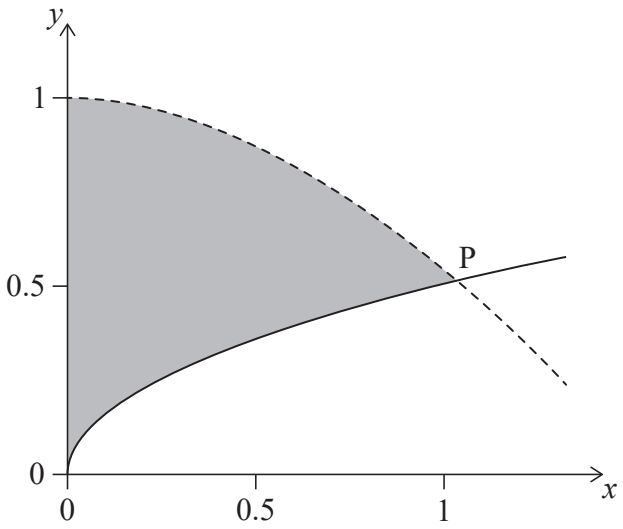
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Question 12 - working space

16. [Maximum mark: 9]

The following diagram shows parts of the curves of $y = \cos x$ and $y = \frac{\sqrt{x}}{2}$.

P is the point of intersection of the two curves.



(a) Use your graphic display calculator to find the coordinates of P. [2]

The shaded region is rotated 360° **about the y-axis** to form a volume of revolution V.

(b) Express V as the sum of two definite integrals. [5]

(c) Hence find the value of V. [2]

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Question 16 - working space

17. [Maximum mark: 6]

Consider the differential equation

$$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{2y-2}\text{, for } x\geq 0,\ y\geq 1,$$

where $y=1$ when $x=0$.

(a) Explain why Euler's method cannot be used to find an approximate value for y when $x=0.1$. [1]

(b) By solving the differential equation, show that $y=1+\sqrt{\frac{\ln\left(x^2+1\right)}{2}}$. [4]

(c) Hence deduce the value of y when $x=0.1$. [1]

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References:

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Question 17 - working space

6. [Maximum mark: 15]

A model speedboat has its position, at time t seconds $t \geq 0$, defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5y - 0.05x\,,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 0.05y\,,$$

where x metres is the distance east and y metres is the distance north of a fixed point O.

(a) Find the eigenvalues of $\boldsymbol{A} = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$, giving your answers in the form $a + b\mathrm{i}$, where $a \neq 0$, $b \neq 0$. [4]

(b) (i) State what $a \neq 0$ indicates about the path of the speedboat.

(ii) State what the sign of a indicates about the path of the speedboat. [2]

At time $t = 0$, the speedboat has position $(20, 0)$.

(c) At time $t = 0$, find the value of

(i) $\frac{\mathrm{d}y}{\mathrm{d}t}$.

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x}$. [5]

(d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat. [4]

Turn over

Question 6 - working space

3. [Maximum mark: 16]

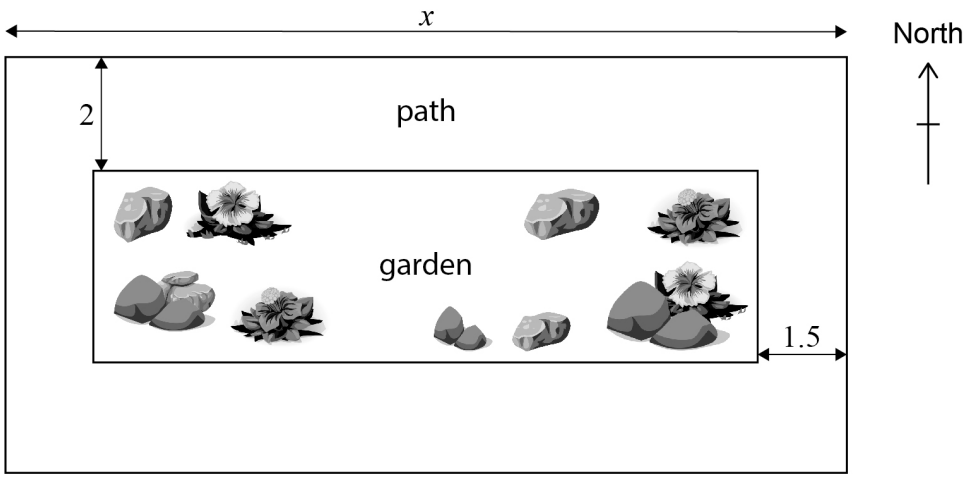
A particular park consists of a rectangular garden, of area $A\,\text{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\text{m}^2$.

The width of the path at the north and south side of the park is $2\,\text{m}$.

The width of the path at the west and east side of the park is $1.5\,\text{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \frac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\text{m}^2$. [4]

(c) Find an expression for $\frac{\text{d}A}{\text{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

Turn over

Question 3 - working space

7. [Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 6y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 9x - y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{p}_1 \mathrm{e}^{\lambda_1 t} + B\boldsymbol{p}_2 \mathrm{e}^{\lambda_2 t}$, where A, B, λ_1, λ_2 $(\lambda_2 > \lambda_1)$ are scalar constants and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

(c) Determine the value of A and the value of B. [2]

(d) State the long-term ratio of male fruit flies to female fruit flies. [1]

(e) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at time $t = 0$. [3]

(f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]

References:

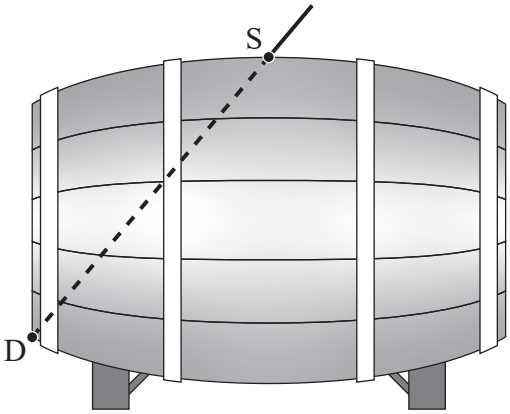
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Question 7 - working space

1. [Maximum mark: 26]

In this question you will use a historic method of calculating the cost of a barrel of wine to determine which shape of barrel gives the best value for money.

In Austria in the 17th century, one method for measuring the volume of a barrel of wine, and hence determining its cost, was by inserting a straight stick into a hole in the side, as shown in the following diagram, and measuring the length SD. The longer the length, the greater the cost to the customer.



Let SD be d metres and the cost be C gulden (the local currency at the time). When the length of SD was 0.5 metres, the cost was 0.80 gulden.

(a) Given that C was directly proportional to d, find an equation for C in terms of d. [3]

A particular barrel of wine cost 0.96 gulden.

(b) Show that $d = 0.6$. [1]

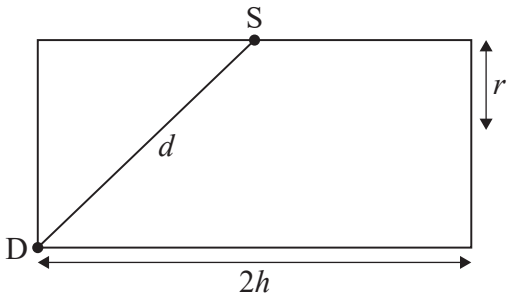
(This question continues on the following page)

(Question 1 continued)

This method of determining the cost was noticed by a mathematician, Kepler, who decided to try to calculate the dimensions of a barrel which would give the maximum volume of wine for a given length SD.

Initially he modelled the barrel as a cylinder, with S at the midpoint of one side. He took the length of the cylinder as $2h$ metres and its radius as r metres, as shown in the following diagram of the cross-section.

diagram not to scale



(c) Find an expression for r^2 in terms of d and h. [3]

Let the volume of this barrel be $V\,\text{m}^3$.

(d) Show that $V = \frac{\pi}{2}\left(d^2h - h^3\right)$. [2]

The remainder of this question considers the shape of barrel that gives the best value when $d = 0.6$.

(e) (i) Use the formula from part (d) to find the volume of this barrel when $h = 0.4$. [2]

(ii) Use differentiation to show that $h = \sqrt{0.12}$ when $\frac{\text{d}V}{\text{d}h} = 0$. [3]

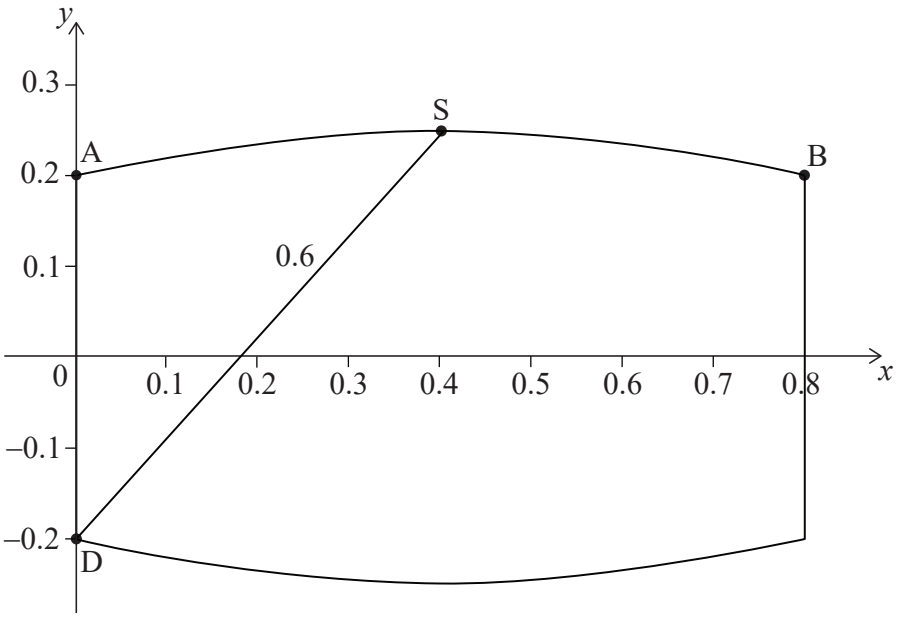
(iii) Given that this value of h maximizes the volume, find the largest possible volume of this barrel. [2]

(This question continues on the following page)

(Question 1 continued)

Kepler then considered a non-cylindrical barrel whose base and lid are circles with radius $0.2\,\text{m}$ and whose length is $0.8\,\text{m}$.

He modelled the curved surface of this barrel by rotating a quadratic curve, ASB, with equation $y = ax^2 + bx + c$, $0 \le x \le 0.8$, about the x-axis. The origin of the coordinate system is at the centre of one of the circular faces as shown in the following diagram. S is at the vertex of the quadratic curve and $\text{SD} = 0.6$.



Kepler wished to find out if his barrel would give him more wine than any cylindrical barrel with $d = 0.6$.

The coordinates of A and B are $(0,\ 0.2)$ and $(0.8,\ 0.2)$ respectively.

(f) Find the equation of the quadratic curve, ASB. [6]

(g) Show that the volume of this barrel is greater than the maximum volume of any cylindrical barrel with $d = 0.6$. [3]

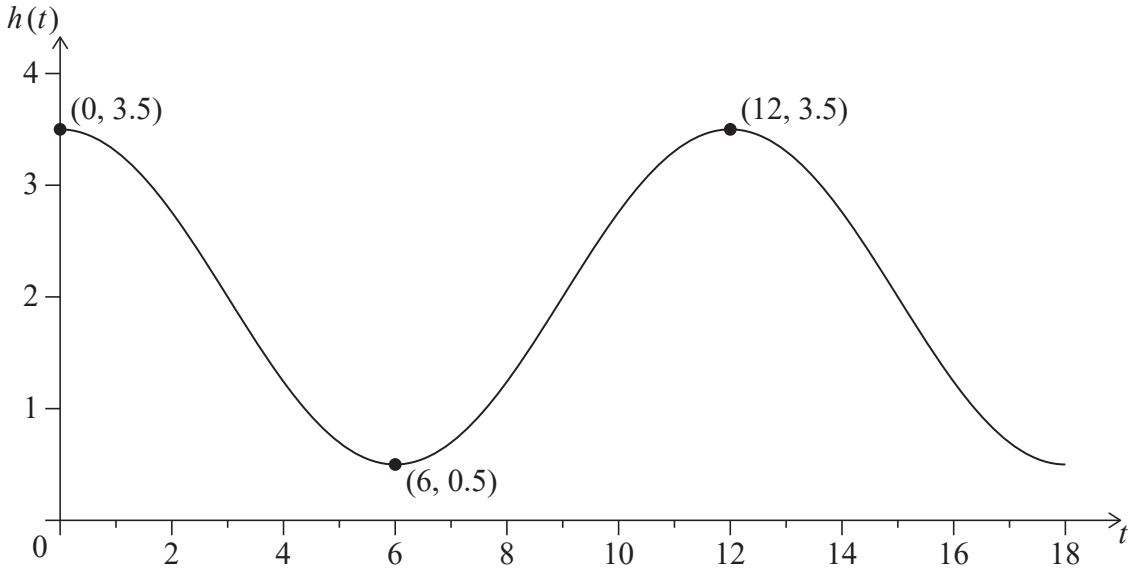
(h) State one assumption, not already given, that has been made in using these models to find the shape of the barrel that gives the best value. [1]

Turn over

Question 1 - working space

9. [Maximum mark: 8]

Joon is a keen surfer and wants to model waves passing a particular point P, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and t, the time in seconds from when he begins recording the height of the water at point P.



The function has the form $h(t) = p\cos\left(\frac{\pi}{6}t\right) + q$, $t \geq 0$.

(a) Find the values of p and q. [2]

(b) Find

(i) $h'(t)$.

(ii) $h''(t)$. [3]

Joon will begin to surf the wave when the rate of change of h with respect to t, at P, is at its maximum. This will first occur when $t = k$.

(c) (i) Find the value of k.

(ii) Find the height of the water at this time. [3]

(This question continues on the following page)

(Question 9 continued)

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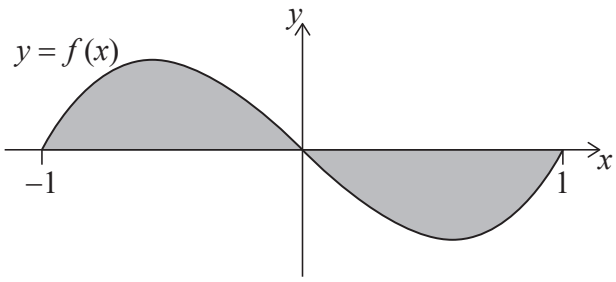
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Question 9 - working space

11. [Maximum mark: 7]

Consider the function $f(x) = x^3 - x$, for $-1 \leq x \leq 1$. The shaded region, R, is bounded by the graph of $y = f(x)$ and the x-axis.

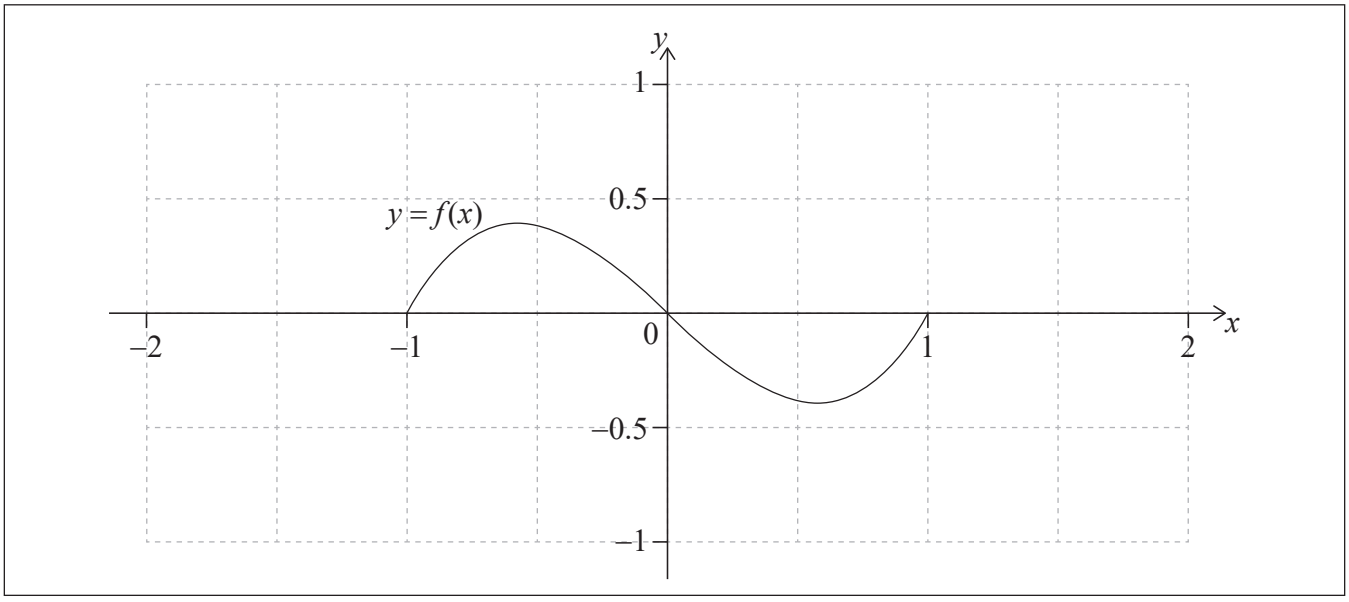


(a) (i) Write down an integral that represents the area of R.

(ii) Find the area of R. [2]

Another function, g, is defined such that $g(x) = 2f(x-1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$. [2]



The region R from the original graph $y = f(x)$ is rotated through 2π radians about the x-axis to form a solid.

(c) Find the volume of the solid. [3]

(This question continues on the following page)

(Question 11 continued)

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Question 11 - working space

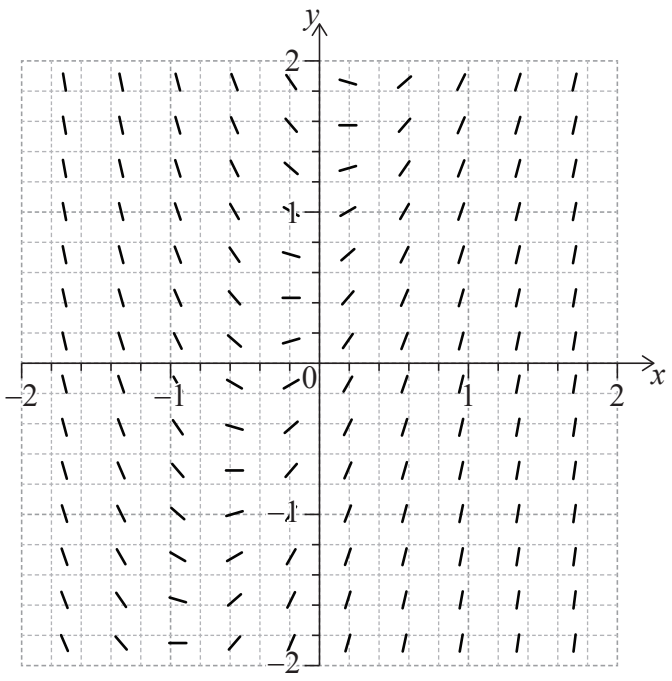
12. [Maximum mark: 4]

Consider the differential equation $\frac{dy}{dx} = 3x - y + 1$.

(a) Find the equation of the tangent to the solution curve at the point $(-1, -1)$ in the form $ax + by + c = 0$. [2]

The slope field for this differential equation is shown in the following diagram.

(b) Sketch the solution curve that passes through the point $(-1, -1)$. [2]



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Question 12 - working space

13. [Maximum mark: 8]

The velocity v of a particle at time t, as it moves along a straight line, can be modelled by the piecewise function

$$v(t) = \begin{cases} u_1(t), & 0 \le t \le T \\ u_2(t), & t \ge T \end{cases}$$

where $u_1(t) = 2t^2 - t^3$ and $u_2(t) = 8 - 4t$. It is required that $u_1(T) = u_2(T)$.

(a) Find the value of T. [2]

(b) Show that $u_1'(T) = u_2'(T)$. [2]

The displacement of the particle at time $t = 0$ is zero.

(c) Find the time when the particle returns to its initial position. [4]

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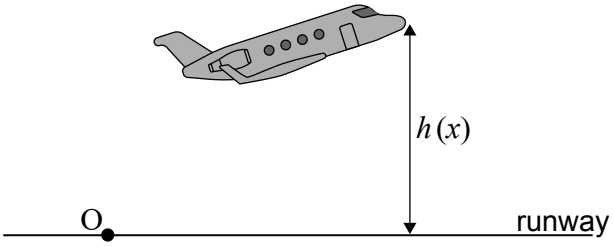
Question 13 - working space

4. [Maximum mark: 12]

A plane takes off from a horizontal runway. Let point O be the point where the plane begins to leave the runway and x be the horizontal distance, in km, of the plane from O. The function h models the vertical height, in km, of the nose of the plane from the horizontal runway, and is defined by

$$h(x) = \frac{10}{1+150\mathrm{e}^{-0.07x}} - 0.06,\ x \geq 0.$$

diagram not to scale



(a) (i) Find $h(0)$.

(ii) Interpret this value in terms of the context. [2]

(b) (i) Find the horizontal asymptote of the graph of $y = h(x)$.

(ii) Interpret this value in terms of the context. [2]

(c) Find $h'(x)$ in terms of x. [4]

A safety regulation recommends that $h'(x)$ never exceed 0.2.

(d) Given that this plane flies a distance of at least $200\,\mathrm{km}$ horizontally from point O, determine whether the plane is following this safety regulation. [4]

Turn over

Question 4 - working space

1. [Maximum mark: 28]

This question uses differential equations to model the maximum velocity of a skydiver in free fall.

In 2012, Felix Baumgartner jumped from a height of $40\,000\,\text{m}$. He was attempting to travel at the speed of sound, $330\,\text{m s}^{-1}$, whilst free-falling to the Earth.

Before making his attempt, Felix used mathematical models to check how realistic his attempt would be. The simplest model he used suggests that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g$$

where $v\,\text{m s}^{-1}$ is Felix's velocity and $g\,\text{m s}^{-2}$ is the acceleration due to gravity. The time since he began to free-fall is t seconds and the displacement from his initial position is s metres.

Throughout this question, the direction towards the centre of the Earth is taken to be positive and v is a positive quantity.

When $s = 0$, it is given that Felix jumps with an initial velocity $v = 10$.

(a) (i) Use the chain rule to show that $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$. [1]

(ii) Assuming that g is a constant, solve the differential equation $v\frac{\mathrm{d}v}{\mathrm{d}s} = g$ to find v as a function of s. [4]

(iii) Using $g = 9.8$, determine whether the model predicts that Felix will succeed in travelling at the speed of sound at some point before $s = 40\,000$. Justify your answer. [3]

(b) To test the model

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\,,$$

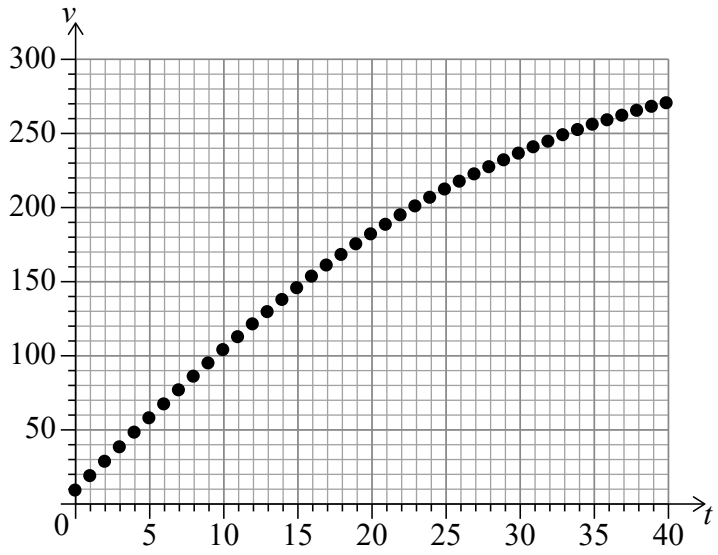
Felix conducted a trial jump from a lower height, and data for v against t was found.

(i) If the model is correct, describe the shape of the graph of v against t. [2]

(This question continues on the following page)

(Question 1 continued)

Felix's data are plotted on the following graph.



(ii) Use the plot to comment on the validity of the model in part (a). [1]

(c) An improved model considers air resistance, using

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2$$

where k is a positive constant. You are reminded that initially $s = 0$ and $v = 10$.

(i) By using $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$, solve the differential equation to find v in terms of s, g and k. You may assume that $g - kv^2 > 0$. [5]

Felix uses the graph of v against t shown in part (b) to estimate the value of k.

(ii) The gradient is estimated to be 9.672 when $v = 40$. Taking g to be 9.8, use this information to show that Felix found that $k = 8 \times 10^{-5}$. [2]

(iii) Hence, find the value of v predicted by this model, as s tends to infinity. [2]

(iv) Find the upper bound for the velocity according to this model, given that $0 < s \leq 40\,000$. Give your answer to four significant figures. [2]

(This question continues on the following page)

Turn over

(Question 1 continued)

The assumption that the value of g is constant is not correct. It can be shown that

$$g = \frac{3.98\times10^{14}}{\left(6.41\times10^{6} - s\right)^{2}}.$$

Hence, the new model is given by

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{3.98\times10^{14}}{\left(6.41\times10^{6} - s\right)^{2}} - \left(8\times10^{-5}\right)v^{2}.$$

When $s = 0$, it is known that $v = 10$.

(d) Use Euler's method with a step length of 4000 to estimate the value of v when $s = 40\,000$. [4]

(e) After Felix completed his record-breaking jump, he found that the answer from part (d) was not supported by data collected during the jump.

(i) Suggest **one** improvement to the use of Euler's method which might increase the accuracy of the prediction of the model. [1]

(ii) Suggest **one** factor **not** explicitly considered by the model in part (d) which might lead to a difference between the model's prediction and the data collected. [1]

Question 1 - working space

4. [Maximum mark: 4]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \log_{10}(x+y)$, where $x \geq 0$ and $y > 0$.

Given that $y = 1$ when $x = 0$, use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 2$.

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Question 4 - working space

9. [Maximum mark: 7]

(a) Find $\int \frac{8}{2x+3}\mathrm{d}x$. [3]

(b) Hence find the exact area between the curve $y = \frac{8}{2x+3}$, the x-axis and the lines $x = 0$ and $x = 6$. Give your answer in the form $a \ln b$, where $a, b \in \mathbb{N}$. [4]

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Question 9 - working space

15. [Maximum mark: 6]

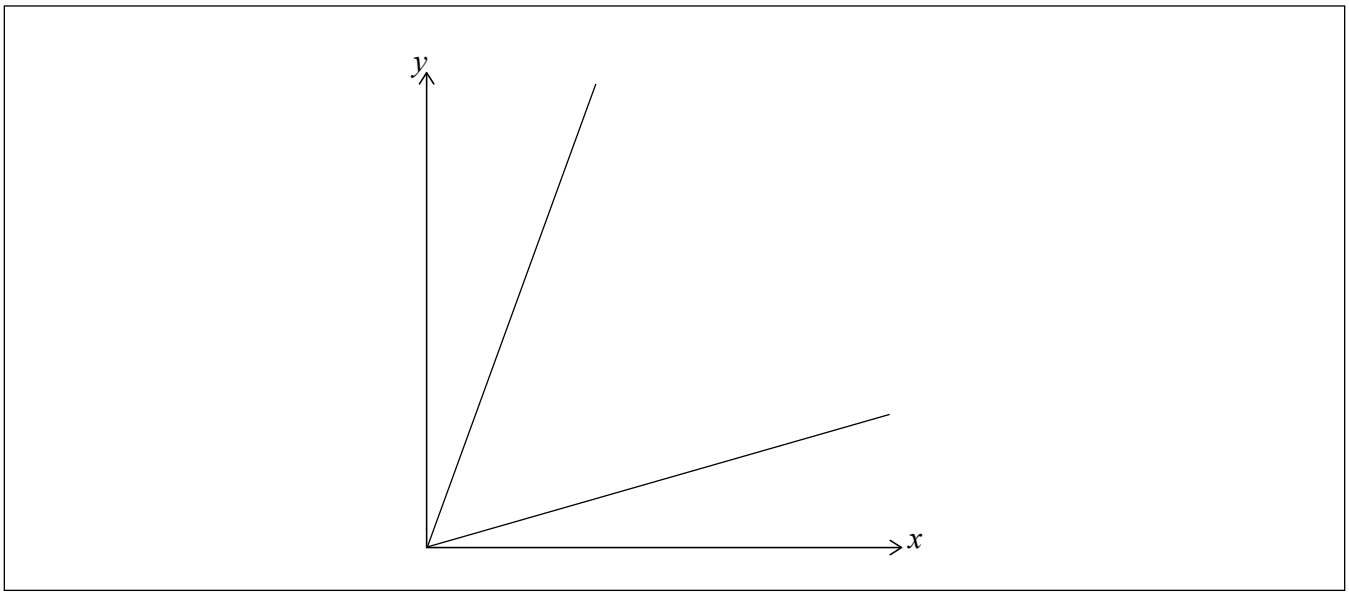
A system of differential equations of the form $\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by$, $\frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$ has eigenvalues $\lambda = -1$ and $\lambda = 2$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

The following incomplete phase portrait for this system, with $x, y \geq 0$, shows lines through $(0, 0)$ parallel to the eigenvectors.

(a) On the phase portrait

(i) show the direction of motion along the eigenvectors.

(ii) sketch one trajectory in each of the three regions. [3]



In the system described above, x and y are the population sizes of two species, X and Y. The population of Y is vulnerable, so it will be increased by adding more animals from a different area. Currently, $x = 252$ and $y = 60$.

(b) Find the minimum number of new animals from species Y that need to be added for the population not to reduce to 0 over time. [3]

(This question continues on the following page)

(Question 15 continued)

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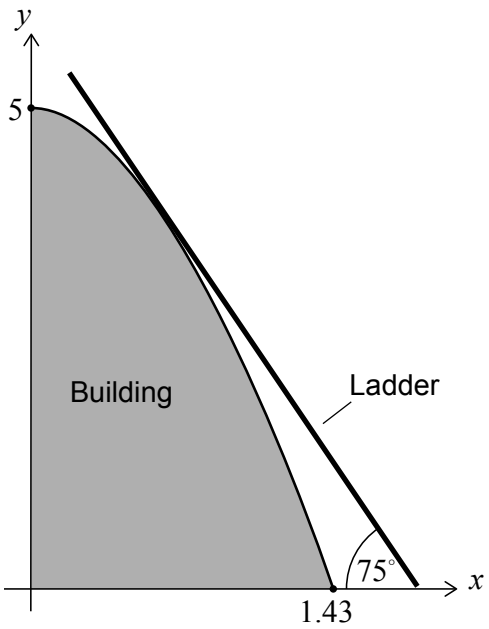
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Question 15 - working space

16. [Maximum mark: 8]

The cross section of the side of a building can be modelled by a curve with equation $y = 5\cos(1.1x), 0 \le x \le 1.43$, as shown in the following diagram. Distances are measured in metres.

diagram not to scale



A builder leans a straight ladder against the building to do repairs. For safety reasons, the angle between the ladder and the horizontal ground must be $75°$.

Find the height above the ground at which the ladder touches the building.

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Turn over

Question 16 - working space

7. [Maximum mark: 4]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy - 1$, given that $y = 2$ when $x = 1$.

Use Euler's method with step size 0.1 to find the approximate value of y when $x = 1.5$.

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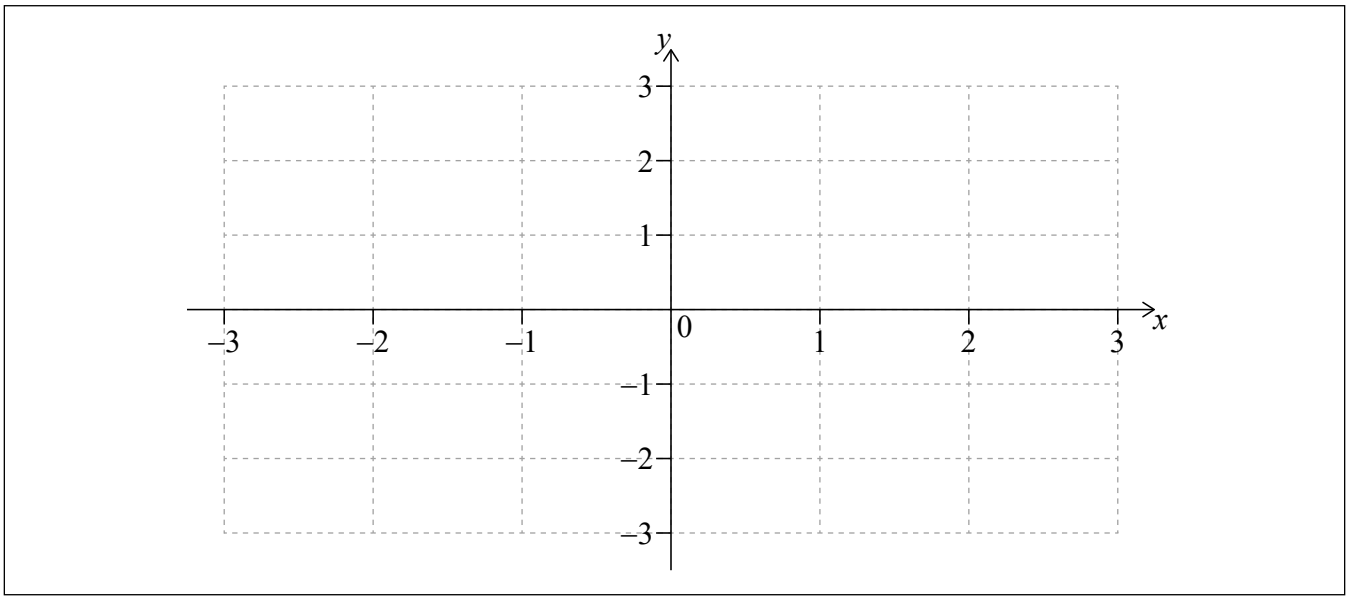
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Question 7 - working space

10. [Maximum mark: 9]

Consider the function $f(x) = x\sqrt{3 - x^2}$, $-\sqrt{3} \leq x \leq \sqrt{3}$.

(a) Sketch the graph of $y = f(x)$ on the following pair of axes. [2]



The area between the graph of $y = f(x)$ and the x-axis is rotated through 360° about the x-axis.

(b) (i) Write down an integral that represents this volume.

(ii) Calculate the value of this integral. [4]

The graph of the function f is transformed, to give the graph of the function g, in the following way:

- It is first stretched by scale factor 2, parallel to the x-axis with the y-axis invariant.
- It is then stretched by scale factor 0.5, parallel to the y-axis with the x-axis invariant.

(c) Find the volume obtained when the area between the graph of $y = g(x)$ and the x-axis is rotated through 360° about the x-axis. [3]

(This question continues on the following page)

(Question 10 continued)

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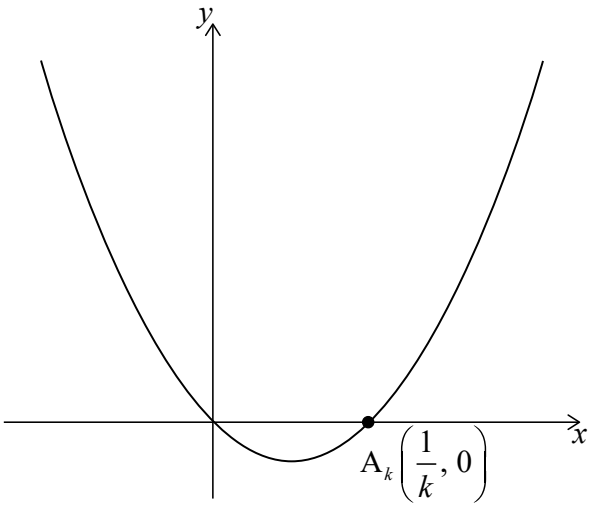
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Question 10 - working space

11. [Maximum mark: 9]

The diagram shows the curve with equation $y_k = kx^2 - x,\ k > 0$, which intersects the x-axis at the origin and at the point $\mathrm{A}_k\left(\frac{1}{k}, 0\right)$.



The normal to the curve at A_k intersects the curve again at point B_k.

(a) Show that the x-coordinate of B_k is $-\frac{1}{k}$. [6]

Consider the case where $k = 2$.

(b) Calculate the finite area of the region between the curve with equation $y_2 = 2x^2 - x$ and the normal at A_2. [3]

(This question continues on the following page)

(Question 11 continued)

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Question 11 - working space

13. [Maximum mark: 8]

A particle starts from rest at point O and moves in a straight line with velocity, v, given by

$$v = 3\sin(t)\left(1+\cos(t)\right),\ t \geq 0$$

where v is measured in metres per second and time, t (radians), is measured in seconds.

The particle next comes to instantaneous rest when $t = a$.

(a) Determine the value of a. [2]

(b) Find the maximum velocity of the particle during the interval $0 \leq t \leq a$. [2]

(c) By finding the total distance travelled between $t = 0$ and $t = a$, find the average speed of the particle during the interval $0 \leq t \leq a$. [4]

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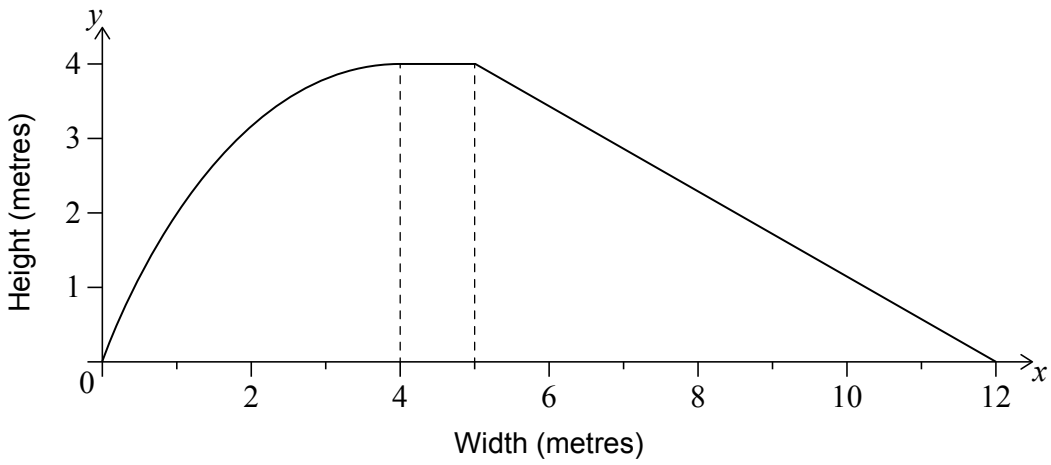
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Question 13 - working space

2. [Maximum mark: 15]

The following diagram shows a model of the side view of a water slide. All lengths are measured in metres.



The curved edge of the slide is modelled by

$$f(x) = -\frac{1}{4}x^2 + 2x \text{ for } 0 \le x \le 4.$$

The remainder of the slide is modelled by

$$g(x) = \begin{cases} 4, \text{ for } 4 \le x \le 5 \\ \frac{48}{7} - \frac{4x}{7}, \text{ for } 5 \le x \le 12 \end{cases}$$

(a) Use the trapezoidal rule with an interval width of 1 to calculate the approximate area under the model of the slide in the interval $0 \le x \le 4$. [5]

(b) Find $\int\left(-\frac{1}{4}x^2 + 2x\right)\mathrm{d}x$. [3]

(c) Calculate the exact area under the entire model of the slide, for $0 \le x \le 12$. [4]

(d) Find the percentage error in the **total** area under the entire model of the slide when using the approximate value from part (a). [3]

Turn over

Question 2 - working space

2. [Maximum mark: 31]

The following question explores a possible method of drawing phase portraits for non-linear coupled systems, taking a predator-prey model as a particular example.

Gander Green wildlife park contains a population of Czech geese (x, measured in hundreds), and a population of gray foxes (y, measured in hundreds).

Research indicates that the population growth of both geese and foxes can be modelled by the following differential equations, in which t is measured in years.

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2x - \frac{xy}{2} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -3y + xy\end{aligned}\right\} \text{ for } x,\ y \geq 0$$

(a) At a specific time, there are 500 geese and 500 foxes, represented here by the coordinate pair $(5, 5)$. At this time, determine the rate of change of

(i) geese. [2]

(ii) foxes. [1]

There are two equilibrium points for the populations: $\mathrm{A}(0, 0)$ and $\mathrm{B}(p, q)$.

(b) (i) Explain why A is an equilibrium point. [1]

(ii) Find the value of p and the value of q. [3]

At points close to $\mathrm{A}(0, 0)$, we can ignore the xy terms, so that the system can be approximated by:

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2x \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -3y\end{aligned}\right\} \text{ for } x,\ y \geq 0.$$

(c) By solving these two differential equations,

(i) find an expression for x in terms of t. [4]

(ii) find an expression for y in terms of t. [1]

(d) (i) Using your answers from part (c), show that phase portrait trajectories close to A may be given by the equation $x^3y^2 = k$, where k is a positive constant. [3]

(ii) Hence sketch, on a phase portrait, one possible trajectory for small values of x and y. [3]

(This question continues on the following page)

(Question 2 continued)

Now consider points (x, y) close to B on the phase plane. These coordinates can be rewritten as $x = p + X$ and $y = q + Y$, where p and q are the values from part (b)(ii).

(e) By substituting into the original model, show that, for small values of X and Y:

$$\dot{X} \approx -\frac{3Y}{2}.$$ [3]

Similarly, it can be shown that $\dot{Y} \approx 4X$.

(f) Given that $\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} X \\ Y \end{pmatrix}$, where $\boldsymbol{M}$ is a square matrix, write down $\boldsymbol{M}$. [1]

(g) By finding the eigenvalues of $\boldsymbol{M}$, describe the path of the trajectories close to point B. [4]

(h) Hence sketch a complete set of trajectories in the phase plane for the original model, clearly indicating both equilibrium points. [3]

(i) In this wildlife park, at a specific time, there are 500 Czech geese and 500 gray foxes.

Based on the values found in part (a), the park's wildlife keeper is worried and assumes that the geese will quickly die out. Suggest whether this assumption is supported by the model. Justify your answer. [2]

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References:

1. Panimoni, 2022. *Volume Target icon in flat style on color background. Darts game. Arrow in the center aim. Vector design element for you business projects*. [image online] Available at: https://www.gettyimages.co.uk/detail/illustration/volume-target-icon-in-flat-style-on-color-royalty-free-illustration/1044319572 [Accessed 21 February 2022]. Source adapted.

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Question 2 - working space

14. [Maximum mark: 8]

While playing on the beach, Sabine builds a mound of sand. The shape of the cross section through the centre of this mound can be modelled by the function

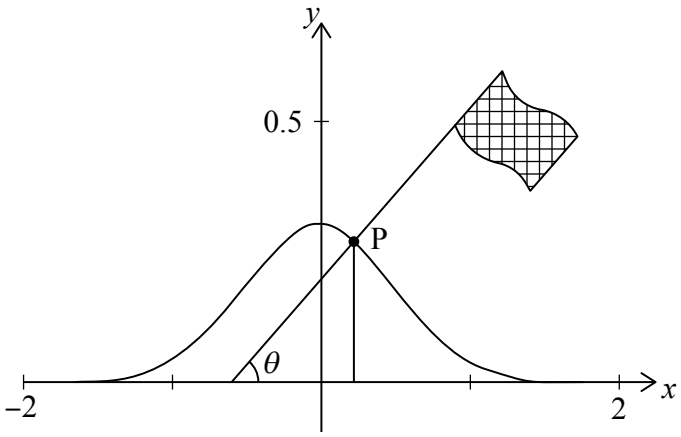
$$f(x) = 0.3e^{-2x^2},$$

where $-2 < x < 2$.

Sabine places a flagpole into this mound at a point, P, which is $0.25\,\mathrm{m}$ above the level beach. The pole is perpendicular to the mound of sand at point P.

This information is shown in the diagram, where the x-axis represents the level beach and the y-axis represents the height above the beach. The scale on the axes is 1 unit $= 1$ metre.

diagram not to scale



The coordinates of P are $(a, 0.25)$, where $a > 0$.

(a) Calculate the value of a. [2]

(b) Find an expression for $f'(x)$. [2]

(c) Hence, or otherwise, find the angle, θ, the flagpole makes with the level beach. [4]

(This question continues on the following page)

(Question 14 continued)

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Question 14 - working space

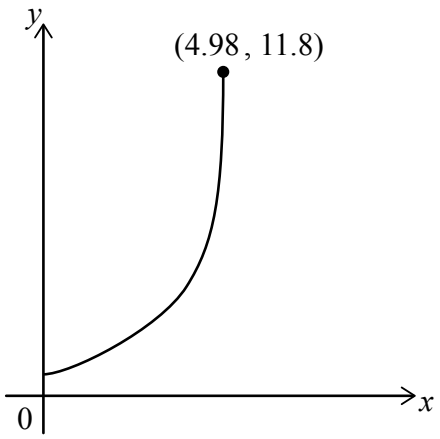
16. [Maximum mark: 7]

As part of his mathematics exploration, Jules models the shape of part of a wine glass to find the capacity (volume) of the glass.

He finds that the edge of half the glass can be modelled by the function

$$f(x) = 4 - 2\ln(5 - x), \text{ where } 0 \leq x \leq 4.98.$$

A graph of $y = f(x)$ is shown, with a scale of 1 unit $= 1\,\text{cm}$.



(a) Find the y-intercept. [1]

The point $(4.98, 11.8)$ represents a point at the top of the glass.

Let R be the region enclosed by the graph of f, the y-axis and the line $y = 11.8$.

Jules finds the capacity by rotating the region, R, $360°$ about the y-axis.

(b) Calculate the capacity of the glass that Jules obtains. [6]

(This question continues on the following page)

(Question 16 continued)

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Question 16 - working space

17. [Maximum mark: 7]

The fish in a lake feed on insects. Let F be the population of fish and L be the population of insects at time t (weeks).

The populations can be modelled using the coupled differential equations

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 0.000004FL - 0.2F$$
$$\frac{\mathrm{d}L}{\mathrm{d}t} = 0.06L - 0.00003FL$$

where $F > 0$ and $L > 0$.

When $t = 0$, it is estimated that $F = 5000$ and $L = 80\,000$.

(a) For $0 \leq t \leq 52$, use Euler's method, with a step length $h = 4$, to find an approximate value for

(i) the maximum value of F

(ii) the minimum value of L. [5]

(b) (i) State what will happen to the number of fish in the lake, as predicted by the model.

(ii) Suggest a reason why this prediction may not occur. [2]

(This question continues on the following page)

(Question 17 continued)

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Figure 1 animatedfunk, n.d. *Beaver Illustrations – stock illustration* [image online] Available at: https://www.gettyimages.co.uk/detail/illustration/beaver-illustrations-royalty-free-illustration/1453231718 [Accessed 30 November 2023]. Source adapted.

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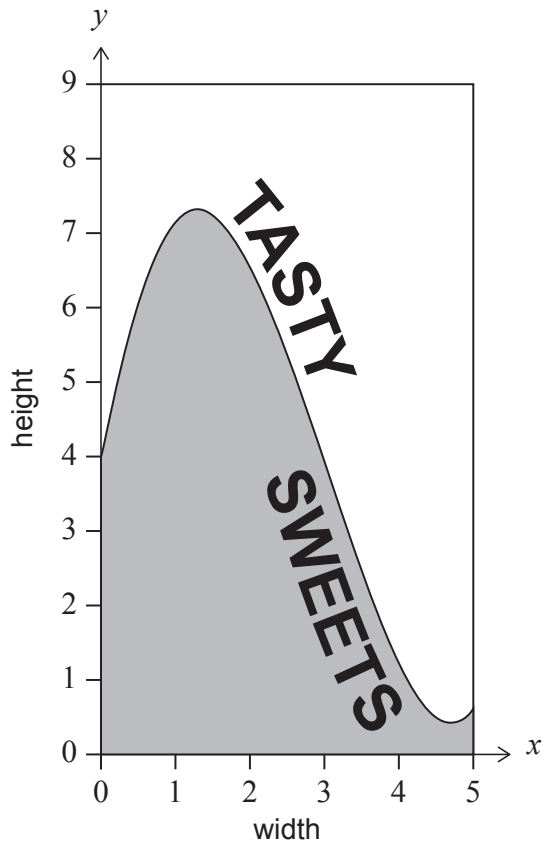
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Question 17 - working space

2. [Maximum mark: 18]

Sweets are sold in cylindrical containers. A new label for the container is being considered. The label will be a rectangle that is $5\,\text{cm}$ wide and $9\,\text{cm}$ high.

The design on the label is a curve, as shown on the following axes, where one unit represents $1\,\text{cm}$ for both axes.



The values in the table approximate points on the curve, correct to one decimal place.

Width, x	0	1	2	3	4	5
Height, y	4	7.3	6.7	4.0	1.3	0.7

(a) Use the trapezoidal rule with five intervals, and the values given in the table, to estimate the shaded area below the curve. [3]

The curve used in the label design can be modelled by:

$$y = \frac{x^3}{3} - 3x^2 + 6x + 4 \text{, for } 0 \le x \le 5 .$$

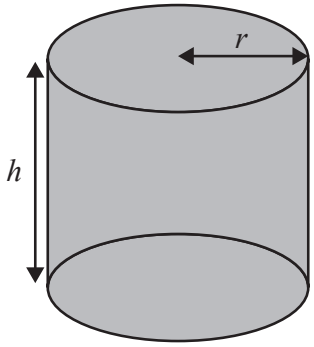
(b) Use this equation to find the area of the shaded region. [2]

(This question continues on the following page)

(Question 2 continued)

The sweets are sold in closed cylindrical containers, with radius r and height h.

diagram not to scale



The whole container is made from one type of material, and it is assumed that the thickness of the material is negligible.

Each container has a volume of $600\,\text{cm}^3$.

(c) Write down an equation, in terms of r and h, that shows this information. [1]

The amount of material used for each container can be modelled by the external surface area of the container.

The external surface area, A, of the container can be expressed as

$$A = 2\pi r^2 + \frac{k}{r}, \text{ where } r > 0.$$

(d) Find the value of k. [4]

(e) (i) Find $\frac{\mathrm{d}A}{\mathrm{d}r}$.

(ii) Given that A has a minimum value, find the value of r that will minimize the material used. [5]

The containers are made so that the surface area is minimized. The $5\,\text{cm}$ by $9\,\text{cm}$ rectangular label is to be glued to the curved surface of the container.

(f) Show that the label will fit on the container. [3]

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Question 2 - working space

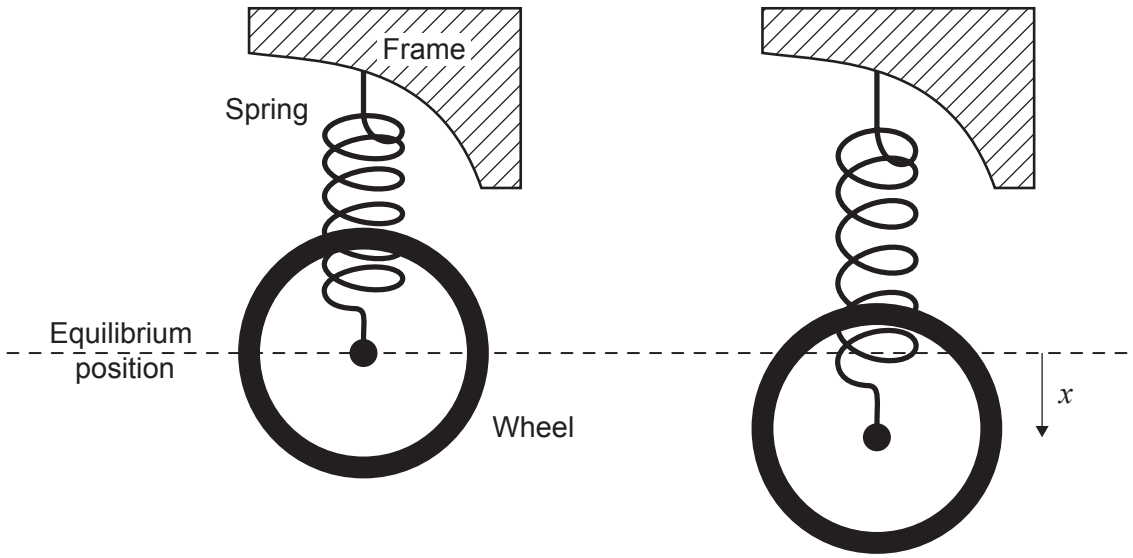
7. [Maximum mark: 18]

The wheel on a motorbike is attached to the frame by a spring. The movement of the spring acts as a **shock absorber**. When the rider sits on the motorbike, the spring compresses, and this position is called the equilibrium position.

When the wheel goes into holes or over bumps in the road, the spring will extend or compress to ensure a smooth ride.

Let x denote the vertical displacement, in centimetres, of the centre of the wheel below the equilibrium position, as shown in the diagram.

diagram not to scale



The vertical displacement of the centre of the wheel, at time t seconds, can be modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = 0 \text{, where } a, b \in \mathbb{R}.$$

Let $y = \frac{\mathrm{d}x}{\mathrm{d}t}$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay$. [2]

(This question continues on the following page)

(Question 7 continued)

The equations $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay$ can be written in matrix form as

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}.$$

A manufacturer wants to compare two springs, Spring 1 and Spring 2, that could be used as shock absorbers.

The differential equation for Spring 1 has $a = 18$ and $b = 77$.

For these values, the eigenvalues of $\boldsymbol{M}$ are -7 and -11.

(b) Find the corresponding eigenvectors. [3]

The manufacturer models **both** springs using the same initial conditions:

$$t = 0,\ x = 5\,\text{cm} \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = 2\,\text{cm}\,\text{s}^{-1}.$$

(c) Hence, for Spring 1

(i) find the exact solution for $x(t)$

(ii) sketch the graph of $x(t)$, for $0 \leq t \leq 1$. [7]

The differential equation for Spring 2 has $a = 18$ and $b = 85$.

For these values, the eigenvalues of $\boldsymbol{M}$ are $-9 \pm 2\mathrm{i}$.

(d) (i) Sketch the phase portrait for Spring 2, indicating the direction of the trajectory.

(ii) Hence, sketch the graph of x against t. [5]

(e) Using your answers to parts (c)(ii) and (d)(ii), give a reason why Spring 1 might make a better shock absorber than Spring 2. [1]

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References:

1. InterNations. 2022. *Expat Insider 2022*. [online] Available at: https://www.internations.org/expat-insider/2022/. Source adapted.

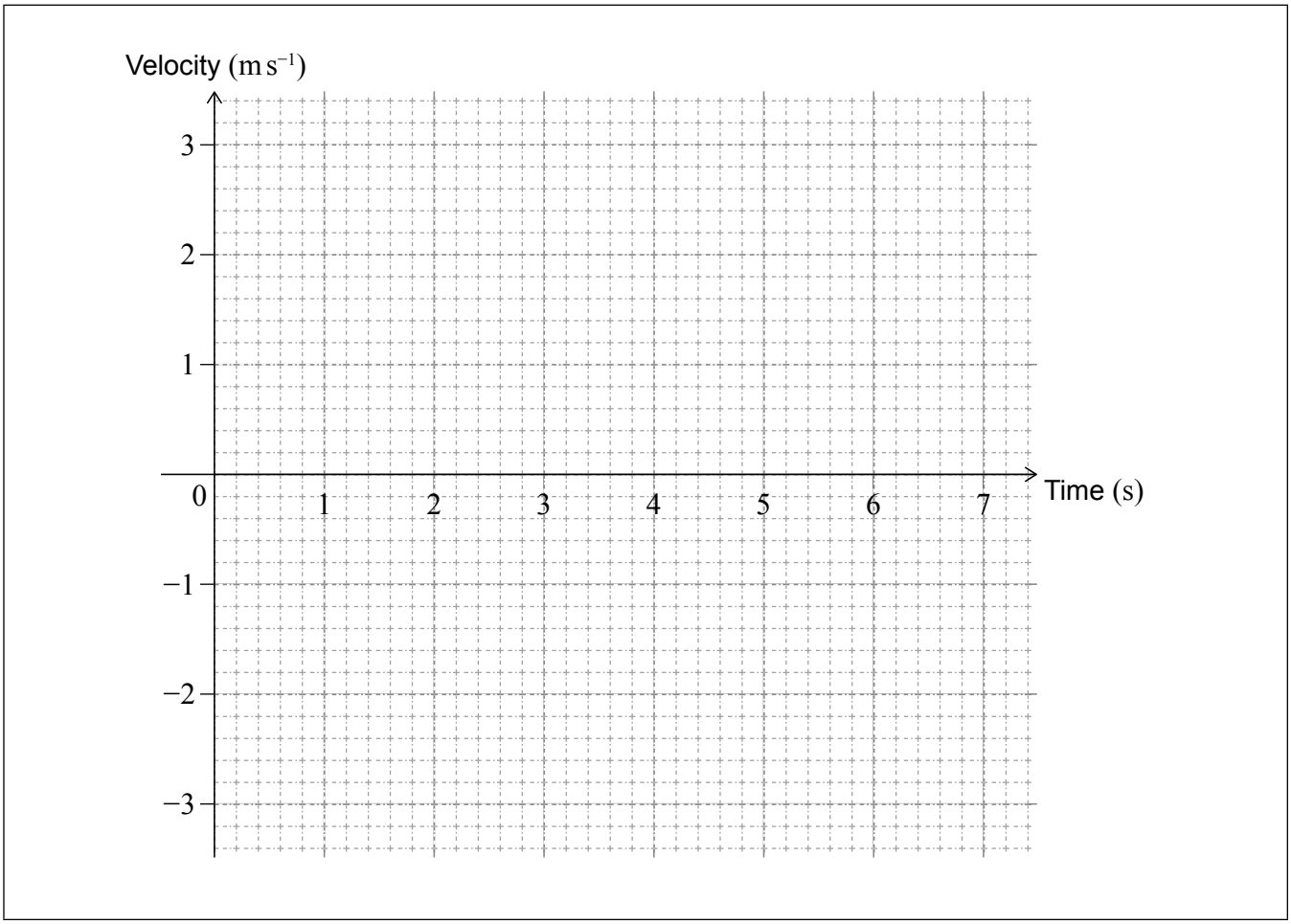
Heliwell, J. F., Huang, H., Wang, S. and Norton, M., 2022. Statistical Appendix for “Happiness, benevolence, and trust during COVID-19 and beyond,” Chapter 2 of *World Happiness Report 2022*. [pdf online] Available at: https://worldhappiness.report/ed/2022/happiness-benevolence-and-trust-during-covid-19-and-beyond/ [Accessed 13 November 2023]. Source adapted.

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Question 7 - working space

7. [Maximum mark: 8]

Conrad is investigating the motion of a particle. The velocity of the particle, in m s^{-1}, is given by $v(t) = 2\cos t + \sin 2t - 0.2$ where t is the time, in seconds, after the investigation begins.



(a) On the axes,

(i) sketch the graph of $v(t)$ for $0 \leq t \leq 6$.

(ii) label, with coordinates, the point(s) where the particle has zero acceleration. [5]

(b) (i) Write down an integral expression for the distance the particle travels during the first 6 seconds of the investigation.

(ii) Hence, find the distance the particle travels during the first 6 seconds of the investigation. [3]

(This question continues on the following page)

(Question 7 continued)

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Question 7 - working space

8. [Maximum mark: 7]

Consider the function $f(x) = x^3 - 9x^2 + 23x - 15$ for $0 \le x \le 5$.

The graph of $f(x)$ crosses the x-axis when $x = 1$, $x = 5$ and $x = a$.

(a) Find the value of a. [2]

(b) Use your graphic display calculator to find the area enclosed by the curve $y = f(x)$ and the x-axis, when $1 \le x \le a$. [2]

Another function is defined as $g(x) = -x^2 + 6x - 5$.

The area enclosed by the curve $y = f(x)$ and the x-axis, when $1 \le x \le b$, is equal to the area enclosed by the curve $y = g(x)$ and the x-axis, when $1 \le x \le b$.

(c) Find the value of b, where $1 < b < 3$. [3]

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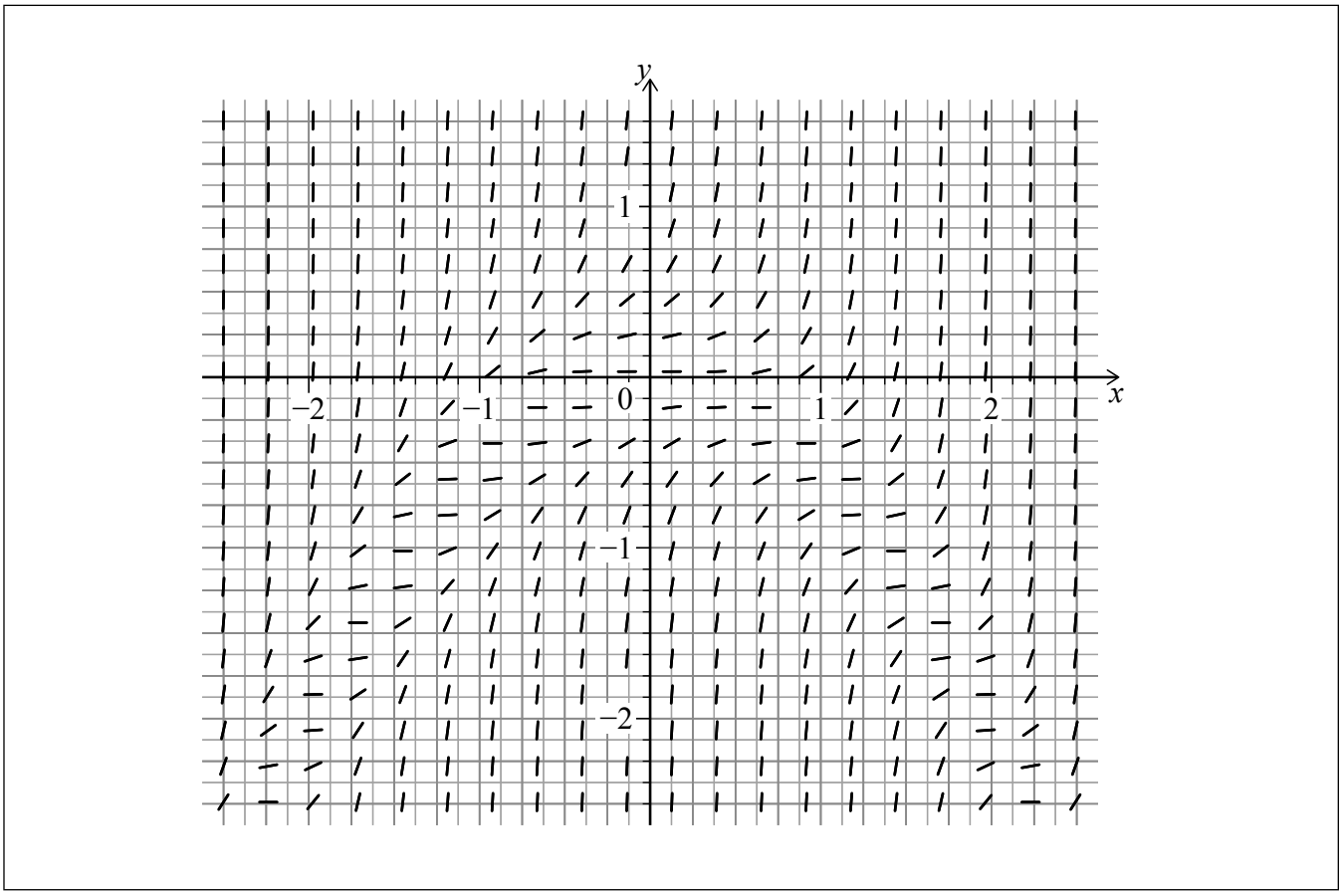
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Question 8 - working space

12. [Maximum mark: 7]

The following graph shows the slope field for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^4 + 4x^2y + 4y^2.$$



(a) On the slope field, sketch the solution to the differential equation that passes through the point $(0, 1)$. [2]

Some of the solutions to the differential equation have a point at which the gradient is zero. The curve on which all these points lie is given by the equation $y = ax^2$, where a is a constant to be found.

(b) Sketch this curve on the slope field. [2]

(c) Use the differential equation to find the value of a. [3]

(This question continues on the following page)

(Question 12 continued)

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Question 12 - working space

15. [Maximum mark: 8]

Quentin is investigating the motion of a mass hanging on a large spring.

From research, Quentin thinks the displacement, x, in metres, of the mass at time t, in seconds, will satisfy the differential equation

$$9.8 = 2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2.4\frac{\mathrm{d}x}{\mathrm{d}t} + 8x\,.$$

Quentin begins his investigation when the mass is at a displacement of 0.5 metres and is moving such that $\frac{\mathrm{d}x}{\mathrm{d}t} = 9.8$.

(a) By using Euler's method with $h = 0.1$ to estimate the motion of the mass after being released, find:

(i) the displacement of the mass after 1 second,

(ii) the first estimated time the mass reaches $x = 0$. [5]

Using technology, Quentin correctly measures the displacement of the mass, after 1 second. His software gives him a value of 385 centimetres, which is accurate to the nearest centimetre.

(b) Find the **maximum possible** percentage error between your estimate from part (a)(i) and the exact displacement after 1 second. [3]

(This question continues on the following page)

(Question 15 continued)

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References:

2. fitie, 2017. *Vector illustration of dolphins* [image online] Available at: https://www.gettyimages.co.uk/detail/illustration/vector-illustration-of-dolphins-royalty-free-illustration/638431854 [Accessed 19 April 2024] Source adapted.

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Question 15 - working space

7. [Maximum mark: 7]

(a) Find the indefinite integral $\int xe^{-x^2}\,dx$. [4]

(b) Hence find the area bounded by the x-axis, the curve $y = xe^{-x^2}$ and the line $x = k$. Give your answer in terms of k. [3]

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Question 7 - working space

9. [Maximum mark: 7]

A climate scientist is modelling an ice sheet as a rectangle.

She believes that the width $(x\,\text{km})$ is increasing at a constant rate of $10\,\text{km}$ per year and the length $(y\,\text{km})$ is decreasing at a constant rate of $5\,\text{km}$ per year.

The time, t, is measured in years, and the area, A, is measured in km^2.

When $t = 0$ then $x = 75$ and $y = 40$.

(a) Find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $t = 0$. [3]

(b) State, with justification, whether the area of the ice sheet is increasing or decreasing when $t = 0$. [1]

(c) Find the change in the area of the ice sheet between $t = 0$ and $t = 1$. [3]

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Question 9 - working space

12. [Maximum mark: 9]

An engineer's model for an object's motion is that its acceleration, $\frac{\mathrm{d}v}{\mathrm{d}t}$, is proportional to $v^{1.5}$, where v is its velocity measured in $\mathrm{m\,s^{-1}}$.

(a) Write down a differential equation based on the engineer's belief. [1]

The initial velocity of the object is $4\,\mathrm{m\,s^{-1}}$ and its initial acceleration is $-3\,\mathrm{m\,s^{-2}}$.

(b) Use the engineer's model to find an expression for the velocity of the object after t seconds. [8]

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Question 12 - working space

15. [Maximum mark: 8]

An astronomer models the shape of a parabolic mirror using the equation $y = x^2$.

(a) Find the equation of the normal to the mirror at the point $(2, 4)$. [3]

A ray of light comes from an object at coordinates $(0, 10)$ and hits the mirror at the point $(2, 4)$.

(b) Find the gradient of the ray of light. [2]

(c) Find the angle between the ray of light and the normal to the mirror. [3]

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Question 15 - working space

9. [Maximum mark: 6]

Consider the curve $y = 4x\sqrt{3x-2}$ for $x \geq \frac{2}{3}$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [3]

(b) Hence show that the equation of the tangent to the curve at the point $(6, 96)$ is $y = 25x - 54$. [3]

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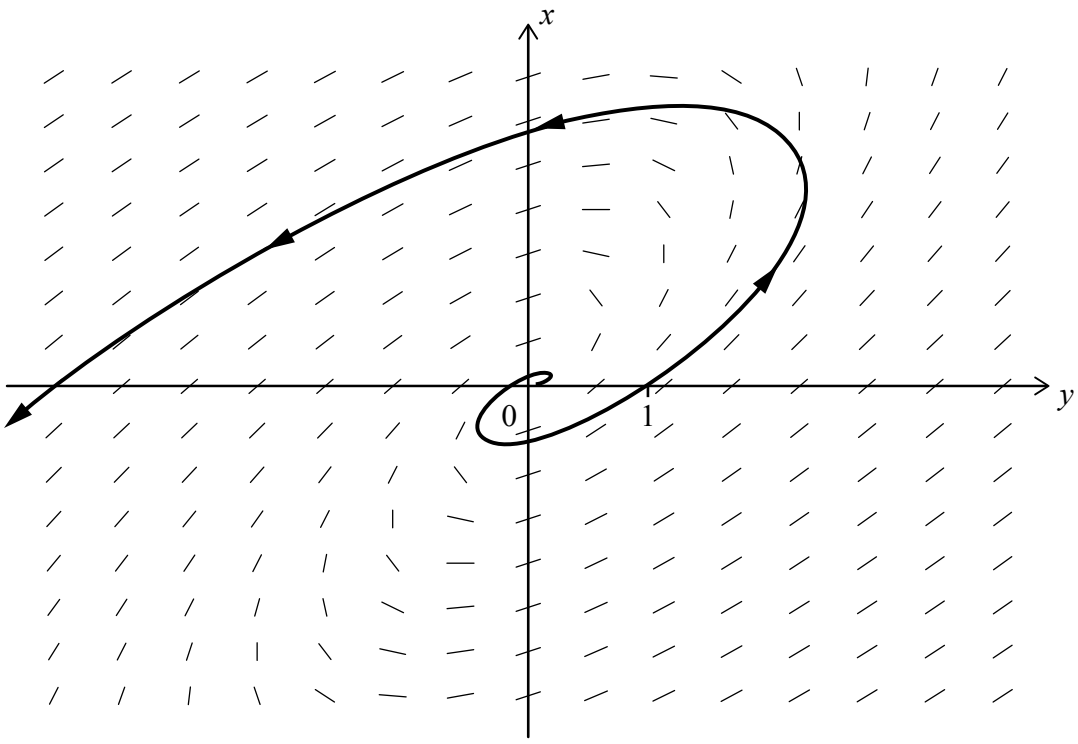
Question 9 - working space

11. [Maximum mark: 6]

Consider the system given by the coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - y$$

The following phase portrait shows the trajectory that passes through the point $(1, 0)$ at time $t = 5$.



(a) Write down which one of the following could be an eigenvalue for the system. [1]

A. 1

B. $2\mathrm{i}$

C. $1 + 2\mathrm{i}$

D. $-1 + 2\mathrm{i}$

(b) Use Euler's method with a step length of 0.1 to find the approximate values of x and y when $t = 5.5$. [5]

(This question continues on the following page)

(Question 11 continued)

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Question 11 - working space

13. [Maximum mark: 9]

The growth of a fungus on a dish can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.2(1-P),\ 0 \leq P < 1$$

where P is the proportion of the dish that is covered with fungus and t is the time in days from when the fungus began to form.

(a) Assuming that $P = 0$ when $t = 0$, show that $P = 1 - \mathrm{e}^{-0.2t}$. [7]

(b) Find the time at which the fungus covers 90% of the dish. [2]

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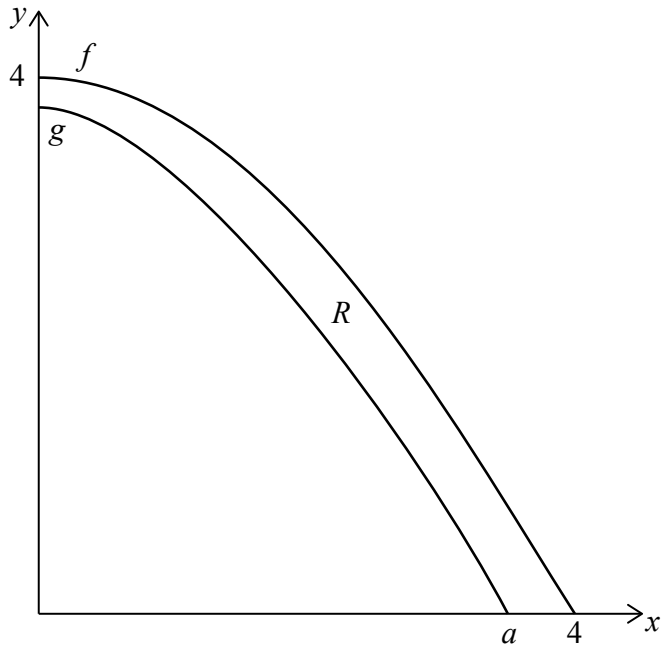
Question 13 - working space

16. [Maximum mark: 8]

The graph of f is defined by $f(x) = 4\cos\left(\frac{\pi}{8}x\right)$, $0 \le x \le 4$.

The graph of $y = g(x)$ is formed by translating the graph of f 0.3 units to the left and 0.2 units down. The graph of g intersects the x-axis at $(a, 0)$ and its domain is then restricted to $0 \le x \le a$.

These curves are shown on the following graph where units are in centimetres.



(a) Write down an expression for $g(x)$. [2]

(b) Find the value of a. [1]

A wall-mounted light shade can be modelled by the solid formed when the region between the two curves, R, is rotated through π radians about the x-axis.

(c) (i) Write down an expression for the volume of the solid formed.

(ii) Hence find the volume of material used in the light shade. [5]

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(Question 16 continued)

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Question 16 - working space

3. [Maximum mark: 17]

Ethan and Avery are researching population data for the city of Los Angeles to create a model predicting population values. They collect the following data.

Population data for the city of Los Angeles

Year	Population (thousands)
1900	102
1920	577
1940	1504
1960	2479
1980	2967
2000	3685
2020	3899

Ethan proposes the population can be modelled using quadratic regression to find a function of the form $f(x) = ax^2 + bx + c$, where x is the number of years after 1900.

(a) Find the equation of Ethan's model. [3]

Ethan finds the coefficient of determination for his model is 0.98843 to five significant figures.

(b) State whether the coefficient of determination supports Ethan's proposal. Justify your answer. [2]

(c) Comment on the validity of Ethan's model with reference to one of the parameters in the equation. [1]

(d) (i) Find the value of $f'(110)$ and interpret this value in context.

(ii) By considering the population changes in the table, use the value found in part (d)(i) to comment on the validity of Ethan's model. [4]

(This question continues on the following page)

(Question 3 continued)

Avery proposes that the population instead follows a logistic model of the form

$$g(x) = \frac{4000}{1 + 14\ \mathrm{e}^{-0.05x}}$$

where x is the number of years after 1900.

(e) State a reason why it may be valid to use Avery's proposal to predict the future population. [1]

(f) (i) Find $g'(x)$.

(ii) Hence find the year, according to Avery's model, during which the greatest population growth rate occurred. [6]

Turn over

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Question 3 - working space

6. [Maximum mark: 14]

Rea is investigating how the number of healthy skin cells changes following infection by a virus. At the beginning of the investigation, she inserts $27\,000$ virus particles into a sample of 1350 healthy skin cells.

Let h be the number of healthy skin cells at time t, where t is the number of hours after the investigation has begun. Let v be the number of virus particles, **in thousands**, at time t.

Rea finds that the rate of change, $\frac{\mathrm{d}h}{\mathrm{d}t}$, is proportional to $\frac{h}{v}$.

(a) Given that $\frac{\mathrm{d}h}{\mathrm{d}t} = -5$ at time $t = 0$, show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{h}{10v}$. [3]

Rea models the number of virus particles, in thousands, after t hours as $v = 27 + 0.3t$.

(b) Find an expression for the number of healthy skin cells at time t. [7]

Rea asks her colleague Artem to attempt the same investigation. Artem's models differ from those of Rea.

Artem models the number of virus particles, in thousands, after t hours as $v = 27 + 0.28t$ and finds that $h = 4380(27 + 0.28t)^{-0.36}$.

(c) Using **Artem's** models, predict how many hours it will take for the number of virus particles to be at least 100 times the number of healthy skin cells. [4]

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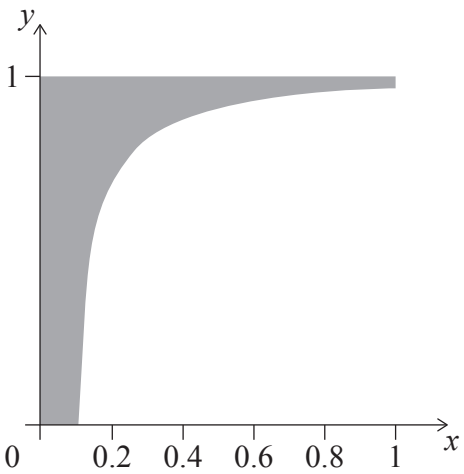
Question 6 - working space

7. [Maximum mark: 12]

Bobby is designing a table that will be 1 metre high. The table will have a flat circular top with a radius of 1 metre. To begin the design, he uses the curve of

$$y = \frac{2}{3}\cos^{-1}\left(\frac{1}{10x}\right), \text{ for } 0.1 \leq x \leq 1,$$

to shape the underside of the table, where x and y are measured in metres, as shown.

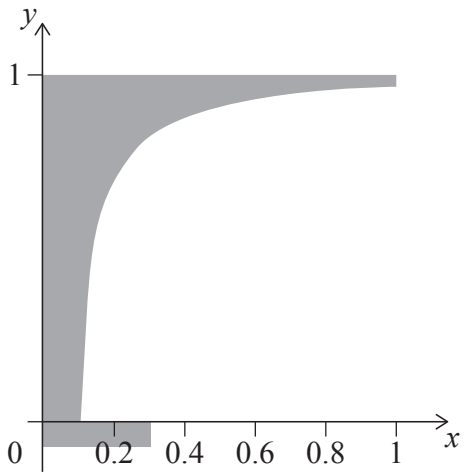


(a) Find the area of the shaded region. [4]

(This question continues on the following page)

(Question 7 continued)

To create the base of the table, Bobby adds a rectangle with a height of 0.075 metres and a width of 0.3 metres to the bottom of the figure, as shown.



To finish the design of the table, he rotates the entire shaded region through 2π about the y-axis.

(b) Find the three expressions necessary to determine the total volume of this table. [6]

(c) Hence find the total volume of the table. [2]

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References:

3. Los Angeles Almanac, n.d. *Historical General Population City & County of Los Angeles, 1850 to 2020* [online] Available at: https://www.laalmanac.com/population/po02.php [Accessed 4 June 2024]. Source adapted.

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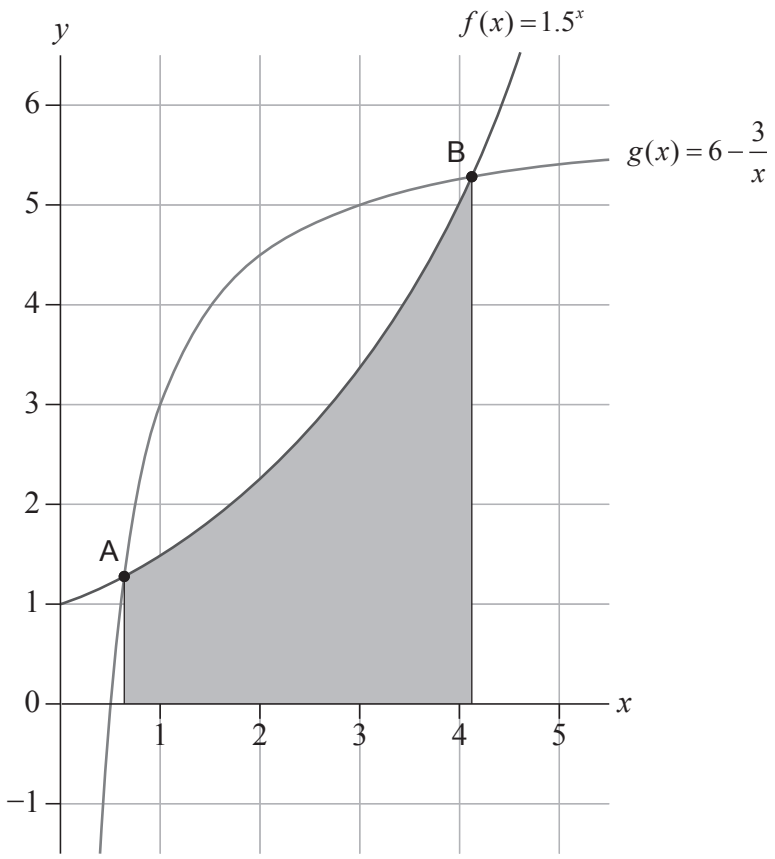
Question 7 - working space

2. [Maximum mark: 12]

The diagram shows part of the graphs of the functions

$$f(x) = 1.5^x \qquad x \geq 0$$

$$g(x) = 6 - \frac{3}{x} \qquad x > 0.$$



(a) Solve $f(x) = g(x)$. [3]

(b) (i) Write down the integral that represents the area of the shaded region.

(ii) Calculate the area of this shaded region.

(iii) Hence, or otherwise, calculate the area of the region enclosed between the curves $y = f(x)$ and $y = g(x)$. [6]

The tangent to the graph of $y = f(x)$ is parallel to the tangent to the graph of $y = g(x)$ at $x = k$.

(c) Find the value of k. [3]

Turn over

Question 2 - working space

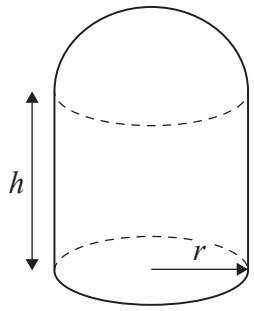
3. [Maximum mark: 16]

Ju Shen designs a plastic cover, in the shape of a cylinder combined with a hemisphere on top, as shown in the diagram.

The plastic used to make the cover forms the curved surface of both the hemisphere and the cylinder; there is no bottom to the cover, however it stands on a flat horizontal surface.

Let the height of the cylinder be $h\,\text{cm}$ and the internal radius of its base be $r\,\text{cm}$.

diagram not to scale



(a) Find an expression for the total internal surface area, $A\,\text{cm}^2$, of the cover in terms of r and h. [2]

(b) Show that the total volume, $V\,\text{cm}^3$, enclosed by the cover and the horizontal surface is given by the expression

$$V = \frac{2\pi r^3 + 3\pi r^2 h}{3}.$$ [2]

The total volume enclosed by the cover is $10\,000\,\text{cm}^3$.

(c) Hence show that $h = \frac{30\,000 - 2\pi r^3}{3\pi r^2}$. [2]

Ju Shen uses the total internal surface area to model the amount of plastic used to construct the cover.

(d) Show that A is given by the expression

$$A = \frac{2\pi r^2}{3} + \frac{20\,000}{r}.$$ [2]

(This question continues on the following page)

(Question 3 continued)

Ju Shen wants to use the minimum amount of plastic in the construction of the cover.

(e) Find an expression for $\frac{dA}{dr}$. [3]

(f) Find the value of r and the value of h that minimizes the use of plastic. [4]

(g) By interpreting your answer to part (f), suggest the best shape for Ju Shen's plastic cover. [1]

Question 3 - working space

6. [Maximum mark: 21]

A financial analyst models the change in value of one share, x dollars at time t minutes, after a report is released. She uses the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0 .$$

This equation can be written as the coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x - 4y .$$

(a) Find the general solution for x. [5]

Initially $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -1$.

(b) (i) Find an expression for x in terms of t.

(ii) Sketch x against t in the interval $0 \leq t \leq 4$. [6]

Once the report has been released, the analyst is going to buy some shares and then sell them later.

(c) (i) Use your graph to find how long after the report is released the analyst should wait to buy the shares in order to maximize her profit.

(ii) Find the upper limit of the profit the analyst can make per share. [4]

An improved model is written as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = x\sin t .$$

The same initial conditions as above apply.

(d) Use Euler's method with a t-interval of 0.1 to predict the value of x when $t = 1$. [6]

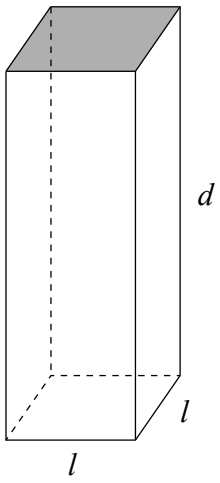
Turn over

Question 6 - working space

1. [Maximum mark: 20]

Kailash manufactures drink containers in the shape of a cuboid. The container has a square top and a square base of length, $l\,\text{cm}$. Its height, $d\,\text{cm}$, is three times the length of the base.

diagram not to scale



(a) Write down an expression for d in terms of l. [1]

The container can hold $375\,\text{cm}^3$ of drink.

(b) Find the value of l and d. [3]

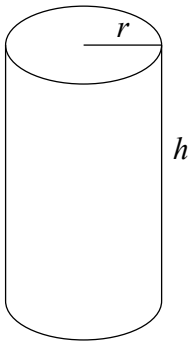
(c) Calculate the total external surface area of the container. [3]

(This question continues on the following page)

(Question 1 continued)

To reduce environmental impact, Kailash is trying to minimize the amount of material needed for the production of the $375\,\text{cm}^3$ container.

He is willing to change the shape to a cylinder with radius $r\,\text{cm}$, and height $h\,\text{cm}$, as shown below.



The cylindrical container of drink must also hold $375\,\text{cm}^3$.

(d) Find an expression for the height, h, of the container in terms of r. [2]

Let the total external surface area be $A\,\text{cm}^2$.

(e) Show that $A = 2\pi r^2 + \frac{750}{r}$. [2]

(f) Find $\frac{\text{d}A}{\text{d}r}$. [3]

(g) Hence or otherwise

(i) find the value of r that will minimize A.

(ii) find the minimum value of A needed for the cylinder. [3]

(h) (i) Find $\frac{\text{d}^2A}{\text{d}r^2}$.

(ii) Hence determine whether the graph of A is concave-up or concave-down for $r > 0$. Justify your answer. [3]

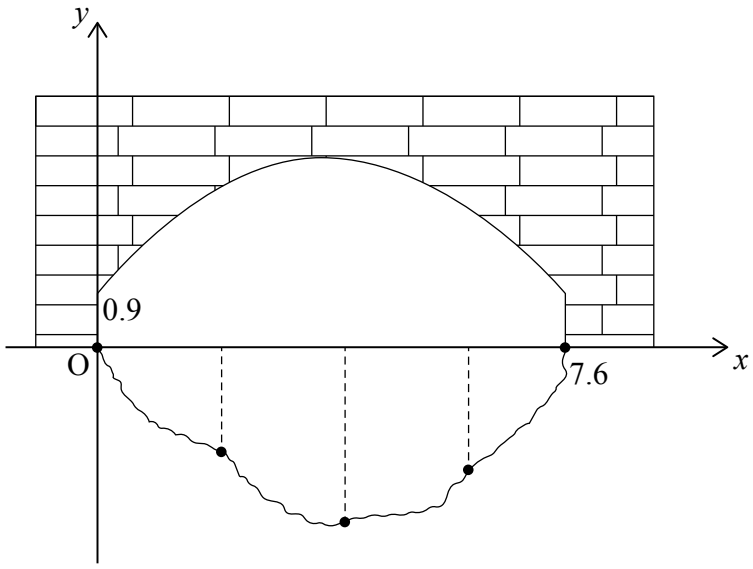
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Question 1 - working space

3. [Maximum mark: 12]

The diagram shows the cross-section of a bridge and a river. A coordinate system has been added with the origin, O, at the point where the bridge meets the water on one side. All units are in metres.

diagram not to scale



A researcher wants to calculate the volume of water that flows under the bridge. To do this he takes measurements of the depth every $1.9\,\mathrm{m}$ from O. The depths are shown in the following table.

Horizontal distance from O in metres	0	1.9	3.8	5.7	7.6
Vertical depth of water in metres	0	1.68	2.81	2.32	0

(a) Use the trapezoidal rule to find the cross-sectional area of the river as it passes under the bridge. [3]

The water flows under the bridge at a rate of $0.3\,\mathrm{m\,s^{-1}}$.

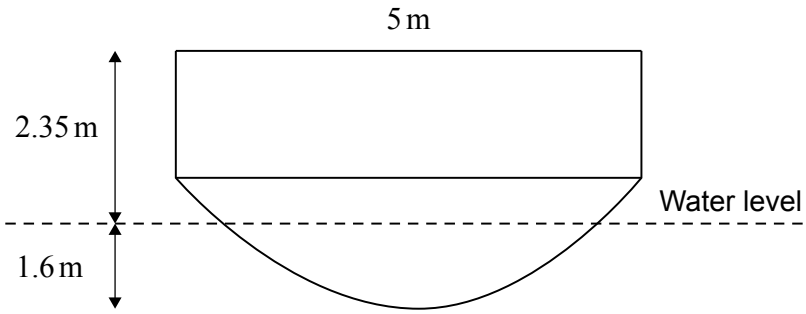
(b) Find the volume of water that passes under the bridge each second. [2]

(This question continues on the following page)

(Question 3 continued)

A boat is travelling along the river. The cross-section of the boat and the water level is shown in the following diagram.

The top of the boat is parallel to the water level and has a width of $5\,\text{m}$. The height of the boat is $2.35\,\text{m}$ above the water level and the lowest part of the boat is $1.6\,\text{m}$ below the water level.



The boat is travelling down the centre of the river.

(c) Find the vertical distance between the lowest part of the boat and the bottom of the river as it passes under the bridge. [1]

The curved arch of the bridge can be modelled by the equation

$$y = -0.15x^2 + 1.14x + 0.9,\ 0 \le x \le 7.6.$$

(d) Find the maximum height of the curved arch above the water level. [2]

(e) Determine whether the top of the boat will be able to pass under the bridge. [4]

Question 3 - working space

7. [Maximum mark: 8]

Consider the function $B(x) = 20 + \frac{32}{x} + 2x$, where $x \geq 1$, $x \in \mathbb{R}$.

(a) (i) Find $B'(x)$.

(ii) Use your answer to part (a)(i) to find the minimum value of $B(x)$. You may assume that $B(x)$ has no local maximum point over the given domain. [6]

Kim goes to a restaurant that offers a large family banquet. The time, T, in minutes, to prepare the family banquet depends on the number of chefs, n.

To predict the value of T, Kim uses the model $T(n) = 20 + \frac{32}{n} + 2n$, where $n \in \mathbb{Z}^+$.

The restaurant informs Kim that the time taken to prepare the family banquet is less than 40 minutes. There are k chefs preparing the family banquet.

(b) Find the possible values of k. [2]

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Question 7 - working space

10. [Maximum mark: 5]

The function f is defined by $f(x) = \frac{6}{2x^2 + 5x + 4}$.

(a) Find $f'(x)$. [2]

(b) Hence or otherwise, find the x-coordinates of the points of inflexion of f. [3]

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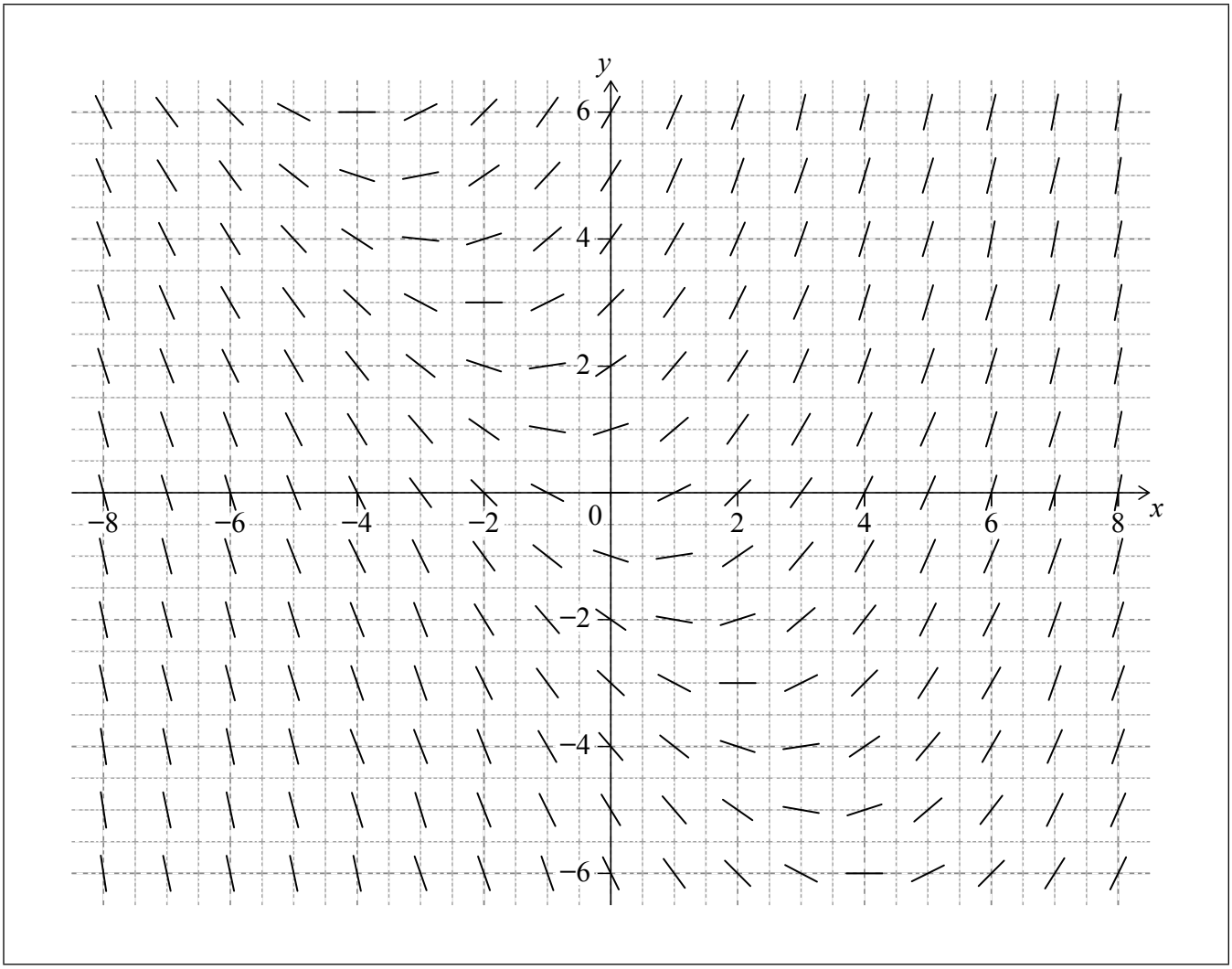
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Question 10 - working space

11. [Maximum mark: 4]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2} + \frac{y}{3}$.

The slope field for the differential equation is shown in the diagram.



All the solutions to $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ lie on a straight line.

(a) (i) Find the equation of this line.

(ii) Sketch this line on the slope field. [2]

One solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2} + \frac{y}{3}$ passes through the point $(6, 4)$.

(b) Sketch this solution on the slope field. [2]

(This question continues on the following page)

(Question 11 continued)

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Question 11 - working space

14. [Maximum mark: 6]

Consider the system of coupled differential equations given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2.2x - 2.6y$$

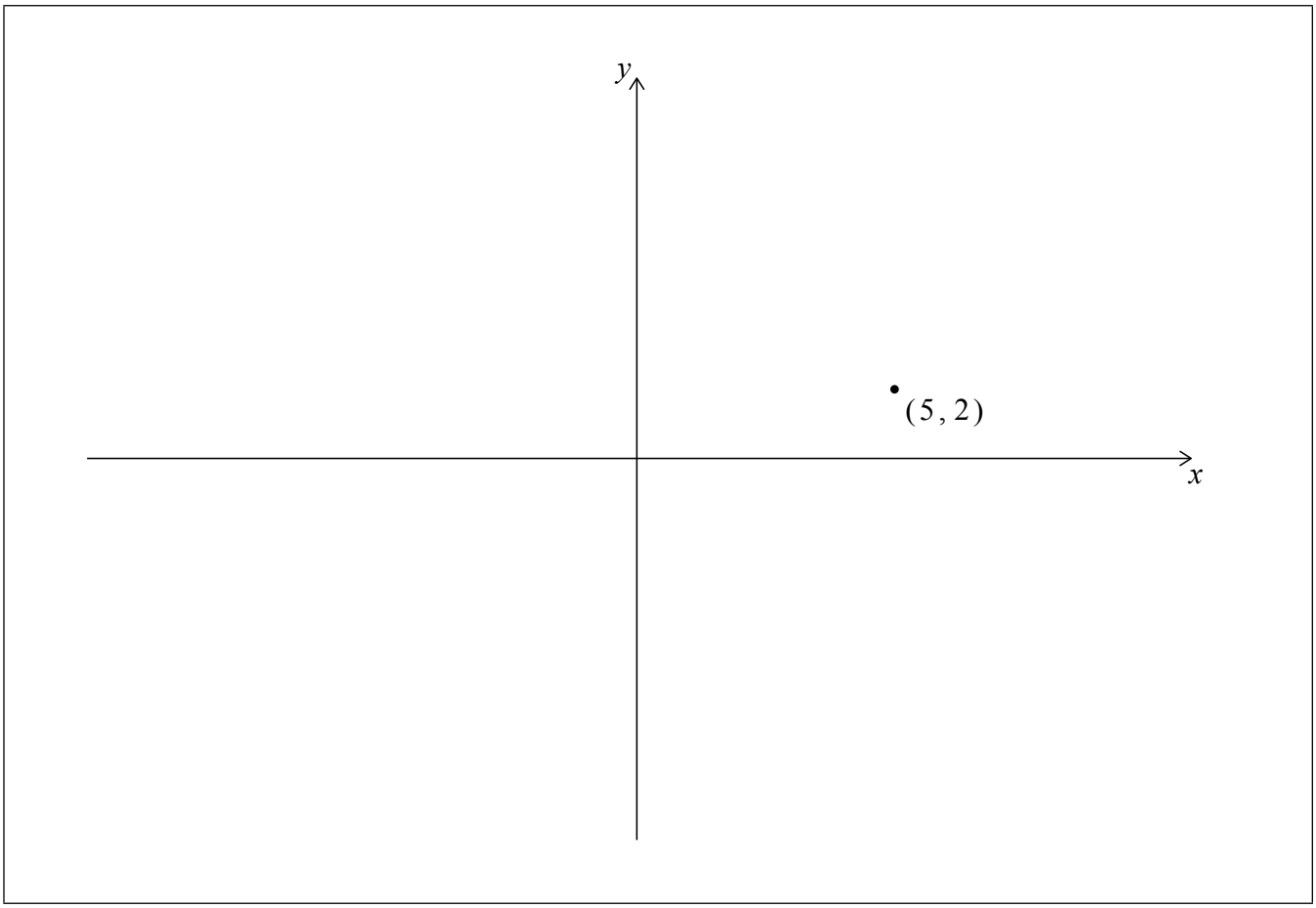
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3.4x - 2.2y\,.$$

When $t = 0$, $x = 5$ and $y = 2$.

(a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t = 0$. [3]

The eigenvalues for this system are $\pm 2\mathrm{i}$.

(b) On the following phase portrait, sketch the trajectory that passes through the point $(5, 2)$. Clearly indicate the direction of this trajectory. [3]



(This question continues on the following page)

(Question 14 continued)

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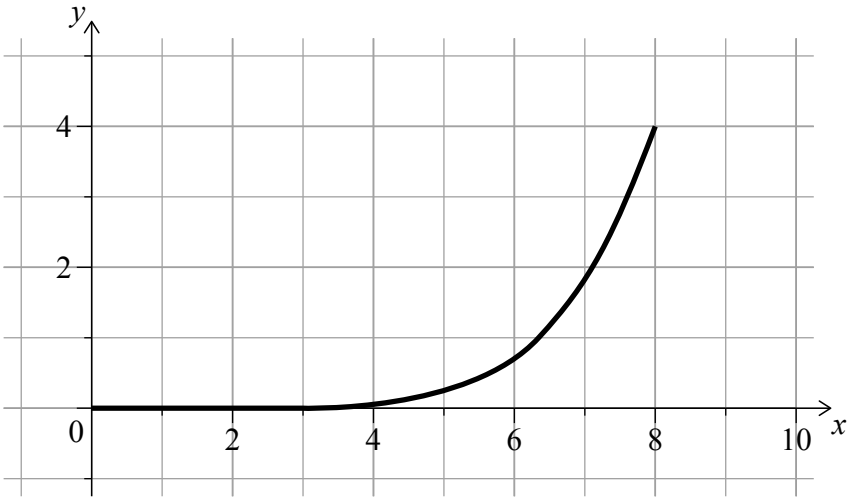
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Question 14 - working space

15. [Maximum mark: 8]

Jun Ho models the cross-section of a bowl, in order to calculate its volume.

His model for part of the cross-section is $y = \frac{x^6}{65536}$, where $0 \leq x \leq 8$, as shown in the following graph. One unit represents one centimetre.



Let R be the region enclosed by this graph, the line $y = 4$ and the line $x = 0$.

He obtains the volume of the bowl by rotating R, 2π radians about the y-axis.

(a) Find the volume of the bowl. [5]

Jun Ho pours $250\,\text{cm}^3$ of water into the bowl.

(b) Find the depth of the water in the bowl. [3]

(This question continues on the following page)

(Question 15 continued)

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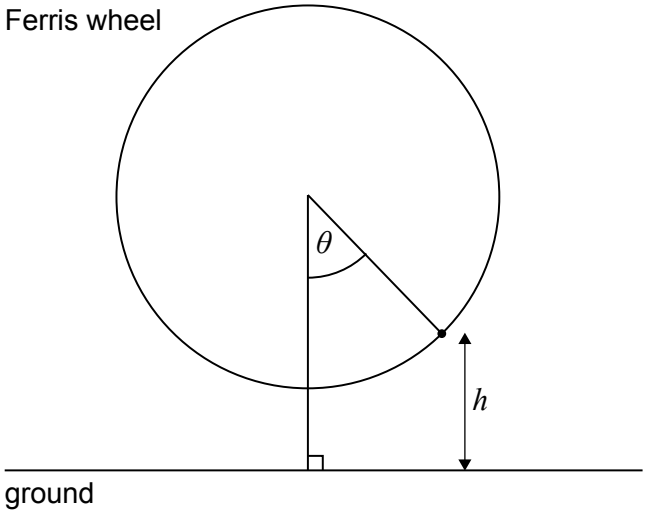
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Question 15 - working space

17. [Maximum mark: 8]

Younsue rides on a Ferris wheel. As the wheel rotates, her height above the ground, h metres, can be modelled in terms of the angle, θ radians, that the wheel has rotated, using $h = 16 - 15\cos\theta$. The value of θ is measured from Younsue's starting position at the bottom of the wheel.

diagram not to scale



(a) Write down the radius of the Ferris wheel. [1]

The speed at which the wheel rotates changes over time and is given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\pi^2}{24}\left|\sin\left(\frac{\pi}{2}t\right)\right|$$

where t is the time, in minutes, after Younsue starts her ride.

(b) (i) Find the angle rotated by the Ferris wheel between $t = 0$ and $t = 4.5$.

(ii) Find the value of $\frac{\mathrm{d}h}{\mathrm{d}t}$, when $t = 4.5$. [7]

(This question continues on the following page)

(Question 17 continued)

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Question 17 - working space

2. [Maximum mark: 16]

The height of a plant, $h\,\text{cm}$, depends on the time, t days, after it is planted.

For the first 15 days, the rate of change of the height of the plant is modelled as

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 1.2 - 0.1t \text{, where } 0 \leq t \leq 15.$$

(a) (i) Find the value of $\frac{\mathrm{d}h}{\mathrm{d}t}$ at $t = 3$.

(ii) Interpret the meaning of your answer to part (a)(i) in context. [2]

(b) Use $\frac{\mathrm{d}h}{\mathrm{d}t}$ to find the value of t when the plant reaches its maximum height. [2]

Two days after it is planted, the height of the plant is $6.5\,\text{cm}$.

(c) Find an expression for h in terms of t, for $0 \leq t \leq 15$. [5]

(d) Hence, find the maximum height of the plant. [2]

A second plant was planted at the same time as the first plant. Its height, $H\,\text{cm}$, is modelled by $H = 2(1.2)^t$, where $0 \leq t \leq 15$.

(e) Write down the initial height of the second plant. [1]

Each day, the height of the second plant increases by $p\%$.

(f) Find the value of p. [2]

(g) Find the value of t when the two plants have the same height. [2]

Turn over

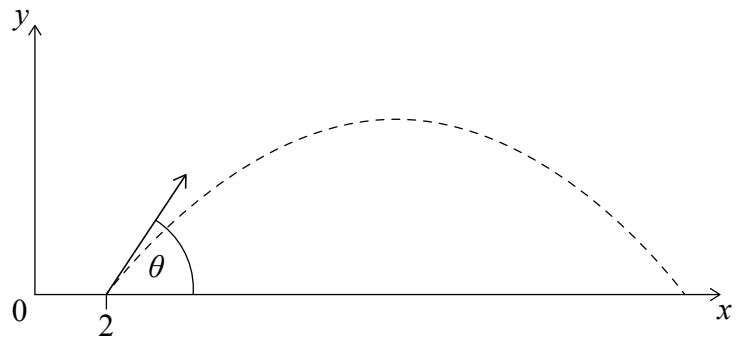
Question 2 - working space

6. [Maximum mark: 14]

At an outdoor ballpark for children, a ball is launched from the ground, with initial speed $12\,\text{m}\,\text{s}^{-1}$ and initial launch angle θ to the horizontal ground, where $0 < \theta < \frac{\pi}{2}$.

The position of the ball, t seconds after it is launched, is given by $\begin{pmatrix} x \\ y \end{pmatrix}$ where x is the horizontal displacement from an origin and y is the vertical displacement from the ground. Distances are measured in metres.

The ball's initial position is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and its path is shown in the diagram.



The ball can be modelled as a projectile, using $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 12\cos\theta \\ 12\sin\theta - 9.8t \end{pmatrix}$.

(a) Find an expression for x, in terms of θ and t. [3]

It is given that $y = (12\sin\theta)t - 4.9t^2$.

(b) When the ball hits the ground, show that $t = \frac{120\sin\theta}{49}$. [3]

Let x_g be the value of x when the ball hits the ground.

(c) Find an expression for x_g in terms of θ only. [2]

(d) Hence, find the value of θ which maximizes the value of x_g. [2]

The model is adapted to account for a horizontal wind with speed $3\,\text{m}\,\text{s}^{-1}$.

The new model for the ball is $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 12\cos\theta - 3 \\ 12\sin\theta - 9.8t \end{pmatrix}$.

In the new model, the time taken for the ball to hit the ground remains $t = \frac{120\sin\theta}{49}$.

(e) (i) Find an expression for x_g in terms of θ only.

(ii) Hence, find the value of θ which maximizes the value of x_g. [4]

Turn over

Question 6 - working space

2. [Maximum mark: 29]

In this question researchers are trying to find the most accurate model to use when modelling a population of wolves.

Historically, a population of wolves in an area had a stable size of 200. After some years of disruption, the population was reduced to 40 wolves. At this point, the area became a protected space and the population began to grow again.

Researchers in the area wish to model the size of the wolf population, x, as a function of t, where t is the time, in years, since the area became protected.

(a) Initially, the researchers consider using the logistic model

$$x = \frac{L}{1 + C\mathrm{e}^{-kt}}, \text{ where } L, C, k \in \mathbb{R}^+.$$

The researchers decide to let $L = 200$.

(i) State the assumption being made in assuming $L = 200$. [1]

At $t = 0$, the population of wolves is 40.

(ii) Find the value of C. [2]

At $t = 5$, the population of wolves is found to have increased to 70.

(iii) Find the value of k. [2]

(iv) Use your model to predict the size of the wolf population in the area 10 years after it became protected. Give your answer correct to the nearest whole number. [2]

(b) An alternative model for population growth is called the Gompertz model. When applied by the researchers to the wolf population, this model satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax\ln\left(\frac{200}{x}\right),\ a \in \mathbb{R}^+.$$

(i) Write down the value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 200$. [1]

(ii) Interpret your answer to part (b)(i) in context. [1]

(This question continues on the following page)

(Question 2 continued)

Consider the function $f(x) = \ln(\ln 200 - \ln x)$, where $0 < x < 200$.

(iii) Show that $f'(x) = \dfrac{-1}{x \ln\left(\dfrac{200}{x}\right)}$. [2]

(iv) Hence, use separation of variables to show that the general solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax \ln\left(\frac{200}{x}\right), \text{ where } 0 < x < 200,$$

can be written as

$$\ln x = \ln 200 - A\mathrm{e}^{-at},$$

where A is an arbitrary positive constant. [5]

(v) Use the size of the wolf population at $t = 0$ to find the value of A.
Give your answer in the form $A = \ln p$, where $p \in \mathbb{Z}^+$. [2]

(vi) Use the size of the wolf population at $t = 5$, given in part (a), to show that $a = 0.0855$, correct to three significant figures. [2]

(vii) Use the Gompertz model to predict the size of the wolf population at $t = 10$.
Give your answer correct to the nearest whole number. [3]

After 10 years, the wolf population is measured and is found to be 85.

(c) Comment on the predictions made by the two models. [1]

By tracking individual wolves, the researchers find that about 3% of the wolf population emigrate from the protected area each year.

They decide to adapt the Gompertz model to allow for this. The new model will satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.0855x \ln\left(\frac{200}{x}\right) - 0.03x.$$

(d) (i) Use Euler's method, with a step size of 0.5 years and an initial value of $x_0 = 70$ when $t = 5$, to find an estimate for the size of the wolf population when $t = 10$.
Give your answer correct to the nearest whole number. [4]

(ii) Comment on your answer. [1]

Question 2 - working space