

APOLLO

Geometry and trigonometry

HL

Mathematics: applications and interpretation

First assessment 2021

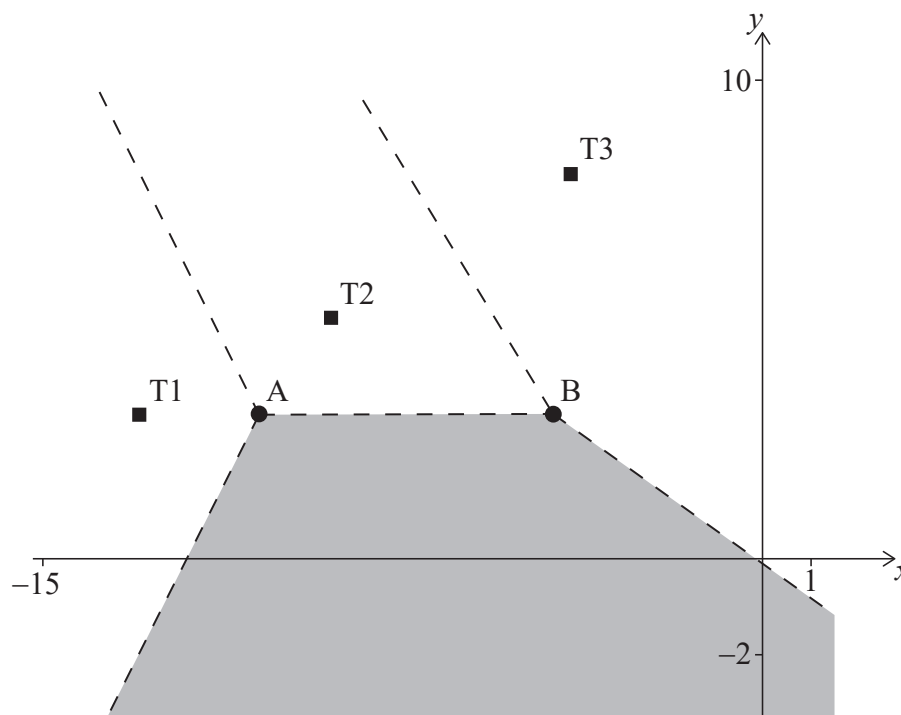
101 questions | questions

2. [Maximum mark: 6]

The Voronoi diagram below shows three identical cellular phone towers, T1, T2 and T3. A fourth cellular phone tower, T4 is located in the shaded region. The dashed lines in the diagram below represent the edges in the Voronoi diagram.

Horizontal scale: 1 unit represents 1 km.

Vertical scale: 1 unit represents 1 km.



Tim stands inside the shaded region.

- (a) Explain why Tim will receive the strongest signal from tower T4. [1]

Tower T2 has coordinates $(-9, 5)$ and the edge connecting vertices A and B has equation $y = 3$.

- (b) Write down the coordinates of tower T4. [2]

Tower T1 has coordinates $(-13, 3)$.

- (c) Find the gradient of the edge of the Voronoi diagram between towers T1 and T2. [3]

(This question continues on the following page)



(Question 2 continued)

[illegible]

Turn over

5. [Maximum mark: 6]

A garden has a triangular sunshade suspended from three points $A(2, 0, 2)$, $B(8, 0, 2)$ and $C(5, 4, 3)$, relative to an origin in the corner of the garden. All distances are measured in metres.

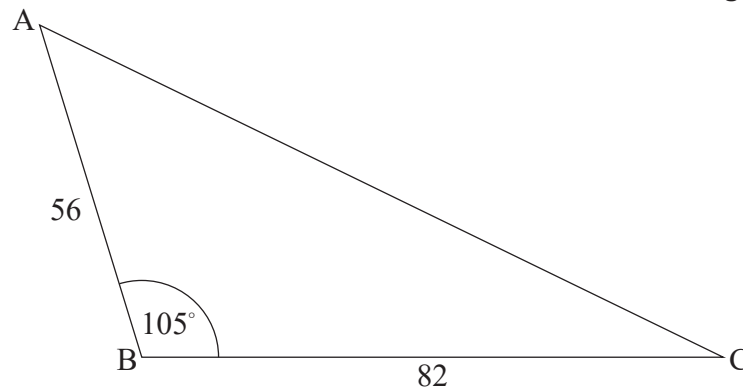
- (a) (i) Find $\overrightarrow{CA}$.
- (ii) Find $\overrightarrow{CB}$. [2]
- (b) Find $\overrightarrow{CA} \times \overrightarrow{CB}$. [2]
- (c) Hence find the area of the triangle ABC. [2]

[illegible]

6. [Maximum mark: 5]

A triangular field ABC is such that $AB = 56\text{ m}$ and $BC = 82\text{ m}$, each measured correct to the nearest metre, and the angle at B is equal to 105° , measured correct to the nearest 5° .

diagram not to scale



Calculate the maximum possible area of the field.

[illegible]

Turn over

10. [Maximum mark: 7]

An engineer plans to visit six oil rigs (A–F) in the Gulf of Mexico, starting and finishing at A. The travelling time, in minutes, between each of the rigs is shown in the table.

	A	B	C	D	E	F
A		55	63	79	87	93
B	55		46	58	88	92
C	63	46		87	77	66
D	79	58	87		23	70
E	87	88	77	23		47
F	93	92	66	70	47	

The data above can be represented by a graph G .

- (a) (i) Use Prim's algorithm to find the weight of the minimum spanning tree of the subgraph of G obtained by deleting A and starting at B. List the order in which the edges are selected.

- (ii) Hence find a lower bound for the travelling time needed to visit all the oil rigs.

- (b) Describe how an improved lower bound might be found.

[illegible]

Turn over

13. [Maximum mark: 7]

A submarine is located in a sea at coordinates $(0.8, 1.3, -0.3)$ relative to a ship positioned at the origin O . The x direction is due east, the y direction is due north and the z direction is vertically upwards.

All distances are measured in kilometres.

The submarine travels with direction vector $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$.

- (a) Assuming the submarine travels in a straight line, write down an equation for the line along which it travels. [2]

The submarine reaches the surface of the sea at the point P .

- (b) (i) Find the coordinates of P .
- (ii) Find OP . [5]

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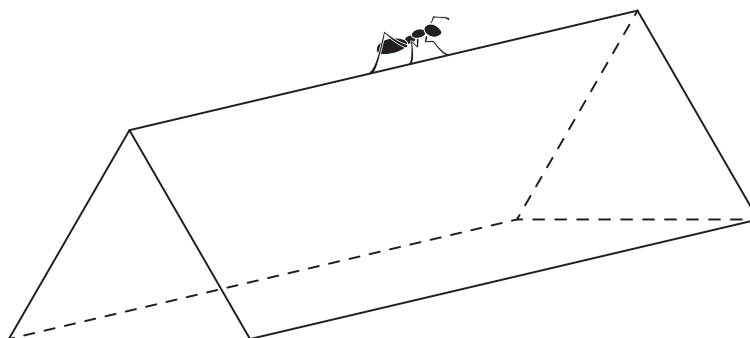
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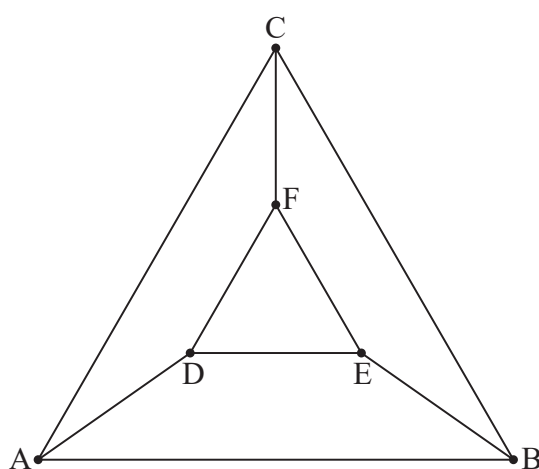


16. [Maximum mark: 5]

An ant is walking along the edges of a wire frame in the shape of a triangular prism.



The vertices and edges of this frame can be represented by the graph below.



- (a) Write down the adjacency matrix, M , for this graph. [3]
- (b) Find the number of ways that the ant can start at the vertex A, and walk along exactly 6 edges to return to A. [2]

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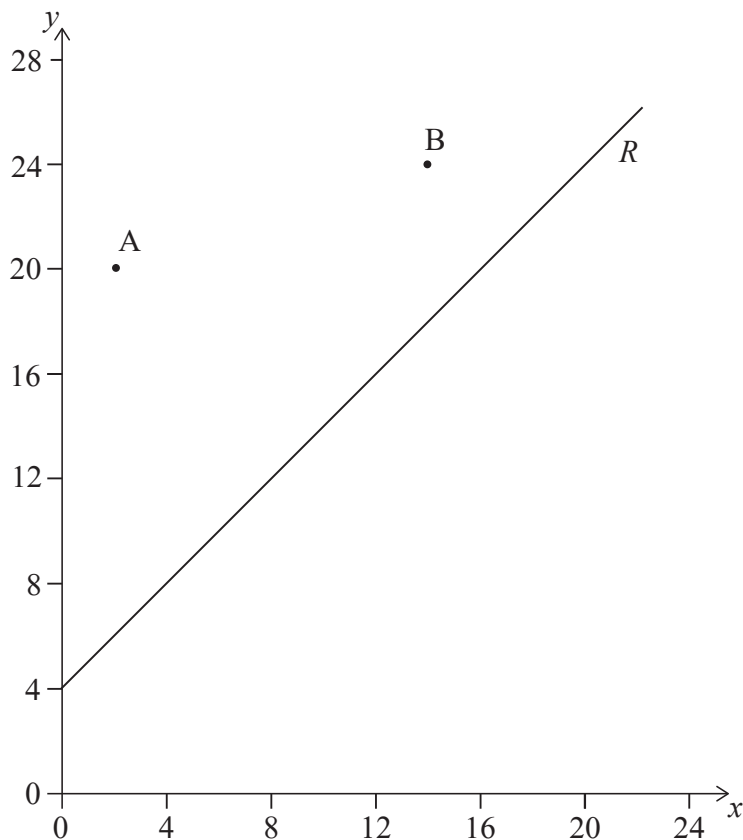
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[illegible]

Turn over

1. [Maximum mark: 7]

Two schools are represented by points $A(2, 20)$ and $B(14, 24)$ on the graph below. A road, represented by the line R with equation $-x + y = 4$, passes near the schools. An architect is asked to determine the location of a new bus stop on the road such that it is the same distance from the two schools.



- (a) Find the equation of the perpendicular bisector of $[AB]$. Give your equation in the form $y = mx + c$. [5]

- (b) Determine the coordinates of the point on R where the bus stop should be located. [2]

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(Question 1 continued)

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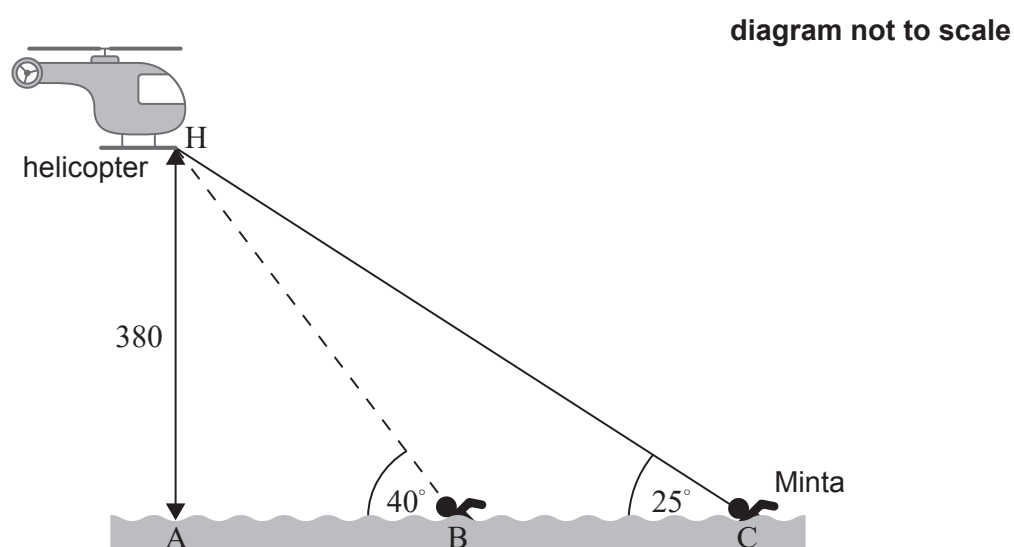
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Turn over

4. [Maximum mark: 6]

The diagram below shows a helicopter hovering at point H, 380m vertically above a lake. Point A is the point on the surface of the lake, directly below the helicopter.



Minta is swimming at a constant speed in the direction of point A. Minta observes the helicopter from point C as she looks upward at an angle of 25° . After 15 minutes, Minta is at point B and she observes the same helicopter at an angle of 40° .

- Find the distance from A to C. [2]
- Find the distance from B to C. [3]
- Find Minta's speed, in metres per hour. [1]

[illegible]

8. [Maximum mark: 7]

Two lines L_1 and L_2 are given by the following equations, where $p \in \mathbb{R}$.

$$L_1: r = \begin{pmatrix} 2 \\ p+9 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} p \\ 2p \\ 4 \end{pmatrix}$$

$$L_2: r = \begin{pmatrix} 14 \\ 7 \\ p+12 \end{pmatrix} + \mu \begin{pmatrix} p+4 \\ 4 \\ -7 \end{pmatrix}$$

It is known that L_1 and L_2 are perpendicular.

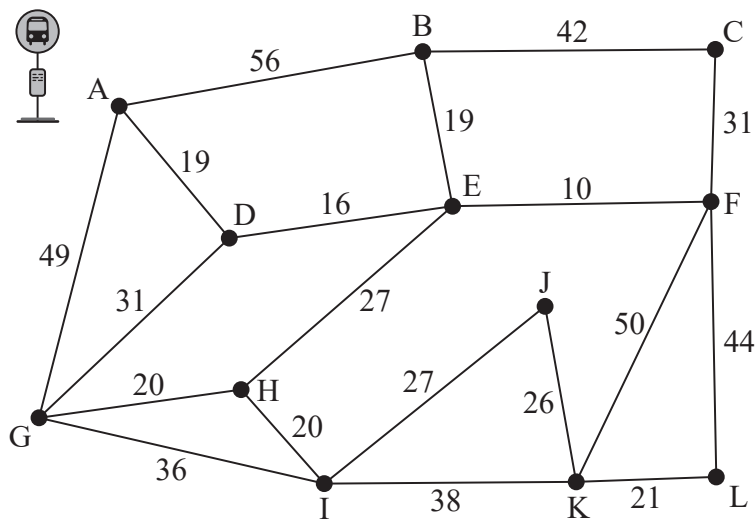
- (a) Find the possible value(s) for p . [3]
- (b) In the case that $p < 0$, determine whether the lines intersect. [4]

This image shows a full page of primary-ruled notebook paper. It features ten sets of horizontal lines across the page. Each set consists of three lines: a solid top line, a dashed middle line, and a dotted bottom line. The lines are evenly spaced and extend from the left margin to the right edge of the page. There are no margins or other markings present.

Turn over

11. [Maximum mark: 7]

The diagram below shows a network of roads in a small village with the weights indicating the distance of each road, in metres, and junctions indicated with letters.



Musab is required to deliver leaflets to every house on each road. He wishes to minimize his total distance.

- (a) Musab starts and finishes from the village bus-stop at A. Determine the total distance Musab will need to walk. [5]

Instead of having to catch the bus to the village, Musab's sister offers to drop him off at any junction and pick him up at any other junction of his choice.

- (b) Explain which junctions Musab should choose as his starting and finishing points. [2]

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14. [Maximum mark: 7]

A geometric transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix}$ is defined by

$$T: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 7 & -10 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 4 \end{pmatrix}.$$

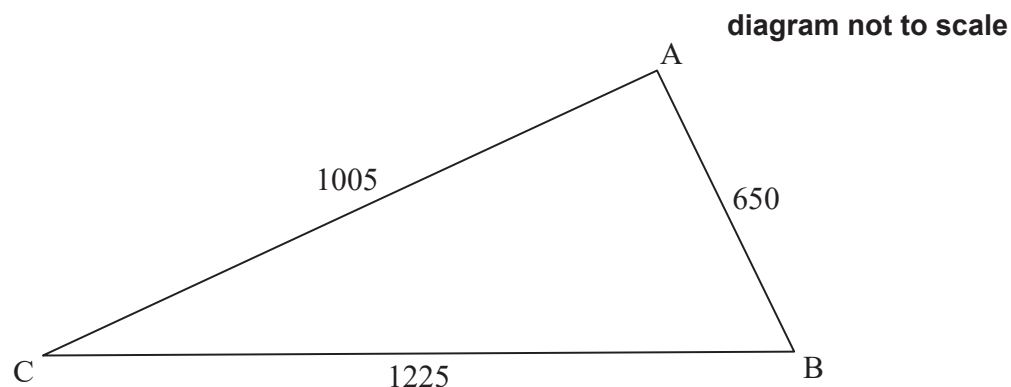
- Find the coordinates of the image of the point $(6, -2)$. [2]
- Given that $T: \begin{pmatrix} p \\ q \end{pmatrix} \mapsto 2\begin{pmatrix} p \\ q \end{pmatrix}$, find the value of p and the value of q . [3]
- A triangle L with vertices lying on the xy plane is transformed by T .

Explain why both L and its image will have exactly the same area. [2]

[illegible]

1. [Maximum mark: 15]

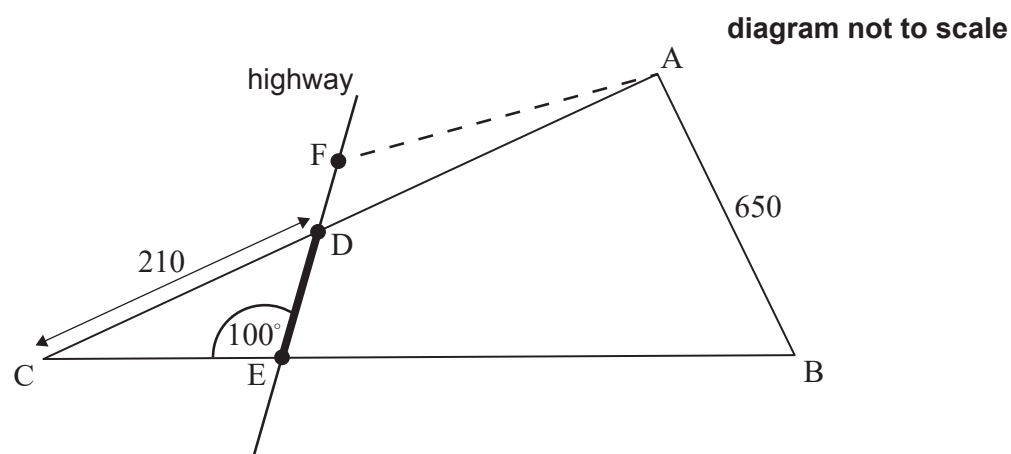
A farmer owns a field in the shape of a triangle ABC such that $AB = 650$ m, $AC = 1005$ m and $BC = 1225$ m.



(a) Find the size of $\hat{ACB}$.

[3]

The local town is planning to build a highway that will intersect the borders of the field at points D and E, where $DC = 210$ m and $\hat{CED} = 100^\circ$, as shown in the diagram below.



(b) Find DE.

[3]

The town wishes to build a carpark here. They ask the farmer to exchange the part of the field represented by triangle DCE. In return the farmer will get a triangle of equal area ADF, where F lies on the same line as D and E, as shown in the diagram above.

(c) Find the area of triangle DCE.

[5]

(d) Estimate DF. You may assume the highway has a width of zero.

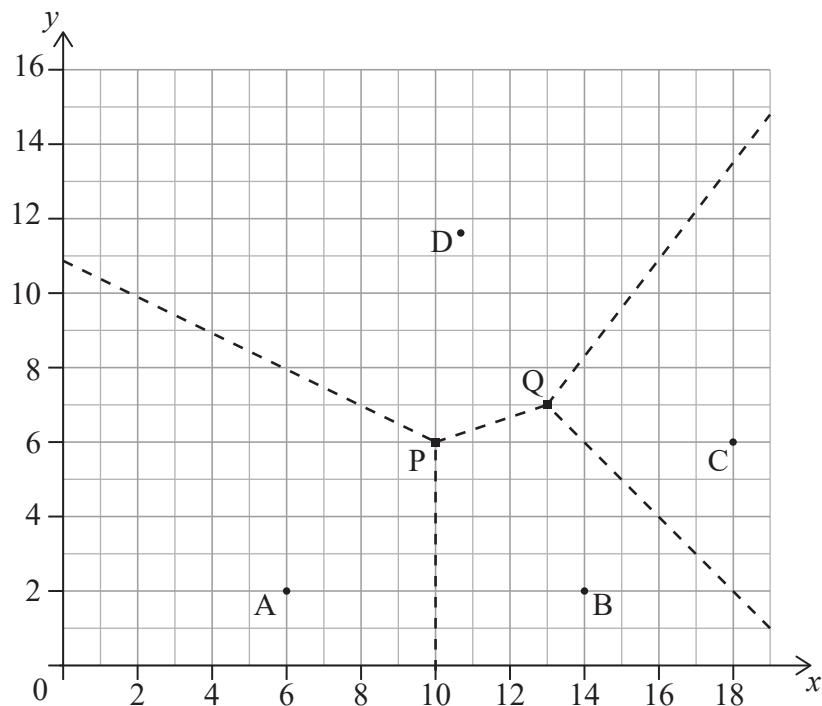
[4]

4. [Maximum mark: 6]

There are four stations used by the fire wardens in a national forest.

On the following Voronoi diagram, the coordinates of the stations are $A(6, 2)$, $B(14, 2)$, $C(18, 6)$ and $D(10.8, 11.6)$ where distances are measured in kilometres.

The dotted lines represent the boundaries of the regions patrolled by the fire warden at each station. The boundaries meet at $P(10, 6)$ and $Q(13, 7)$.



To reduce the areas of the regions that the fire wardens patrol, a new station is to be built within the quadrilateral $ABCD$. The new station will be located so that it is as far as possible from the nearest existing station.

(a) Show that the new station should be built at P .

[3]

The Voronoi diagram is to be updated to include the region around the new station at P . The edges defined by the perpendicular bisectors of $[AP]$ and $[BP]$ have been added to the following diagram.

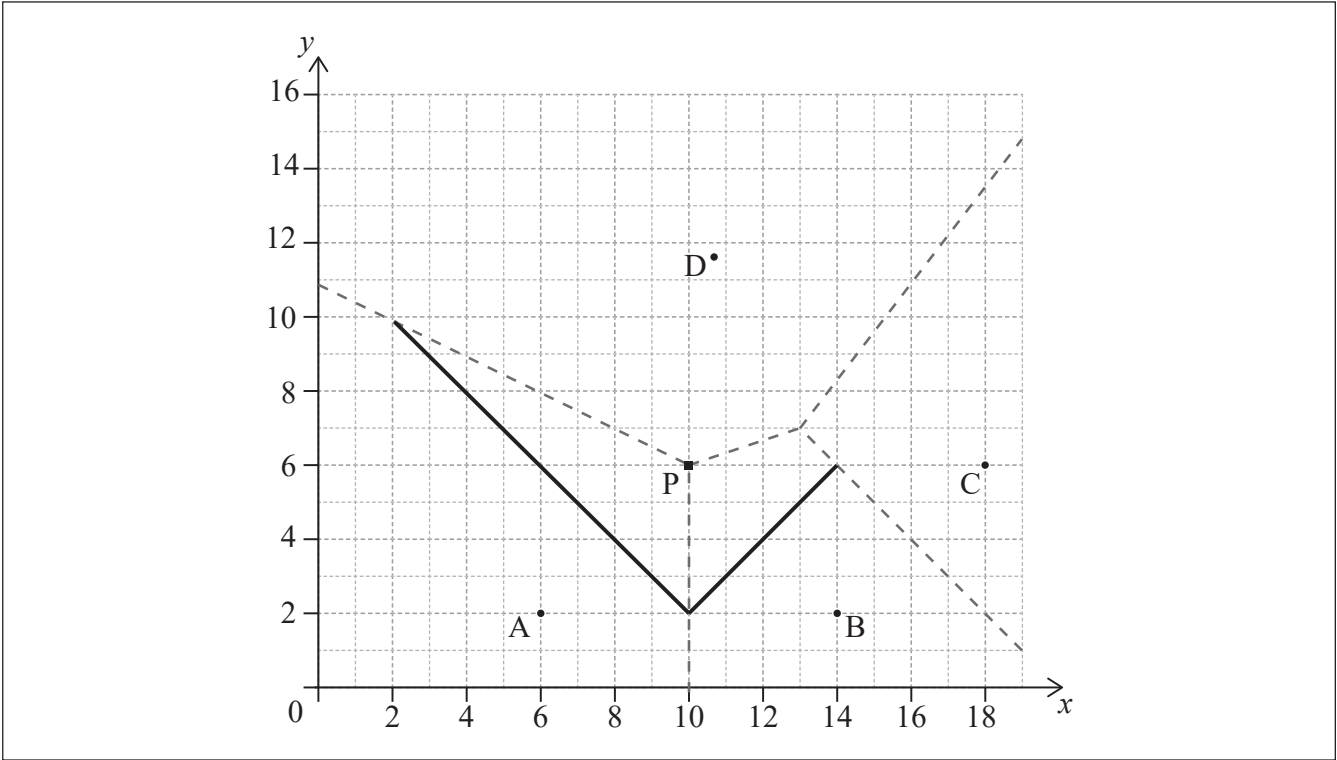
(b) (i) Write down the equation of the perpendicular bisector of $[PC]$.

(ii) Hence draw the missing boundaries of the region around P on the following diagram. [3]

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(Question 4 continued)



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Turn over

16. [Maximum mark: 9]

A ship S is travelling with a constant velocity, $\mathbf{v}$, measured in kilometres per hour, where

$$\mathbf{v} = \begin{pmatrix} -12 \\ 15 \end{pmatrix}.$$

At time $t = 0$ the ship is at a point A(300, 100) relative to an origin O, where distances are measured in kilometres.

- (a) Find the position vector $\overrightarrow{OS}$ of the ship at time t hours. [1]

A lighthouse is located at a point (129, 283).

- (b) Find the value of t when the ship will be closest to the lighthouse. [6]

An alarm will sound if the ship travels within 20 kilometres of the lighthouse.

- (c) State whether the alarm will sound. Give a reason for your answer. [2]

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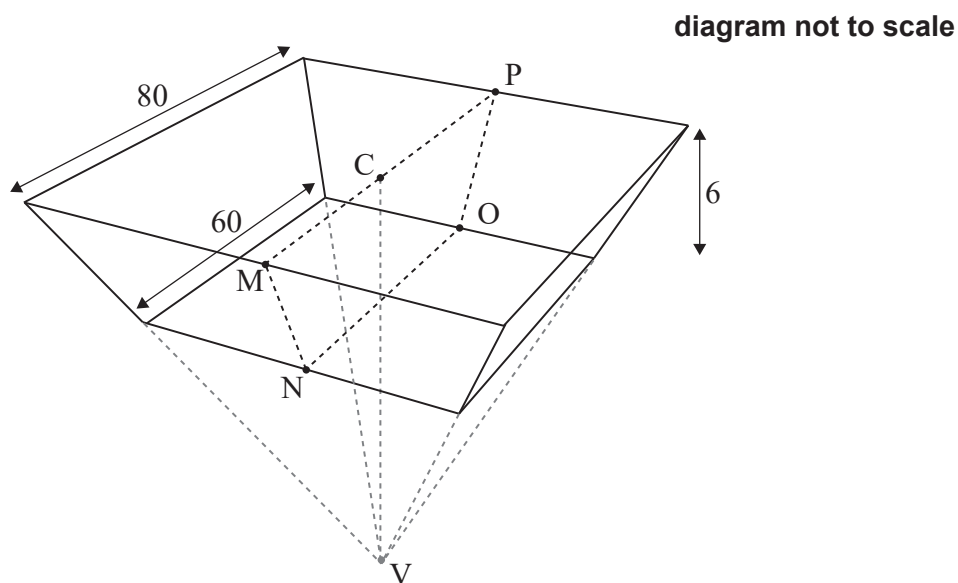
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1. [Maximum mark: 14]

A large water reservoir is built in the form of part of an upside-down right pyramid with a horizontal square base of length 80 metres. The point C is the centre of the square base and point V is the vertex of the pyramid.

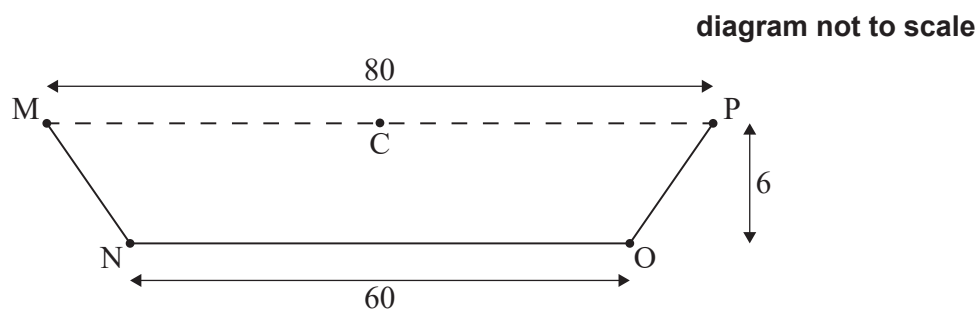


The bottom of the reservoir is a square of length 60 metres that is parallel to the base of the pyramid, such that the depth of the reservoir is 6 metres as shown in the diagram.

(This question continues on the following page)

(Question 1 continued)

The second diagram shows a vertical cross section, $MNOPC$, of the reservoir.



- (a) Find the angle of depression from M to N . [2]
- (b) (i) Find CV .
- (ii) Hence or otherwise, show that the volume of the reservoir is $29\,600\text{ m}^3$. [5]

Every day 80 m^3 of water from the reservoir is used for irrigation.

Joshua states that, if no other water enters or leaves the reservoir, then when it is full there is enough irrigation water for at least one year.

- (c) By finding an appropriate value, determine whether Joshua is correct. [2]

To avoid water leaking into the ground, the five interior sides of the reservoir have been painted with a watertight material.

- (d) Find the area that was painted. [5]

Turn over

4. [Maximum mark: 18]

A flying drone is programmed to complete a series of movements in a horizontal plane relative to an origin O and a set of x - y -axes.

In each case, the drone moves to a new position represented by the following transformations:

- a rotation anticlockwise of $\frac{\pi}{6}$ radians about O
- a reflection in the line $y = \frac{x}{\sqrt{3}}$.
- a rotation clockwise of $\frac{\pi}{3}$ radians about O .

All the movements are performed in the listed order.

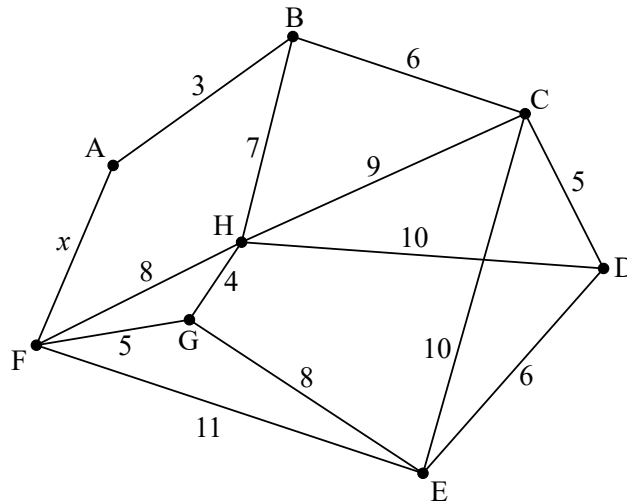
- (a) (i) Write down each of the transformations in matrix form, clearly stating which matrix represents each transformation.
- (ii) Find a single matrix $\mathbf{P}$ that defines a transformation that represents the overall change in position.
- (iii) Find $\mathbf{P}^2$.
- (iv) Hence state what the value of $\mathbf{P}^2$ indicates for the possible movement of the drone. [12]
- (b) Three drones are initially positioned at the points A , B and C . After performing the movements listed above, the drones are positioned at points A' , B' and C' respectively.
- Show that the area of triangle ABC is equal to the area of triangle $A'B'C'$. [2]
- (c) Find a single transformation that is equivalent to the three transformations represented by matrix $\mathbf{P}$. [4]

Turn over

1. [Maximum mark: 25]

This question explores how graph algorithms can be applied to a graph with an unknown edge weight.

Graph W is shown in the following diagram. The vertices of W represent tourist attractions in a city. The weight of each edge represents the travel time, to the nearest minute, between two attractions. The route between A and F is currently being resurfaced and this has led to a variable travel time. For this reason, AF has an unknown travel time x minutes, where $x \in \mathbb{Z}^+$.



- (a) Write down a Hamiltonian cycle in W .

[1]

Daniel plans to visit all the attractions, starting and finishing at A. He wants to minimize his travel time.

To find a lower bound for Daniel's travel time, vertex A and its adjacent edges are first deleted.

- (b) (i) Use Prim's algorithm, starting at vertex B, to find the weight of the minimum spanning tree of the remaining graph. You should indicate clearly the order in which the algorithm selects each edge.
- (ii) Hence, for the case where $x < 9$, find a lower bound for Daniel's travel time, in terms of x .

[5]

[2]

(This question continues on the following page)

(Question 1 continued)

Daniel makes a table to show the minimum travel time between each pair of attractions.

	A	B	C	D	E	F	G	H
A		3	9	14	19 or $(11 + x)$	18 or x	14 or $(5 + x)$	10 or $(8 + x)$
B			6	11	16 or $(14 + x)$	15 or $(3 + x)$	11 or $(8 + x)$	7
C				5	10	17 or $(9 + x)$	p	9
D					6	q or $(r + x)$	14	10
E						11	8	12
F							5	8
G								4
H								

(c) Write down the value of

(i) p ; [1]

(ii) q ; [1]

(iii) r . [1]

To find an upper bound for Daniel's travel time, the nearest neighbour algorithm is used, starting at vertex A.

(d) Consider the case where $x = 3$.

(i) Use the nearest neighbour algorithm to find two possible cycles. [3]

(ii) Find the best upper bound for Daniel's travel time. [2]

(e) Consider the case where $x > 3$.

(i) Find the least value of x for which the edge AF will definitely not be used by Daniel. [2]

(ii) Hence state the value of the upper bound for Daniel's travel time for the value of x found in part (e)(i). [2]

The tourist office in the city has received complaints about the lack of cleanliness of some routes between the attractions. Corinne, the office manager, decides to inspect all the routes between all the attractions, starting and finishing at H. The sum of the weights of all the edges in graph W is $(92 + x)$.

Corinne inspects all the routes as quickly as possible and takes 2 hours.

(f) Find the value of x during Corinne's inspection. [5]

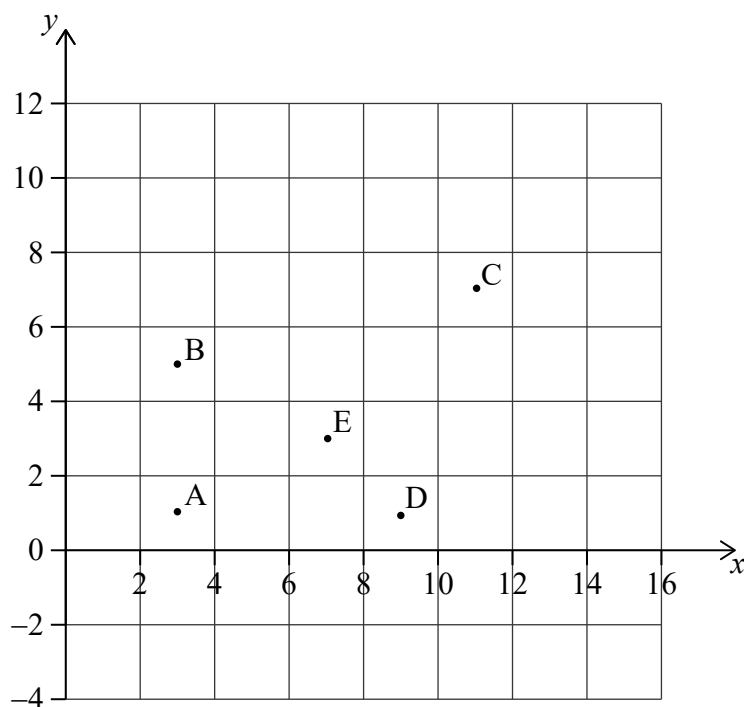
Turn over

4. [Maximum mark: 6]

Points A(3, 1), B(3, 5), C(11, 7), D(9, 1) and E(7, 3) represent snow shelters in the Blackburn National Forest. These snow shelters are illustrated in the following coordinate axes.

Horizontal scale: 1 unit represents 1 km.

Vertical scale: 1 unit represents 1 km.



(a) Calculate the gradient of the line segment AE.

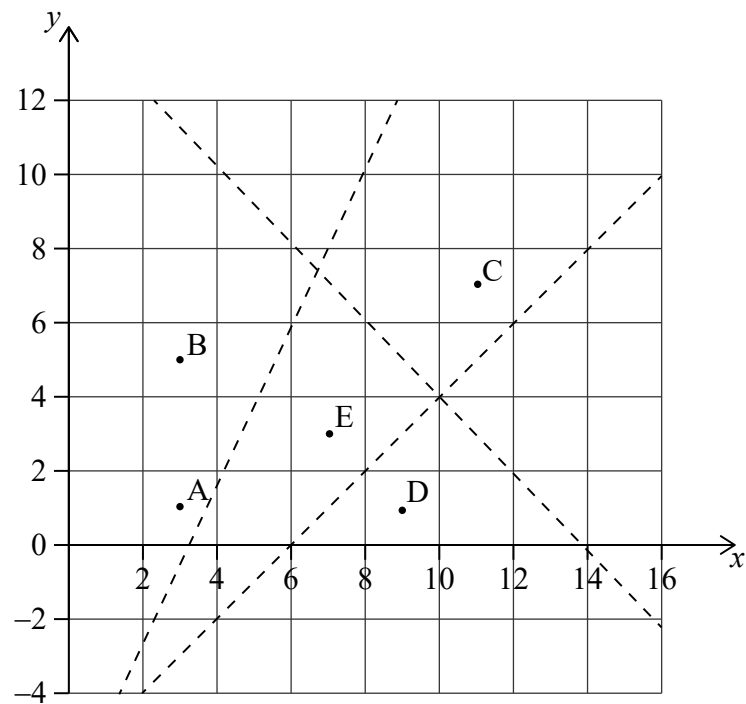
[2]

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(Question 4 continued)

The Park Ranger draws three straight lines to form an incomplete Voronoi diagram.



- (b) Find the equation of the line which would complete the Voronoi cell containing site E. Give your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$. [3]
- (c) In the context of the question, explain the significance of the Voronoi cell containing site E. [1]

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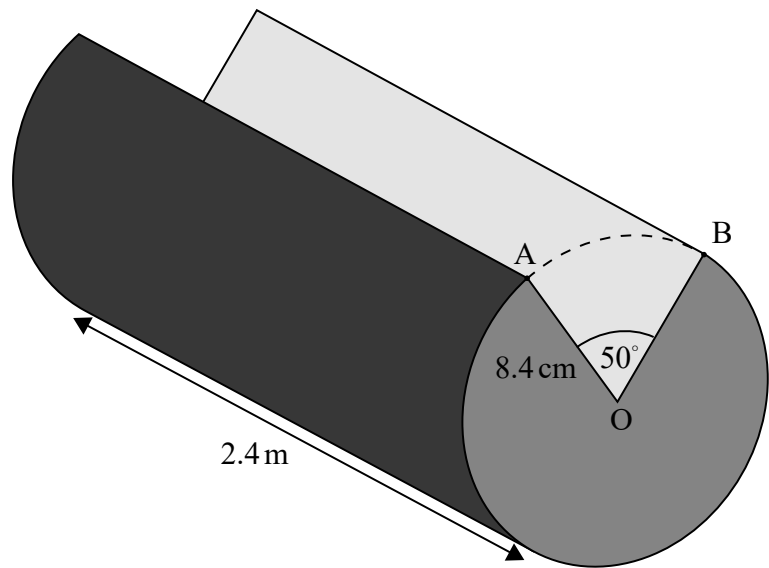
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5. [Maximum mark: 5]

Helen is building a cabin using cylindrical logs of length 2.4 m and radius 8.4 cm. A wedge is cut from one log and the cross-section of this log is illustrated in the following diagram.

diagram not to scale



- (a) Find 50° in radians. [1]
- (b) Find the volume of this log. [4]

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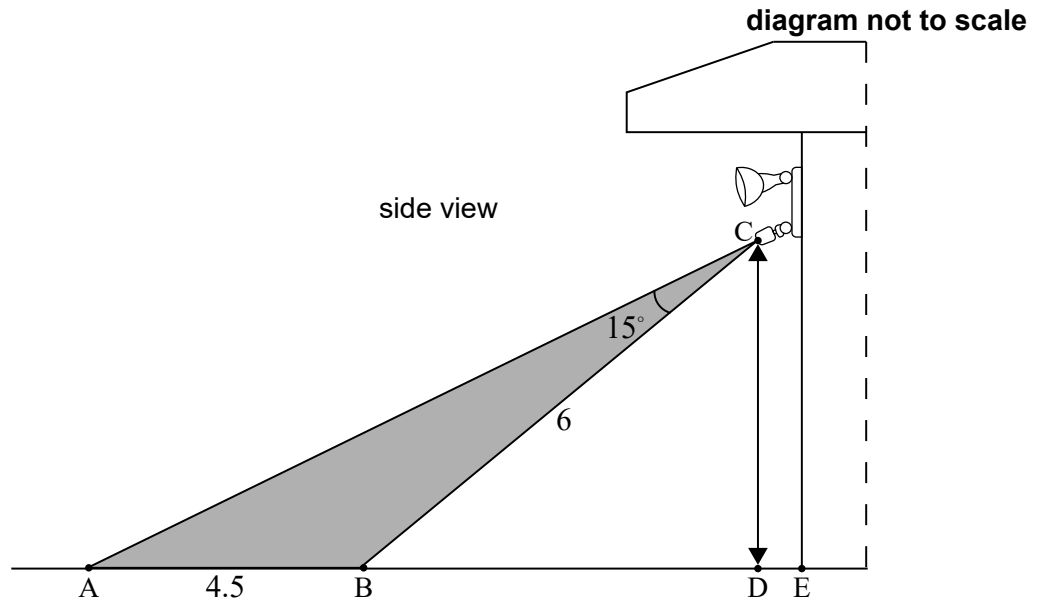
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8. [Maximum mark: 8]

Ollie has installed security lights on the side of his house that are activated by a sensor. The sensor is located at point C directly above point D. The area covered by the sensor is shown by the shaded region enclosed by triangle ABC. The distance from A to B is 4.5 m and the distance from B to C is 6 m. Angle $\widehat{ACB}$ is 15° .



- (a) Find $\widehat{CAB}$. [3]

Point B on the ground is 5 m from point E at the entrance to Ollie’s house. He is 1.8 m tall and is standing at point D, below the sensor. He walks towards point B.

- (b) Find the distance Ollie is **from the entrance to his house** when he first activates the sensor. [5]

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Turn over

11. [Maximum mark: 6]

A particle P moves with velocity $\mathbf{v} = \begin{pmatrix} -15 \\ 2 \\ 4 \end{pmatrix}$ in a magnetic field, $\mathbf{B} = \begin{pmatrix} 0 \\ d \\ 1 \end{pmatrix}$, $d \in \mathbb{R}$.

(a) Given that $\mathbf{v}$ is perpendicular to $\mathbf{B}$, find the value of d . [2]

The force, $\mathbf{F}$, produced by P moving in the magnetic field is given by the vector equation $\mathbf{F} = a\mathbf{v} \times \mathbf{B}$, $a \in \mathbb{R}^+$.

(b) Given that $|\mathbf{F}| = 14$, find the value of a . [4]



4. [Maximum mark: 14]

An aircraft's position is given by the coordinates (x, y, z) , where x and y are the aircraft's displacement east and north of an airport, and z is the height of the aircraft above the ground. All displacements are given in kilometres.

The velocity of the aircraft is given as $\begin{pmatrix} -150 \\ -50 \\ -20 \end{pmatrix} \text{ km h}^{-1}$.

At 13:00 it is detected at a position 30 km east and 10 km north of the airport, and at a height of 5 km. Let t be the length of time in hours from 13:00.

- (a) Write down a vector equation for the displacement, $\mathbf{r}$ of the aircraft in terms of t . [2]
- (b) If the aircraft continued to fly with the velocity given
- (i) verify that it would pass directly over the airport;
 - (ii) state the height of the aircraft at this point;
 - (iii) find the time at which it would fly directly over the airport. [4]

When the aircraft is 4 km above the ground it continues to fly on the same bearing but adjusts the angle of its descent so that it will land at the point $(0, 0, 0)$.

- (c) (i) Find the time at which the aircraft is 4 km above the ground.
- (ii) Find the direct distance of the aircraft from the airport at this point. [5]
- (d) Given that the velocity of the aircraft, after the adjustment of the angle of descent, is $\begin{pmatrix} -150 \\ -50 \\ a \end{pmatrix} \text{ km h}^{-1}$, find the value of a . [3]

Turn over

5. [Maximum mark: 17]

The following table shows the costs in US dollars (US\$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	B	C	D	E	F
Departure city	A		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

- (a) Show the direct flights between the cities as a graph. [2]
- (b) Write down the adjacency matrix for this graph. [2]
- (c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]
- (d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

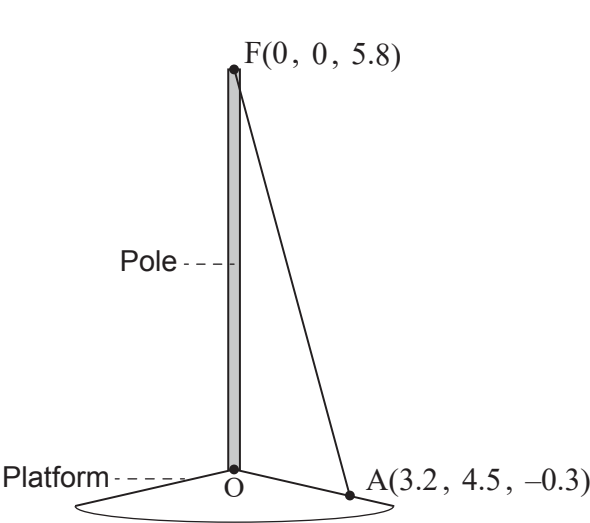
The following table shows the least cost to travel between the cities.

		Destination city					
		A	B	C	D	E	F
Departure city	A	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

- (e) Find the values of a and b . [2]
- A travelling salesman has to visit each of the cities, starting and finishing at city A.
- (f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]
- (g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

6. [Maximum mark: 8]

A vertical pole stands on a sloped platform. The bottom of the pole is used as the origin, O , of a coordinate system in which the top, F , of the pole has coordinates $(0, 0, 5.8)$. All units are in metres.



The pole is held in place by ropes attached at F .

One of these ropes is attached to the platform at point $A(3.2, 4.5, -0.3)$. The rope forms a straight line from A to F .

- (a) Find $\vec{AF}$. [1]
- (b) Find the length of the rope. [2]
- (c) Find $\hat{FAO}$, the angle the rope makes with the platform. [5]



Turn over

13. [Maximum mark: 5]

At 1:00 pm a ship is 1 km east and 4 km north of a harbour. A coordinate system is defined with the harbour at the origin. The position vector of the ship at 1:00 pm is given by $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

The ship has a constant velocity of $\begin{pmatrix} 1.2 \\ -0.6 \end{pmatrix}$ kilometres per hour (km h^{-1}).

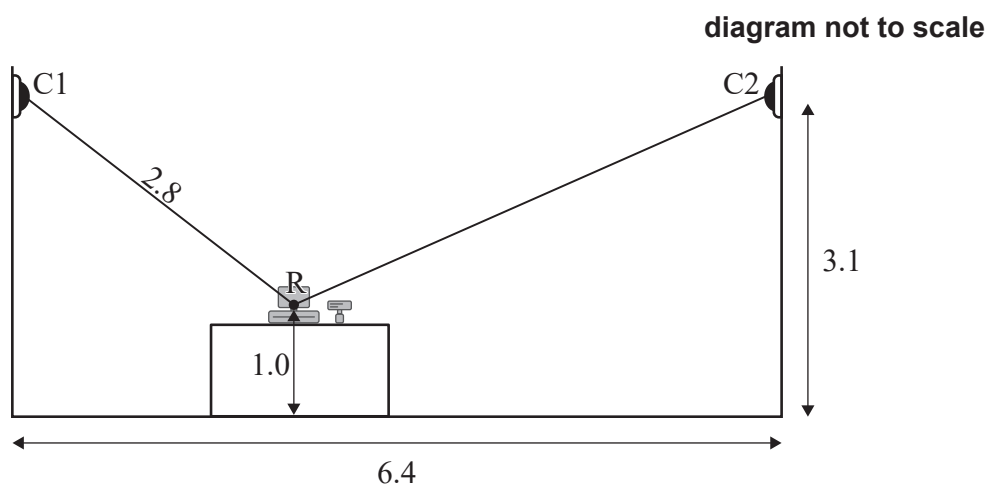
- (a) Write down an expression for the position vector $\mathbf{r}$ of the ship, t hours after 1:00 pm. [1]
- (b) Find the time at which the bearing of the ship from the harbour is 045° . [4]



2. [Maximum mark: 8]

The owner of a convenience store installs two security cameras, represented by points C1 and C2. Both cameras point towards the centre of the store's cash register, represented by the point R.

The following diagram shows this information on a cross-section of the store.



The cameras are positioned at a height of 3.1 m, and the horizontal distance between the cameras is 6.4 m. The cash register is sitting on a counter so that its centre, R, is 1.0 m above the floor.

The distance from Camera 1 to the centre of the cash register is 2.8 m.

- Determine the angle of depression from Camera 1 to the centre of the cash register. Give your answer in degrees. [2]
- Calculate the distance from Camera 2 to the centre of the cash register. [4]
- Without further calculation, determine which camera has the largest angle of depression to the centre of the cash register. Justify your response. [2]

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(Question 2 continued)

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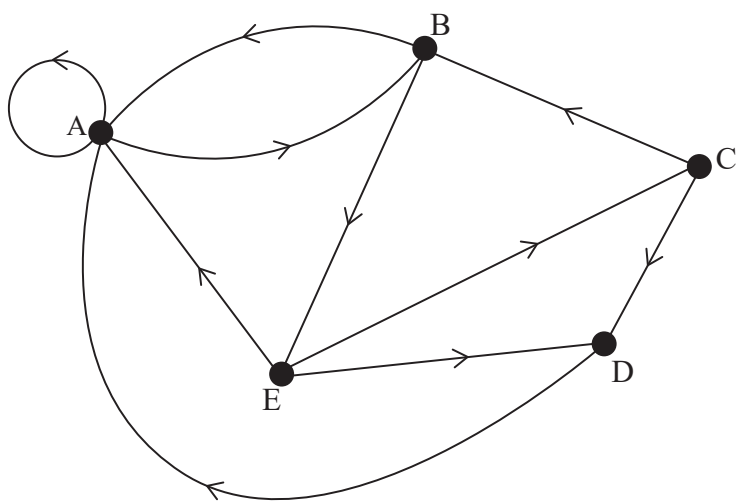
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6. [Maximum mark: 5]

Consider the following directed network.

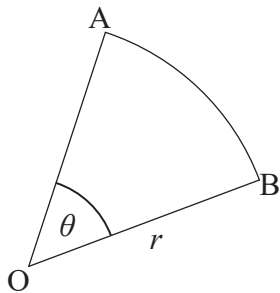


- (a) Write down the adjacency matrix for this network. [2]
- (b) Determine the number of different walks of length 5 that start and end at the same vertex. [3]

[illegible]

8. [Maximum mark: 8]

The diagram shows a sector, OAB, of a circle with centre O and radius r , such that $\angle AOB = \theta$.



Sam measured the value of r to be 2 cm and the value of θ to be 30° .

- (a) Use Sam’s measurements to calculate the area of the sector. Give your answer to four significant figures.

[2]

It is found that Sam’s measurements are accurate to only one significant figure.

- (b) Find the upper bound and lower bound of the area of the sector.

[3]
- (c) Find, with justification, the largest possible percentage error if the answer to part (a) is recorded as the area of the sector.

[3]

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15. [Maximum mark: 7]

The equation of the line $y = mx + c$ can be expressed in vector form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$.

- (a) Find the vectors $\mathbf{a}$ and $\mathbf{b}$ in terms of m and/or c . [2]

The matrix $\mathbf{M}$ is defined by $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$.

- (b) Find the value of $\det \mathbf{M}$. [1]

The line $y = mx + c$ (where $m \neq -2$) is transformed into a new line using the transformation described by matrix $\mathbf{M}$.

- (c) Show that the equation of the resulting line does not depend on m or c . [4]



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will not be marked.



16. [Maximum mark: 7]

The position vector of a particle, P , relative to a fixed origin O at time t is given by

$$\vec{\text{OP}} = \begin{pmatrix} \sin(t^2) \\ \cos(t^2) \end{pmatrix}.$$

- (a) Find the velocity vector of P . [2]
- (b) Show that the acceleration vector of P is never parallel to the position vector of P . [5]

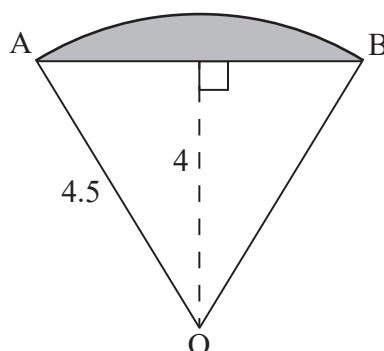
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2. [Maximum mark: 17]

A sector of a circle, centre O and radius 4.5 m, is shown in the following diagram.

diagram not to scale



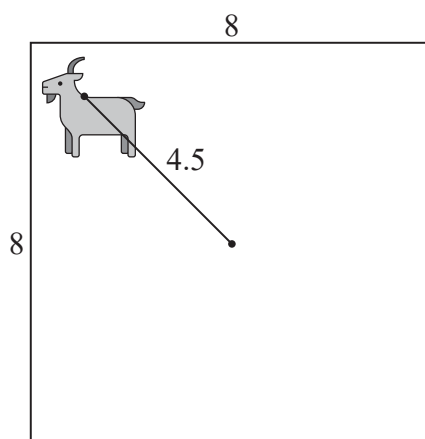
(a) (i) Find the angle $\hat{AOB}$.

(ii) Find the area of the shaded segment.

[8]

A square field with side 8 m has a goat tied to a post in the centre by a rope such that the goat can reach all parts of the field up to 4.5 m from the post.

diagram not to scale



(b) Find the area of the field that can be reached by the goat.

[4]

Let V be the volume of grass eaten by the goat, in cubic metres, and t be the length of time, in hours, that the goat has been in the field.

The goat eats grass at the rate of $\frac{dV}{dt} = 0.3te^{-t}$.

(c) Find the value of t at which the goat is eating grass at the greatest rate.

[2]

The goat is tied in the field for 8 hours.

(d) Find the total volume of grass eaten by the goat during this time.

[3]

7. [Maximum mark: 18]

A transformation, T , of a plane is represented by $\mathbf{r}' = \mathbf{P}\mathbf{r} + \mathbf{q}$, where $\mathbf{P}$ is a 2×2 matrix, $\mathbf{q}$ is a 2×1 vector, $\mathbf{r}$ is the position vector of a point in the plane and $\mathbf{r}'$ the position vector of its image under T .

The triangle OAB has coordinates $(0, 0)$, $(0, 1)$ and $(1, 0)$. Under T , these points are transformed to $(0, 1)$, $\left(\frac{1}{4}, 1 + \frac{\sqrt{3}}{4}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$ respectively.

(a) (i) By considering the image of $(0, 0)$, find $\mathbf{q}$.

(ii) By considering the image of $(1, 0)$ and $(0, 1)$, show that

[6]

$$\mathbf{P} = \begin{pmatrix} \frac{\sqrt{3}}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}.$$

$\mathbf{P}$ can be written as $\mathbf{P} = \mathbf{R}\mathbf{S}$, where $\mathbf{S}$ and $\mathbf{R}$ are matrices.

$\mathbf{S}$ represents an enlargement with scale factor 0.5, centre $(0, 0)$.

$\mathbf{R}$ represents a rotation about $(0, 0)$.

(b) Write down the matrix $\mathbf{S}$.

[1]

(c) (i) Use $\mathbf{P} = \mathbf{R}\mathbf{S}$ to find the matrix $\mathbf{R}$.

(ii) Hence find the angle and direction of the rotation represented by $\mathbf{R}$.

[7]

The transformation T can also be described by an enlargement scale factor $\frac{1}{2}$, centre (a, b) , followed by a rotation about the same centre (a, b) .

(d) (i) Write down an equation satisfied by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(ii) Find the value of a and the value of b .

[4]

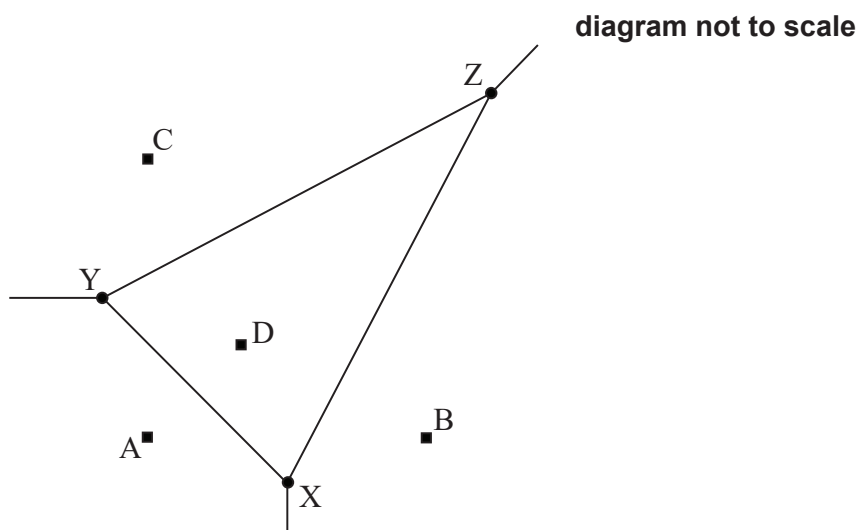
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2. mynamepong, n.d. Goat [image online] Available at: <https://thenounproject.com/term/goat/1761571/>
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3. [Maximum mark: 18]

The Voronoi diagram below shows four supermarkets represented by points with coordinates $A(0, 0)$, $B(6, 0)$, $C(0, 6)$ and $D(2, 2)$. The vertices X , Y , Z are also shown. All distances are measured in kilometres.



(a) Find the midpoint of $[BD]$. [2]

(b) Find the equation of (XZ) . [4]

The equation of (XY) is $y = 2 - x$ and the equation of (YZ) is $y = 0.5x + 3.5$.

(c) Find the coordinates of X . [3]

The coordinates of Y are $(-1, 3)$ and the coordinates of Z are $(7, 7)$.

(d) Determine the exact length of $[YZ]$. [2]

(e) Given that the exact length of $[XY]$ is $\sqrt{32}$, find the size of $\hat{XYZ}$ in degrees. [4]

(f) Hence find the area of triangle XYZ . [2]

A town planner believes that the larger the area of the Voronoi cell XYZ , the more people will shop at supermarket D .

(g) State one criticism of this interpretation. [1]

6. [Maximum mark: 21]

At an archery tournament, a particular competition sees a ball launched into the air while an archer attempts to hit it with an arrow.

The path of the ball is modelled by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} u_x \\ u_y - 5t \end{pmatrix}$$

where x is the horizontal displacement from the archer and y is the vertical displacement from the ground, both measured in metres, and t is the time, in seconds, since the ball was launched.

- u_x is the horizontal component of the initial velocity
- u_y is the vertical component of the initial velocity.

In this question both the ball and the arrow are modelled as single points. The ball is launched with an initial velocity such that $u_x = 8$ and $u_y = 10$.

- (a) (i) Find the initial speed of the ball.
- (ii) Find the angle of elevation of the ball as it is launched. [4]
- (b) Find the maximum height reached by the ball. [3]
- (c) Assuming that the ground is horizontal and the ball is not hit by the arrow, find the x coordinate of the point where the ball lands. [3]
- (d) For the path of the ball, find an expression for y in terms of x . [3]

An archer releases an arrow from the point $(0, 2)$. The arrow is modelled as travelling in a straight line, in the same plane as the ball, with speed 60 m s^{-1} and an angle of elevation of 10° .

- (e) Determine the two positions where the path of the arrow intersects the path of the ball. [4]
- (f) Determine the time when the arrow should be released to hit the ball before the ball reaches its maximum height. [4]

Turn over

2. [Maximum mark: 27]

This question is about a metropolitan area council planning a new town and the location of a new toxic waste dump.

A metropolitan area in a country is modelled as a square. The area has four towns, located at the corners of the square. All units are in kilometres with the x -coordinate representing the distance east and the y -coordinate representing the distance north from the origin at $(0, 0)$.

- Edison is modelled as being positioned at $E(0, 40)$.
- Fermi town is modelled as being positioned at $F(40, 40)$.
- Gaussville is modelled as being positioned at $G(40, 0)$.
- Hamilton is modelled as being positioned at $H(0, 0)$.

- (a) The model assumes that each town is positioned at a single point. Describe possible circumstances in which this modelling assumption is reasonable. [1]
- (b) Sketch a Voronoi diagram showing the regions within the metropolitan area that are closest to each town. [1]

The metropolitan area council decides to build a new town called Isaacopolis located at $I(30, 20)$.

A new Voronoi diagram is to be created to include Isaacopolis. The equation of the perpendicular bisector of $[IE]$ is $y = \frac{3}{2}x + \frac{15}{2}$.

- (c) (i) Find the equation of the perpendicular bisector of $[IF]$. [4]
- (ii) Given that the coordinates of one vertex of the new Voronoi diagram are $(20, 37.5)$, find the coordinates of the other two vertices within the metropolitan area. [4]
- (iii) Sketch this new Voronoi diagram showing the regions within the metropolitan area which are closest to each town. [2]

The metropolitan area is divided into districts based on the Voronoi regions found in part (c).

- (d) A car departs from a point due north of Hamilton. It travels due east at constant speed to a destination point due North of Gaussville. It passes through the Edison, Isaacopolis and Fermi town districts. The car spends 30% of the travel time in the Isaacopolis district.

Find the distance between Gaussville and the car's destination point. [4]

(This question continues on the following page)

(Question 2 continued)

A toxic waste dump needs to be located within the metropolitan area. The council wants to locate it as far as possible from the nearest town.

- (e) (i) Find the location of the toxic waste dump, given that this location is not on the edge of the metropolitan area. [4]
- (ii) Make one possible criticism of the council's choice of location. [1]
- (f) The toxic waste dump, T, is connected to the towns via a system of sewers.

The connections are represented in the following matrix, M , where the order of rows and columns is (E, F, G, H, I, T).

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

A leak occurs from the toxic waste dump and travels through the sewers. The pollution takes one day to travel between locations that are directly connected.

The digit 1 in M represents a direct connection. The values of 1 in the leading diagonal of M mean that once a location is polluted it will stay polluted.

- (i) Find which town is last to be polluted. Justify your answer. [3]
- (ii) Write down the number of days it takes for the pollution to reach the last town. [1]
- (iii) A sewer inspector needs to plan the shortest possible route through each of the connections between different locations. Determine an appropriate start point and an appropriate end point of the inspection route.

Note that the fact that each location is connected to itself does not correspond to a sewer that needs to be inspected. [2]

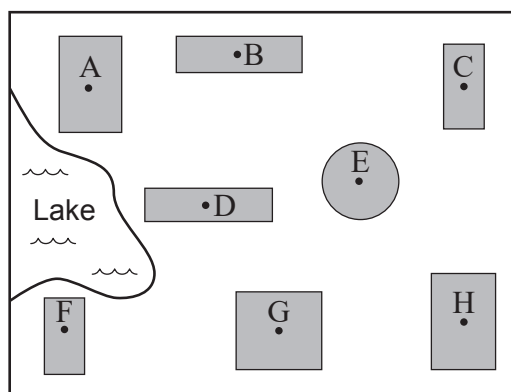
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2. [Maximum mark: 28]

This question compares possible designs for a new computer network between multiple school buildings, and whether they meet specific requirements.

A school's administration team decides to install new fibre-optic internet cables underground. The school has eight buildings that need to be connected by these cables. A map of the school is shown below, with the internet access point of each building labelled A–H.



Jonas is planning where to install the underground cables. He begins by determining the distances, in metres, between the underground access points in each of the buildings.

He finds $AD = 89.2\text{ m}$, $DF = 104.9\text{ m}$ and $\angle ADF = 83^\circ$.

- (a) Find AF . [3]

The cost for installing the cable directly between A and F is \$21 310.

- (b) Find the cost per metre of installing this cable. [2]

Jonas estimates that it will cost \$110 per metre to install the cables between all the other buildings.

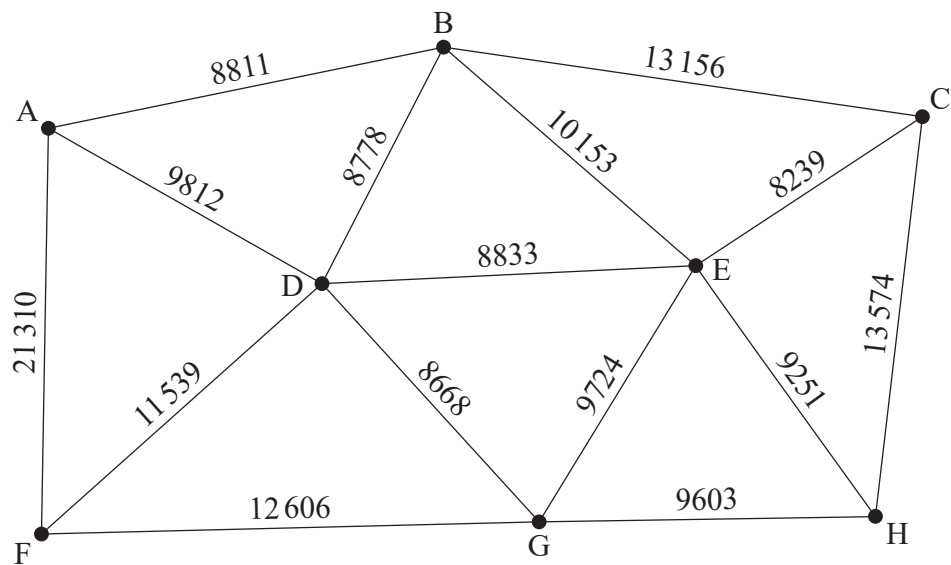
- (c) State why the cost for installing the cable between A and F would be higher than between the other buildings. [1]

(This question continues on the following page)

Turn over

(Question 2 continued)

Jonas creates the following graph, S , using the cost of installing the cables between two buildings as the weight of each edge.



The computer network could be designed such that each building is directly connected to at least one other building and hence all buildings are indirectly connected.

- (d) (i) By using Kruskal's algorithm, find the minimum spanning tree for S , showing clearly the order in which edges are added. [3]
- (ii) Hence find the minimum installation cost for the cables that would allow all the buildings to be part of the computer network. [2]

The computer network fails if any part of it becomes unreachable from any other part. To help protect the network from failing, every building could be connected to at least two other buildings. In this way if one connection breaks, the building is still part of the computer network. Jonas can achieve this by finding a Hamiltonian cycle within the graph.

- (e) State why a path that forms a Hamiltonian cycle does not always form an Eulerian circuit. [1]
- (f) Starting at D, use the nearest neighbour algorithm to find the upper bound for the installation cost of a computer network in the form of a Hamiltonian cycle.

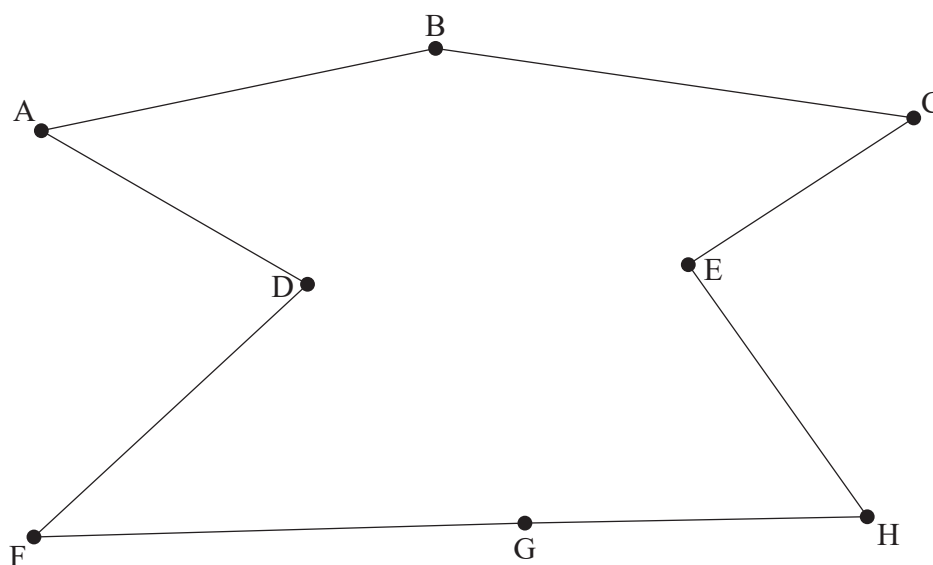
Note: Although the graph is not complete, in this instance it is not necessary to form a table of least distances. [5]

- (g) By deleting D, use the deleted vertex algorithm to find the lower bound for the installation cost of the cycle. [6]

(This question continues on the following page)

(Question 2 continued)

After more research, Jonas decides to install the cables as shown in the diagram below.



Each individual cable is installed such that each end of the cable is connected to a building's access point. The connection between each end of a cable and an access point has a 1.4% probability of failing after a power surge.

For the network to be successful, each building in the network must be able to communicate with every other building in the network. In other words, there must be a path that connects any two buildings in the network. Jonas would like the network to have less than a 2% probability of failing to operate after a power surge.

- (h) Show that Jonas's network satisfies the requirement of there being less than a 2% probability of the network failing after a power surge.

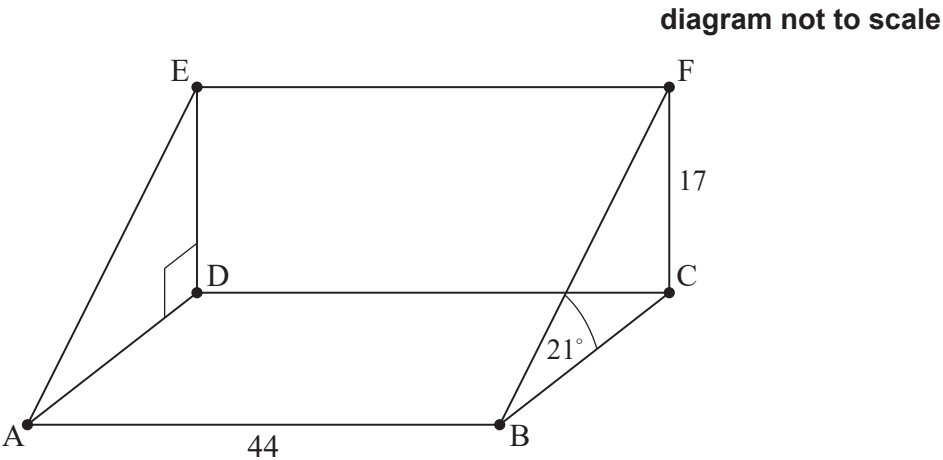
[5]

References:

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3. [Maximum mark: 5]

An artificial ski slope can be modelled as a triangular prism, as shown in the diagram. Rectangle ABCD is horizontal, and rectangle CDEF is vertical.



The maximum height of the ski slope, CF, is 17 metres and the steepest angle of the ski slope, $\hat{FBC}$, is 21° .

- (a) Calculate the length of [BF]. [2]

The width of the base of the ski slope, AB, is 44 metres. Mayumi skis in a straight line, starting from point E and finishing at the base of the ski slope.

- (b) Find the value of the least steep angle that Mayumi can ski. [3]

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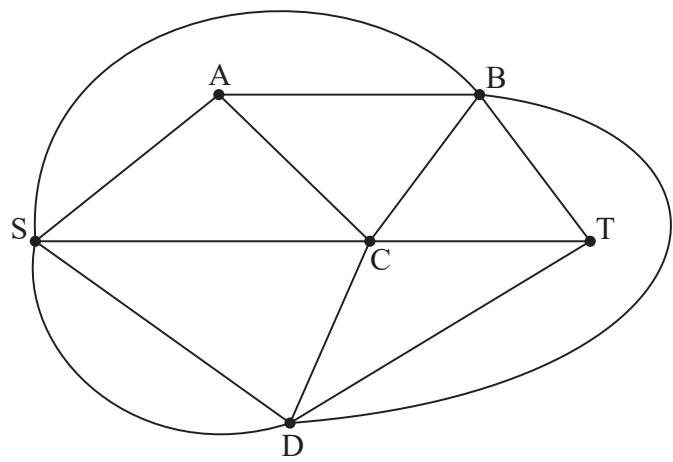
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4. [Maximum mark: 7].

In a competition, a contestant has to move through a maze to find treasure. A graph of the maze is shown below, where each edge represents a corridor in the maze. The contestant starts at S and the treasure is located at T.



(a) Complete the adjacency matrix, M , for the graph. [2]

	S	A	B	C	D	T
S	0	1	1	1	☐	0
A	1	0	1	1	☐	0
B	1	1	0	1	1	1
C	1	1	1	0	1	1
D	☐	☐	1	1	0	1
T	0	0	1	1	1	0

The competition rules state that the contestant can walk along a maximum of four corridors.

(b) Find the number of walks from S to T with a maximum of 4 edges. [4]

(c) Explain why the number of ways the contestant can reach the treasure is less than the answer to part (b). [1]

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(Question 4 continued)

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8. [Maximum mark: 7]

Line L_1 has a vector equation $\mathbf{r} = \begin{pmatrix} 3p+4 \\ 2p-1 \\ p+9 \end{pmatrix}$, where $p \in \mathbb{R}$.

Line L_2 has a vector equation $\mathbf{r} = \begin{pmatrix} q-2 \\ 1-q \\ 2q+1 \end{pmatrix}$, where $q \in \mathbb{R}$.

The two lines intersect at point M.

- (a) Find the coordinates of M.

[3]
- (b) Find the acute angle between the two lines.

[4]

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The transformation T is represented by the matrix $\mathbf{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

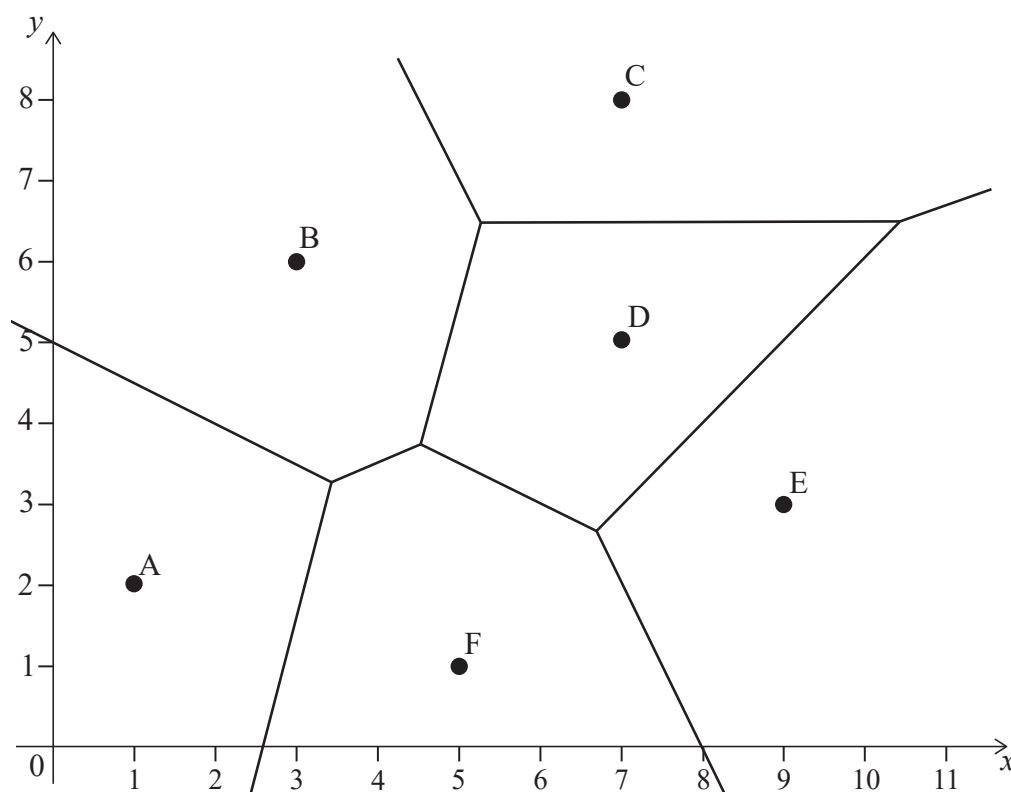
(a) Find the area of the image of the pentagon. [2]

(b) Find, in terms of t , the coordinates of X. [6]

[illegible]

2. [Maximum mark: 13]

Six restaurant locations (labelled A, B, C, D, E and F) are shown, together with their Voronoi diagram. All distances are measured in kilometres.



- (a) Elena wants to eat at the closest restaurant to her. Write down the restaurant she should go to, if she is at

(i) $(2, 7)$.

(ii) $(0, 1)$, when restaurant A is closed.

[2]

Restaurant C is at $(7, 8)$ and restaurant D is at $(7, 5)$.

- (b) Find the equation of the perpendicular bisector of CD.

[2]

Restaurant B is at $(3, 6)$.

- (c) Find the equation of the perpendicular bisector of BC.

[5]

- (d) Hence find

(i) the coordinates of the point which is of equal distance from B, C and D.

(ii) the distance of this point from D.

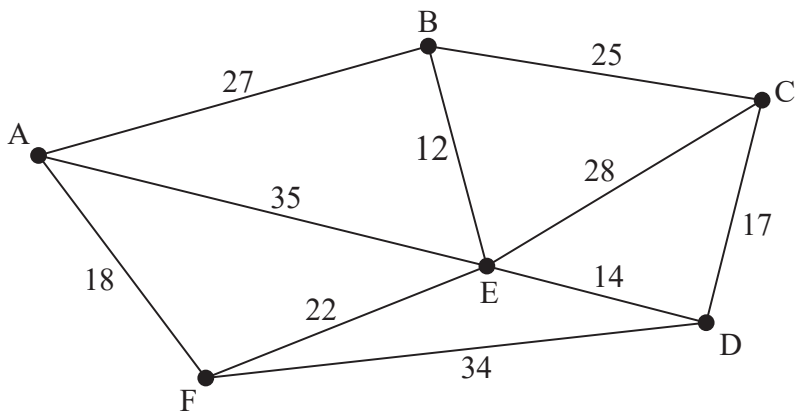
[4]

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4. [Maximum mark: 14]

A company has six offices, A, B, C, D, E and F. One of the company managers, Nanako, needs to visit the offices. She creates the following graph that shows the distances, in kilometres, between some of the offices.

diagram not to scale



- (a) Write down a Hamiltonian cycle for this graph.

[1]
- (b) State, with a reason, whether the graph contains an Eulerian circuit.

[1]

Nanako wishes to find the shortest cycle to visit all the offices. She decides to complete a weighted adjacency table, showing the least distance between each pair of offices.

	A	B	C	D	E	F
A		27	52	p	35	18
B			25	26	12	q
C				17	28	r
D					14	34
E						22
F						

- (c) Write down the value of
- (i) p .
- (ii) q .
- (iii) r .

[3]

(This question continues on the following page)

(Question 4 continued)

- (d) Starting at vertex E, use the nearest neighbour algorithm to find an upper bound for Nanako's cycle. [3]
- (e) By deleting vertex F, find a lower bound for Nanako's cycle. [4]
- (f) Explain, with a reason, why the answer to part (e) might not be the best lower bound. [2]

Turn over

7. [Maximum mark: 20]

The position vector of a particle at time t is given by $\mathbf{r} = 3 \cos(3t)\mathbf{i} + 4 \sin(3t)\mathbf{j}$.
Displacement is measured in metres and time is measured in seconds.

- (a) (i) Find an expression for the velocity of the particle at time t .
(ii) Hence find the speed when $t = 3$. [4]

- (b) (i) Find an expression for the acceleration of the particle at time t .
(ii) Hence show that the acceleration is always directed towards the origin. [4]

The position vector of a second particle is given by $\mathbf{r} = -4 \sin(4t)\mathbf{i} + 3 \cos(4t)\mathbf{j}$.

- (c) For $0 \leq t \leq 10$, find the time when the two particles are closest to each other. [5]

At time k , where $0 < k < 1.5$, the second particle is moving parallel to the first particle.

- (d) (i) Find the value of k .
(ii) At time k , show that the two particles are moving in the opposite direction. [7]

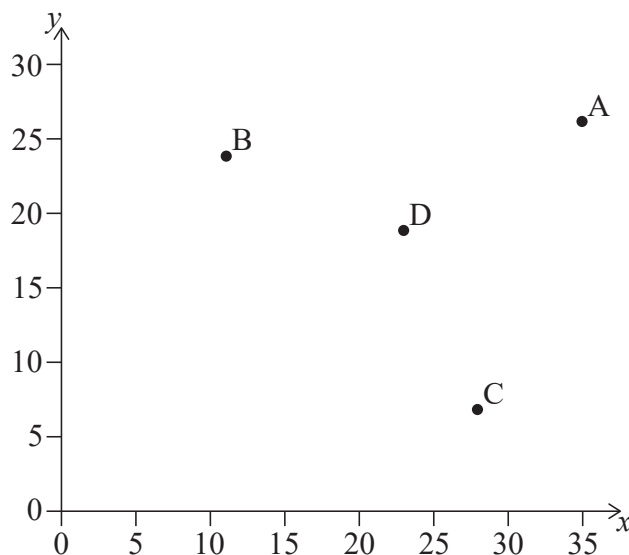
References:

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3. [Maximum mark: 5]

Three towns have positions $A(35, 26)$, $B(11, 24)$, and $C(28, 7)$ according to the coordinate system shown where distances are measured in miles.

Dominique's farm is located at the position D(24, 19).



(a) Find AD .

[2]

On a particular day, the mean temperatures recorded in each of towns A, B and C are 34°C , 29°C and 30°C respectively.

(b) Use nearest neighbour interpolation to estimate the temperature at Dominique's farm on that particular day.

[3]

[illegible]

Turn over

5. [Maximum mark: 5]

A boat travels 8 km on a bearing of 315° and then a further 6 km on a bearing of 045° .
Find the bearing on which the boat should travel to return directly to the starting point.



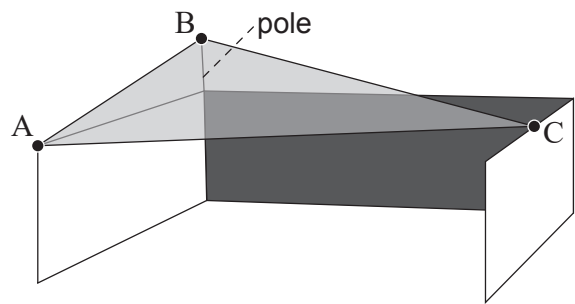
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7. [Maximum mark: 9]

A triangular cover is positioned over a walled garden to provide shade. It is anchored at points A and C, located at the top of a 2 m wall, and at a point B, located at the top of a 1 m vertical pole fixed to a top corner of the wall.

The three edges of the cover can be represented by the vectors

$\vec{AB} = \begin{pmatrix} 0 \\ 6 \\ 1 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$ and $\vec{BC} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix}$, where distances are measured in metres.



- (a) Calculate the vector product $\vec{AB} \times \vec{AC}$.

[2]
- (b) Hence find the area of the triangular cover.

[2]

The point X on [AC] is such that [BX] is perpendicular to [AC].

- (c) Use your answer to part (b) to find the distance BX.

[3]
- (d) Find the angle the cover makes with the horizontal plane.

[2]

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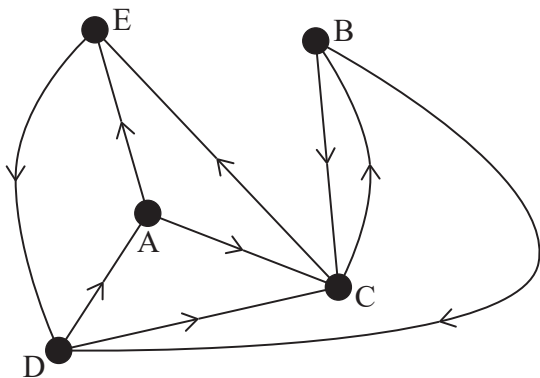
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8. [Maximum mark: 7]

The following directed, unweighted, graph shows a simplified road network on an island, connecting five small villages marked A to E.



- (a) Construct the adjacency matrix M for this network. [3]

Beatriz the bus driver starts at village E and drives to seven villages, such that the seventh village is A.

- (b) (i) Determine how many possible routes Beatriz could have taken, to travel from E to A.
(ii) Describe one possible route taken by Beatriz, by listing the villages visited in order. [4]



Turn over

13. [Maximum mark: 7]

The matrices $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\mathbf{P}$, and this image is then transformed by $\mathbf{Q}$ to form a new triangle, T' .

- (a) Find the single matrix that represents the transformation $T' \rightarrow T$, which will undo the transformation described above.

[4]

The area of T' is 273 cm^2 .

- (b) Using your answer to part (a), or otherwise, determine the area of T .

[3]

[illegible]

Turn over

14. [Maximum mark: 8]

In this question, i denotes a unit vector due east, and j denotes a unit vector due north.

Two ships, A and B, are each moving with constant velocities.

The position vector of ship A, at time t hours, is given as $\mathbf{r}_A = (1 + 2t)\mathbf{i} + (3 - 3t)\mathbf{j}$.

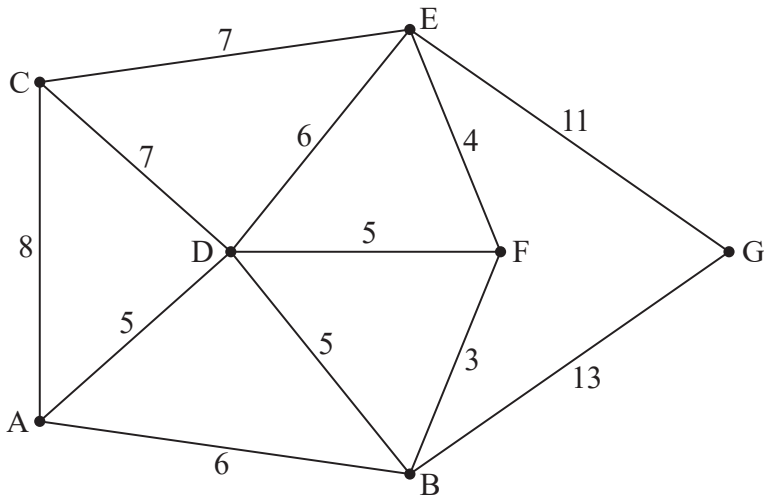
The position vector of ship B, at time t hours, is given as $\mathbf{r}_B = (-2 + 4t)\mathbf{i} + (-4 + t)\mathbf{j}$.

- (a) Find the bearing on which ship A is sailing. [3]
- (b) Find the value of t when ship B is directly south of ship A. [2]
- (c) Find the value of t when ship B is directly south-east of ship A. [3]

[illegible]

4. [Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



- (a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer.

[2]

The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A		6	8	5	11	9	19
B	6		12	5	7	3	13
C	8	12		7	7	<i>a</i>	<i>b</i>
D	5	5	7		6	5	<i>c</i>
E	11	7	7	6		4	11
F	9	3	<i>a</i>	5	4		<i>d</i>
G	19	13	<i>b</i>	<i>c</i>	11	<i>d</i>	

(This question continues on the following page)

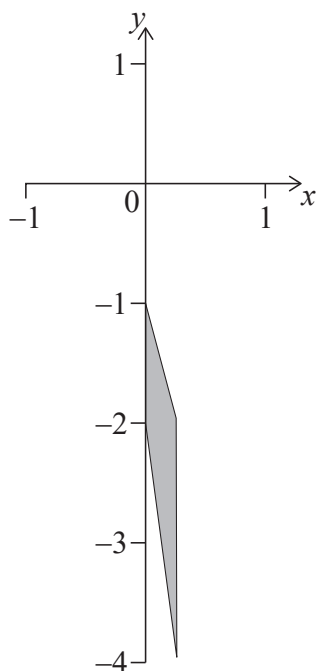
(Question 4 continued)

- (b) Find the value of
- (i) a ;
 - (ii) b ;
 - (iii) c ;
 - (iv) d . [2]
- (c) Use the nearest neighbour algorithm, starting at vertex G , to find an upper bound for the travelling salesman problem. [3]
- (d) (i) Sketch a minimum spanning tree for the subgraph with vertices A, B, C, D, E, F .
- (ii) Write down the total weight of the minimum spanning tree. [4]
- (e) Hence find a lower bound for the travelling salesman problem. [2]
- (f) Explain one way in which an improved lower bound could be found. [1]
- It is found that the optimum solution starting at A is actually $A-C-E-G-B-F-D-A$.
- (g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]

Turn over

7. [Maximum mark: 18]

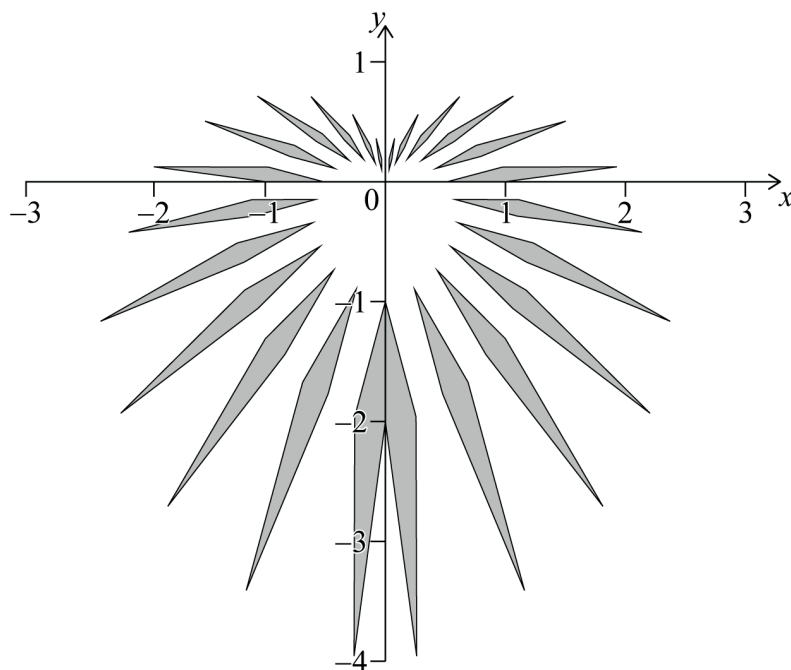
A trapezoid, Q , has vertices $(0, -1)$, $(0, -2)$, $(\sin 15^\circ, -3 - \cos 15^\circ)$, $(\sin 15^\circ, -1 - \cos 15^\circ)$ as shown.



- (a) Show that the area of the trapezoid is $\frac{3}{2}\sin 15^\circ$.

[2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid Q . These elements are shaded in the following diagram such that the y -axis is a line of symmetry.



(This question continues on the following page)

(Question 7 continued)

The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$\mathbf{M}_k = \begin{pmatrix} \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) & -\left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) \\ \left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) & \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) \end{pmatrix}$$

where $k = 0, 1, 2, 3, \dots, 11$.

- (b) (i) Find the matrix $\mathbf{M}_6$. Give your answer in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Q}$.
 (ii) Hence find the coordinates of the image of the vertex $(0, -1)$ after it is transformed by the matrix $\mathbf{M}_6$. [4]

The matrix $\mathbf{M}_k$ can be expressed as the product of a rotation matrix and an enlargement matrix.

- (c) Write down, in terms of k ,
 (i) the rotation matrix;
 (ii) the enlargement matrix;
 (iii) the angle of the rotation;
 (iv) the scale factor of the enlargement. [4]
 (d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix $\mathbf{M}_k$ in terms of k . [2]
 (e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix $\mathbf{N}_k$, where $k = 0, 1, 2, 3, \dots, 11$.

- (f) Write down the matrix $\mathbf{N}_k$ as a product of two matrices. [2]

References:

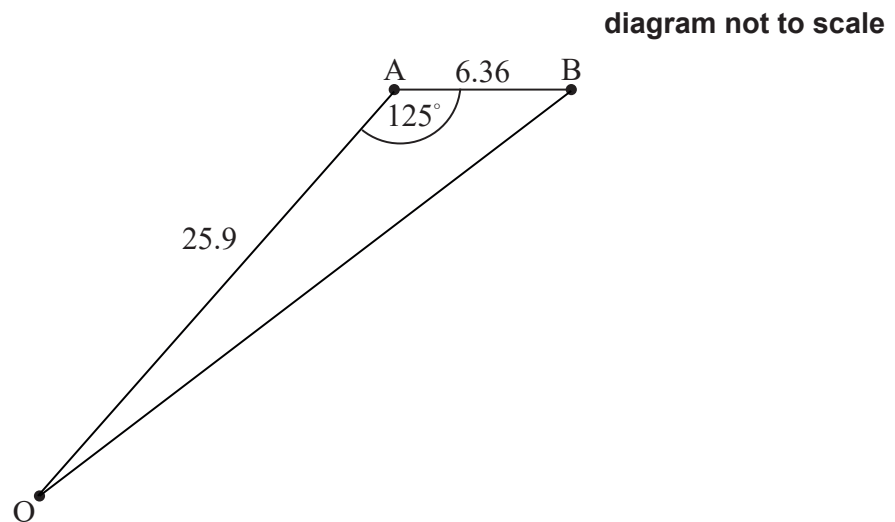
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1. [Maximum mark: 13]

The diagram shows points in a park viewed from above, at a specific moment in time.

The distance between two trees, at points A and B, is 6.36 m.

Odette is playing football in the park and is standing at point O, such that $OA = 25.9$ m and $\angle AOB = 125^\circ$.



- (a) Calculate the area of triangle AOB.

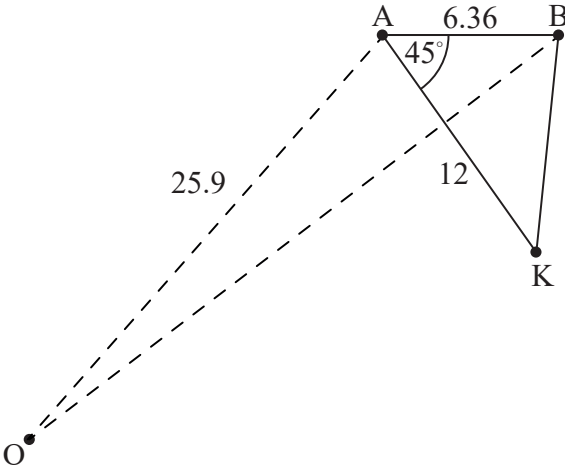
[3]

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(Question 1 continued)

Odette’s friend, Khemil, is standing at point K such that he is 12 m from A and $\widehat{KAB} = 45^\circ$.

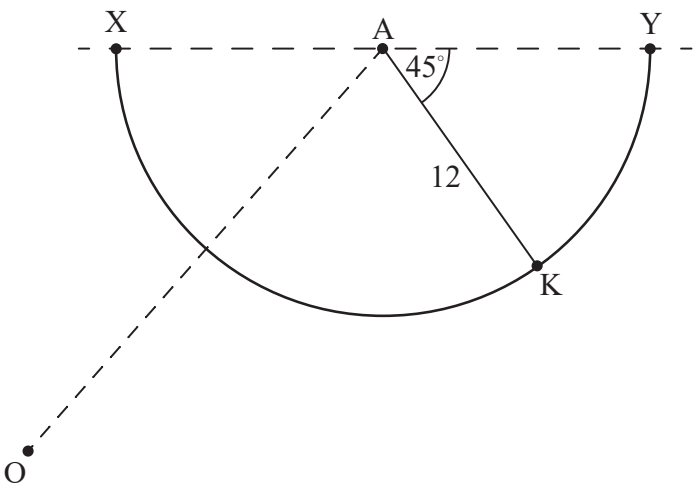
diagram not to scale



- (b) Calculate Khemil’s distance from B. [3]

XY is a semicircular path in the park with centre A, such that $\widehat{KAY} = 45^\circ$. Khemil is standing on the path and Odette’s football is at point X. This is shown in the diagram below.

diagram not to scale



The length $KX = 22.2$ m, $\widehat{KOX} = 53.8^\circ$ and $\widehat{OKX} = 51.1^\circ$.

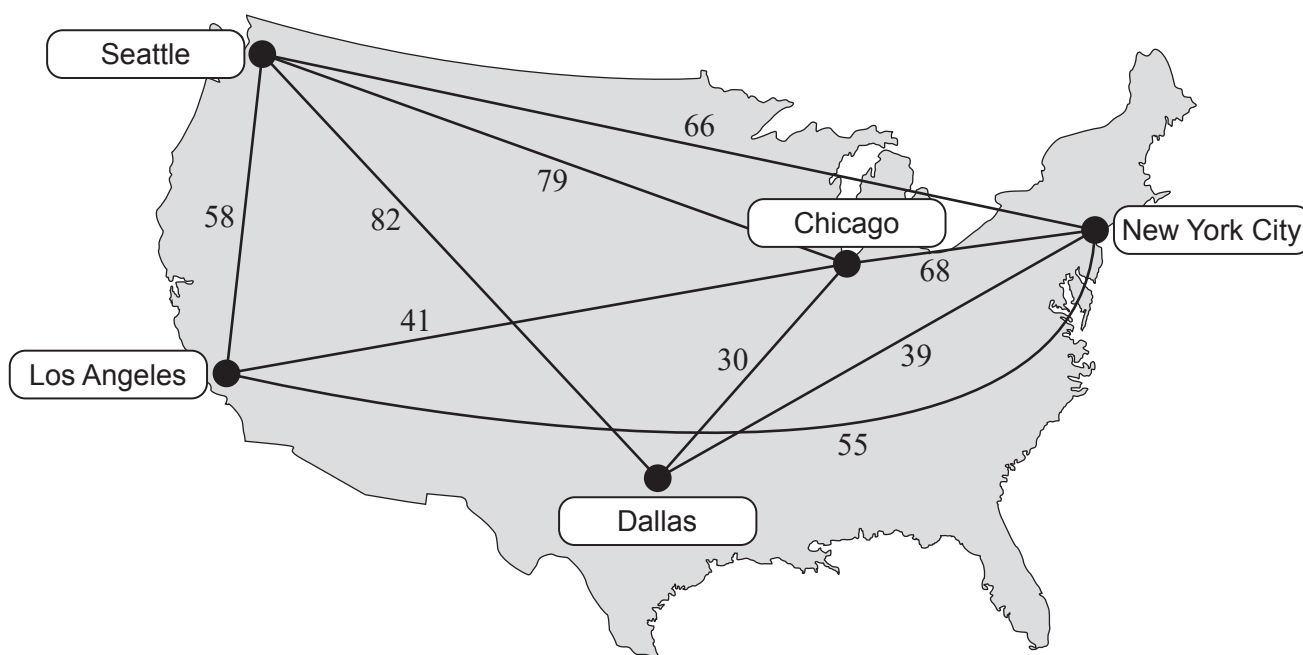
- (c) Find whether Odette or Khemil is closer to the football. [4]

Khemil runs along the semicircular path to pick up the football.

- (d) Calculate the distance that Khemil runs. [3]

4. [Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars (\$) between cities.



(a) Explain why the graph can be described as “connected”, but not “complete”. [2]

(b) Find a minimum spanning tree for the graph using Kruskal’s algorithm. [3]

State clearly the order in which your edges are added, and draw the tree obtained.

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing \$26. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c). [3]

State clearly the order in which you are adding the vertices.

(This question continues on the following page)

(Question 4 continued)

- (e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem. [5]
- (ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]
- (f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]
- (g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

Turn over

2. [Maximum mark: 29]

In this question you will use vector methods to determine whether aircraft are obeying air traffic regulations.

The base of an air traffic control tower at an airport is taken as the origin of a coordinate system. An aircraft's position is given by the coordinates (x, y, z) , where x and y are respectively the aircraft's displacement east and north of the tower, and z is the vertical displacement of the aircraft above the base of the tower. All displacements are measured in kilometres.

At 12:00 two aircraft, A and B, are at the points P(100, -82, 10.7) and Q(215, -197, 10.7) respectively.

- (a) Find the distance between the two aircraft at 12:00. [2]

The two aircraft are flying along the same straight line (flight path), with B behind A.

They both have the same constant velocity of $\begin{pmatrix} -640 \\ 640 \\ 0 \end{pmatrix}$ kilometres per hour.

- (b) Find the speed of both aircraft. [2]

Air traffic regulations state that if two aircraft are on the same flight path then they must always maintain at least a 10 minute gap between them. If at any time two aircraft are too close they are said to be "in conflict".

- (c) Find the length of time it takes B to reach point P from point Q, and hence state whether the two aircraft are in conflict. [3]

- (d) Write down, $\mathbf{r}_A$, the position vector of A, t hours after 12:00. [1]

If two aircraft are not on the same flight path, air traffic regulations state:

When the vertical distance between the two aircraft is less than 300 m the aircraft must be more than 10 km apart.

When the vertical distance between the two aircraft is at least 300 m there are no restrictions.

The air traffic controller notices an aircraft, C, flying on a different flight path but close to A. The position of C, t hours after 12:00, is given by

$$\mathbf{r}_C = \begin{pmatrix} -400 \\ -41 \\ 9.1 \end{pmatrix} + t \begin{pmatrix} -140 \\ 604 \\ 2 \end{pmatrix}.$$

(This question continues on the following page)

(Question 2 continued)

- (e) (i) Find the two values of t at which the distance between A and C is 10 km. [5]

It is given that the distance between A and C is less than 10 km, only between these two values of t .

- (ii) Determine whether the two aircraft, A and C, will break the air traffic regulations if they continue with their current velocities. Justify your answer. [5]

A new coordinate system, (x, y) , is defined with an origin R at a point 2 km directly above the air traffic control tower. When an aircraft is flying in the horizontal plane containing R, the values of x and y represent its displacement, in km, east and north of point R respectively.

A fourth aircraft, D, is flying at a constant height of 2 km near the airport while waiting for permission to land. Its position at time t is given by

$$\vec{RD} = \begin{pmatrix} 6.4 \cos(38t) \\ 6.4 \sin(38t) \end{pmatrix}.$$

- (f) Describe the path followed by D. [3]

Around the same time a small aircraft, E, is flying at the same height as D and along the line with vector equation

$$\vec{RE} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- (g) Let $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

- (i) Find $\vec{RE} \cdot \mathbf{b}$ in terms of λ . [2]
- (ii) Hence find the value of λ for which the distance from R to the line is minimum. [2]
- (iii) Find this minimum distance. [2]
- (iv) Show that D and E will not break the air traffic regulations for any value of t . [2]

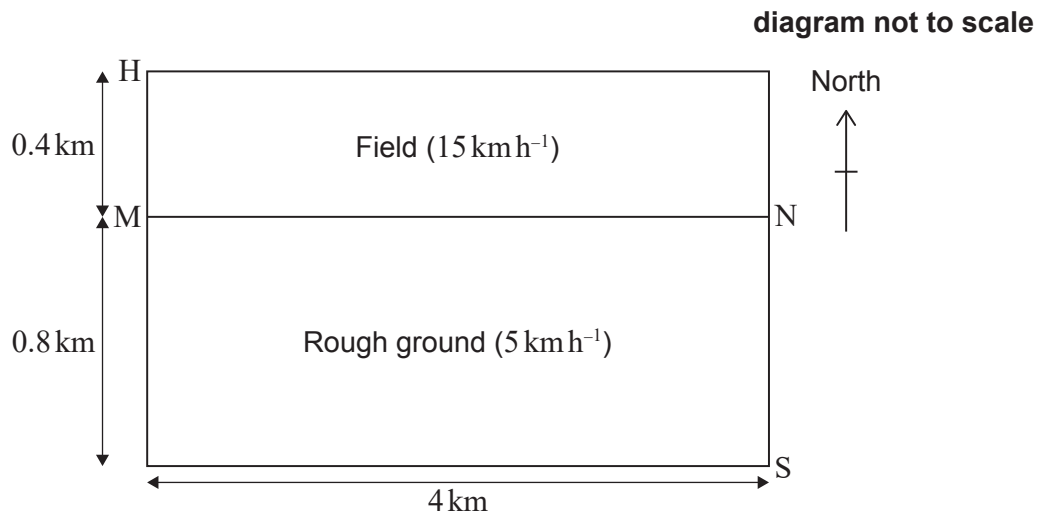
References:

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1. [Maximum mark: 26]

This question considers the optimal route between two points, separated by several regions where different speeds are possible.

Huw lives in a house, H, and he attends a school, S, where H and S are marked on the following diagram. The school is situated 1.2 km south and 4 km east of Huw's house. There is a boundary [MN], going from west to east, 0.4 km south of his house. The land north of [MN] is a field over which Huw runs at 15 kilometres per hour (km h^{-1}). The land south of [MN] is rough ground over which Huw walks at 5 km h^{-1} . The two regions are shown in the following diagram.



- (a) Huw travels in a straight line from H to S. Calculate the time that Huw takes to complete this journey. Give your answer correct to the nearest minute. [6]
- (b) Huw realizes that his journey time could be reduced by taking a less direct route. He therefore defines a point P on [MN] that is x km east of M. Huw decides to run from H to P and then walk from P to S. Let $T(x)$ represent the time, in hours, taken by Huw to complete the journey along this route.

(i) Show that $T(x) = \frac{\sqrt{0.4^2 + x^2} + 3\sqrt{0.8^2 + (4-x)^2}}{15}$. [3]

(ii) Sketch the graph of $y = T(x)$. [2]

(This question continues on the following page)

(Question 1 continued)

(iii) Hence determine the value of x that minimizes $T(x)$. [1]

(iv) Find by how much Huw's journey time is reduced when he takes this optimal route, compared to travelling in a straight line from H to S. Give your answer correct to the nearest minute. [2]

(c) (i) Determine an expression for the derivative $T'(x)$. [3]

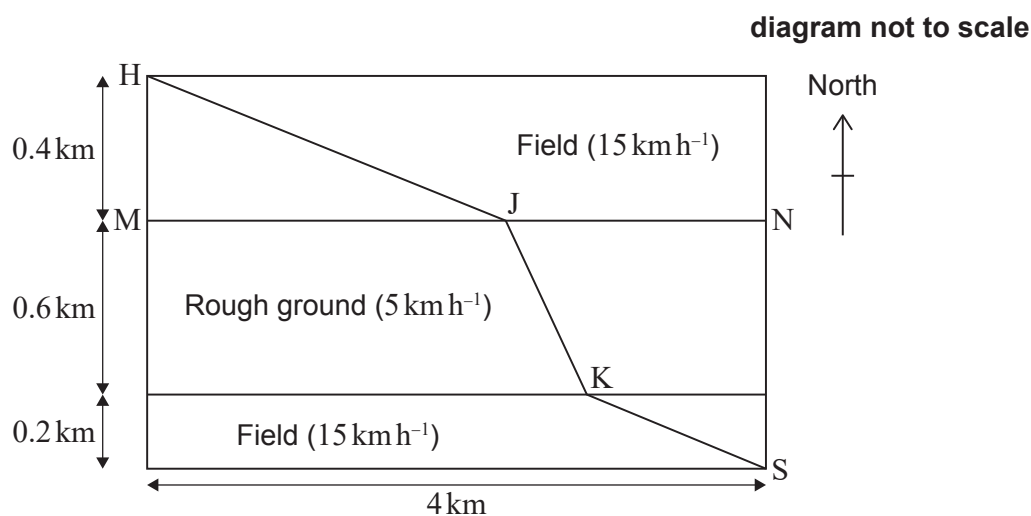
(ii) Hence show that $T(x)$ is minimized when

$$\frac{x}{\sqrt{0.16 + x^2}} = \frac{3(4 - x)}{\sqrt{0.64 + (4 - x)^2}}. \quad [1]$$

(iii) For the optimal route, verify that the equation in part (c)(ii) satisfies the following result:

$$\frac{\cos \hat{H}PM}{\cos \hat{S}PN} = \frac{\text{speed over field}}{\text{speed over rough ground}}. \quad [2]$$

(d) The owner of the rough ground converts the southern quarter into a field over which Huw can run at 15 km h^{-1} . The following diagram shows the optimal route, HJKS, in this new situation. You are given that [HJ] is parallel to [KS].



Using a similar result to that given in part (c)(iii), at the point J, determine MJ. [6]

Turn over

3. [Maximum mark: 6]

The Great Pyramid of Giza is the oldest of the Seven Wonders of the Ancient World. When it was built, 4500 years ago, the measurements of the pyramid were in Royal Egyptian Cubits (REC).



[Source: Nina Aldin Thune. https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza#/media/File:Kheops-Pyramid.jpg. Licensed under CC BY 2.5 <https://creativecommons.org/licenses/by/2.5/#>. Image adapted.]

Viktor reads online that 1 REC is equal to 0.52 metres, rounded to two decimal places.

- (a) Write down the upper and lower bounds of 1 REC in metres. [2]

The Great Pyramid of Giza has a square base with side lengths of 440 REC and a height of 280 REC. Viktor assumes that these two measurements are exact and that the Great Pyramid can be modelled as a square-based pyramid with smooth faces.

- (b) Find the minimum possible volume of the pyramid in cubic metres. [4]

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Jason sails his boat from point A for a distance of 100 km, on a bearing of 112° , to arrive at point B. He then sails on a bearing of 041° to point C. Jason's journey is shown in the diagram.

- Point C is directly east of point A.

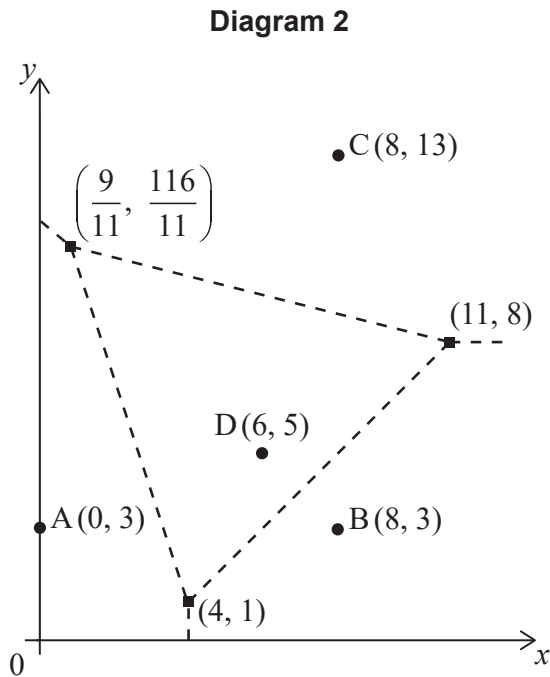
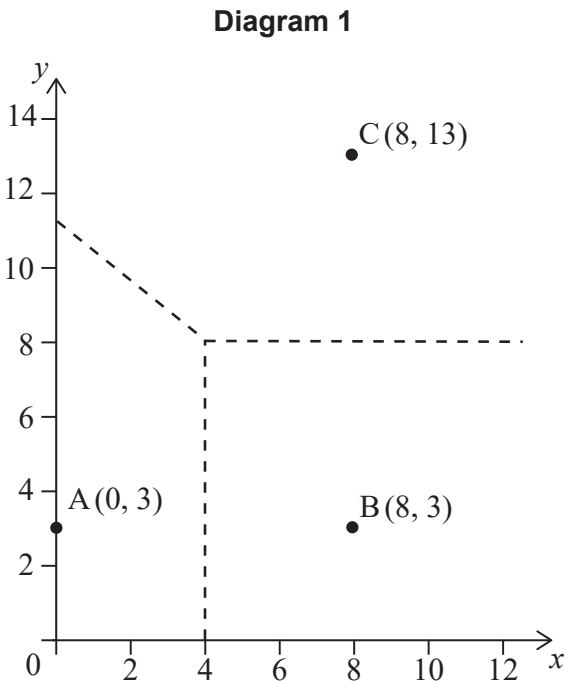
- (b) Calculate the distance that Jason sails to return directly from point C to point A. [4]

[illegible]

Turn over

6. [Maximum mark: 8]

On the following Voronoi diagram, the coordinates of three farmhouses are $A(0, 3)$, $B(8, 3)$ and $C(8, 13)$, where distances are measured in kilometres. Each farmhouse owns the land that is closest to it, and their boundaries are defined by the dotted lines on **Diagram 1**.



To provide water to the farms it is decided to construct a well at the point where the boundaries meet on **Diagram 1**.

- (a) Write down the coordinates of this point. [1]
- (b) Find the equation of the perpendicular bisector of $[AC]$. [3]

An additional farmhouse $D(6, 5)$ is built on the land. The Voronoi diagram has been redrawn to show the new boundaries. The coordinates of the vertices of these boundaries are indicated on **Diagram 2**.

A wind turbine is to be built at one of the vertices.

- (c) The wind turbine should be as far from the nearest farmhouses as possible.
 - (i) By calculating appropriate distances, find the location of the wind turbine.
 - (ii) Hence, write down the distance of the wind turbine to the nearest farmhouse. [4]

(This question continues on the following page)



(Question 6 continued)

[illegible]

Turn over

14. [Maximum mark: 7]

A straight line L has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and point Q has coordinates $(11, -1, 3)$.

Point P is the point on L closest to Q.

- (a) Find the coordinates of P. [4]
- (b) Find a vector that is perpendicular to both L and the line passing through points P and Q. [3]

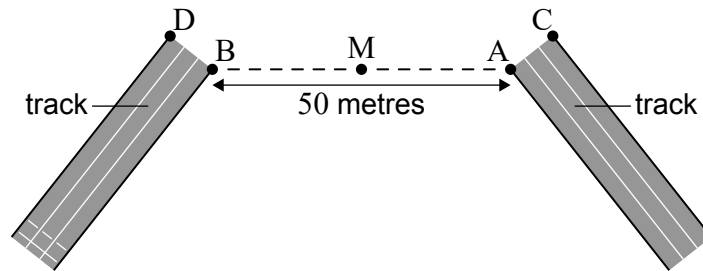


1. [Maximum mark: 15]

Madhu is designing a jogging track for the campus of her school. The following diagram shows an incomplete portion of the track.

Madhu wants to design the track such that the inner edge is a smooth curve from point A to point B, and the other edge is a smooth curve from point C to point D. The distance between points A and B is 50 metres.

diagram not to scale



To create a smooth curve, Madhu first walks to M, the midpoint of [AB].

- (a) Write down the length of [BM].

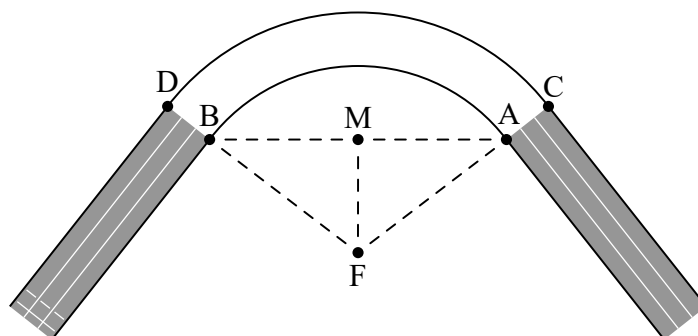
[1]

(This question continues on the following page)

(Question 1 continued)

Madhu then walks 20 metres in a direction perpendicular to $[AB]$ to get from point M to point F . Point F is the centre of a circle whose arc will form the smooth curve between points A and B on the track, as shown in the following diagram.

diagram not to scale



- (b) (i) Find the length of $[BF]$.
 (ii) Find $\widehat{BFM}$. [4]
 (c) Hence, find the length of arc AB . [3]

The outer edge of the track, from C to D , is also a circular arc with centre F , such that the track is 2 metres wide.

- (d) Calculate the area of the curved portion of the track, $ABDC$. [4]

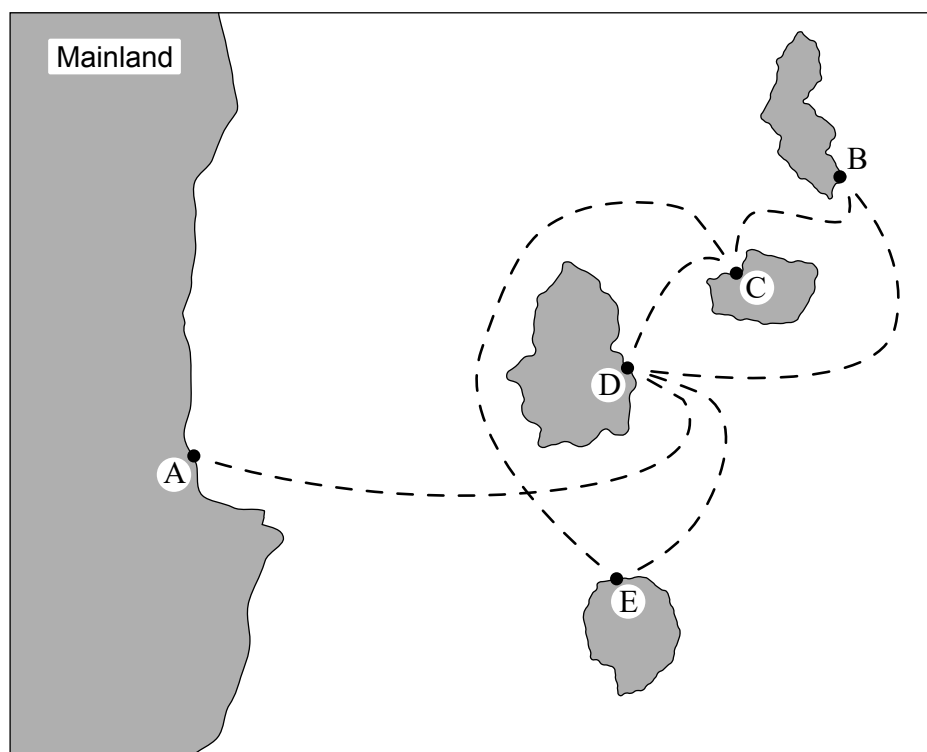
The base of the track will be made of concrete that is 12 cm deep.

- (e) Calculate the volume of concrete needed to create the curved portion of the track. [3]

Turn over

5. [Maximum mark: 18]

The following diagram is a map of a group of four islands and the closest mainland. Travel from the mainland and between the islands is by boat. The scheduled boat routes between the ports A, B, C, D and E are shown as dotted lines on the map.



Let the undirected graph G represent the boat routes between the ports A, B, C, D and E.

(a) Draw graph G . [1]

(b) Graph G can be represented by an adjacency matrix $\mathbf{P}$, where the rows and columns represent the ports in alphabetical order.

(i) Given that $\mathbf{P}^3 = \begin{pmatrix} 0 & 1 & 2 & 4 & 1 \\ 1 & 2 & 5 & a & 2 \\ 2 & 5 & 4 & 6 & 5 \\ 4 & a & 6 & 4 & 6 \\ 1 & 2 & 5 & 6 & 2 \end{pmatrix}$, find the value of a .

(ii) Hence, write down the number of different ways that someone could start at port B and end at port C, using three boat route journeys. [3]

(c) Find a possible Eulerian trail in G , starting at port A. [2]

(This question continues on the following page)

(Question 5 continued)

The cost of a journey on the different boat routes is given in the following table; all prices are given in USD. The cost of a journey is the same in either direction between two ports.

	A	B	C	D	E
A				10	
B			20	25	
C		20		50	45
D	10	25	50		30
E			45	30	

Sofia wants to make a trip where she travels on each of the boat routes at least once, beginning and ending at port A.

- (d) Find the minimum cost of Sofia's trip. [3]

The boat company decides to add an additional boat route to make it possible to travel on each boat route **exactly** once, starting and ending at the same port.

- (e) (i) Identify between which two ports the additional boat route should be added.
(ii) Determine the cost of the additional boat route such that the overall cost of the trip is the same as your answer to part (d). [2]

The boat company plans to redesign which ports are connected by boat routes. Their aim is to have a single boat trip that visits all the islands and minimizes the total distance travelled, starting and finishing at the mainland, A.

The following table shows the distances in kilometres between the ports A, B, C, D and E.

	A	B	C	D	E
A		80	90	50	60
B	80		30	70	120
C	90	30		45	100
D	50	70	45		55
E	60	120	100	55	

- (f) (i) Use the nearest neighbour algorithm to find an upper bound for the minimum total distance.
(ii) Use the deleted vertex algorithm on port A to find a lower bound for the minimum total distance. [7]

Turn over

6. [Maximum mark: 14]

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O . In one of his games, an object travels on a straight line L_1 with vector equation

$$\mathbf{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

- (a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where $l, m \in \mathbb{Z}$. [1]

François uses the matrix $\mathbf{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2 . The object will then travel along L_2 .

- (b) Find the vector equation of L_2 . [4]

François knows that the transformation given by matrix $\mathbf{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan \alpha$, $0 \leq \alpha < \pi$

- (c) Write down the matrix that represents

(i) the rotation.

(ii) the enlargement. [2]

- (d) The matrix $\mathbf{R}$ represents the reflection. Write down $\mathbf{R}$ in terms of α . [1]

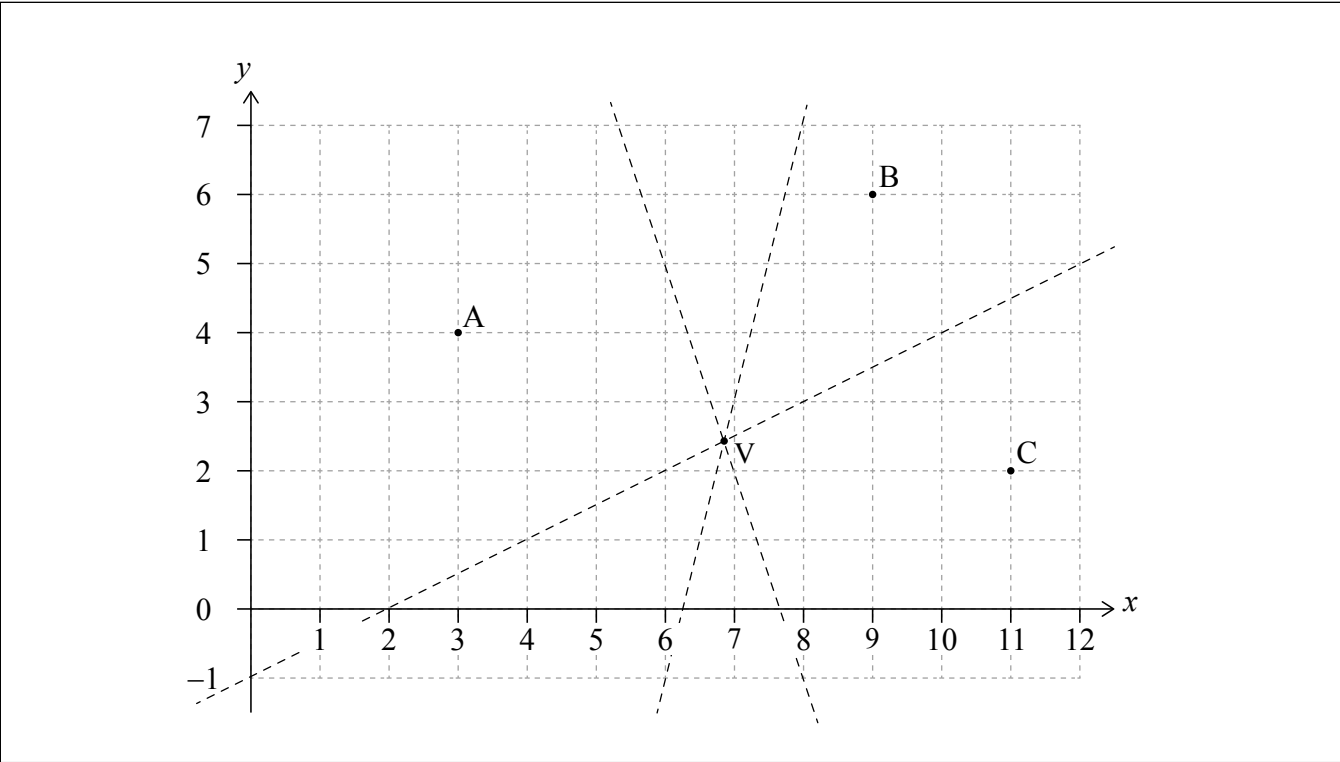
- (e) Given that $\mathbf{T} = \mathbf{R}\mathbf{X}$,

(i) use your answers to part (c) to find matrix $\mathbf{X}$.

(ii) hence, find the value of α . [6]

5. [Maximum mark: 6]

Points A(3, 4), B(9, 6) and C(11, 2) are shown on the following diagram, along with the perpendicular bisectors of [AB], [AC] and [BC].



The perpendicular bisector of [BC] intercepts the axes at coordinates (0, −1) and (2, 0).

- (a) Write down the equation of the perpendicular bisector of [BC]. [2]

The equation of the perpendicular bisector of [AB] is $y = -3x + 23$.

- (b) Find the coordinates of point V where the perpendicular bisectors meet. Give your answer to four significant figures. [2]

A Voronoi diagram is constructed with points A, B and C as the three sites.

- (c) Draw, clearly, the edges of the Voronoi diagram on the given diagram. [2]

(This question continues on the following page)



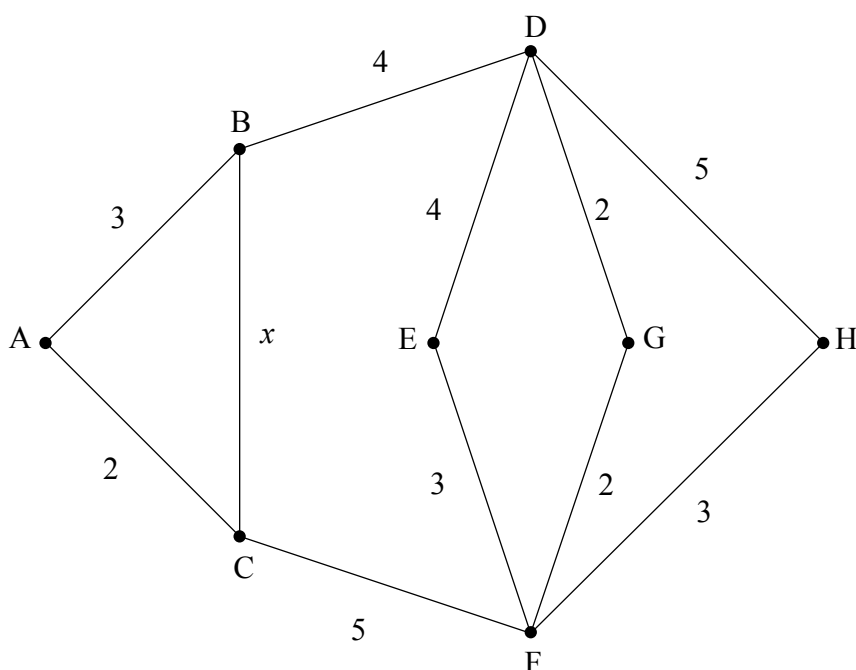
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Apollo | Original source crop | 91

8. [Maximum mark: 5]

The weights on the following graph represent the lengths of different roads in kilometres.

diagram not to scale



- (a) Write down the vertices with odd degree.

[1]

The total length of the roads is $33 + x$ km.

- (b) Find **two** expressions, in terms of x , for the shortest distance required to walk along all of the paths, beginning and ending at the same vertex.
Include in your answer the interval of values of x for which each expression is valid.

[4]

(This question continues on the following page)



[illegible]

11. [Maximum mark: 8]

Let $R(\alpha)$ be the matrix representing a rotation, counter-clockwise (anticlockwise) about the origin, through an angle of α .

- (a)

Write down $R(2\alpha)$ as a 2×2 matrix.

[2]
- (b)

Calculate $R(\alpha) \times R(\alpha)$.

[2]
- (c)

Use your answers from part (a) and part (b) to

(i)

explain why $\sin(2\alpha) = 2\sin(\alpha) \cos(\alpha)$.

(ii)

show that $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$.

[4]

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6. [Maximum mark: 6]

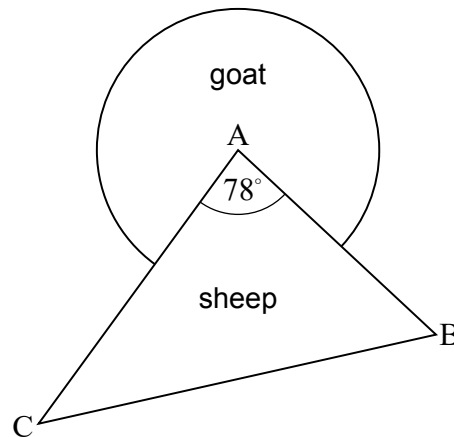
A sheep is in a field in the shape of a triangle, ABC.

AC = 21 metres, AB = 15 metres and $\hat{CAB} = 78^\circ$.

A goat is in an adjacent field in the shape of a sector of a circle with centre, A, and radius 8 metres.

The fields are shown in the diagram.

diagram not to scale



Determine which animal, the sheep or the goat, is in the field with the larger area, and state how many extra square metres are in this larger field.

[illegible]

8. [Maximum mark: 10]

The quadrilateral ABCD has coordinates A(1, 3, 5), B(4, 7, 5), C(5, 8, 7) and D(2, 4, 7).

- (a) Write down $\vec{AD}$. [1]
- (b) Calculate
- (i) the size of $\hat{BAD}$.
- (ii) the area of triangle BAD. [7]
- (c) Show that ABCD is a parallelogram. [2]

[illegible]

12. [Maximum mark: 7]

A duck is sitting in a duck pond at point A(7, 4, 0) relative to an origin O, where lengths are measured in metres and time, t , is measured in seconds. It takes off and flies in a straight line with vector equation

$$\mathbf{d} = \begin{pmatrix} 7 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}.$$

- (a) Find the speed of the duck through the air (in m s^{-1}). [2]

A hawk hovering at position vector $\begin{pmatrix} -38 \\ 134 \\ 315 \end{pmatrix}$, relative to O, sees the duck take off and immediately dives from its position with constant velocity vector $\begin{pmatrix} 15 \\ -20 \\ -60 \end{pmatrix}$ to intercept the duck.

- (b) Write down the vector equation for $\mathbf{h}$, that models the flight of the hawk. [1]
- (c) Find the position vector at which the hawk intercepts the duck. [4]

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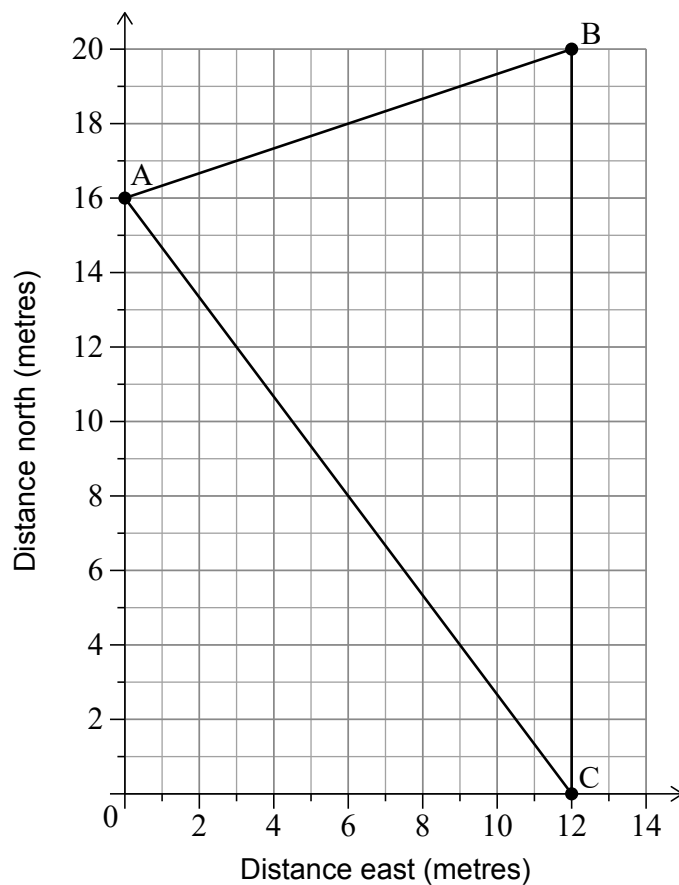
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1. [Maximum mark: 14]

Mai is at an amusement park. A map of part of the amusement park is represented on the following coordinate axes.

Mai's favourite three attractions are positioned at $A(0, 16)$, $B(12, 20)$ and $C(12, 0)$. All measurements are in metres.



(a) Write down the distance between B and C.

[1]

(b) Calculate the distance between A and B.

[2]

(This question continues on the following page)

(Question 1 continued)

Mai is standing at the attraction at B and wants to walk directly to the attraction at A.

- (c) Calculate the bearing of A from B. [3]

A drinking fountain is to be installed at a point that is an equal distance from each of the attractions at A, B and C.

- (d) (i) Write down the gradient of [AC].
(ii) Write down the mid-point of [AC].
(iii) Hence calculate the coordinates of the drinking fountain. [8]

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3. [Maximum mark: 22]

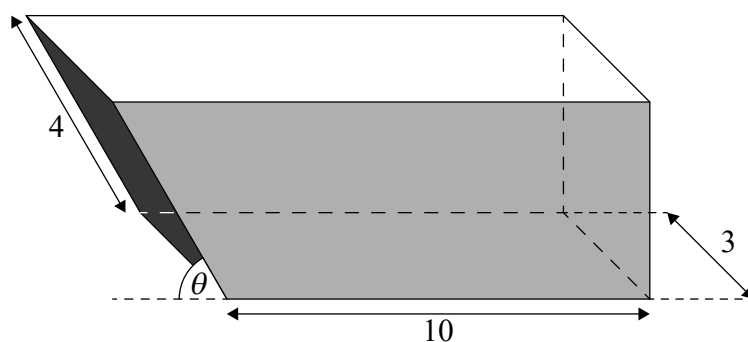
A skip is a container used to carry garbage away from a construction site. For safety reasons the garbage must not extend beyond the top of the skip. The maximum volume of garbage to be removed is therefore equal to the volume of the skip.



A particular design of skip can be modelled as a prism with a trapezoidal cross section. For the skip to be transported, it must have a rectangular base of length 10m and width 3m. The length of the sloping edge is fixed at 4m, and makes an angle of θ with the horizontal.

The following diagram shows such a skip.

diagram not to scale



(This question continues on the following page)

(Question 3 continued)

- (a) Find the volume of this skip,
- (i) if the length of the top edge of the skip is 11 m.
 - (ii) if the height of the skip is 3.2 m.
 - (iii) if θ is $\frac{\pi}{3}$. [9]
- (b) Show that the volume, $V \text{ m}^3$, of the skip is given by
- $$24 \sin(\theta)(5 + \cos(\theta)).$$
- [2]
- (c) Explain, in context, why $\theta \neq 0$. [1]
- (d) (i) Sketch the graph of $V = 24 \sin(\theta)(5 + \cos(\theta))$, $0 < \theta < \frac{\pi}{2}$.
- (ii) Find the maximum volume of the skip and the value of θ for which this maximum volume occurs. [4]
- (e) Show, by differentiation, that the maximum volume occurs at a value of θ that satisfies the equation $2 \cos^2 \theta + 5 \cos \theta - 1 = 0$. [6]

Turn over

4. [Maximum mark: 12]

A hygiene inspector lives in Town A and must visit restaurants in five towns (B–F), before returning to A. The inspector must not repeat any of the towns. The distances, in km, between the six towns are shown in the table.

	A	B	C	D	E	F
A		31	28	26	22	23
B	31		25	20	27	25
C	28	25		19	22	24
D	26	20	19		21	22
E	22	27	22	21		24
F	23	25	24	22	24	

- (a) Starting at A, use the nearest neighbour algorithm to find an upper bound for the length of the journey the inspector must take. State the order in which the towns are to be visited.

[4]
- (b) By deleting A, use Prim’s algorithm starting at B to find a lower bound for the length of the inspector’s journey.

[5]
- (c) By considering the minimum spanning tree found in part (b), determine whether the journey given by this lower bound is an achievable solution.

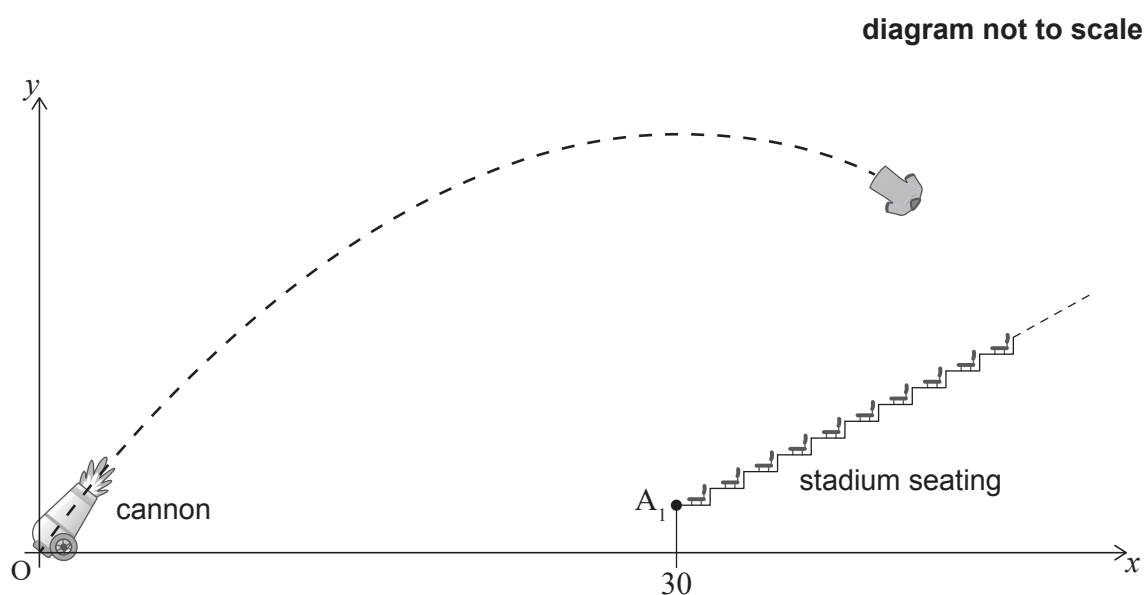
[3]

2. [Maximum mark: 29]

A sports stadium has a T-shirt cannon which is used to launch T-shirts into the crowd. The purpose of this question is to determine whether a person sitting in a particular seat will ever receive a T-shirt.

A T-shirt cannon is placed on the horizontal ground of a stadium playing area. A coordinate system is created such that the origin, O , is the point on the ground from where the T-shirts are launched. In this coordinate system, x and y represent the horizontal and vertical displacement from O , and are measured in metres.

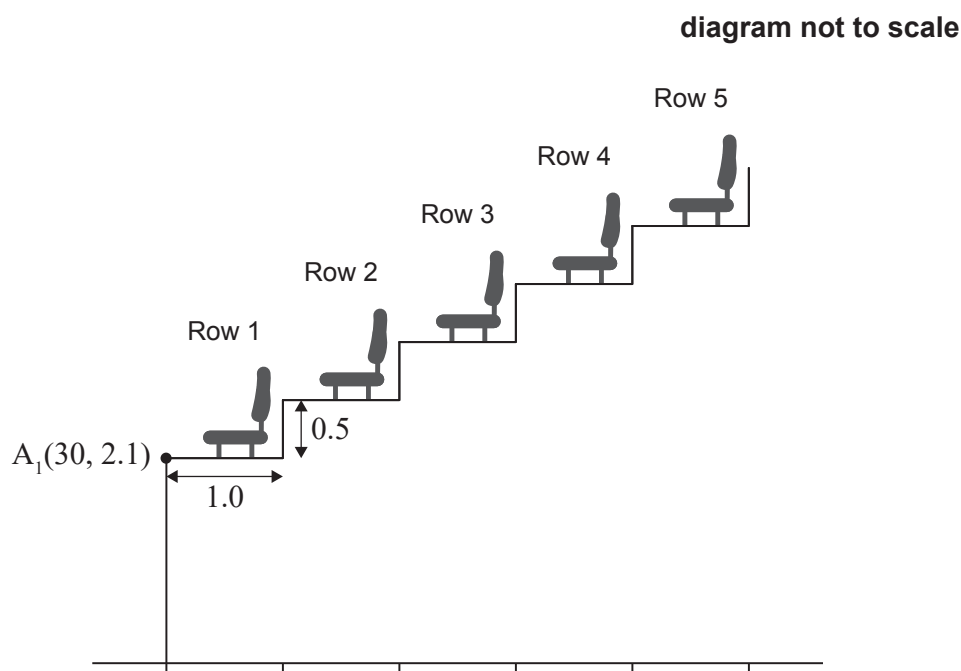
Seat A_1 is the nearest seat to the T-shirt cannon. The coordinates of the front of the foot space for seat A_1 are $(30, 2.1)$.



(This question continues on the following page)

(Question 2 continued)

Each seat behind seat A_1 is 1.0 m further from O horizontally and 0.5 m higher than the seat in the row below it, as shown on the diagram.



Seat A_1 is in row 1. Let seat A_n be the seat directly behind A_1 in row n .

- (a) (i) Write down the coordinates of the front of the foot space of seat A_5 . [2]
- (ii) Find, in terms of n , the coordinates for the front of the foot space of seat A_n . [3]

While in motion, the T-shirt can be treated as a projectile.

Let t be the time, in seconds, after a T-shirt is launched.

At any time $t > 0$, the acceleration of the T-shirt, in m s^{-2} , is given by the vector

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}.$$

The initial velocity, in m s^{-1} , of the T-shirt is given as $\begin{pmatrix} 29.4 \cos \theta \\ 29.4 \sin \theta \end{pmatrix}$, where θ is the angle to the ground at which the T-shirt is launched and $0^\circ < \theta \leq 90^\circ$.

- (b) (i) Find an expression for the velocity, $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, at time t . [3]
- (ii) Hence show that when the T-shirt is launched vertically, the time for it to reach its maximum height is 3 seconds. [3]

(This question continues on the following page)

Turn over

(Question 2 continued)

The displacement of the T-shirt, t seconds after it is launched, is given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 29.4(\cos \theta)t \\ 29.4(\sin \theta)t - 4.9t^2 \end{pmatrix}.$$

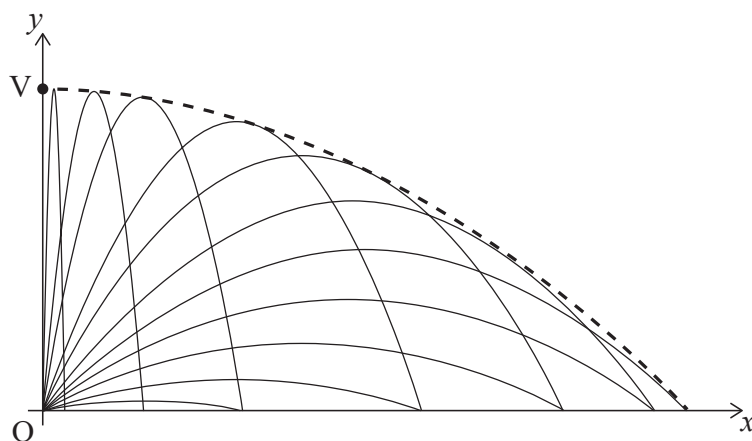
- (c) Using the given answer to part (b)(ii) or otherwise, find the maximum height reached by a T-shirt when it is launched vertically. [2]

- (d) (i) If there was no seating, and the T-shirt was launched at an angle θ , show that the value of x when it would hit the ground is given by the expression

$$x = 176.4 \sin \theta \cos \theta. \quad [3]$$

- (ii) Hence find the maximum possible value for x if there was no seating to block the path of the T-shirt. [2]

In order to calculate the seats in the stadium which can be reached by a T-shirt it is required to find the equation of the curve that forms the boundary of all the points that can be reached. This boundary is represented by the dashed curve in the following diagram, while the solid curves represent some of the possible trajectories for the T-shirts.



It is given that the boundary curve is the parabola $y = ax^2 + bx + c$, with its vertex V on the y -axis.

- (e) Using your answers to parts (c) and (d)(ii), or otherwise, find
- (i) the value of c . [1]
 - (ii) the value of b . [2]
 - (iii) the value of a . [3]

A spectator is sitting in seat A_{40} .

- (f) Show that it is not possible for the spectator to ever get a T-shirt. [5]

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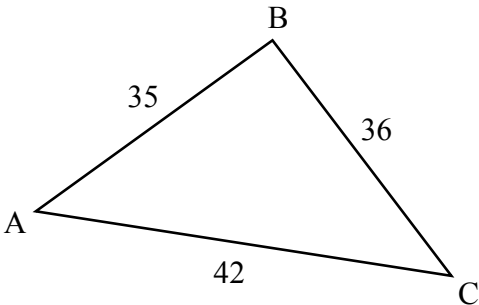
References:

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1. [Maximum mark: 5]

Consider the following triangle, ABC , such that $AB = 35\text{ cm}$, $BC = 36\text{ cm}$, and $CA = 42\text{ cm}$.

diagram not to scale



- (a) Find the value of $\hat{CAB}$.

[3]
- (b) Find the area of the triangle ABC .

[2]

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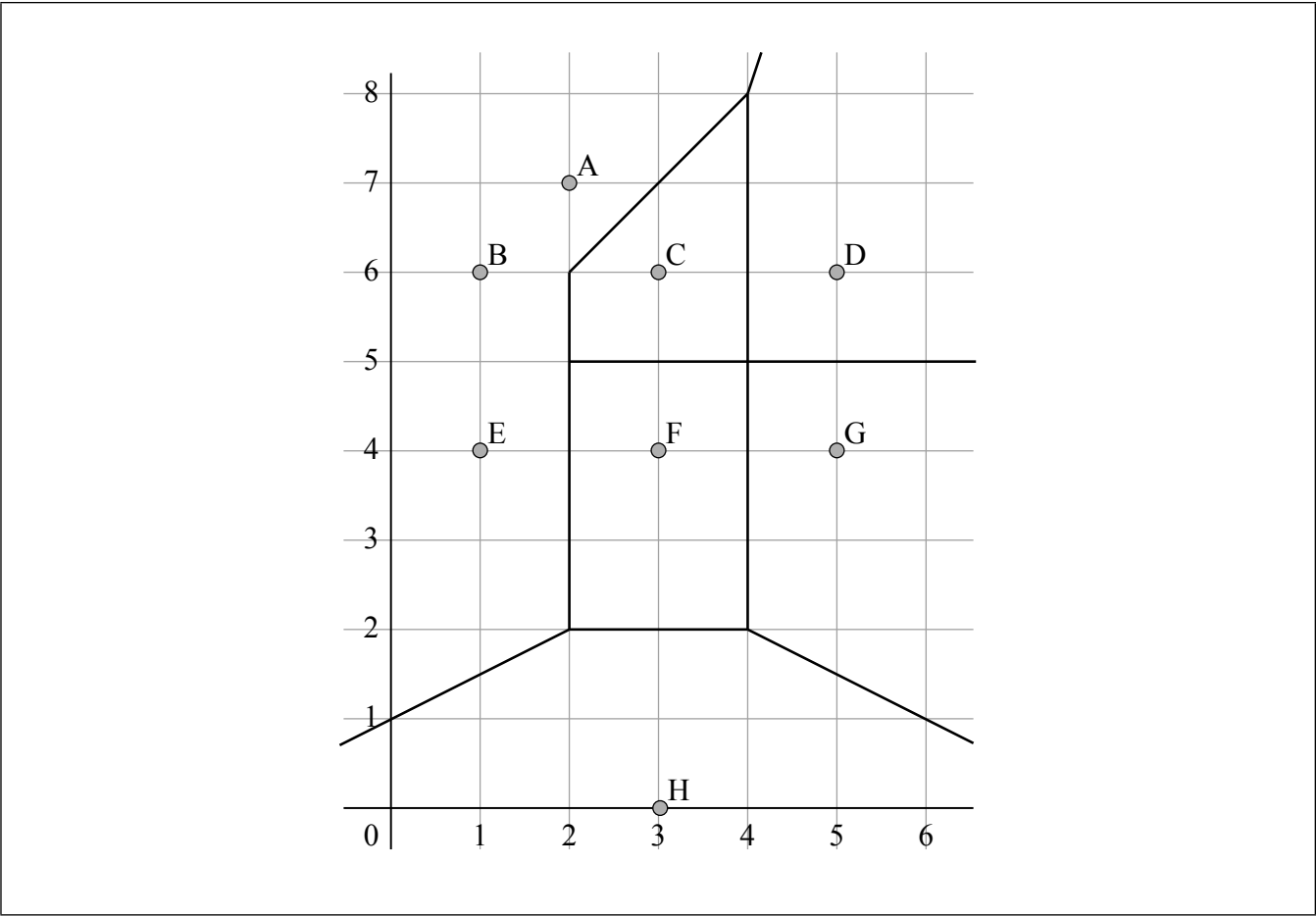
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5. [Maximum mark: 6]

The sites in the Voronoi diagram represent eight hospitals in a city.



Each site has integer coordinates. Two edges are missing from the diagram.

- (a) Draw the missing edges on the diagram. [2]

One square unit on the diagram represents 4 km^2 in the city.

- (b) Find the area, in km^2 , of the cell containing
- (i) site F
 - (ii) site C. [3]

The hospitals at sites C and F have the same number of patients each year.

- (c) Suggest a reason why the number of patients is not proportional to the area of the cell. [1]

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(Question 5 continued)

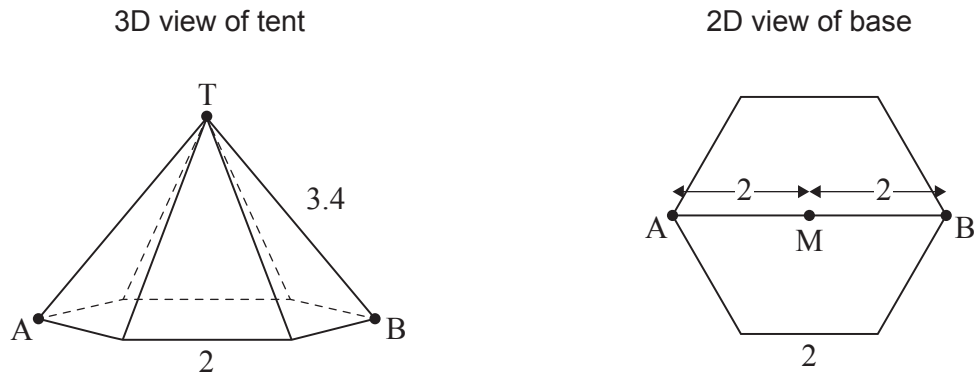
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4. [Maximum mark: 17]

Gaurika is designing a tent in the shape of a right pyramid with a regular hexagonal base, centre M . The length of each side of the base is 2 m , the length of each sloping edge is 3.4 m , and the distance between each vertex on the base and M is 2 m , as shown in the diagrams.

diagrams not to scale

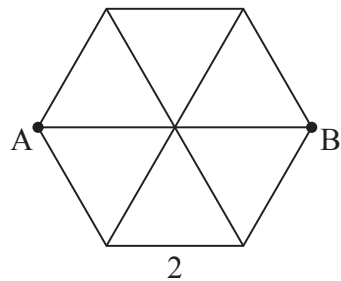


The top of the tent, T , will be supported by a vertical pole from M .

- (a) Find the length of the pole, MT .
- [2]

The hexagonal base can be divided into six equilateral triangles.

diagram not to scale



- (b) Find
- (i) the area of the base

(ii) the volume of the tent.
- [5]
- (c) Find the value of $\hat{MT}$.
- [2]

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(Question 4 continued)

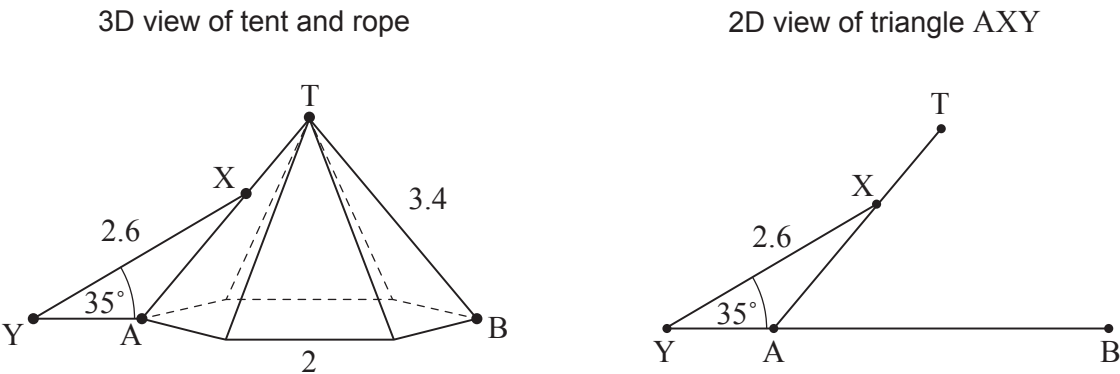
For extra support, Gaurika decides to attach a rope, with length 2.6 m, to the tent at a point, X, on the edge AT.

The rope will be fixed to the ground at point Y, such that:

- the rope, [XY], is straight
- points Y, A and B lie on a straight line
- $\widehat{AYX} = 35^\circ$.

This is shown in the diagrams.

diagrams not to scale



(d) Find AY. [4]

For decoration at night, Gaurika wants to fix a strip of lights from point A to a point, Z, along the rope [XY].

The strip of lights, [AZ], is straight and has length 0.9 m.

(e) Find the two possible values of YZ. [4]

5. [Maximum mark: 15]

In this question, all distances are in kilometres and t is in hours.

Let $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be a displacement of 1 km due east, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ be a displacement of 1 km due north,
and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be a vertical displacement of 1 km.

Highway 85 in Saudi Arabia is a long, straight, flat road.

Relative to the centre of the town Arar, point O, the position vector of a car, C, travelling along this road is given by:

$$\vec{OC} = \begin{pmatrix} 10 \\ -5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 50 \\ -33 \\ 0 \end{pmatrix}.$$

(a) Find the speed of the car.

[2]

The police are testing a long-range drone, D, to monitor cars travelling along this road. The

drone is launched at $t = 0$ from the point with position vector $\begin{pmatrix} 200 \\ -100 \\ 0.02 \end{pmatrix}$ and flies in a straight

line with a constant height of 0.02 km and a constant velocity of $\begin{pmatrix} -15 \\ -20 \\ 0 \end{pmatrix}$.

(b) Find the angle between the path of the car and the path of the drone.

[3]

(c) Write down the position vector, $\vec{OD}$, of the drone at time t .

[1]

(This question continues on the following page)

(Question 5 continued)

- (d) At time t_1 , the drone passes through the point with position vector $\begin{pmatrix} 152 \\ p \\ 0.02 \end{pmatrix}$.
Find the value of

(i) t_1

(ii) p . [3]

- (e) (i) Find an expression for $\vec{CD}$, the relative position of the drone from the car.

(ii) Hence, find the shortest distance between the car and the drone. [6]

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6. [Maximum mark: 13]

Juan is creating animations for a website. He uses matrices to transform objects relative to the origin, O .

One matrix that he uses is $A = \begin{pmatrix} \cos(15^\circ) & -\sin(15^\circ) \\ \sin(15^\circ) & \cos(15^\circ) \end{pmatrix}$.

(a) Describe fully the transformation represented by matrix A . [1]

(b) Find the smallest value of n such that $A^n = I$, $n \in \mathbb{Z}^+$. [2]

Juan also uses matrix B , which represents an enlargement with a scale factor of 1.05, centre $(0, 0)$.

(c) (i) Write down matrix B .

(ii) Describe fully the transformation represented by B^n , where n is the value found in part (b). [3]

Juan creates a new matrix, $C = AB$.

(d) Find matrix C . [2]

Juan creates an animation by repeatedly transforming an object by C .

A point, P , on the object is initially at $(1, 0)$. Juan sets the speed of the animation to 6 transformations per second.

(e) Sketch the path of P for the first 4 seconds of motion **and** label the coordinates of the start and end points. [2]

Juan uses a different transformation, T , defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To create his animation, he repeatedly transforms an object by T . After many transformations, he notices that all points, (x, y) , on the object tend towards a single point, (p, q) , such that

$$\lim_{a \rightarrow \infty} T^a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \text{ where } a \in \mathbb{Z}^+.$$

(f) Find $\begin{pmatrix} p \\ q \end{pmatrix}$, where $p, q \in \mathbb{R}$. [3]

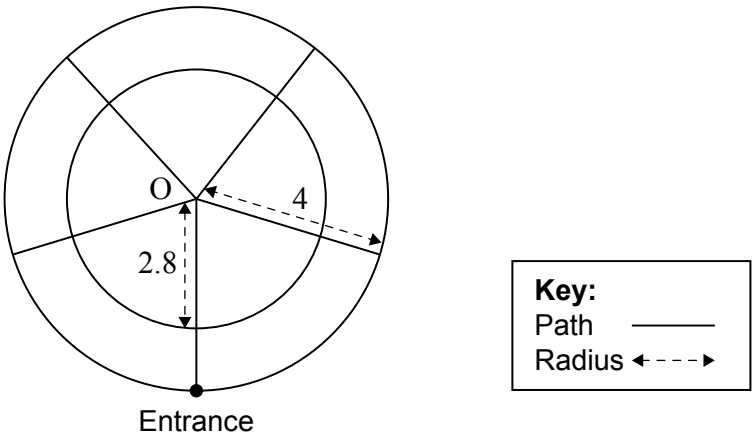
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1. [Maximum mark: 6]

A child's game is played by making paths in the snow. First, two circular paths are made using the same centre, O . The radius of the smaller circle is 2.8 m , and the radius of the larger circle is 4 m . Additional paths are then made from O to the outer edge of the larger circle, dividing each circle into 5 equal sectors, as shown in the following diagram.

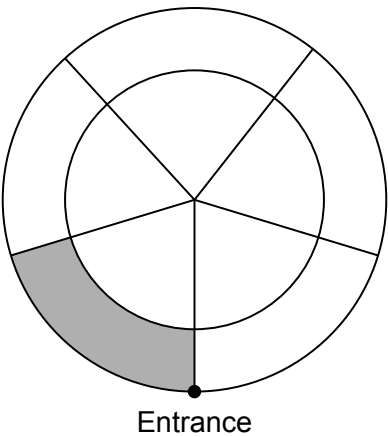
For your calculations, ignore the widths of the paths.

diagram not to scale



Each region **between** paths will be coloured with environmentally friendly dye. The shaded region below will be coloured orange.

diagram not to scale



Maureen has enough orange dye to cover an area of 6 m^2 .

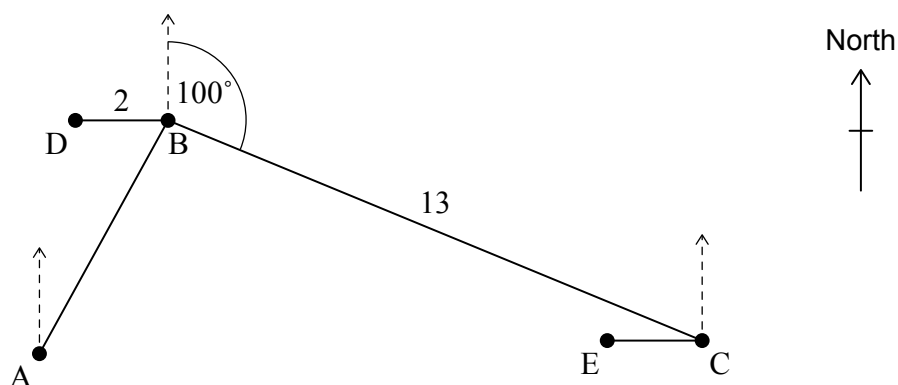
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5. [Maximum mark: 9]

Ticky is sailing a boat in the ocean. For the first part of her trip, she plans to start at point A and travel to point B. She then plans to turn and travel 13 km on a bearing of 100° to point C as shown in the following diagram.

diagram not to scale



Due to navigational error in the first part of the trip, Ticky arrives at point D instead of point B. Point D is 2 km due west of point B.

(a) Determine

(i) the distance from point D to point C.

(ii) the bearing Ticky must use to travel directly from point D to point C.

[7]

Due to navigational error in the second part of the trip, Ticky arrives at point E instead of point C. Point E is 2 km due west of point C.

(b) Write down the distance between point D and point E. Justify your answer.

[2]

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(Question 5 continued)

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13. [Maximum mark: 7]

The points $R(-2, 2)$, $S(3, -5)$, and $T(5, 1)$ are the vertices of a triangle. The triangle goes through a sequence of transformations in the following order:

I. a transformation using the matrix $A = \begin{bmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{bmatrix}$

II. a transformation using the matrix $B = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$.

- (a) Describe the nature of the transformation represented by A . [2]

Instead of using this sequence of transformations, the image of triangle RST can be found using only a reflection of triangle RST in line $y = ax$.

- (b) Given that $a > 0$, find the value of a . [5]



2. [Maximum mark: 9]

The point A has coordinates $(1, 2, 1)$ and the point B has coordinates $(3, 5, 2)$.

(a) Find AB. [2]

Triangle ABC is right-angled with its right angle at B. The point C has coordinates $(2, 8, k)$.

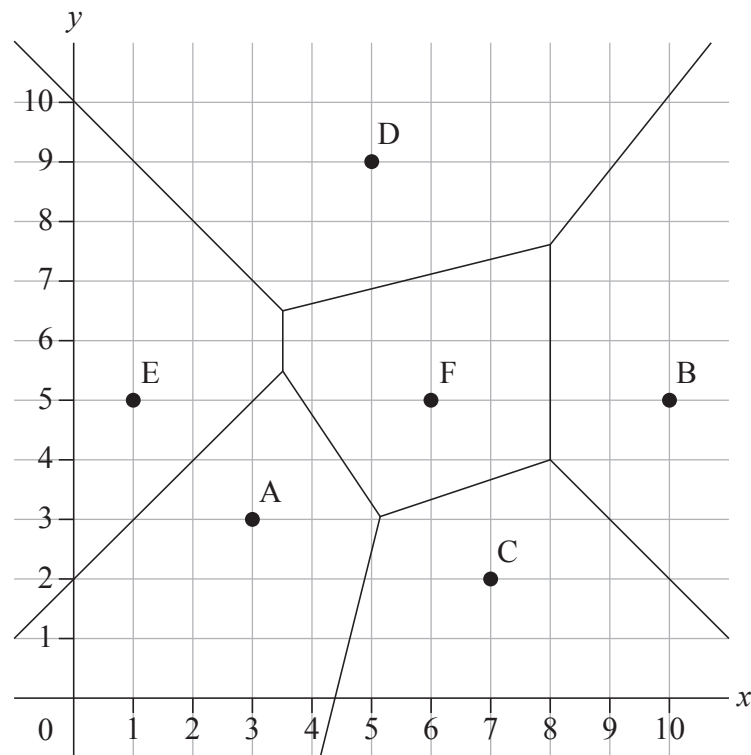
(b) Find the value of k . [4]

(c) Calculate the size of $\hat{BAC}$. [3]



4. [Maximum mark: 7]

Consider the Voronoi diagram which shows the sites $A(3, 3)$, $B(10, 5)$, $C(7, 2)$, $D(5, 9)$, $E(1, 5)$ and $F(6, 5)$. The diagram also shows the cells formed by each site and their boundaries.



Vertex X is equidistant from sites B , C and F .

(a) (i) Write down the coordinates of X .

(ii) The exact value of BX is $\sqrt{n}$. Write down the value of n . [2]

Vertex $Y(a, b)$ is equidistant from sites B , D and F .

(b) (i) Write down the value of a .

(ii) Find the exact value of b . [5]

(This question continues on the following page)





A mapping system stores the connections between 5 towns, labelled A, B, C, D and E, in an adjacency matrix. The adjacency matrix, with rows and columns in alphabetical order, is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

- Draw and label a graph to represent the adjacency matrix. [2]
- Determine the number of walks of length 4 which start and end at the same town. [4]

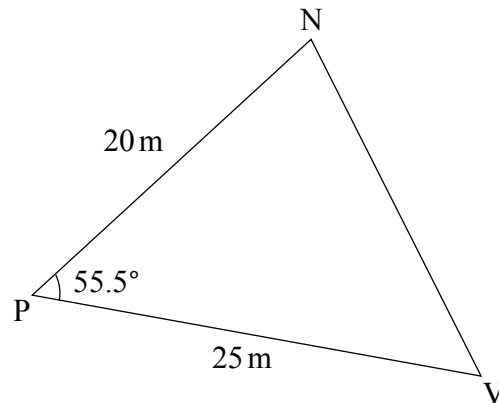
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1. [Maximum mark: 9]

Three points N, P, and V are shown on the following diagram. NP is 20 metres, PV is 25 metres and $\angle VP\hat{N}$ is 55.5° .

diagram not to scale



- (a) Find NV. [3]
- (b) Find $\angle P\hat{N}V$. [3]
- (c) Hence or otherwise, find the shortest distance between P and [NV]. [3]

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(Question 1 continued)

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8. [Maximum mark: 5]

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ transforms a point A to a point B.

The matrix $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ transforms the point B to a point C with coordinates (4, 7).

Find the coordinates of A.

[illegible]

14. [Maximum mark: 8]

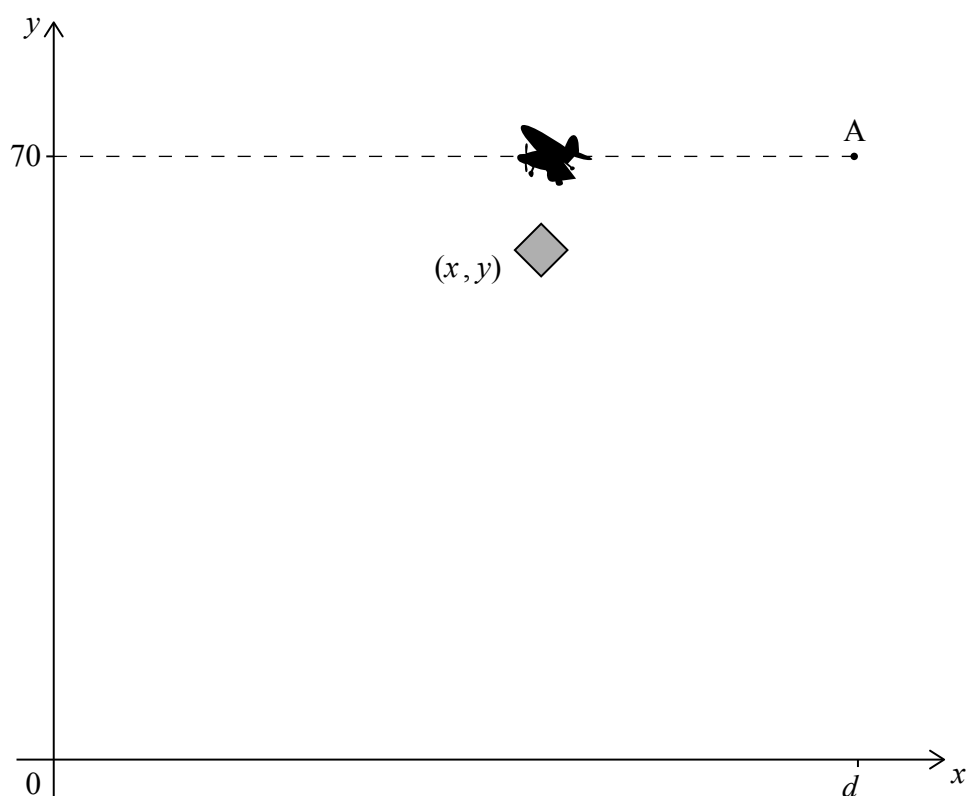
A plane needs to drop a relief package for a remote village. The villagers have marked out a point O in the middle of a field where the package should land.

The package is released from the plane at point A when it is a horizontal distance of d metres from O .

The position vector of the package t seconds after release is given by the vector $\begin{pmatrix} x \\ y \end{pmatrix}$,

where x is the horizontal displacement of the package from O and y is the vertical height of the package above the ground. This is shown in the following diagram.

diagram not to scale



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(Question 14 continued)

The acceleration of the package is $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \text{ms}^{-2}$ and its velocity at the point of release is $\begin{pmatrix} -20 \\ 0 \end{pmatrix} \text{ms}^{-1}$.

- (a) Find the velocity vector of the package $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at time t . [2]

The vertical height of the plane above the ground at the moment the package is released is 70m.

- (b) Find the value of d , given that the package lands at O. [6]



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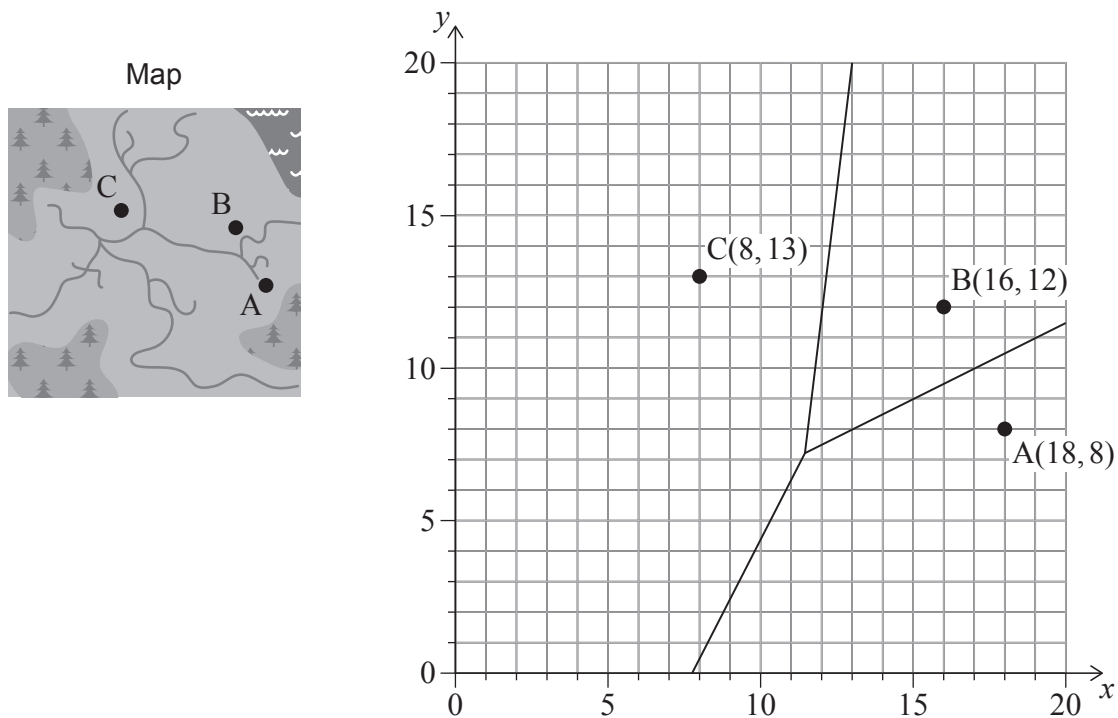
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2. [Maximum mark: 19]

The locations of three fire stations within a 50 by 50 kilometre rural area of Japan are shown on the following map.

A Voronoi diagram can be used to determine the part of the rural area for which each fire station is responsible. The Voronoi diagram for the three fire stations is shown on the coordinate grid beside the map, where 1 unit represents 2.5 km.



- (a) (i) Find the midpoint of $[BC]$.
- (ii) Find the gradient of the perpendicular bisector of $[BC]$.
- (iii) Hence find the equation of the edge between sites B and C. [6]
- (b) (i) Identify the fire station that is expected to respond, based on the Voronoi diagram, if a fire is reported at a location with coordinates $(14, 10)$. Justify your response.
- (ii) Suggest a reason why a different fire station might respond to this fire. [2]

(This question continues on the following page)

(Question 2 continued)

A fire is reported at a location D with coordinates $\left(11\frac{1}{3}, 7\frac{1}{6}\right)$. The distance of D from A on the Voronoi diagram, to six significant figures, is 6.71855 units (16.7964 km on the map).

- (c) (i) Show that the distance of D from B, to six significant figures, is also 6.71855 units.
- (ii) Show that any of the three fire stations would be expected to respond to the fire reported at D. [4]

The equation of the edge between fire stations A and B is $y = 0.5x + 1.5$. The edge between fire stations A and C has an x -intercept of 7.75.

- (d) (i) Determine the area, to the nearest square unit, of the region within the given Voronoi diagram for which fire station A is responsible.
- (ii) Hence, calculate the actual area, in square kilometres, of this region. [7]

Turn over

4. [Maximum mark: 17]

Lizjerika works in an air traffic control tower. At 10:00 am, she detects a passenger airplane and a flock of birds whose flight paths are given by the following equations:

$$\text{Airplane: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + t_1 \begin{pmatrix} -1.4 \\ 1.65 \\ 0 \end{pmatrix} \quad \text{Birds: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \\ 6 \end{pmatrix} + t_1 \begin{pmatrix} -0.40 \\ -0.35 \\ 0.1 \end{pmatrix}$$

where the x -coordinate represents displacement east of the control tower, the y -coordinate represents displacement north of the control tower, and the z -coordinate represents height (above sea level).

Distances are measured in kilometres and t_1 is the time, in minutes, since 10:00am.

- (a) Determine the speed, in kilometres per minute, at which the flock of birds is travelling. [2]
- (b) By making two statements, describe the path of the airplane in context. [2]
- (c) (i) Find the time at which the birds and the airplane will have the same height.
- (ii) Using your answer to part (c)(i), explain why the birds will not collide with the airplane. [6]

At 10:20 am, Lizjerika detects a jet whose movement is given by the following equation,

$$\text{Jet: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -95 \\ 32 \\ 1.5 \end{pmatrix} + t_2 \begin{pmatrix} 1.3 \\ 1.45 \\ 0.2 \end{pmatrix}$$

where t_2 is the time, in minutes, since 10:20am.

- (d) Find the vector equation of the motion of **the passenger airplane** in terms of t_2 . [2]

An air traffic law requires that the jet and the airplane must be more than 5 kilometres from each other at all times.

- (e) Determine whether the jet and the airplane will break this law if they follow their current paths. [5]

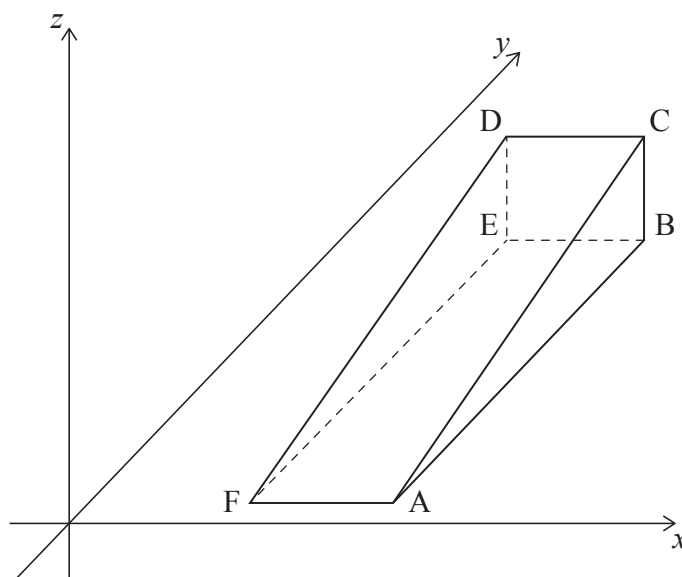
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5. [Maximum mark: 16]

AirFlow Industries is designing a doorstop in the shape of a triangular prism, $ABCDEF$.

The points $A(13, 1, 0)$, $B(13, 25, 0)$ and $C(13, 25, 7)$ are the coordinates of the vertices of one of the triangular faces of the prism. Point $D(4, 25, 7)$ is another vertex as shown in the following diagram. All the measurements are in centimetres.

diagram not to scale



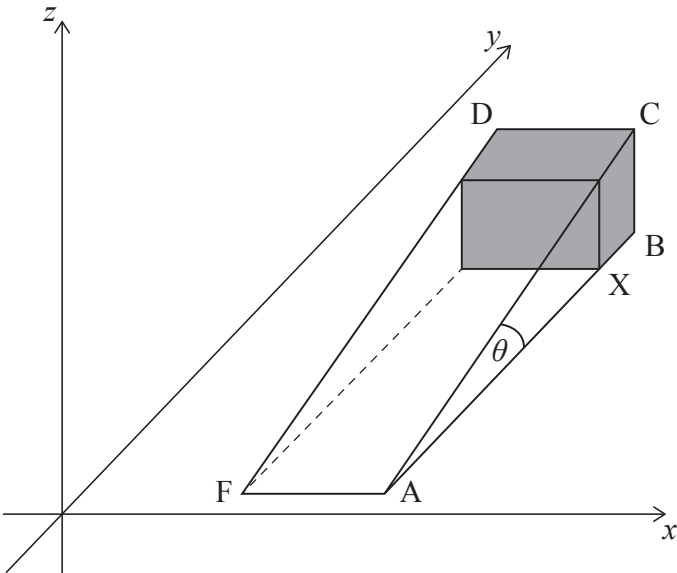
- (a) Find AC . [2]
- (b) Show that triangle ABC is a right-angled triangle. [2]
- (c) Find the volume of material needed to make the doorstop. [3]

(This question continues on the following page)

(Question 5 continued)

To lower the cost of the doorstop, AirFlow decides to reduce its volume to 625 cm^3 by removing the shaded section shown in the following diagram. The remaining triangular faces are still right-angled triangles.

diagram not to scale



- (d) (i) Find the value of θ .
- (ii) Hence, or otherwise, find AX . [6]

The material needed to make the doorstop costs 0.025 US dollars (USD) per cubic centimetre. The production of each doorstop requires 10% more material than its final volume, due to wastage in the production process.

To make a profit, the company will sell the doorstop for 20% more than the cost of the material.

- (e) Determine the price, to two decimal places, the company will charge for the **new** doorstop. [3]

Turn over

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4. [Maximum mark: 13]

A wind farm consists of five wind turbines, located at points A to E.

The table below shows the distances, in kilometres, between each pair of turbines.

	A	B	C	D	E
A		0.90	0.88	1.56	0.86
B	0.90		0.74	0.94	1.28
C	0.88	0.74		0.78	0.62
D	1.56	0.94	0.78		1.36
E	0.86	1.28	0.62	1.36	

The turbines must all be connected by cables. However, there does not need to be a direct connection between every pair.

- (a) Use Prim’s algorithm, starting with vertex A, to find the minimum total length of cable required to connect the turbines. Show the order in which you added the vertices.
- [4]

The supervisor of the wind farm has a monitoring cabin located at point F. The distances from F to each turbine are shown in the table.

Turbine	A	B	C	D	E
Distance from F (km)	0.96	1.82	1.57	2.24	1.14

The supervisor wants to visit every turbine exactly once for inspection, starting and finishing at the cabin, and using the route of shortest possible length.

- (b) By deleting vertex F, find a lower bound for the length of the shortest route.
- [2]
- (c) Use the nearest neighbour algorithm starting at F to find an upper bound for the length of the shortest route.
- [3]

(This question continues on the following page)

(Question 4 continued)

The table below shows the lower bounds found by deleting each of the other five vertices, and the upper bounds found by starting at each of the other five vertices.

Vertex	A	B	C	D	E
Lower bound	5.02	4.86	5.02	4.90	4.84
Upper bound	6.36	6.36	7.13	7.22	6.82

The supervisor travels between the turbines at a constant speed of 28 km/h and spends 12 minutes inspecting each turbine.

- (d) Based on the information given in the table, find the best possible upper and lower bounds for the shortest amount of time, T hours, required for the inspection. Write your answer as an inequality.

[4]

4. [Maximum marks: 16]

The ferry routes between five ports, A to E, can be represented by a graph, G .

The adjacency matrix M for the graph G is shown below.

$$M = \begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

- (a) State which direct route between ports can only be travelled in one direction. [1]
- (b) By finding an appropriate matrix
- show that it is not possible to reach all of the ports from any other port in **exactly** three trips.
 - state whether it is possible to reach all of the ports from any other port in three trips or fewer. Justify your answer. [5]

In the summer there is a new timetable for ferry routes. The least cost, in euros (EUR), to travel between any two of the ports is shown in the table below.

Port	A	B	C	D	E
A	0	3	7	8	5
B	3	0	4	9	6
C	7	4	0	a	10
D	8	9	a	0	3
E	5	6	10	3	0

- (c) Given there is no direct route between D and C or C and D, show that the value of a is 13 EUR. In your answer ensure you have considered all possible routes. [3]
- (d) Starting at A, use the nearest neighbour algorithm to find an upper bound for the cost of visiting all the ports and returning to your starting position. [3]
- (e) By deleting the vertex A, use the deleted vertex algorithm to find a lower bound for the cost of visiting all the ports. [4]

5. [Maximum mark: 15]

The position of a particle, P_1 , t seconds after leaving a point A, is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 4 \\ 7 \end{pmatrix}, \quad t \geq 0.$$

The units of distance are metres.

(a) Write down the coordinates of A.

[1]

Five seconds after leaving A, P_1 is at point B.

(b) Find

(i) $\vec{AB}$.

(ii) $\left| \vec{AB} \right|$.

[4]

A second particle P_2 leaves A at the same time as P_1 . P_2 is moving in the direction of

the vector $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, and passes through a point C.

(c) Find $\hat{CAB}$.

[5]

The particle P_2 has a speed of 12 m s^{-1} .

(d) Find the distance from P_1 to P_2 when $t = 5$.

[5]

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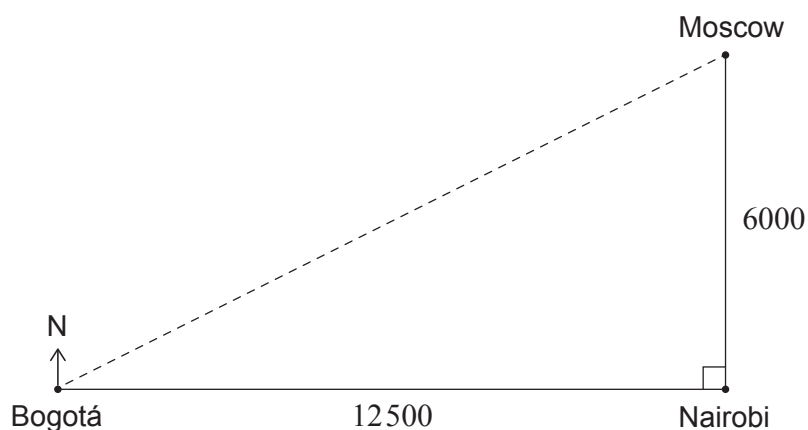
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2. [Maximum mark: 27]

The following question compares the distance and direction between cities on a flat surface to the distance and direction between cities on a sphere.

Consider a model where the cities of Bogotá, Moscow, and Nairobi lie on a flat surface. In this model, Nairobi is 6000 km due south of Moscow and Bogotá is 12 500 km due west of Nairobi, as shown in the following diagram.

diagram not to scale



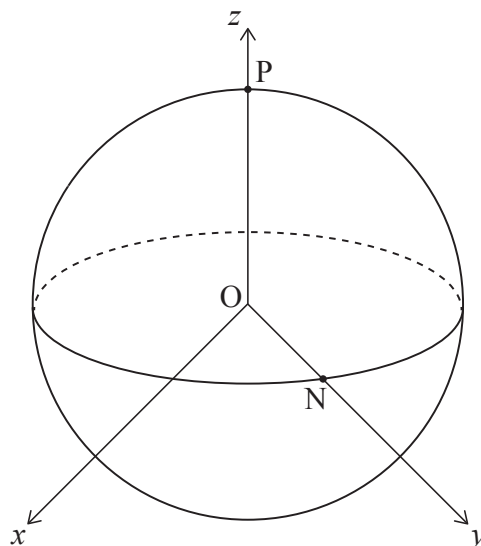
- (a) (i) Find the distance from Bogotá to Moscow. [2]
- (ii) Find the bearing of Moscow from Bogotá. Give your answer in degrees. [3]

(This question continues on the following page)

(Question 2 continued)

In reality, these three cities lie on the curved surface of the Earth which will change the distances and directions found in part (a).

Now consider a curved model using a coordinate system (x, y, z) with its origin, O , at the centre of the Earth. The units of this system are thousands of kilometres and the Earth is modelled as a sphere with radius 6000 km. The North Pole, P , lies on the z -axis, and Nairobi, N , is modelled as being on the equator and lying on the y -axis.



P has position vector $\vec{OP} = \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ and N has position vector $\vec{ON} = \mathbf{n} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$.

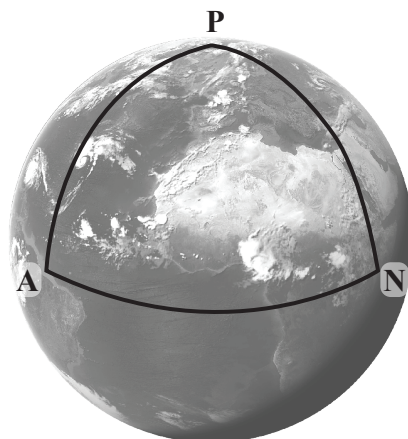
- (b) (i) Use the scalar product to find the angle between $\mathbf{p}$ and $\mathbf{n}$. [2]
- (ii) Show that the distance between P and N along the arc from P to N is 3000π km. [2]

(This question continues on the following page)

Turn over

(Question 2 continued)

Point A, which is also on the equator, has position vector $\mathbf{a} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ as shown in the following diagram.



P, N and A, and the arcs connecting them, form a spherical triangle.

The angle at vertex A is defined as the angle between the vectors $\mathbf{a} \times \mathbf{p}$ and $\mathbf{a} \times \mathbf{n}$.

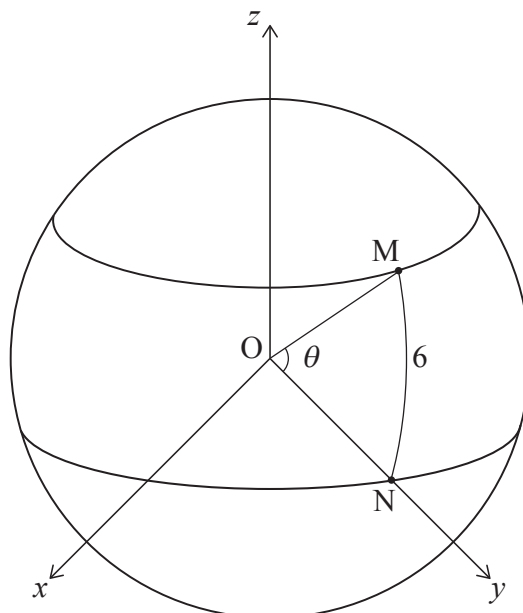
(c) (i) Find the vector $\mathbf{a} \times \mathbf{p}$. [2]

(ii) Show that the angle at vertex A in the spherical triangle is 90° . [3]

(This question continues on the following page)

(Question 2 continued)

Moscow, M, has position vector $\vec{OM} = \mathbf{m} = \begin{pmatrix} 0 \\ 6 \cos \theta \\ 6 \sin \theta \end{pmatrix}$, as shown in the following diagram.



The shortest distance between two points on the sphere lies along an arc of a circle on the sphere with centre O. In this model the shortest distance from Moscow to Nairobi is 6000 km.

- (d) Show that $\theta = 57.3^\circ$, correct to three significant figures. [2]

Bogotá, B, is west of Nairobi and has position vector $\vec{OB} = \mathbf{b} = \begin{pmatrix} 6 \sin 120^\circ \\ 6 \cos 120^\circ \\ 0 \end{pmatrix}$.

- (e) Find the shortest distance from Bogotá to Moscow on the sphere. [5]

The bearing from B to M is defined as the angle at vertex B in the spherical triangle

containing B, M and P. It is given that $\mathbf{b} \times \mathbf{p} = \begin{pmatrix} 36 \cos 120^\circ \\ -36 \sin 120^\circ \\ 0 \end{pmatrix}$.

- (f) Using the method from part (c), find the bearing from Bogotá to Moscow. [6]

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References:

- 2(c)** Google Maps/Google Earth. Data from Data SIO, NOAA, U.S. Navy, NGA, GEBCOLandsat / Copernicus, IBCAO, U.S. Geological Survey, PGC/NASA. Imagery from 14/12/2015. Image available at: https://earth.google.com/web/@12.01529518,-18.56070747,-158.39383184a,23597813.93249989d,30.00008083y,359.99981502h,0t,0r/data=CgRCAggBOgMKATBCAggASg0I_____ARAA.

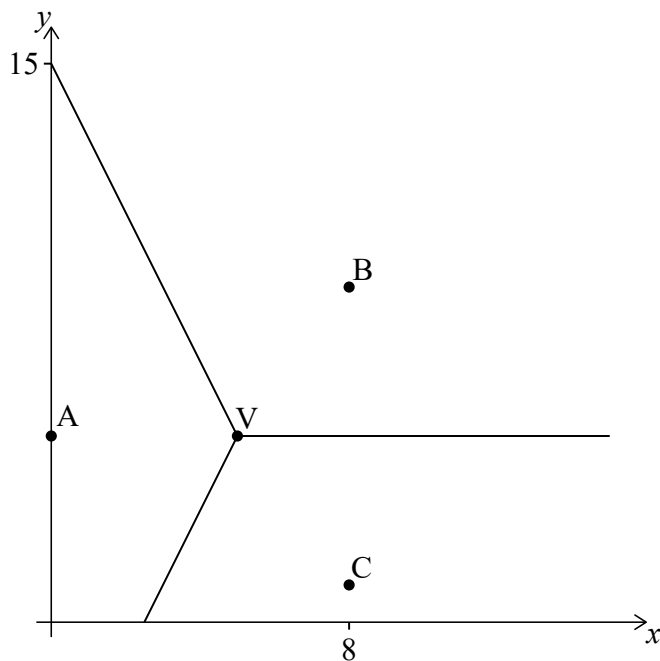
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1. [Maximum mark: 28]

This question considers how the assessment of the Air Quality Index (AQI) for a school depends on the method chosen by the person doing the assessing.

Air quality for a district is measured at three monitoring stations. The positions of these stations on a coordinate system with units in kilometres are $A(0, 5)$, $B(8, 9)$ and $C(8, 1)$.

A Voronoi diagram is constructed with the three stations as sites.



- (a) (i) Find the equation of the perpendicular bisector of $[BC]$. [2]
- (ii) Given that the equation of the perpendicular bisector of $[AB]$ is $y = 15 - 2x$, find the coordinates of vertex V . [2]

(This question continues on the following page)

(Question 1 continued)

A school, S , is situated in the district at the point with coordinates $(5, 6)$.

- (b) State which station is closest to the school. [1]

The principal of the school is concerned about the air quality in the area. Air quality is measured by the Air Quality Index (AQI). In this district, values less than 50 are taken to indicate good air quality.

The principal contacts the local environmental agency requesting an AQI value for her school. They tell her the mean AQI reading from the closest station to the school.

- (c) Write down the type of interpolation being used by the environmental agency. [1]

The principal obtains the mean AQI value from each of the three stations.

Station	Mean AQI value
A	132
B	49
C	125

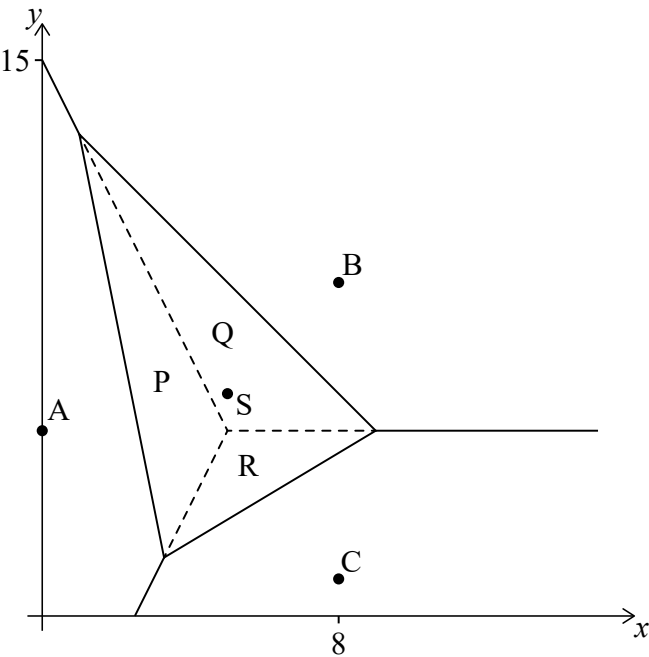
- (d) Explain why the principal might not accept that the air quality around the school can be classed as “good”. [1]

(This question continues on the following page)

(Question 1 continued)

The principal decides to obtain an expected value for the AQI at the school that uses all the available information. To do this, she uses an alternative method: the natural neighbour algorithm. This algorithm has two stages.

The first stage is to create a new Voronoi diagram with the school as an extra site. This is shown in the following diagram with the edges of the previous diagram shown by dashed lines.



The second stage is to estimate the AQI value at the school, W , by using the formula

$$W = \frac{w_A a_A + w_B a_B + w_C a_C}{T}.$$

In the formula, a_A is the area within the new cell that has been taken from the cell surrounding site A, shown as region P on the diagram, and w_A is the mean AQI value from site A. This is given as 132 in the table of mean AQI values above. Similarly for sites B and C. T is the total area of the new cell around S.

- (e) (i) Find the mid-point of [SB]. [2]
- (ii) Show that the equation of the perpendicular bisector of [SB] is $y = 14 - x$. [3]
- (iii) Hence find a_B (the area of region Q on the diagram). [5]

(This question continues on the following page)

(Question 1 continued)

The areas of regions P and R are $a_A = 13.7 \text{ km}^2$ and $a_C = 6.9 \text{ km}^2$ respectively.

- (f) Use the natural neighbour algorithm to show that an estimate for the expected AQI value at the school, $\bar{W}$, is 94.4. [2]

The principal is still concerned that this method is underestimating the AQI value at the school, as the school is situated close to a busy traffic intersection. She decides to take her own readings (x) over a period of 60 days. Her results are summarized as

$$\bar{x} = 97.8, s_{n-1} = 17.2, n = 60.$$

The principal assumes that the daily AQI values at the school can be modelled by a normal distribution and that each value is independent of any other value.

- (g) Test at the 10% significance level the hypothesis that the mean AQI value at the school is greater than 94.4. State clearly the null and alternative hypotheses and the conclusion of the test in context. [6]

The principal now assumes that the distribution of AQI values at the school follows a normal distribution with mean 97.8 and standard deviation 17.2.

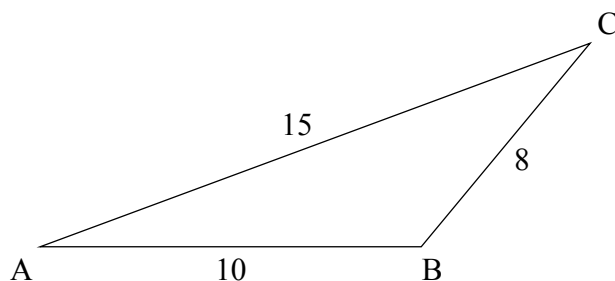
- (h) Use this model to find the expected number of days per year (correct to the nearest day) on which the AQI value can be classed as “good”. [3]

Turn over

4. [Maximum mark: 8]

Triangle ABC has sides of length $AB = 10$ cm, $BC = 8$ cm and $AC = 15$ cm, as shown in the diagram.

diagram not to scale

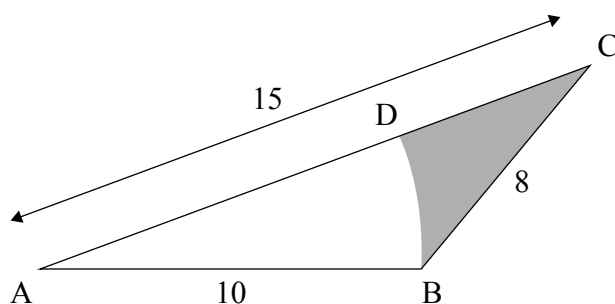


- (a) Find the size of $\hat{BAC}$.

[3]

The circular arc BD, with centre A, is drawn inside the triangle, such that D lies on AC.

diagram not to scale



- (b) Find the area of the shaded region.

[5]

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(Question 4 continued)

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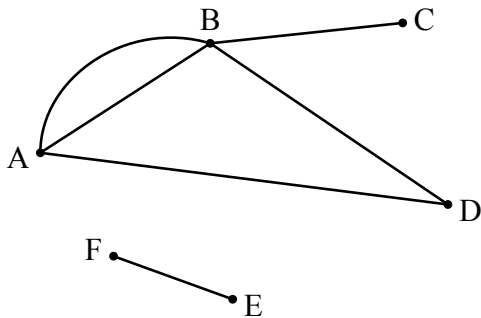
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12. [Maximum mark: 8]

The graph G has six vertices and is shown in the diagram.



- (a) State, giving a reason, whether G is

(i) simple;

(ii) connected.

[2]
- (b) Complete the adjacency matrix for G .

[2]

	A	B	C	D	E	F
A	0	☐	0	1	0	0
B	☐	0	☐	1	0	0
C	0	☐	0	0	0	0
D	1	1	0	0	0	0
E	0	0	0	0	0	1
F	0	0	0	0	1	0

- (c) Hence, determine the number of walks from A to B with fewer than four edges.

[4]

(This question continues on the following page)



(Question 12 continued)

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13. [Maximum mark: 8]

Air traffic control begins to monitor two airplanes.

The position of the first airplane is given by

$$\mathbf{r}_1 = \begin{pmatrix} 29 \\ -32 \\ 5.8 \end{pmatrix} + t \begin{pmatrix} -320 \\ 600 \\ 0.5 \end{pmatrix},$$

where t is the time, in hours, after air traffic control begins to monitor the airplane.

Distances are in kilometres and are relative to an origin at air traffic control.

The second airplane is initially at $\begin{pmatrix} -13 \\ 50 \\ 7.2 \end{pmatrix}$ and is travelling at a speed of 800 km h^{-1} in the direction $\begin{pmatrix} 7 \\ -24 \\ 0 \end{pmatrix}$.

- (a) Find an expression for $\mathbf{r}_2$, the position of the second airplane at time t . [3]
- (b) Find the least distance between the two airplanes. [5]



Turn over

1. [Maximum mark: 26]

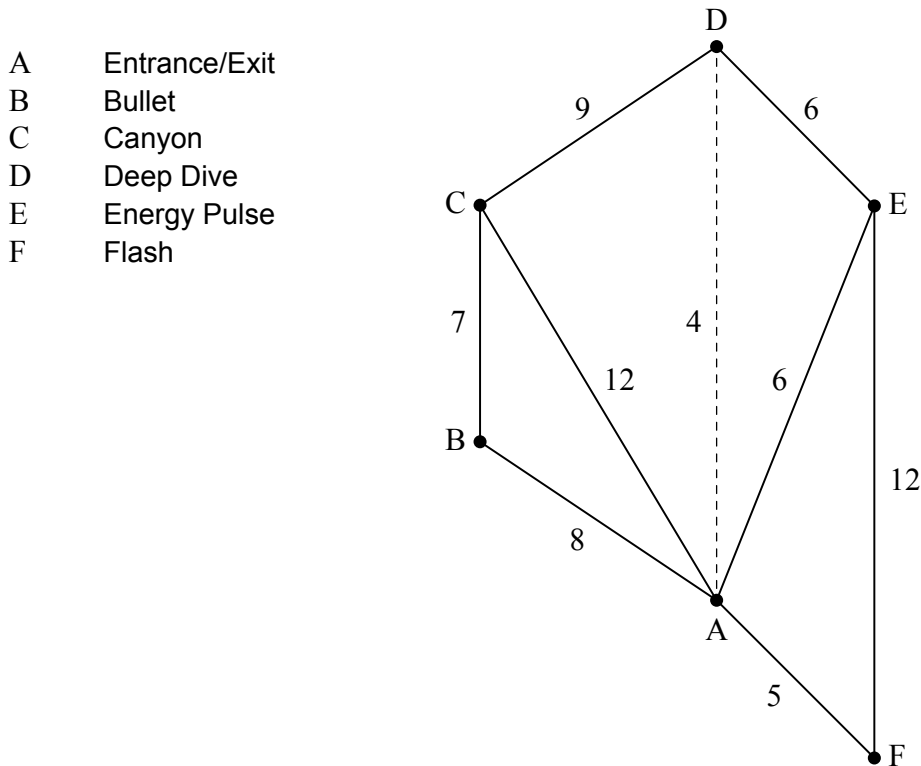
This question considers whether it is reasonable to go on all the rides in a theme park and get back to the entrance in two and a half hours.

Martin is visiting a theme park. He will enter the park at 09:00 and must leave the park by 11:30. He uses information available on the internet to calculate whether he will be able to go on all of the rides in the two and a half hours.

He begins by constructing a graph which shows the main paths between the rides and the route of the cable car between the entrance/exit A and ride D.

His graph and the names of the rides are shown in the following diagram.

diagram not to scale



(This question continues on the following page)

(Question 1 continued)

The weights on the edges of the graph represent the times, in minutes, to walk between the rides and the time to travel by cable car between A and D.

Let T be the shortest possible time, in minutes, taken to visit all the rides, beginning and ending at A.

Martin notices that the graph contains a Hamiltonian cycle. He decides to use the weight of the Hamiltonian cycle as an upper bound for T .

- (a) Find the weight of this Hamiltonian cycle.
- [2]

Martin constructs **Table 1** to show the shortest possible time it takes to travel between any two rides and between the entrance and any ride.

Table 1

	A	B	C	D	E	F
A	0	8	12	4	6	5
B	8	0	7	a	14	13
C	12	7	0	9	15	b
D	4	a	9	0	6	9
E	6	14	15	6	0	11
F	5	13	b	9	11	0

- (b) Write down the value of
- (i) a ;

[1]

(ii) b .

[1]
- (c) Use the nearest neighbour algorithm on **Table 1** to find an upper bound for T .
- [3]

(This question continues on the following page)

(Question 1 continued)

Martin decides to use the deleted vertex algorithm to find a lower bound for T by first deleting vertex A . The shortest possible time to travel between each ride, with vertex A deleted, is given in **Table 2**.

Table 2

	B	C	D	E	F
B	0	7	a	14	13
C	7	0	9	15	b
D	a	9	0	6	9
E	14	15	6	0	11
F	13	b	9	11	0

- (d)

(i)

Use Prim’s algorithm on **Table 2** to find the weight of the minimum spanning tree for the graph with vertices B , C , D , E and F .
Start at vertex B and write down the order in which the edges are selected.

[3]
- (ii)

Hence find a lower bound for T .

[2]

Martin finds more lower bounds for T , by deleting each vertex in turn. The results are shown in the following table.

Deleted vertex	Lower bound
B	39
C	39
D	36
E	36
F	39

Martin finds the smallest possible interval within which T lies, based on his calculated values for upper and lower bounds. He writes his answer in the form $p \leq T \leq q$.

- (e)

Write down the value of
- (i)

p ;

[1]
- (ii)

q .

[1]

(This question continues on the following page)

(Question 1 continued)

Martin's favourite ride is Energy Pulse (E), so he decides to go there first. He plans to begin at A and visit the rides in the order E, D, C, B, F before returning to A. For the rest of the question, assume that Martin is taking this route.

- (f) (i) Find the shortest possible time it would take to complete this route. [1]
- (ii) State the edge which would need to be repeated. [1]

Each of the rides takes 2 minutes to complete.

Let the time spent waiting in the queue for ride B be written as the random variable B_i and similarly for the other rides.

The following distributions model the times spent waiting in the queues. Each waiting time is independent of all other waiting times and the time of day.

$$B_i \sim N(13, 3), C_i \sim N(21, 15), D_i \sim N(16, 8), E_i \sim N(10, 6), F_i \sim N(20, 15)$$

- (g) (i) Find the distribution for the total time spent queuing for all five rides. [3]
- (ii) Find the probability that Martin manages to go on all five rides and return to the entrance in two and a half hours. [4]

Martin enters the park at 09:00 and decides to follow his planned route, but has two consecutive rides on Energy Pulse.

- (h) Find the expected time he will leave the park. [3]

Turn over