

APOLLO

Functions

HL

Mathematics: applications and interpretation

First assessment 2021

51 questions | questions

4. [Maximum mark: 6]

Professor Wei observed that students have difficulty remembering the information presented in his lectures.

He modelled the percentage of information retained, R , by the function $R(t) = 100e^{-pt}$, $t \geq 0$, where t is the number of days after the lecture.

He found that 1 day after a lecture, students had forgotten 50% of the information presented.

- Find the value of p . [2]
- Use this model to find the percentage of information retained by his students 36 hours after Professor Wei's lecture. [2]

Based on his model, Professor Wei believes that his students will always retain some information from his lecture.

- (c) State a mathematical reason why Professor Wei might believe this. [1]
- (d) Write down one possible limitation of the **domain** of the model. [1]

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Turn over

17. [Maximum mark: 7]

The graph of the function $f(x) = \ln x$ is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ so that it then passes through the points $(0, 1)$ and $(e^3, 1 + \ln 2)$.

Find the value of a and the value of b .

[illegible]

References:

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2. [Maximum mark: 7]

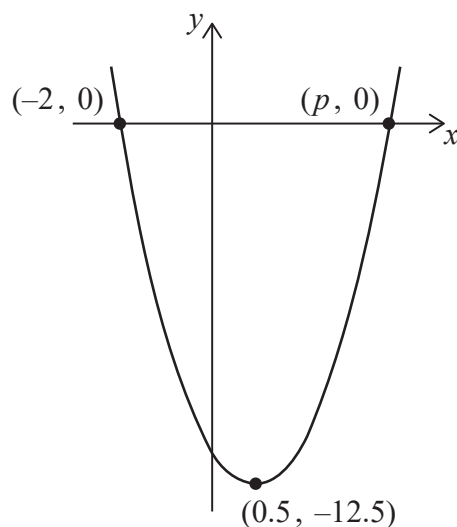
A function is defined by $f(x) = 2 - \frac{12}{x+5}$ for $-7 \leq x \leq 7, x \neq -5$.

- (a) Find the range of f . [3]
- (b) Find an expression for the inverse function $f^{-1}(x)$. The domain is not required. [3]
- (c) Write down the range of $f^{-1}(x)$. [1]



6. [Maximum mark: 7]

Consider the function $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is shown in the diagram. The vertex of the graph has coordinates $(0.5, -12.5)$. The graph intersects the x -axis at two points, $(-2, 0)$ and $(p, 0)$.

diagram not to scale

- (a) Find the value of p . [1]
- (b) Find the value of
- (i) a .
 - (ii) b .
 - (iii) c . [5]
- (c) Write down the equation of the axis of symmetry of the graph. [1]

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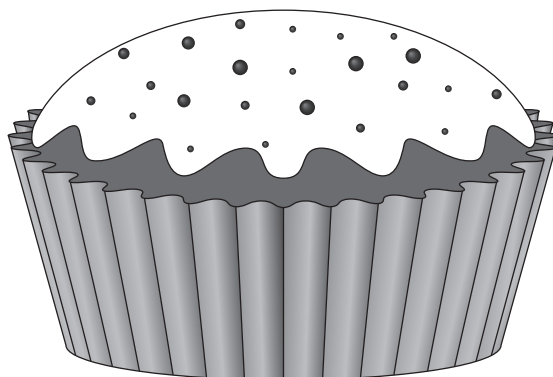
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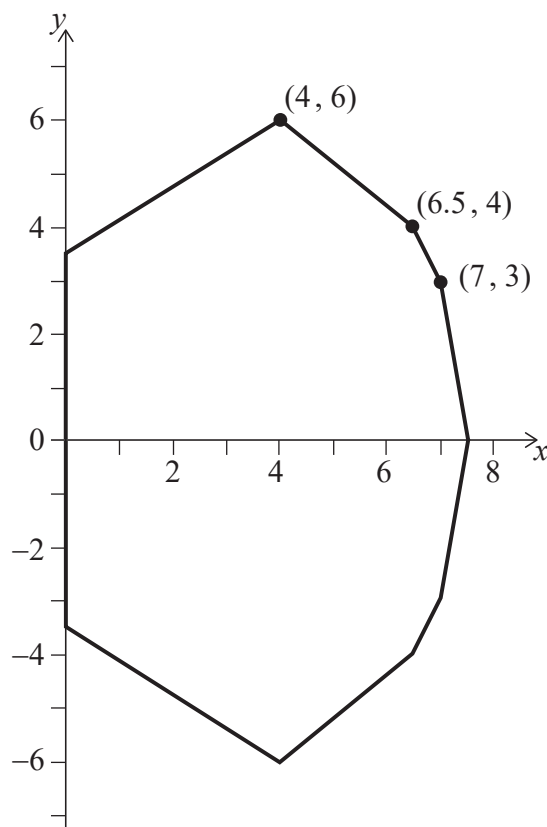
Turn over

4. [Maximum mark: 14]

Charlotte decides to model the shape of a cupcake to calculate its volume.



From rotating a photograph of her cupcake she estimates that its cross-section passes through the points $(0, 3.5)$, $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$, where all units are in centimetres. The cross-section is symmetrical in the x -axis, as shown below:



She models the section from $(0, 3.5)$ to $(4, 6)$ as a straight line.

- (a) Find the equation of the line passing through these two points.

[2]

(This question continues on the following page)

(Question 4 continued)

Charlotte models the section of the cupcake that passes through the points $(4, 6)$, $(6.5, 4)$, $(7, 3)$ and $(7.5, 0)$ with a quadratic curve.

- (b) (i) Find the equation of the least squares regression quadratic curve for these four points.
- (ii) By considering the gradient of this curve when $x = 4$, explain why it may not be a good model. [3]

Charlotte thinks that a quadratic with a maximum point at $(4, 6)$ and that passes through the point $(7.5, 0)$ would be a better fit.

- (c) Find the equation of the new model. [4]

Believing this to be a better model for her cupcake, Charlotte finds the volume of revolution about the x -axis to estimate the volume of the cupcake.

- (d) (i) Write down an expression for her estimate of the volume as a sum of two integrals.
- (ii) Find the value of Charlotte's estimate. [5]

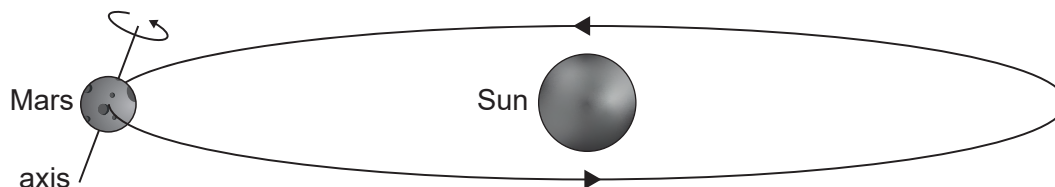
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1. [Maximum mark: 27]

A suitable site for the landing of a spacecraft on the planet Mars is identified at a point, A. The shortest time from sunrise to sunset at point A must be found.

Radians should be used throughout this question. All values given in the question should be treated as exact.

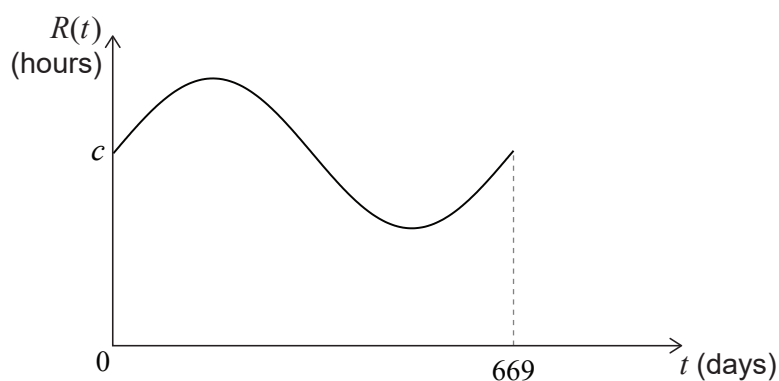
Mars completes a full orbit of the Sun in 669 Martian days, which is one Martian year.



On day t , where $t \in \mathbb{Z}$, the length of time, in hours, from the start of the Martian day until sunrise at point A can be modelled by a function, $R(t)$, where

$$R(t) = a \sin(bt) + c, \quad t \in \mathbb{R}.$$

The graph of R is shown for one Martian year.



- (a) Show that $b \approx 0.00939$.

[2]

Mars completes a full rotation on its axis in 24 hours and 40 minutes.

- (b) Find the angle through which Mars rotates on its axis each hour.

[3]

(This question continues on the following page)

(Question 1 continued)

The time of sunrise on Mars depends on the angle, δ , at which it tilts towards the Sun. During a Martian year, δ varies from -0.440 to 0.440 radians.

The angle, ω , through which Mars rotates on its axis from the start of a Martian day to the moment of sunrise, at point A, is given by $\cos \omega = 0.839 \tan \delta$, $0 \leq \omega \leq \pi$.

- (c) (i) Show that the maximum value of $\omega = 1.98$, correct to three significant figures. [3]
- (ii) Find the minimum value of ω . [1]
- (d) Use your answers to parts (b) and (c) to find
- (i) the maximum value of $R(t)$; [2]
- (ii) the minimum value of $R(t)$. [1]
- (e) Hence show that $a = 1.6$, correct to two significant figures. [2]
- (f) Find the value of c . [2]

Let $S(t)$ be the length of time, in hours, from the start of the Martian day until **sunset** at point A on day t . $S(t)$ can be modelled by the function

$$S(t) = 1.5 \sin(0.00939t + 2.83) + 18.65.$$

The length of time between sunrise and sunset at point A, $L(t)$, can be modelled by the function

$$L(t) = 1.5 \sin(0.00939t + 2.83) - 1.6 \sin(0.00939t) + d.$$

- (g) Find the value of d . [2]

Let $f(t) = 1.5 \sin(0.00939t + 2.83) - 1.6 \sin(0.00939t)$ and hence $L(t) = f(t) + d$.

$f(t)$ can be written in the form $\operatorname{Im}(z_1 - z_2)$, where z_1 and z_2 are complex functions of t .

- (h) (i) Write down z_1 and z_2 in exponential form, with a constant modulus. [3]
- (ii) Hence or otherwise find an equation for L in the form $L(t) = p \sin(qt + r) + d$, where $p, q, r, d \in \mathbb{R}$. [4]
- (iii) Find, in hours, the shortest time from sunrise to sunset at point A that is predicted by this model. [2]

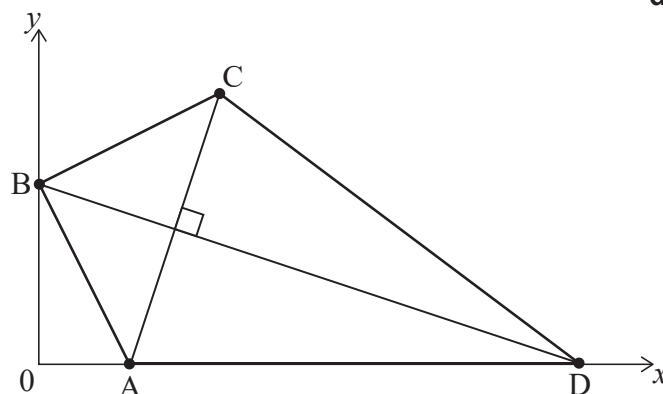
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1. [Maximum mark: 6]

Dilara is designing a kite ABCD on a set of coordinate axes in which one unit represents 10 cm.

The coordinates of A, B and C are (2, 0), (0, 4) and (4, 6) respectively. Point D lies on the x -axis. [AC] is perpendicular to [BD]. This information is shown in the following diagram.

diagram not to scale



- (a) Find the gradient of the line through A and C. [2]
- (b) Write down the gradient of the line through B and D. [1]
- (c) Find the equation of the line through B and D. Give your answer in the form $ax + by + d = 0$, where a , b and d are integers. [2]
- (d) Write down the x -coordinate of point D. [1]

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(Question 1 continued)

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Turn over

3. [Maximum mark: 7]

Let the function $h(x)$ represent the height in centimetres of a cylindrical tin can with diameter x cm.

$$h(x) = \frac{640}{x^2} + 0.5 \text{ for } 4 \leq x \leq 14.$$

- (a) Find the range of h . [3]

The function h^{-1} is the inverse function of h .

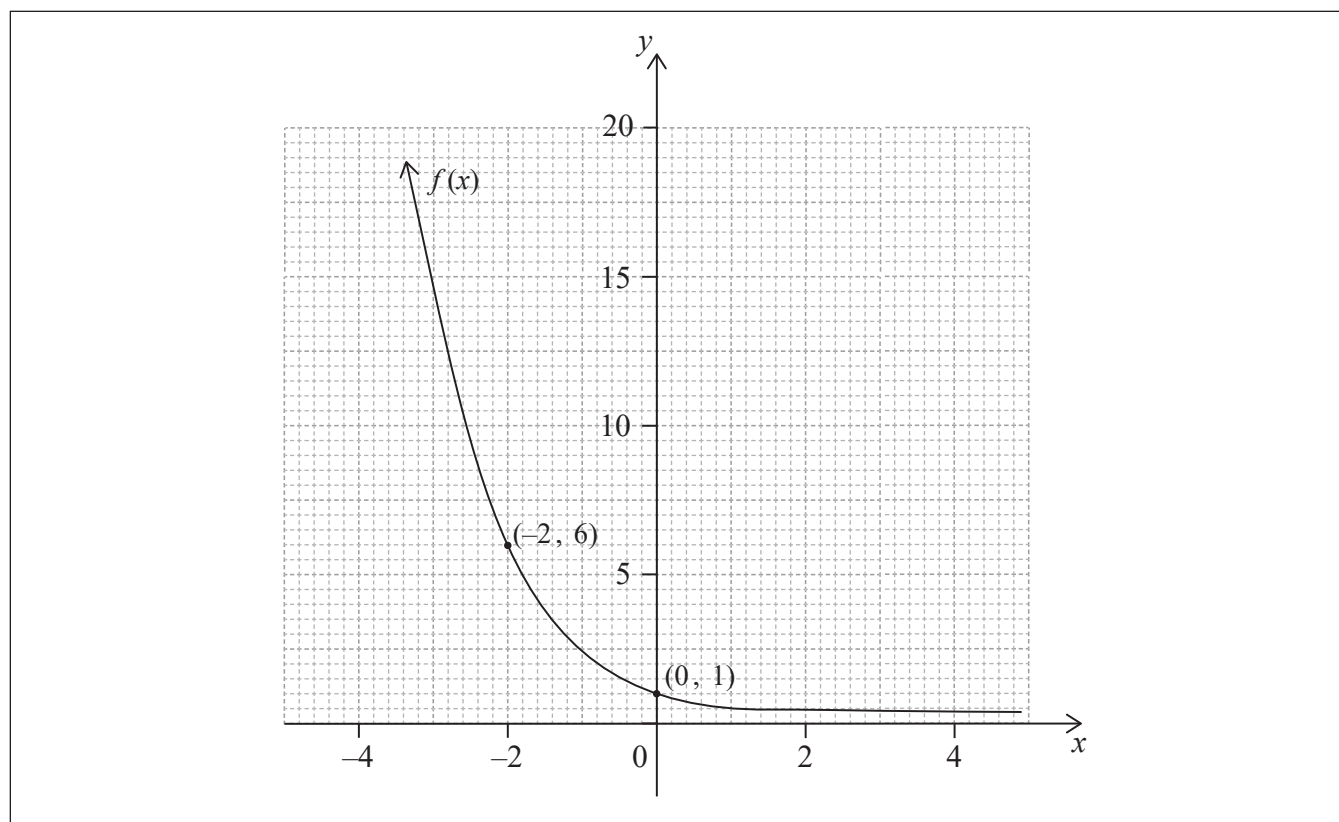
- (b) (i) Find $h^{-1}(10)$.
(ii) In the context of the question, interpret your answer to part (b)(i).
(iii) Write down the range of h^{-1} . [4]



Turn over

10. [Maximum mark: 4]

The graph of $y = f(x)$ is given on the following set of axes. The graph passes through the points $(-2, 6)$ and $(0, 1)$, and has a horizontal asymptote at $y = 0$.



Let $g(x) = 2f(x - 2) + 4$.

(a) Find $g(0)$. [2]

(b) On the same set of axes draw the graph of $y = g(x)$, showing any intercepts and asymptotes. [2]

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(Question 10 continued)

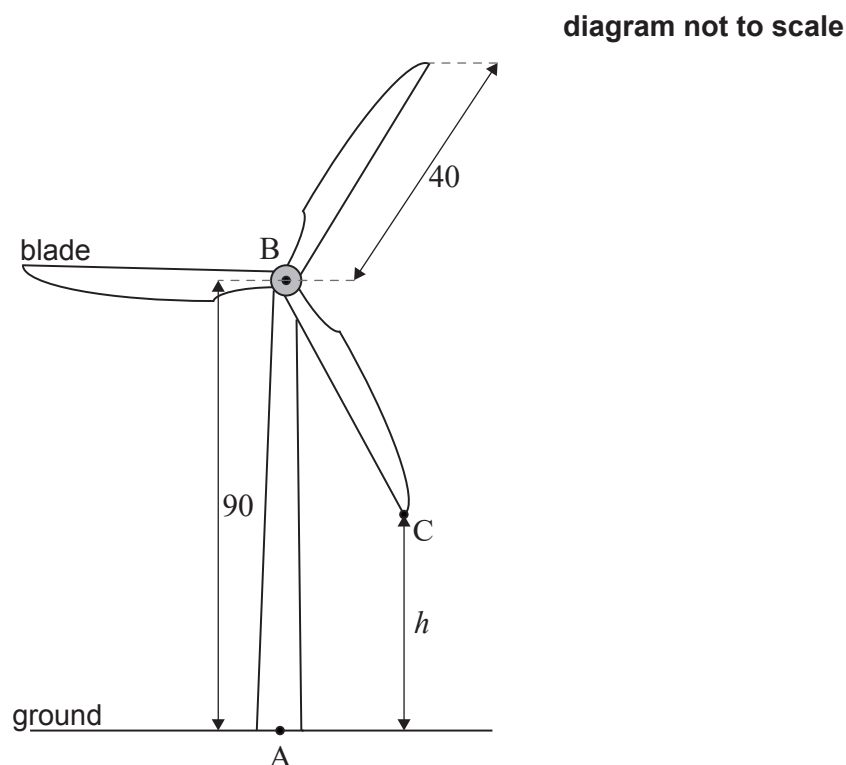
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Turn over

2. [Maximum mark: 20]

A wind turbine is designed so that the rotation of the blades generates electricity. The turbine is built on horizontal ground and is made up of a vertical tower and three blades.

The point A is on the base of the tower directly below point B at the top of the tower. The height of the tower, AB , is 90 m . The blades of the turbine are centred at B and are each of length 40 m . This is shown in the following diagram.



The end of one of the blades of the turbine is represented by point C on the diagram. Let h be the height of C above the ground, measured in metres, where h varies as the blade rotates.

(a) Find the

(i) maximum value of h .

(ii) minimum value of h .

[2]

The blades of the turbine complete 12 rotations per minute under normal conditions, moving at a constant rate.

(b) (i) Find the time, in seconds, it takes for the blade $[BC]$ to make one complete rotation under these conditions.

(ii) Calculate the angle, in degrees, that the blade $[BC]$ turns through in one second.

[3]

(This question continues on the following page)

(Question 2 continued)

The height, h , of point C can be modelled by the following function. Time, t , is measured from the instant when the blade [BC] first passes [AB] and is measured in seconds.

$$h(t) = 90 - 40 \cos(72t^\circ), \quad t \geq 0$$

- (c) (i) Write down the amplitude of the function.
- (ii) Find the period of the function. [2]
- (d) Sketch the function $h(t)$ for $0 \leq t \leq 5$, clearly labelling the coordinates of the maximum and minimum points. [3]
- (e) (i) Find the height of C above the ground when $t = 2$.
- (ii) Find the time, in seconds, that point C is above a height of 100 m, during each complete rotation. [5]

The wind speed increases and the blades rotate faster, but still at a constant rate.

- (f) Given that point C is now higher than 110 m for 1 second during each complete rotation, find the time for one complete rotation. [5]

Turn over

3. [Maximum mark: 6]

Professor Vinculum investigated the migration season of the Bulbul bird from their natural wetlands to a warmer climate.

He found that during the migration season their population, P could be modelled by $P = 1350 + 400(1.25)^{-t}$, $t \geq 0$, where t is the number of days since the start of the migration season.

- (a) Find the population of the Bulbul birds,
 - (i) at the start of the migration season.
 - (ii) in the wetlands after 5 days. [3]
- (b) Calculate the time taken for the population to decrease below 1400. [2]
- (c) According to this model, find the smallest possible population of Bulbul birds during the migration season. [1]



Turn over

1. [Maximum mark: 5]

The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

- (a) Write down the height of the ball above the ground at the instant it is hit by the bat. [1]
- (b) Find the value of t when the ball hits the ground. [2]
- (c) State an appropriate domain for t in this model. [2]



References:

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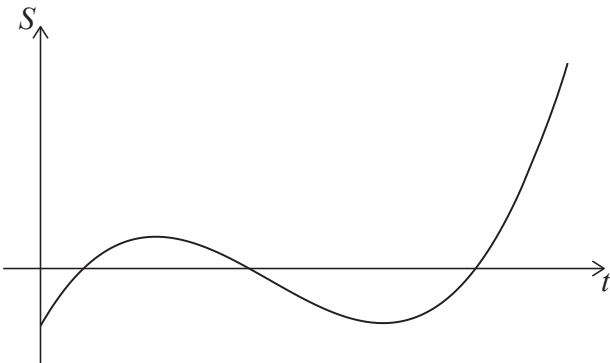
The point A is a local maximum and the point B is a local minimum.

Find the value of p , of q and of r .

[illegible]

5. [Maximum mark: 8]

The graph below shows the average savings, S thousand dollars, of a group of university graduates as a function of t , the number of years after graduating from university.



- (a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario. [1]

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where a , b , c and d are real constants.

The graph of the model must pass through the following four points.

t	0	1	2	3
S	-5	3	-1	-5

- (b) (i) Write down the value of d .
(ii) Write down three simultaneous equations for a , b and c .
(iii) Hence, or otherwise, find the values of a , b and c . [4]

A negative value of S indicates that a graduate is expected to be in debt.

- (c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university. [3]

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[illegible]

10. [Maximum mark: 5]

The function $f(x) = \ln\left(\frac{1}{x-2}\right)$ is defined for $x > 2$, $x \in \mathbb{R}$.

- (a) Find an expression for $f^{-1}(x)$. You are not required to state a domain. [3]
- (b) Solve $f(x) = f^{-1}(x)$. [2]

[illegible]

4. [Maximum mark: 15]

A student investigating the relationship between chemical reactions and temperature finds the Arrhenius equation on the internet.

$$k = Ae^{-\frac{c}{T}}$$

This equation links a variable k with the temperature T , where A and c are positive constants and $T > 0$.

(a) Show that $\frac{dk}{dT}$ is always positive. [3]

(b) Given that $\lim_{T \rightarrow \infty} k = A$ and $\lim_{T \rightarrow 0} k = 0$, sketch the graph of k against T . [3]

The Arrhenius equation predicts that the graph of $\ln k$ against $\frac{1}{T}$ is a straight line.

(c) Write down

(i) the gradient of this line in terms of c ;

(ii) the y -intercept of this line in terms of A . [4]

The following data are found for a particular reaction, where T is measured in Kelvin and k is measured in $\text{cm}^3 \text{mol}^{-1} \text{s}^{-1}$:

T	k
590	5×10^{-4}
600	6×10^{-4}
610	10×10^{-4}
620	14×10^{-4}
630	20×10^{-4}
640	29×10^{-4}
650	36×10^{-4}

(d) Find the equation of the regression line for $\ln k$ on $\frac{1}{T}$. [2]

(e) Find an estimate of

(i) c ;

(ii) A .

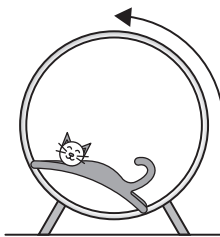
It is not required to state units for these values. [3]

Turn over



6. [Maximum mark: 6]

A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, h cm, of a fixed point, P, on the wheel can be modelled by $h(t) = a \sin(bt) + c$ where t is the time in seconds and $a, b, c \in \mathbb{R}^+$.



When $t = 0$, point P is at a height of 78 cm.

- (a) Write down the value of c . [1]

When $t = 4$, point P first reaches its maximum height of 143 cm.

- (b) Find the value of
- (i) a .
- (ii) b . [3]
- (c) Write down the minimum height of point P. [1]

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

- (d) Write down the new value of b . [1]

[illegible]

Turn over

10. [Maximum mark: 7]

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m , of another star can be modelled as a function of its brightness, b , relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5 \log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

- (a) Find the magnitude of Acubens. [2]

Ceres has a magnitude of 7 and is the least bright star visible without magnification.

- (b) Find the brightness of Ceres. [2]

The star Proxima Centauri has a greater magnitude than the planet Neptune. The difference in their magnitudes is 3.2.

- (c) Find how many times brighter Neptune is compared to Proxima Centauri. [3]



Turn over

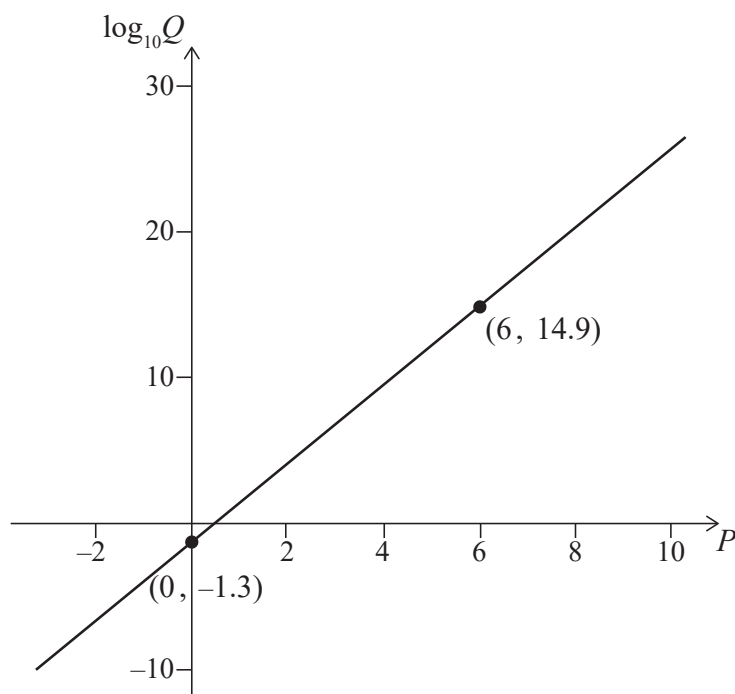
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will not be marked.



13. [Maximum mark: 6]

Gen is investigating the relationship between two sets of data, labelled P and Q , that she collected. She created a scatter plot with P on the x -axis and $\log_{10} Q$ on the y -axis. Gen noticed that the points had a strong linear correlation, so she drew a line of best fit, as shown in the diagram. The line passes through the points $(0, -1.3)$ and $(6, 14.9)$.



- (a) Find an equation for Q in terms of P .

[3]

Gen also investigates the relationship between the same data, Q , and some new data, R . She believes that the data can be modelled by $Q = a \ln R + b$ and she decides to create a scatter plot to verify her belief.

- (b) State what expression Gen should plot on each axis to verify her belief.

[1]

The scatter plot has a linear relationship and Gen finds $a = 4.3$ and $b = 12.1$.

- (c) Find an equation for P in terms of R .

[2]

(This question continues on the following page)



(Question 13 continued)

[illegible]

Turn over

17. [Maximum mark: 8]

The time of sunrise, R hours after midnight, in Taipei can be modelled by

$$R = 1.08 \cos(0.0165t + 0.413) + 4.94,$$

where t is the day of the year 2021 (for example, $t = 2$ represents 2 January 2021).

The time of sunset, S hours after midnight, in Taipei can be modelled by

$$S = 1.15 \cos(0.0165t - 2.97) + 18.9.$$

The number of daylight hours, D , in Taipei during 2021 can be modelled by

$$D = a \cos(0.0165t + b) + c.$$

(a) Find the value of a , of b and of c . [6]

(b) Hence, or otherwise, find the largest number of daylight hours in Taipei during 2021 and the day of the year on which this occurs. [2]

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References:

11. Fleur, 2019. *photo-1560263816-d704d83cce0f*. [image online] Available at: <https://unsplash.com/photos/SE2zTdS1MNo> [Accessed 8 February 2022]. Source adapted.

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6. [Maximum mark: 7]

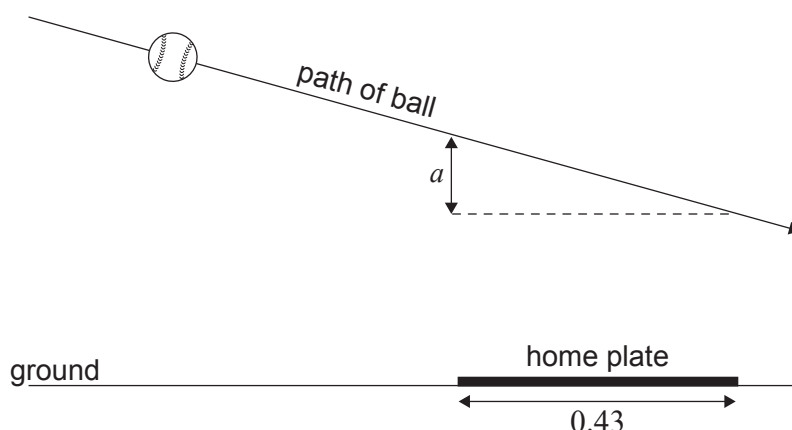
In a baseball game, Sakura is the batter standing beside home plate. The ball is thrown towards home plate along a path that can be modelled by the following function.

$$y = -0.045x + 2$$

In this model, x is the horizontal distance of the ball from the point the ball is thrown and y is the vertical height of the ball above the ground. Both measured in metres.

The outcome of the throw is called a strike if the height of the ball is between 0.53 m and 1.24 m at some point while it travels over home plate. The length of home plate is 0.43 m.

diagram not to scale



When the ball reaches the front of home plate, the height of the ball above the ground is 1.25 m. The height of the ball changes by a metres as the ball travels over the length of home plate.

(a) (i) Find the value of a .

(ii) Justify why this throw is a strike.

[4]

On the next throw, Sakura hits the ball towards a wall that is 5 metres high. The horizontal distance of the wall from the point where the ball was hit is 96 metres. The path of the ball after it is hit can be modelled by the function $h(d)$.

$$h(d) = -0.01d^2 + 1.04d + 0.66, \text{ for } h, d > 0$$

In this model, h is the height of the ball above the ground and d is the horizontal distance of the ball from the point where it was hit. Both h and d are measured in metres.

(b) Determine whether the ball will go over the wall. Justify your answer.

[3]

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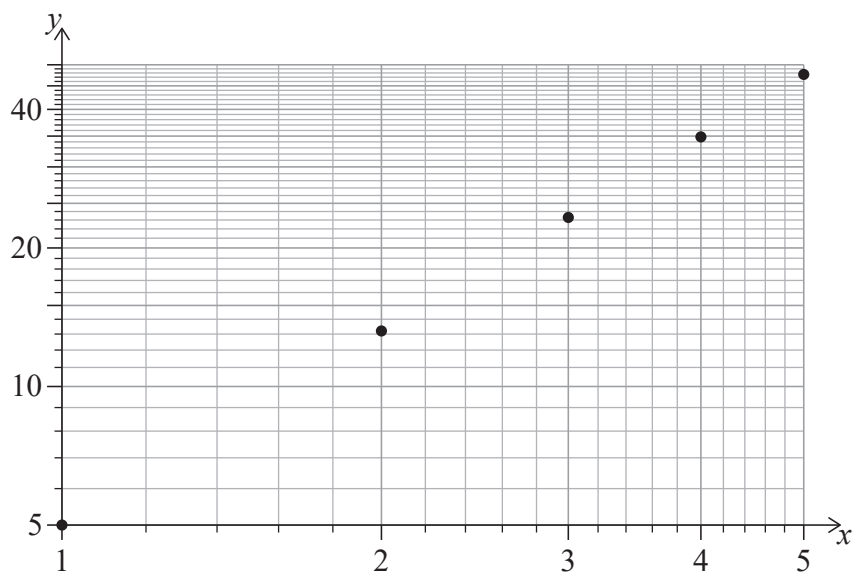
(Question 6 continued)

[illegible]

Turn over

14. [Maximum mark: 6]

Petra examines two quantities, x and y , and plots data points on a log-log graph.



She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2, 13.1951)$ and $(4, 34.822)$, find the equation of the relationship connecting x and y . Your final answer should not include logarithms.

[illegible]

Turn over

A player throws a basketball. The height of the basketball is modelled by

where h is the height of the basketball above the ground, in metres, and t is the time, in seconds, after it was thrown.

- Another player catches the basketball when it is at a height of 1.2 metres.

- (c) Find the value of t when this player catches the basketball. [2]
- (d) Write down one limitation of using $h(t)$ to model the height of the basketball. [1]

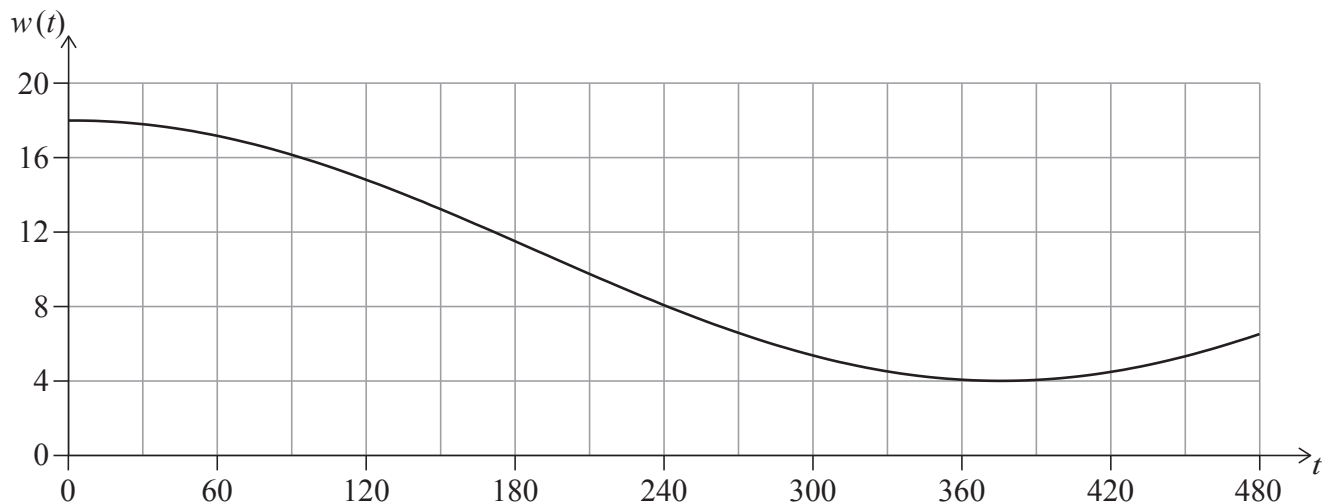
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Turn over

2. [Maximum mark: 15]

The depth of water, w metres, in a particular harbour can be modelled by the function $w(t) = a \cos(bt^\circ) + d$ where t is the length of time, in minutes, after 06:00.

On 20 January, the first high tide occurs at 06:00, at which time the depth of water is 18 m. The following low tide occurs at 12:15 when the depth of water is 4 m. This is shown in the diagram.



- (a) Find the value of a . [2]
- (b) Find the value of d . [2]
- (c) Find the period of the function in minutes. [3]
- (d) Find the value of b . [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least 6 m.

- (e) Find the latest time before 12:00, to the nearest minute, that Naomi can enter the harbour. [4]
- (f) Find the length of time (in minutes) between 06:00 and 15:00 on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

2. [Maximum mark: 12]

A scientist is conducting an experiment on the growth of a certain species of bacteria.

The population of the bacteria, P , can be modelled by the function

$$P(t) = 1200 \times k^t, \quad t \geq 0,$$

where t is the number of hours since the experiment began, and k is a positive constant.

(a) (i) Write down the value of $P(0)$.

(ii) Interpret what this value means in this context. [2]

3 hours after the experiment began, the population of the bacteria is 18 750.

(b) Find the value of k . [2]

(c) Find the population of the bacteria 1 hour and 30 minutes after the experiment began. [2]

The scientist conducts a second experiment with a different species of bacteria.

The population of this bacteria, S , can be modelled by the function

$$S(t) = 5000 \times 1.65^t, \quad t \geq 0,$$

where t is the number of hours since both experiments began.

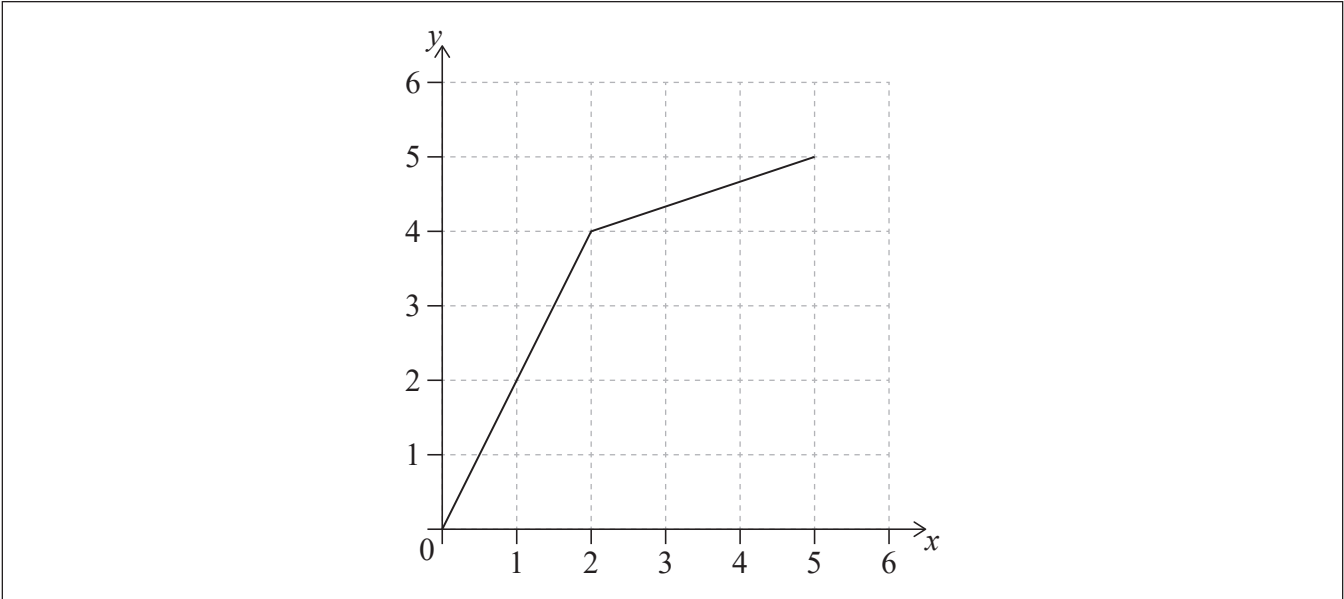
(d) Find the value of t when the two populations of bacteria are equal. [2]

It takes 2 hours and m minutes for the number of bacteria in the second experiment to reach 19 000.

(e) Find the value of m , giving your answer as an integer value. [4]

2. [Maximum mark: 7]

The graph of the function f is given in the following diagram.



- (a) Write down $f(2)$. [1]
- (b) On the axes, sketch $y = f^{-1}(x)$. [2]

The function g is defined as $g(x) = 3x - 1$.

- (c) Find an expression for $g^{-1}(x)$. [2]
- (d) Find a value of x where $f^{-1}(x) = g^{-1}(x)$. [2]

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Turn over

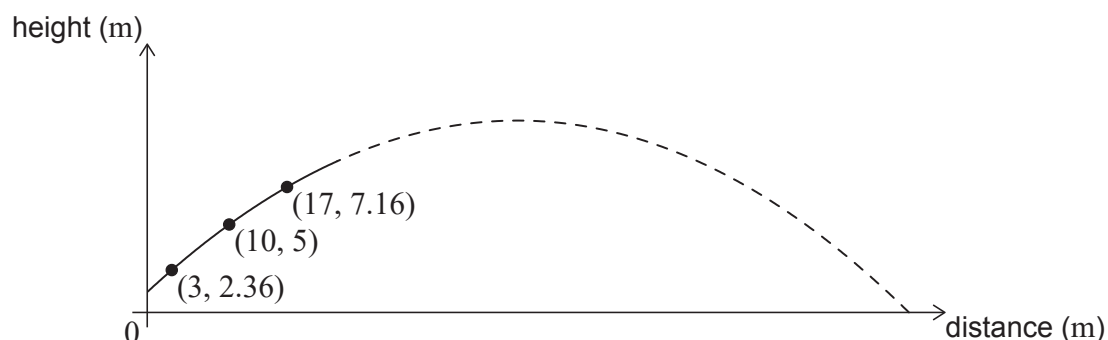
5. [Maximum mark: 9]

A sports player on a horizontal athletic field hits a ball. The height of the ball above the field, in metres, after it is hit can be modelled using a quadratic function of the form $f(x) = ax^2 + bx + c$, where x represents the horizontal distance, in metres, that the ball has travelled from the player.

A specialized camera tracks the initial path of the ball after it is hit by the player. The camera records that the ball travels through the three points $(3, 2.36)$, $(10, 5)$ and $(17, 7.16)$, as shown in **Diagram 1**.

diagram not to scale

Diagram 1

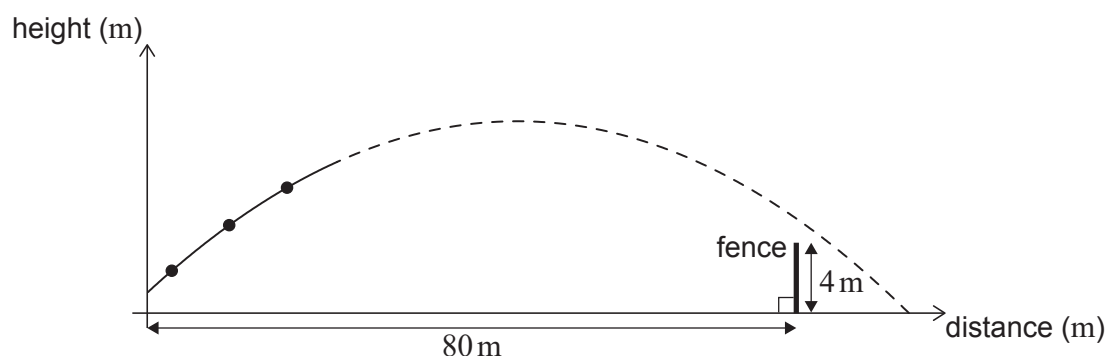


- (a) Use the coordinates $(3, 2.36)$ to write down an equation in terms of a , b , and c . [1]
- (b) Use your answer to part (a) and two similar equations to find the equation of the quadratic model for the height of the ball. [3]

A 4-metre-high fence is 80 metres from where the player hit the ball, as shown in **Diagram 2**.

diagram not to scale

Diagram 2



- (c) Show that the model predicts that the ball will go over the fence. [3]
- (d) Find the horizontal distance that the ball will travel, from the player until it first hits the ground, according to this model. [2]

(This question continues on the following page)



(Question 5 continued)

[illegible]

Turn over

7. [Maximum mark: 8]

When studying big cats, researchers use a model in which the mass (m kilograms) of an animal is directly proportional to the cube of its shoulder height (h metres).



Cheetah

Lion

A cheetah has a mass of 64 kg and shoulder height of 0.8 metres.

- (a) (i) Use the model to find an expression for m in terms of h .
- (ii) Hence find the mass of a different cheetah, with a shoulder height of 0.75 metres. [4]

‘Rubner’s law’ states that the energy needs of an animal (E) are directly proportional to the square of h .

The energy needs of a lion of mass 220 kg are k times the energy needs of a cheetah of mass 64 kg.

- (b) Find the value of k . [4]



Turn over

14. [Maximum mark: 4]

Two model cars are driving along the same straight track. Car A leaves point O at $t = 0$ and its displacement (x_A metres) from O at t seconds is given by the equation

$$x_A = 4\sqrt{t}, \text{ for } 0 \leq t \leq 9.$$

A faster car, B, leaves O at $t = 3$.

After starting its motion, the time taken by car B to reach any point on the track is half the time that was taken by A.

- (a) Write down the equation for the displacement of B, x_B , in terms of t . [2]
- (b) Find the value of t at which B catches up with A. [2]

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Turn over

3. [Maximum mark: 7]

The pH scale is a measure of the acidity of a solution. Its value is given by the formula

$$\text{pH} = -\log_{10} [\text{H}^+],$$

where $[\text{H}^+]$ is the concentration of hydrogen ions in the solution (measured in moles per litre).

- (a) Calculate the pH value if the concentration of hydrogen ions is 0.0003. [2]

The pH of milk is 6.6.

- (b) Calculate the concentration of hydrogen ions in milk. [2]

The strength of an acid is measured by its concentration of hydrogen ions.

A lemon has a pH value of 2 and a tomato has a pH value of 4.5.

- (c) Calculate how many times stronger the acid in a lemon is when compared to the acid in a tomato. [3]



Turn over

2. [Maximum mark: 6]

Radioactive carbon is a material that decays over time.

The mass, $m(t)$ (in nanograms), of radioactive carbon in a fossil of a plant, after t years, can be modelled by the function

$$m(t) = 120e^{-0.000121t}$$

where t is the time since the plant died.

- Write down the initial mass of the radioactive carbon. [1]
- Find the mass of the radioactive carbon after 20 000 years. [2]
- Calculate the smallest number of complete years it takes for more than half the sample to decay. [3]

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Turn over

6. [Maximum mark: 4]

A museum has an annual membership fee of \$200, which includes 10 free visits. Any additional visits are charged at \$30 each. The total cost, \$ C , of n visits during the year can be modelled by

$$C(n) = \begin{cases} 200, & n < p \\ an + b, & n \geq p \end{cases}, \text{ where } a, b, p, n \in \mathbb{Z}.$$

(a) Write down the value of

(i) a

(ii) p . [2]

(b) Find the value of b . [2]



8. [Maximum mark: 7]

The amount of daylight, L (in hours), in London in 2024 can be modelled by

$$L = a \sin(b(t - c)) + d,$$

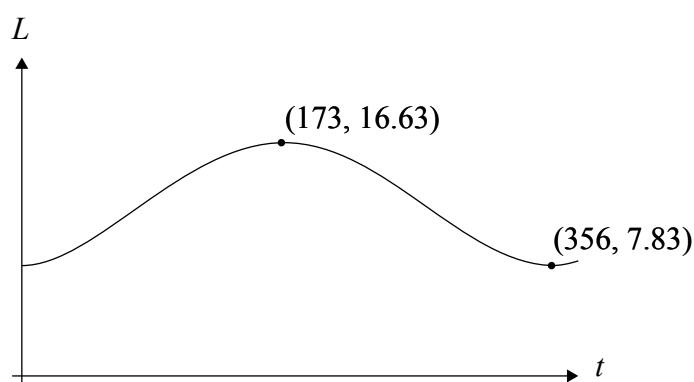
where $a, b, c, d > 0$ and t is the day of the year.

For example, day 1 = 1 January, day 2 = 2 January, and so on.

The maximum value of L is 16.63 hours on day 173 (21 June 2024).

The minimum value of L is 7.83 hours on day 356 (21 December 2024).

This information is shown in the following diagram.



Find the value of

- | | |
|-----------|-----|
| (a) d | [2] |
| (b) a | [1] |
| (c) b | [2] |
| (d) c . | [2] |

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Turn over

10. [Maximum mark: 8]

The function f is defined by $f(x) = 4 \ln(2x - 3)$, where $x > \frac{3}{2}$.

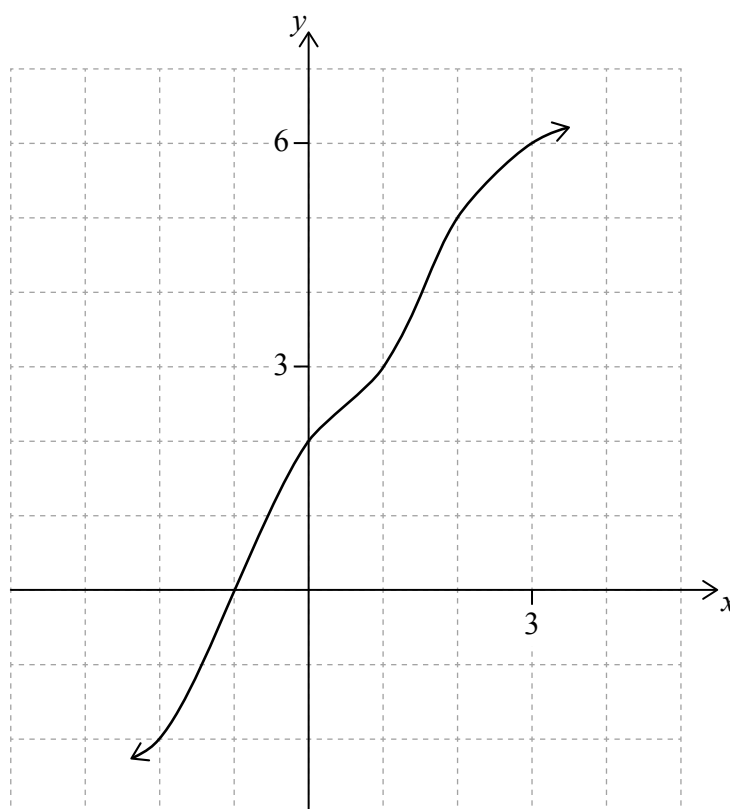
The graph of $y = f(x)$ is obtained from the graph of $y = \ln x$ by a sequence of three transformations.

(a) Describe fully these transformations, including the order in which they occur.

[3]

The graph of $y = g(x)$ is shown.

The graph passes through the points $(-2, -2)$, $(-1, 0)$, $(0, 2)$, $(1, 3)$, $(2, 5)$ and $(3, 6)$.



(b) Find $(f \circ g)(2)$.

[2]

(c) Solve $(f \circ g)(x) = 2 \ln 9$.

[3]

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(Question 10 continued)

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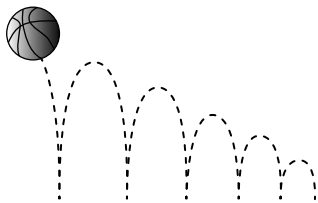
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Answers written on this page
will not be marked.



12. [Maximum mark: 7]

Andreas drops a ball and records a video of the ball bouncing.



He uses the video to find the maximum height, in metres, after each of the first four bounces. His results are shown in the table.

Bounce number, n	Maximum height, h
1	0.613
2	0.514
3	0.439
4	0.377

Andreas thinks the maximum height can be modelled by the function

$$h(n) = 0.613 \left(\frac{0.514}{0.613} \right)^{n-1}, \text{ where } n \in \mathbb{Z}^+.$$

- (a) Complete the following table. [2]

Bounce number, n	Maximum height, h , according to the model
1	0.613
2	0.514
3	
4	

- (b) Hence, calculate the sum of square residuals (SS_{res}). [2]

Andreas' friend thinks a better model for h could be found using an exponential least squares regression curve.

- (c) (i) Find the equation of this model.
- (ii) Use this model to estimate the height from which the ball was originally dropped. [3]

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(Question 12 continued)

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13. [Maximum mark: 6]

In 2008, there were no beavers in Scotland, as they had become extinct. In 2009, beavers were introduced from Norway. Surveys were carried out in the region of Tayside to monitor the beaver population.

Figure 1: A beaver



It is believed that the population, P , grows according to a logistic model

$$P = \frac{L}{1 + Ce^{-0.28t}} ,$$

where t is the number of years after 2008, $t \geq 1$.

A survey in 2017 estimated the population to be 433.

A survey in 2020 estimated the population to be 954.

Find the carrying capacity, L , for the number of beavers in Tayside, according to this model.

[illegible]

Turn over

2. [Maximum mark: 7]

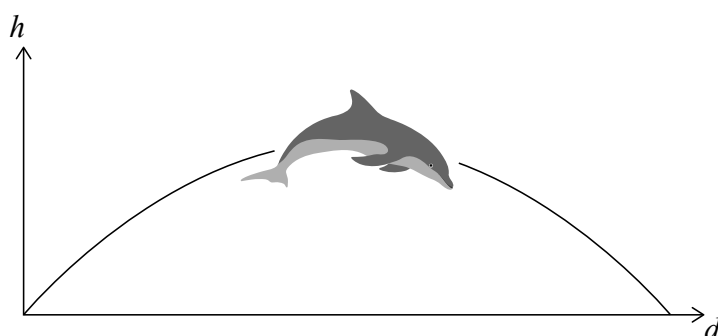
A dolphin jumps out of water. The path of its jump can be modelled by a quadratic function using the following variables:

d is the horizontal distance from the point where the dolphin leaves the water

h is the height of the dolphin above the surface of the water.

All distances are measured in metres, from the point where the dolphin leaves the water.

diagram not to scale



On a particular jump, the dolphin first reaches a height of 5.1 m above the surface of the water when the horizontal distance is 3 m and again when the horizontal distance is 8.5 m.

- (a) Write down the equation of the axis of symmetry of the quadratic function. [2]

The quadratic function that models the dolphin's jump is $h(d) = -0.2d^2 + bd$, where $d > 0$ and b is a constant.

- (b) Calculate the value of b . [2]
- (c) Find the horizontal distance when the dolphin re-enters the water. [2]
- (d) Explain what a negative h -value would mean in this context. [1]

(This question continues on the following page)



[illegible]

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4. [Maximum mark: 7]

A manufacturer designs hats that businesses can order.

For an order of 15 hats or fewer, the price per hat is p euros (EUR). For larger orders, the price of **each** hat after the first 15 ordered is reduced by 5 EUR.

$C(n)$ represents the total cost of purchasing n hats from the manufacturer.

- (a) Write down an expression in terms of p for the total cost of ordering 15 hats.

[1]
- (b) Write down an equation for $C(n)$, in terms of p and n , for the total cost of ordering n hats when $n > 15$.

[3]

A company decides to order 100 hats. The mean price per hat for this order is 25.73 EUR.

- (c) (i) Calculate the exact total cost of the order.
- (ii) Hence, determine the value of p , correct to 2 decimal places.

[3]

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Turn over

6. [Maximum mark: 6]

Walkea Tree is a furniture store. They offer a 30 % discount on all chairs priced over 250 Canadian dollars (CAD).

A different furniture store, LuxeCraft, offer customers a reduction of 150 CAD on all chairs priced over 250 CAD.

- (a) Given that x is the original price of a chair and $x > 250$,
- (i) write down a function $f(x)$ that models the final price of a chair at Walkea Tree, after the discount is applied.
 - (ii) write down a function $g(x)$ that models the final price of a chair at LuxeCraft, after the reduction is applied. [2]

A third store, Opulent Furnishings, decides to offer both a 30 % discount **and** a reduction of 150 CAD on all their chairs priced over 250 CAD.

- (b) Describe the meaning of $(f \circ g)(x)$, in context. [2]
- (c) State, with justification, whether $(f \circ g)(x)$ or $(g \circ f)(x)$ is better for the **customer** when they buy a chair with an original price of 450 CAD. [2]



Turn over

10. [Maximum mark: 5]

Consider the following function, $f(x)$, defined on the domain of integers from 0 to 4 inclusive.

x	0	1	2	3	4
$f(x)$	3	1	0	4	2

- (a) Solve $f(x) = 4$. [1]
- (b) Solve $f(x) = x$. [1]
- (c) Complete the following table.

x	0	1	2	3	4
$f^{-1}(x)$					

[3]

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Turn over

11. [Maximum mark: 6]

A biologist believes that there is a relationship between the possible population size of a group of birds (p thousand) and the population of a colony of wasps (w thousand). Based on her research she believes that the relationship is

$$w = p^3 - 4p^2 + 3p.$$

- (a) When $w = 0$, find the possible values of p . [2]
- (b) Determine the positive values of w for which there is only one positive value of p . [4]

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10. [Maximum mark: 8]

A satellite orbiting the Earth experiences a gravitational force, $F(\text{N})$, which varies inversely as the square of its distance, d (kilometres), from the centre of the Earth.

When $d = 7000$, $F = 40\,000$.

- (a) (i) Find an equation for F in terms of d .
- (ii) Find the value of F when d is 3500 km. [4]
- (b) (i) Identify which **two** of the following graphs would form a straight line.
- A. F against d
 - B. F against d^2
 - C. F against d^{-2}
 - D. $\log_{10} F$ against $\log_{10} d$
 - E. $\log_{10} F$ against d
 - F. F against $\log_{10} d$
- (ii) For each of these two graphs, find the value of the gradient of the line formed. [4]

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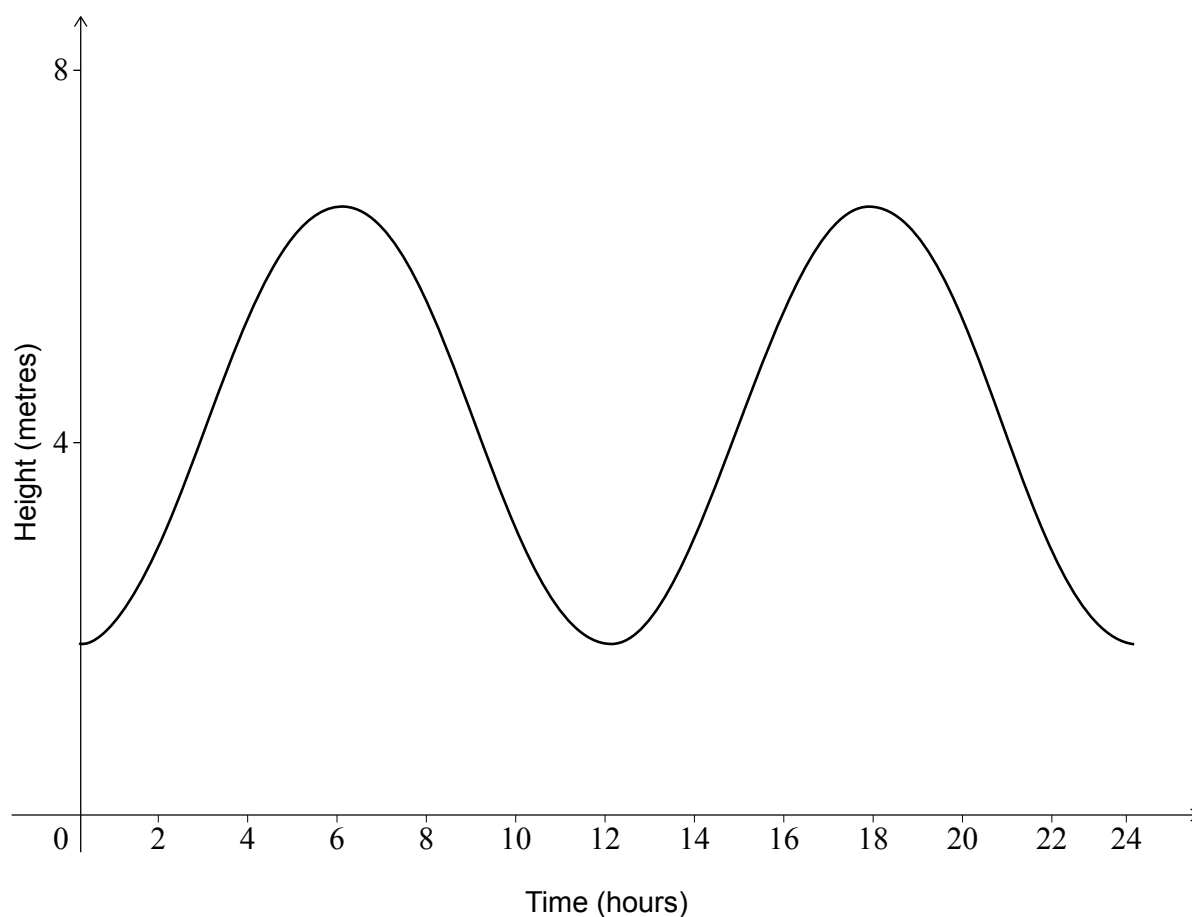
2. [Maximum mark: 15]

On a particular day the height of the tide, h , in metres, at Albion harbour can be modelled by the function

$$h(t) = -2.5 \cos(bt^\circ) + 4.5, \text{ where } b \in \mathbb{R}, 0 \leq t \leq 24$$

and t represents the number of hours after midnight.

The graph of h is shown in the following diagram.



- (a) Show that the value of b is 30. [1]
- (b) Find the height of the tide when $t = 5$. [2]
- (c) Write down
- (i) the amplitude of h .
 - (ii) the equation of the principal axis. [3]

(This question continues on the following page)

(Question 2 continued)

Boats can only leave or return to Albion harbour when the height of the tide is greater than 2.65 m.

Robin knows that due to local weather conditions, there could be possible errors in the height of the tide, h , predicted by the model. The actual height may vary by up to 10 cm above and below the value of h .

Robin wants to leave the harbour to go fishing as soon as possible after the time is 12:00.

- (d) Use the above information to determine the earliest time that it could be possible for Robin to leave the harbour. Give your answer to the nearest minute. [4]

The boat will take 15 minutes to travel from the harbour to the fishing site. Robin intends to return to the harbour on the same day. He wants to be certain he will be able to get back into the harbour.

- (e) Determine the maximum possible length of time he could spend at the fishing site, in hours, and still be certain he will be able to enter the harbour on his return. [5]

Turn over

2. [Maximum mark: 27]

George is researching the growth in the number of electric vehicles (EVs) in the European Union in order to investigate some of the difficulties that might arise if the target of banning sales of all petrol and diesel cars in 2035 is to be met.

George begins his research by predicting how many electric vehicles (EVs) will be in the European Union in 2035.

The number of EVs in the European Union, N , measured in millions, is shown in the following table. The time t is measured in years from the **beginning** of 2016 where $t \in \mathbb{R}$.

End of year	2016	2017	2018	2019	2020	2021	2022
t	1	2	3	4	5	6	7
N	0.2	0.3	0.4	0.6	1.1	1.9	3.1

George models this data set using the logistic function $N = \frac{310}{1 + Ce^{-kt}}$, where $C, k \in \mathbb{R}^+$.

- (a) In context, explain the significance of 310 in George's logistic function. [1]
- (b) (i) Use the value of N at $t = 1$ to show that $C = 1549e^k$. [2]
- (ii) Use the value of N at $t = 7$ to find a second expression for C . [3]
- (c) Use your answers to part (b) to find a value for
- (i) k . [2]
- (ii) C . [1]

George uses his model to predict values of N at the end of the years 2017 to 2021. These values are shown in the following table, correct to one decimal place.

End of year	2017	2018	2019	2020	2021
t	2	3	4	5	6
N	0.3	0.4	0.6	1.1	1.9
Predicted values	0.3	0.5	a	1.2	2.0

- (d) Calculate the value of a correct to one decimal place. [2]
- (e) Using the value of a correct to one decimal place, find the sum of square residuals, SS_{res} , when using this model to predict the values of N . [2]

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(Question 2 continued)

As a measure of how well the model fits the data, George uses the error function, E ,

where $E = \sqrt{\frac{SS_{res}}{n}}$, and n is the number of predictions made using the model.

George decides he will use this model if E is less than 0.25.

- (f) By finding the value of E for George's model, show that George will decide that the model can be used. [3]

- (g) Use the model to predict the number of EVs in the European Union at the end of 2035. [2]

One of the main difficulties in meeting the European Union's target is the provision of sufficient public charging points.

The European Union estimates that 80% of car owners will be able to charge their cars at home and the remaining 20% will require public charging points. It is planned to have one public charging point for every 10 cars that require them.

- (h) Use George's model to find an expression for the total number of public charging points required at time t . [2]

At the end of 2020 there were 0.22 million public charging points in the European Union and at the end of 2022 there were 0.54 million.

- (i) Find the average number of public charging points installed per year during this period. [1]

George assumes that the number of public charging points will continue to increase linearly, at the same rate, until 2035.

- (j) Determine the value of t , where $t \geq 5$, at which the number of charging points will first be insufficient to meet demand, and hence the year in which this occurs. [6]

6. [Maximum mark: 7]

At 12:05 pm, Navam starts draining water from a small reservoir into an empty pond.

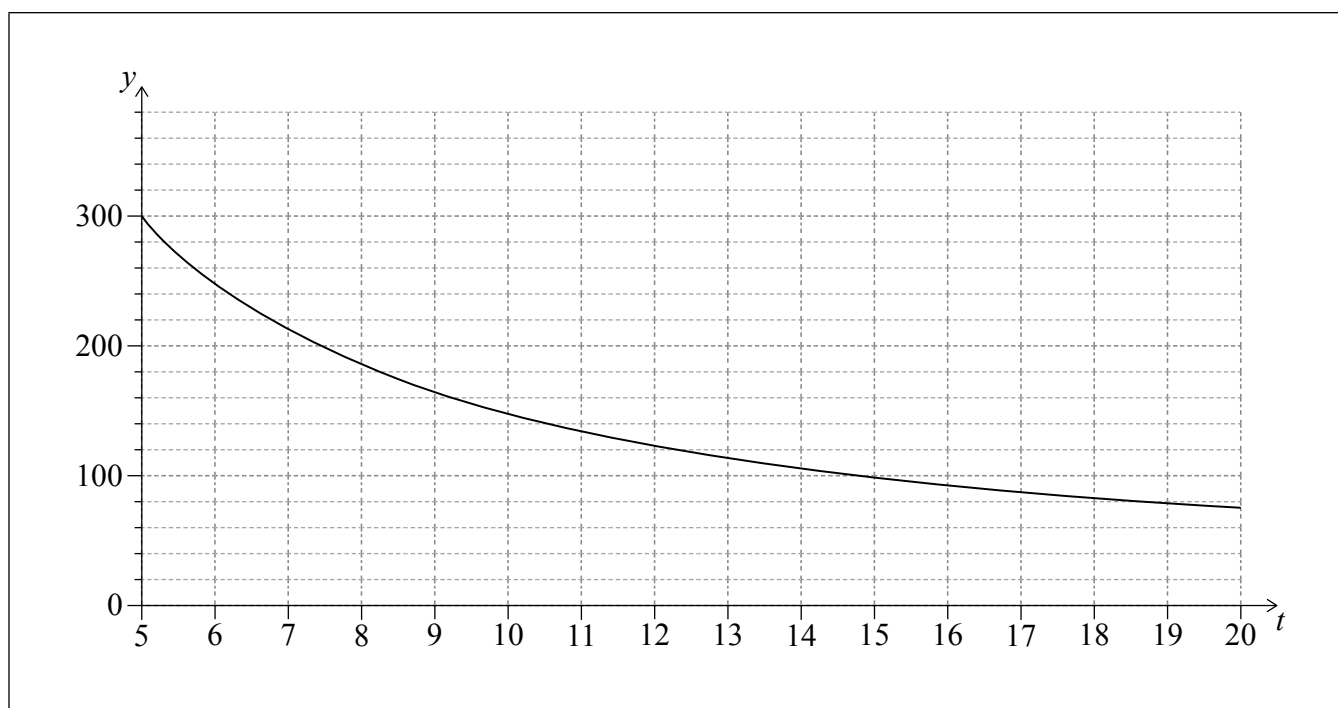
He controls the rate the water is drained so that the volume of water **remaining** in the reservoir, $V\text{m}^3$, varies inversely with the time, t , where t is the number of minutes after 12:00 pm.

At 12:05 pm, the volume of water in the reservoir is 300m^3 .

(a) Show that $V = \frac{1500}{t}$. [2]

(b) Find the value of t when the volume of water remaining in the reservoir equals the volume of water in the pond. [2]

The following diagram shows part of the graph of $y = V(t)$.



(c) On the same diagram, sketch the graph of $y = P(t)$, where P is the volume of water in the pond. [2]

(d) Write down an expression for $P(t)$. [1]

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(Question 6 continued)

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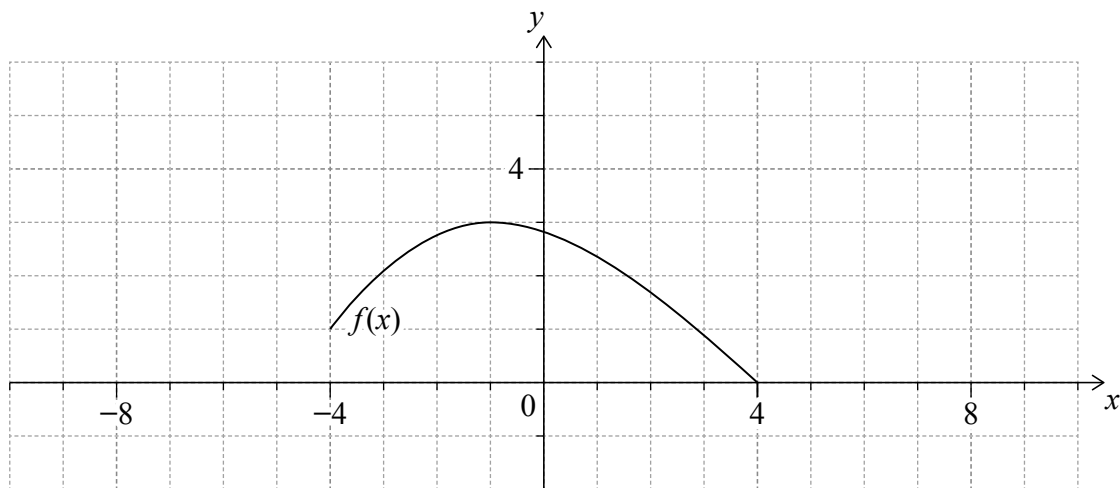
8. [Maximum mark: 4]

The function f has domain $-4 \leq x \leq 4$.

The graph of $y = f(x)$ has endpoints $(-4, 1)$ and $(4, 0)$, and a local maximum at $(-1, 3)$. This is shown in each of the following diagrams.

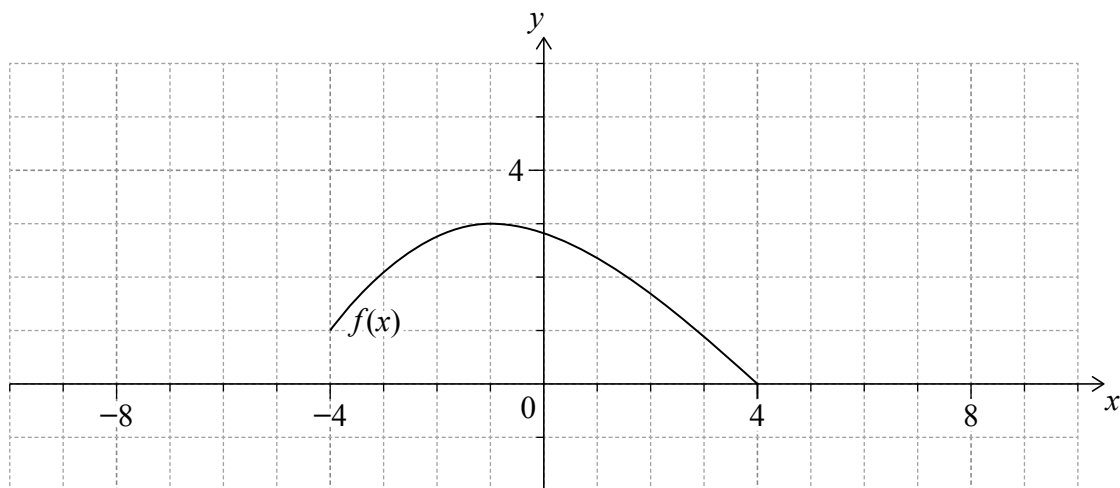
- (a) On the axes provided, sketch the graph of $y = 2f(x) - 1$.

[2]



- (b) On the axes provided, sketch the graph of $y = f(2x - 2)$.

[2]



Turn over

3. [Maximum mark: 16]

Jenny designs a flexible suspension bridge, spanning 10 metres over the Swinburne Creek.

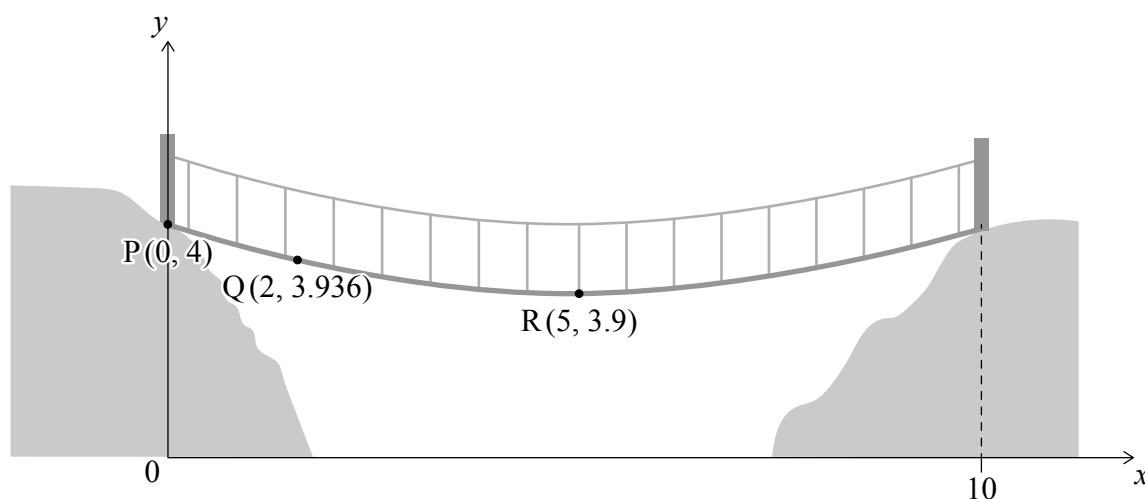
Jenny believes that when no people are on the bridge, the height, y , of the walkway on the bridge above the creek can be modelled by

$$y = ax^2 + bx + c \text{ for } 0 \leq x \leq 10.$$

She uses a coordinate system where the origin is directly below one side of the bridge, as shown in the following diagram. All distances are measured in metres.

According to this model, the points $P(0, 4)$, $Q(2, 3.936)$ and $R(5, 3.9)$ will be located on the walkway of the bridge. Point R is the lowest point of the bridge.

diagram not to scale



- (a) Write down the value of c , the y -intercept of the model.

[1]

Using point Q , Jenny finds the equation $-0.064 = 4a + 2b$.

- (b) (i) Find another equation in terms of a and b .

- (ii) Hence or otherwise, find the value of a and the value of b .

[4]

- (c) For Jenny's model find

- (i) $\frac{dy}{dx}$;

- (ii) the steepest gradient of the walkway of bridge.

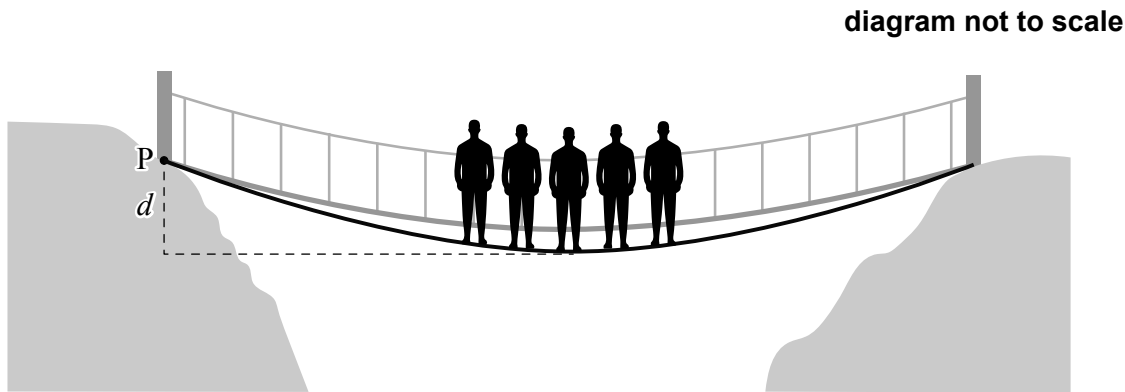
[4]

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(Question 3 continued)

The deflection of the bridge, d , is the vertical distance of the walkway below P at the midpoint of the walkway.

After the bridge is built, Jenny explores the relationship between the number of people standing on the bridge and its deflection.



Jenny collected data on the number of people, n , on the bridge and the deflection, d metres, as shown in the following table.

Number of people, n	1	2	3	4	5	6
Deflection, d (m)	0.215	0.233	0.250	0.269	0.297	0.305

- (d) For this data,
- (i) find the value of Pearson’s product-moment correlation coefficient, r ;
 - (ii) comment on the strength of the correlation. [3]

The equation of the regression line for this data is $d = 0.01889n + 0.1954$ correct to four significant figures.

- (e) Interpret, in context, the meaning of 0.01889 in the equation of the regression line. [1]
- (f) Determine whether Jenny’s **model** for the height of the walkway or the equation of the regression line gives the smaller estimate of the deflection. Justify your answer. [3]

4. [Maximum mark: 14]

Peter likes to exercise and walks for a maximum of 12 hours each day. The distance he walks in one day, d kilometres, is modelled by

$$d = 3\ln(5t + 1), \text{ for } 0 \leq t \leq 12$$

where t is the time he spends walking, in hours, on that day.

- (a) Find the time it takes for Peter to walk 6 km in one day. [2]

Peter uses a motivational fitness app, and earns points each day for walking. The number of points earned, p , is given by

$$p = 20(1 - e^{-0.1d}), \text{ for } p \in \mathbb{R}.$$

- (b) Find the number of points Peter earns for walking 6 km. [1]

- (c) Find an expression for p as a function of t , giving your answer in the form

$$p = 20\left(1 - \frac{1}{(5t+1)^n}\right),$$

where n is a number to be determined. [4]

- (d) Hence or otherwise, find the number of points Peter earns by walking for 2.5 hours. [1]

- (e) Find the greatest number of points Peter can earn in one day. [2]

The fitness app introduces bonus points, b , for the total distance walked in one week, D , using

$$b = 50\left(1 - \frac{1}{D+1}\right).$$

Peter decides to walk for the same length of time, T hours, each day.

He wants to earn a total of 120 points in one week (7 days).

- (f) Find the value of T . [4]