

APOLLO

Calculus

HL

Mathematics: applications and interpretation
First assessment 2021
104 questions | mark schemes

7. (a) (i) $A = \frac{1}{2} \times 6 \times q + \frac{1}{2} \times 8 \times p + 48$ **OR** $A = \frac{1}{2}(p+6)(q+8)$ **OR**
 $A = 3q + 4p + 48$ **A1**

(ii) valid attempt to link p and q , using tangents, similar triangles or other method **(M1)**

eg. $\tan \theta = \frac{8}{p}$ and $\tan \theta = \frac{q}{6}$ **OR** $\tan \theta = \frac{p}{8}$ and $\tan \theta = \frac{6}{q}$ **OR** $\frac{8}{p} = \frac{q}{6}$

correct equation linking p and q **A1**

eg. $pq = 48$ **OR** $p = \frac{48}{q}$ **OR** $q = \frac{48}{p}$

substitute $p = \frac{48}{q}$ into a correct area expression **M1**

eg. $(A =) \frac{1}{2} \times 6 \times q + \frac{1}{2} \times 8 \times \frac{48}{q} + 48$ **OR** $(A =) \frac{1}{2} \left(\frac{48}{q} + 6 \right) (q + 8)$

$$A = 3q + \frac{192}{q} + 48$$

AG

Note: The **AG** line must be seen with no incorrect, intermediate working, for the final **M1** to be awarded.

[4 marks]

(b) $\frac{-192}{q^2} + 3$ **A1A1**

Note: Award **A1** for $\frac{-192}{q^2}$, **A1** for 3. Award **A1A0** if extra terms are seen.

[2 marks]

(c) (i) $\frac{-192}{q^2} + 3 = 0$ **A1**

(ii) $q = 8$ cm **A1**

[2 marks]

Total [8 marks]

12. (a) $\frac{dV}{dt} = -kV^{\frac{1}{2}}$

use of separation of variables

(M1)

$$\Rightarrow \int V^{-\frac{1}{2}} dV = \int -k dt$$

A1

$$2V^{\frac{1}{2}} = -kt (+c)$$

A1

considering initial conditions $40 = c$

A1

$$2\sqrt{324} = -10k + 40$$

$$\Rightarrow k = 0.4$$

A1

$$2\sqrt{V} = -0.4t + 40$$

$$\Rightarrow \sqrt{V} = 20 - 0.2t$$

A1

Note: Award **A1** for any correct intermediate step that leads to the **AG**.

$$\Rightarrow V = \left(20 - \frac{t}{5}\right)^2$$

AG

Note: Do not award the final **A1** if the **AG** line is not stated.

[6 marks]

(b) $0 = \left(20 - \frac{t}{5}\right)^2 \Rightarrow t = 100 \text{ minutes}$

(M1)A1

[2 marks]

Total [8 marks]

$$\begin{pmatrix} 0.8 \\ -2 \end{pmatrix}$$

15. (a) $\sin(x + y) = 0$
 $\Rightarrow x + y = 0$
 (the equation of L_1 is) $y = -x$

A1
(M1)
A1

[3 marks]

- (b) $x + y = \pi$ **OR** $y = -x + \pi$

(M1)A1

[2 marks]

Total [5 marks]

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

16. (a) $x_n = x_{n-1} + h f(x_{n-1}, t_{n-1})$
 $h = 0.1, f(x, t) = x \cos t (e^{-\sin t})$

$$x_n = x_{n-1} + 0.1x_{n-1} \cos t_{n-1} (e^{-\sin t_{n-1}})$$

(M1)

Note: Award **M1** for a valid start.

n	t_n	x_n
0	0	0.367879
1	0.1	0.404667
2	0.2	0.441106
3	0.3	0.476548

(A1)

Note: Award **A1** for a correct x value when $n = 1$.

$$x(0.3) \approx 0.477 \quad (0.476548\dots)$$

A1

[3 marks]

(b) EITHER

$$\int \frac{dx}{x} = \int \cos t (e^{-\sin t}) dt \quad (+c)$$

M1

$$\ln x = -e^{-\sin t} + c$$

A1

$$t = 0, x = \frac{1}{e} \Rightarrow c = 0$$

M1

$$x = e^{(-e^{-\sin t})}$$

$$x(0.3) \approx 0.475140\dots$$

A1

OR

$$\int_{1/e}^x \frac{du}{u} = \int_0^{0.3} \cos t (e^{-\sin t}) dt$$

M1

$$[\ln u]_{1/e}^x = 0.255855\dots \text{ (from GDC)}$$

A1

$$\ln x + 1 = 0.255855\dots$$

$$\ln x = -0.744145\dots$$

A1

$$x = e^{-0.744145} = 0.475140\dots$$

A1

THEN

$$\text{percentage error} = \left| \frac{0.476548\dots - 0.475140\dots}{0.475140\dots} \right| \times 100 = 0.296\% \quad (2.96192\dots)$$

A1

Note: If candidates do not attempt to find c , they may score **M1A0M0A1A1**.

[5 marks]

Total [8 marks]

2. (a) (i) evidence of power rule (at least one correct term seen) (M1)

$$\frac{dy}{dx} = -0.3x^2 + 1.6x \quad \text{A1}$$

- (ii) $-0.3x^2 + 1.6x = 0$ M1

$$x = 5.33 \left(5.33333..., \frac{16}{3} \right) \quad \text{A1}$$

$$y = -0.1 \times 5.33333...^3 + 0.8 \times 5.33333...^2 \quad \text{(M1)}$$

Note: Award **M1** for substituting their zero for $\frac{dy}{dx}$ (5.333...) into y .

$$7.59 \text{ m (7.58519...)} \quad \text{A1}$$

Note: Award **M0A0M0A0** for an unsupported 7.59.
Award at most **M0A0M1A0** if only the last two lines in the solution are seen.
Award at most **M1A0M1A1** if their $x = 5.33$ is not seen.

[6 marks]

- (b) One correct substitution seen (M1)

- (i) 6.4 m A1

- (ii) 7.2 m A1

[3 marks]

continued...

Question 2 continued

(c) $A = \frac{1}{2} \times 2((2.4 + 0) + 2(6.4 + 7.2))$ **(A1)(M1)**

Note: Award **A1** for $h = 2$ seen. Award **M1** for correct substitution into the trapezoidal rule (the zero can be omitted in working).

$$= 29.6 \text{ m}^2$$

A1**[3 marks]**

(d) (i) $A = \int_2^8 -0.1x^3 + 0.8x^2 \, dx$ **OR** $A = \int_2^8 y \, dx$ **A1A1**

Note: Award **A1** for a correct integral, **A1** for correct limits in the correct location. Award at most **A0A1** if dx is omitted.

(ii) $A = 32.4 \text{ m}^2$

A2

Note: As per the marking instructions, **FT** from their integral in part (d)(i). Award at most **A1FTA0** if their area is >48 , this is outside the constraints of the question (a 6×8 rectangle).

[4 marks]**Total [16 marks]**

6. (a) use of product rule (M1)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} abe^{bt} \cos t - ae^{bt} \sin t \\ abe^{bt} \sin t + ae^{bt} \cos t \end{pmatrix} \quad \text{A1A1}$$

[3 marks]

(b) $|\mathbf{v}|^2 = \dot{x}^2 + \dot{y}^2 = [abe^{bt} \cos t - ae^{bt} \sin t]^2 + [abe^{bt} \sin t + ae^{bt} \cos t]^2$ M1

Note: It is more likely that an expression for $|\mathbf{v}|$ is seen.

$\sqrt{\dot{x}^2 + \dot{y}^2}$ is not sufficient to award the **M1**, their part (a) must be substituted.

$$= [a^2 \sin^2 t - 2a^2 b \sin t \cos t + a^2 b^2 \cos^2 t + a^2 \cos^2 t + 2a^2 b \sin t \cos t + a^2 b^2 \sin^2 t] e^{2bt} \quad \text{A1}$$

use of $\sin^2 t + \cos^2 t = 1$ within a factorized expression that leads to the final answer

M1

$$= a^2 (b^2 + 1) e^{2bt} \quad \text{A1}$$

magnitude of velocity is $a e^{bt} \sqrt{(1+b^2)}$ AG

[4 marks]

- (c) when $t = 0$, $ae^{bt} \cos t = 5$

$$a = 5 \quad \text{A1}$$

$$abe^{bt} \cos t - ae^{bt} \sin t = -3.5 \quad \text{(M1)}$$

$$b = -0.7 \quad \text{A1}$$

Note: Use of $a e^{bt} \sqrt{(1+b^2)}$ result from part (b) is an alternative approach.

[3 marks]

(d) $5e^{-0.7 \times 2} \sqrt{(1+(-0.7)^2)}$ (M1)

$$1.51 \quad (1.50504...) \quad \text{A1}$$

[2 marks]

(e) $\dot{x} = 0$ (M1)

$$a e^{bt} (b \cos t - \sin t) = 0$$

$$\tan t = b$$

$$t = 2.53 \quad (2.53086...) \quad \text{(A1)}$$

correct substitution of their t to find x or y (M1)

$$x = -0.697 \quad (-0.696591...) \quad \text{and} \quad y = 0.488 \quad (0.487614...) \quad \text{(A1)}$$

use of Pythagoras / distance formula (M1)

$$OP = 0.850 \text{ m} \quad (0.850297...) \quad \text{A1}$$

[6 marks]

Total [18 marks]

7. (a) $\int \frac{1}{x} dx = \int 2 dt$ (M1)

$\ln x = 2t + c$
 $x = Ae^{2t}$ (A1)

$x(0) = 100 \Rightarrow A = 100$ (M1)

$x = 100e^{2t}$ (A1)

$x(1) = 739$ A1

Note: Accept 738 for the final A1.

[5 marks]

(b) $t_{n+1} = t_n + 0.25$ (A1)

Note: This may be inferred from a correct t column, where this is seen.

$x_{n+1} = x_n + 0.25x_n(2 - 0.01y_n)$ (A1)

$y_{n+1} = y_n + 0.25y_n(0.0002x_n - 0.8)$ (A1)

t	x	y
0	1000	100
0.25	1250	85
0.5	1609	73
0.75	2119	65
1	2836	58

(A1)

Note: Award A1 for whole line correct when $t = 0.5$ or $t = 0.75$. The t column may be omitted and implied by the correct x and y values. The formulas are implied by the correct x and y columns.

(i) 2840 (2836 OR 2837) A1

(ii) 58 OR 59 A1
 [6 marks]

(c) (i) both populations are increasing A1

(ii) rabbits are decreasing and foxes are increasing A1A1
 [3 marks]

(d) setting at least one DE to zero (M1)
 $x = 4000, y = 200$ A1A1

[3 marks]

Total [17 marks]

3. (a) evidence of splitting diagram into equilateral triangles **M1**
- area = $6\left(\frac{1}{2}x^2 \sin 60^\circ\right)$ **A1**
- = $\frac{3\sqrt{3}x^2}{2}$ **AG**

Note: The **AG** line must be seen for the final **A1** to be awarded.

[2 marks]

- (b) total surface area of prism $1200 = 2\left(3x^2 \frac{\sqrt{3}}{2}\right) + 6xh$ **M1A1**

Note: Award **M1** for expressing total surface areas as a sum of areas of rectangles and hexagon(s), and **A1** for a correctly substituted formula, equated to 1200.

[5 marks]

$$h = \frac{400 - \sqrt{3}x^2}{2x} \quad \text{A1}$$

$$\text{volume of prism} = \frac{3\sqrt{3}}{2}x^2h \quad \text{(A1)}$$

$$= \frac{3\sqrt{3}}{2}x^2\left(\frac{400 - \sqrt{3}x^2}{2x}\right) \quad \text{A1}$$

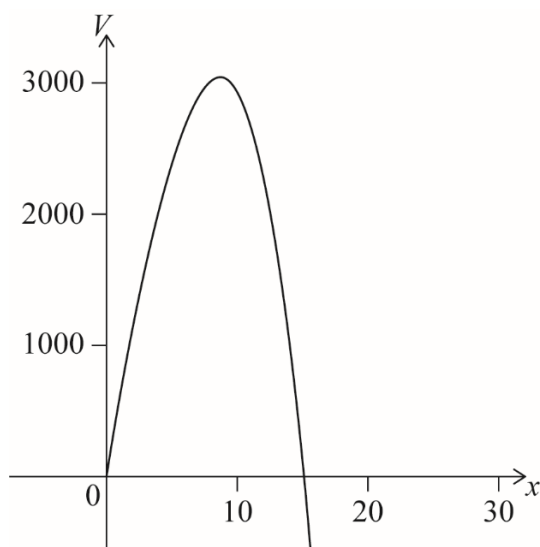
$$= 300\sqrt{3}x - \frac{9}{4}x^3 \quad \text{(AG)}$$

Note: The **AG** line must be seen for the final **A1** to be awarded.

continued...

Question 3 continued

(c)

**A1A1**

Note: Award **A1** for correct shape, **A1** for roots in correct place with some indication of scale (indicated by a labelled point).

[2 marks]

(d) $\frac{dV}{dx} = 300\sqrt{3} - \frac{27}{4}x^2$

A1A1

Note: Award **A1** for a correct term.

[2 marks]

(e) from the graph of V or $\frac{dV}{dx}$ **OR** solving $\frac{dV}{dx} = 0$
 $x = 8.77$ (8.77382...)

(M1)**A1****[2 marks]**

(f) from the graph of V **OR** substituting their value for x into V
 $V_{\max} = 3040 \text{ cm}^3$ (3039.34...)

(M1)**A1****[2 marks]****Total [15 marks]**

6. (a) solving $v = 0$ **M1**
 $t = 2, t = 6$ **A1**
[2 marks]
- (b) use of power rule **(M1)**
 $\frac{dv}{dt} = -4t + 16$ **(A1)**
 $(t = 6)$
 $\Rightarrow a = -8$ **(A1)**
magnitude $= 8 \text{ m s}^{-2}$ **A1**
[4 marks]
- (c) using a sketch graph of v **(M1)**
 24 m s^{-1} **A1**
[2 marks]
- (d) **METHOD ONE**
 $x = \int v \, dt$
attempt at integration of v **(M1)**
 $-\frac{2t^3}{3} + 8t^2 - 24t (+c)$ **A1**
attempt to find c (use of $t = 0, x = 0$) **(M1)**
 $c = 0$ **A1**
 $\left(x = -\frac{2t^3}{3} + 8t^2 - 24t \right)$
- METHOD TWO**
 $x = \int_0^t v \, dt$
attempt at integration of v **(M1)**
 $\left[-\frac{2t^3}{3} + 8t^2 - 24t \right]_0^t$ **A1**
attempt to substituted limits into their integral **(M1)**
 $x = -\frac{2t^3}{3} + 8t^2 - 24t$ **A1**
[4 marks]
- (e) $\int_0^4 |v| \, dt$ **(M1)(A1)**
- Note:** Award **M1** for using the absolute value of v , or separating into two integrals,
A1 for the correct expression.
- $= 32 \text{ m}$ **A1**
[3 marks]
Total [15 marks]

7. (a) $\begin{vmatrix} -4-\lambda & 0 \\ 3 & -2-\lambda \end{vmatrix} = 0$ **(M1)**
 $(-4-\lambda)(-2-\lambda) = 0$ **(A1)**
 $\lambda = -4$ **OR** $\lambda = -2$ **A1**
 $\lambda = -4$

$$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x \\ -4y \end{pmatrix}$$
 (M1)

Note: This **M1** can be awarded for attempting to find either eigenvector.

$$3x - 2y = -4y$$

$$3x = -2y$$

possible eigenvector is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ (or any real multiple) **A1**

$$\lambda = -2$$

$$\begin{pmatrix} -4 & 0 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

$$x = 0, y = 1$$

possible eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (or any real multiple) **A1**

[6 marks]

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-4t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + Be^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **(M1)A1**

Note: Award **M1A1** for $x = -2Ae^{-4t}$, $y = 3Ae^{-4t} + Be^{-2t}$, **M1A0** if LHS is missing or incorrect.

[2 marks]

- (c) two (distinct) real negative eigenvalues **R1**
(or equivalent (eg both $e^{-4t} \rightarrow 0, e^{-2t} \rightarrow 0$ as $t \rightarrow \infty$))
 $\Rightarrow$ stable equilibrium point **A1**

Note: Do not award **R0A1**.

[2 marks]

continued...

Question 7 continued

(d) $\frac{dy}{dx} = \frac{3x-2y}{-4x}$

(M1)

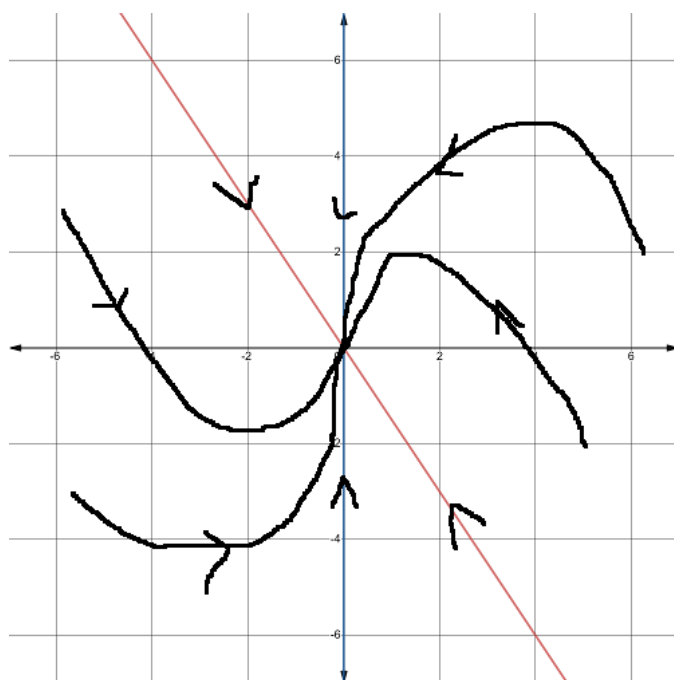
(i) $(4, 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$

A1

(ii) $(-4, 0) \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$

A1**[3 marks]**

(e)

**A1A1A1A1**

Note: Award **A1** for a phase plane, with correct axes (condone omission of labels) and at least three non-overlapping trajectories. Award **A1** for all trajectories leading to a stable node at $(0, 0)$. Award **A1** for showing gradient is negative at $x = 4$ and -4 . Award **A1** for both eigenvectors on diagram.

[4 marks]**Total [17 marks]**

2. (a) (i) population growth rate / birth rate of sharks (due to eating mackerel) **A1** [1 mark]

(ii) (net) death rate of sharks **A1** [1 mark]

(b) (i) $\gamma MS - \delta S = 0$ **A1**
since $S \neq 0$ **R1**

Note: Accept $S > 0$.

getting to given answer without further error by either cancelling or factorizing

A1

$$M = \frac{\delta}{\gamma}$$

AG

[3 marks]

(ii) $\frac{dM}{dt} = 0$
 $\alpha M - \beta MS = 0$ **(M1)**

(since $M \neq 0$) $S = \frac{\alpha}{\beta}$ **A1**

[2 marks]

(c) (i) $M_{eq} = \frac{\delta}{\gamma} \Rightarrow \frac{\delta}{\frac{1}{2}\gamma} = 2M_{eq}$ **M1**

Note: Accept equivalent in words.

Doubles

A1

Note: Do not accept “increases”.

[2 marks]

(ii) $M_{eq} = \frac{\delta}{\gamma}$ is not dependent on α **R1**

Note: Award **R0** for any contextual argument.

no change

A1

Note: Do not award **R0A1**.

[2 marks]

continued...

Question 2 continued

(d) (i) $\frac{dM}{dt} = \alpha M$

A1**[1 mark]**

(ii) $\int \frac{1}{M} dM = \int \alpha dt$

M1

Note: Award **M1** is for an attempt to separate variables. This means getting to the point $\int f(M) dM = \int g(t) dt$ where the integral can be seen or implied by further work.

$$\ln |M| = \alpha t + c$$

A1

Note: Accept $\ln M$. Condone missing constant of integration for this mark.

$$M = ke^{\alpha t}$$

when $t = 0$, $M_0 = k$

M1

Note: Award **M1** for a clear attempt at using initial conditions to find a constant of integration. Only possible if the constant of integration exists. $t = 0$ or “initially” or similar must be seen. Substitution may appear earlier, following the integration.

initial conditions and all other manipulations correct and clearly communicated to get to the final answer

A1

$$M = M_0 e^{\alpha t}$$

AG**[4 marks]**

(iii) $M = 3M_0$ seen anywhere

(A1)

substituting $t = 2$, $M = 3M_0$ into equation $M = M_0 e^{\alpha t}$

(M1)

$$3M_0 = M_0 e^{2\alpha}$$

$$\alpha = \frac{1}{2} \ln 3 \quad \text{OR} \quad 0.549306\dots$$

A1

Note: The **A1** requires either the exact answer or an answer to at least 4 sf.

$$\approx 0.549$$

AG**[3 marks]***continued...*

Question 2 continued

- (e) (i) an attempt to set up one recursive equation

(M1)

Note: Must include **two** given parameters **and** M_n and S_n **and** M_{n+1} or S_{n+1} for the **(M1)** to be awarded.

$$M_{n+1} = M_n + 0.1(0.549M_n - 0.236M_nS_n)$$

A1

$$S_{n+1} = S_n + 0.1(0.244M_nS_n - 1.39S_n)$$

A1**[3 marks]**

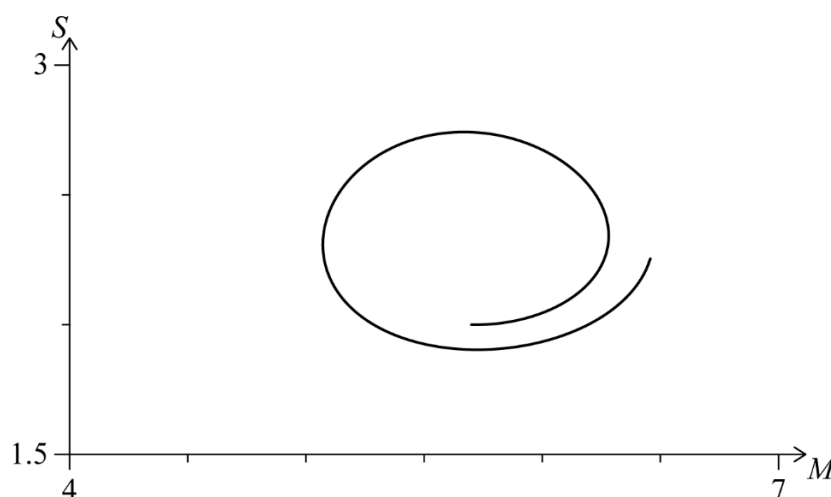
- (ii)
- EITHER**
-
- 6.12 (6.11609...)

A2

OR
6120 (6116.09...) (mackerel per km³)

A2**[2 marks]**

- (f) (i)



spiral or closed loop shape

A1

approximately 1.25 rotations (can only be awarded if a spiral)

A1

correct shape, in approximately correct position (centred at approx. (5.5, 2.5))

A1

Note: Award **A0A0A0** for any plot of S or M against t .

[3 marks]

continued...

Question 2 continued

(ii) **EITHER**approximate minimum is (5.07223...) 5.07 (which is greater than 5) **A1****OR**the line $M = 5$ clearly labelled on their phase portrait **A1****THEN**

(the density will not fall below 5000) hence sufficient for sustainable fishing

A1

Note: Do not award **A0A1**. Only if the minimum point is labelled on the sketch then a statement here that “*the mackerel population is always above 5000*” would be sufficient. Accept the value 5.07 seen within a table of values.

[2 marks]

(iii) Any two from:

A1A1

- Current values / parameters are only an estimate,
- The Euler method is only an approximate method / choosing $h = 0.1$ might be too large.
- There might be random variation / the model has no stochastic component
- Conditions / parameters might change over the nine years,
- A discrete system is being approximated by a continuous system,

Allow any other sensible critique.

If a candidate identifies factors which the model ignores, award **A1** per factor identified. These factors could include:

- Other predators
- Seasonality
- Temperature
- The effect of fishing
- Environmental catastrophe
- Migration

Note: Do not allow:

“You cannot have 5.07 mackerel”.

It is only a model (as this is too vague).

Some factors have been ignored (without specifically identifying the factors).

Values do not always follow the equation / model. (as this is too vague)

[2 marks]**[Total 31 marks]**

2. (a) attempt at using trapezoidal rule formula **(M1)**

$$\frac{1}{2} \left(\frac{2-0}{5} \right) (30 + 50 + 2(50 + 60 + 40 + 20))$$
A1

(total carbon =) 84 tonnes **A1**
[3 marks]

(b) $\left| \frac{84 - 72}{72} \right| \times 100\%$ **(M1)**

Note: Award **(M1)** for correct substitution of final answer in part (a) into percentage error formula.

= 16.7% (16.6666...%) **A1**
[2 marks]
Total: [5 marks]

8. (a) (i) a

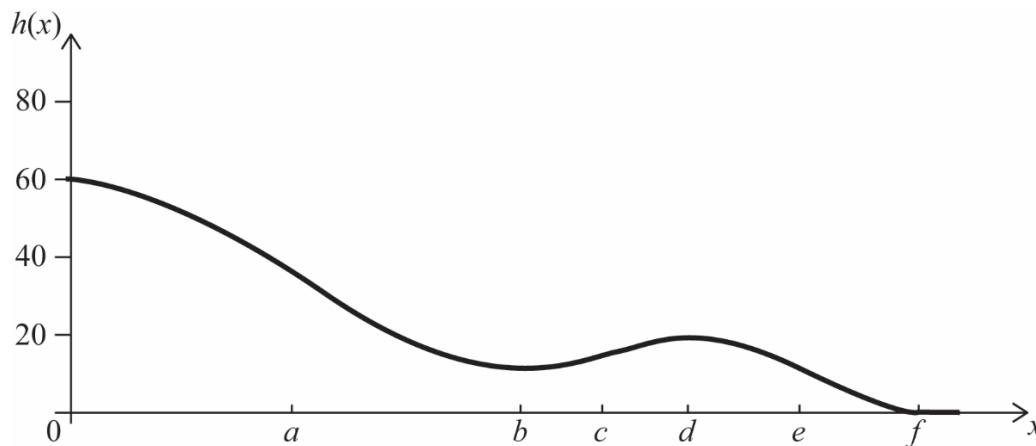
A1

(ii) the hill is at its steepest / largest slope of hill

A1

[2 marks]

(b)



A1A1A1

Note: Award **(A1)** for decreasing function from 0 to b and d to f and increasing from b to d ; **(A1)** for minimum at b and max at d ; **(A1)** for starting at height of 60 and finishing at a height of 0 at f . If reasonable curvature not evident on graph (i.e. only straight lines used) award **A1A0A1**.

[3 marks]

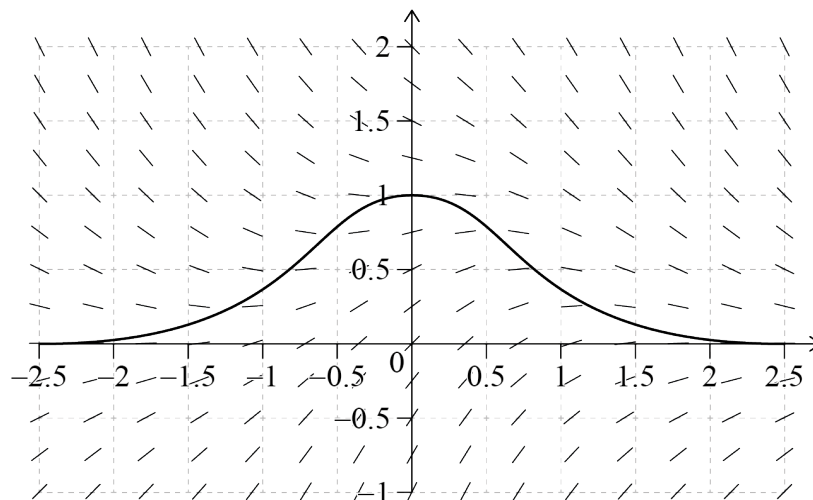
Total: [5 marks]

13. (a) $\left(\frac{dy}{dx} = e^0 - 1\right) = 0$

A1

[1 mark]

(b)



gradient = 0 at (0, 1)
correct shape

A1

A1

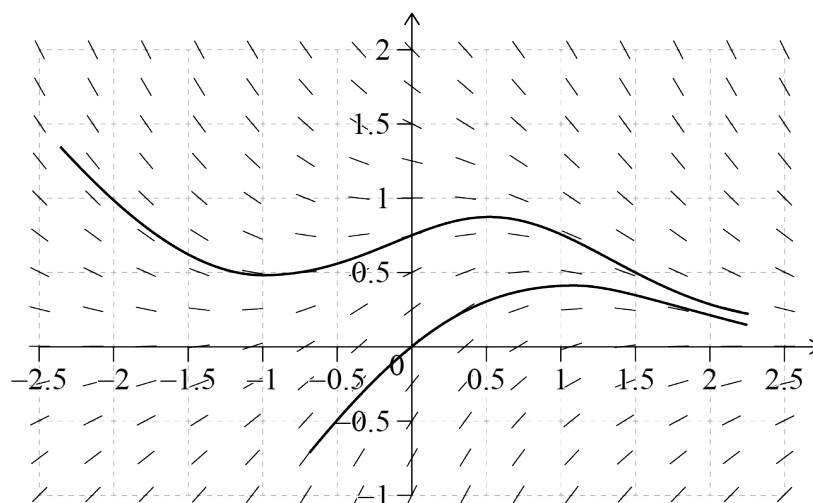
Note: Award second **A1** for horizontal asymptote of $y = 0$, and general symmetry about the y -axis.

[2 marks]

continued...

Question 13 continued

(c)



- (i) positive gradient at origin
correct shape

A1
A1

Note: Award second **A1** for a single maximum in 1st quadrant and tending toward an asymptote.

- (ii) positive gradient at (0, 0.75)
correct shape

A1
A1

Note: Award second **A1** for a single minimum in 2nd quadrant, single maximum in 1st quadrant and tending toward an asymptote.

[4 marks]
Total: [7 marks]

15. (a) $15 = 3 + 4r + 2r\theta$
 $12 = 2r(2 + \theta)$

M1
A1

Note: Award **A1** for any reasonable working leading to expected result e.g, factorizing r .

$$r = \frac{6}{2 + \theta}$$

AG

[2 marks]

(b) (i) attempt to use sector area to find volume

(M1)

$$\text{volume} = \frac{1}{2}r^2\theta \times 1$$

$$= \frac{1}{2} \times \frac{36}{(2 + \theta)^2} \times \theta \quad \left(= \frac{18\theta}{(2 + \theta)^2} \right)$$

A1

(ii) $\frac{dV}{d\theta} = \frac{(2 + \theta)^2 \times 18 - 36\theta(2 + \theta)}{(2 + \theta)^4}$

M1A1A1

$$\frac{dV}{d\theta} = \frac{36 - 18\theta}{(2 + \theta)^3}$$

(iii) $\frac{dV}{d\theta} = \frac{36 - 18\theta}{(2 + \theta)^3} = 0$

M1

Note: Award this **M1** for simplified version equated to zero.
 The simplified version may have been seen in part (b)(ii).

$$\theta = 2$$

A1

[7 marks]

Total: [9 marks]

✓ ✓ ✓ ✓ ✓

17. (a) attempt to use $V = \pi \int_a^b x^2 \, dy$ (M1)

$x = e^{\frac{y}{6}}$ or any reasonable attempt to find x in terms of y (M1)

$V = \pi \int_0^h e^{\frac{y}{3}} \, dy$ A1

Note: Correct limits must be seen for the **A1** to be awarded.

$= \pi \left[3e^{\frac{y}{3}} \right]_0^h$ (A1)

Note: Condone the absence of limits for this **A1** mark.

$= 3\pi \left[e^{\frac{h}{3}} - e^0 \right]$ A1

$= 3\pi \left[e^{\frac{h}{3}} - 1 \right]$ AG

Note: If the variable used in the integral is x instead of y (i.e. $V = \pi \int_0^h e^{\frac{x}{3}} \, dx$) and the candidate has not stated that they are interchanging x and y then award at most **M1M1A0A1A1AG**.

[5 marks]

(b) maximum volume when $h = 9$ cm (M1)

max volume = 180 cm^3 A1

[2 marks]

Total: [7 marks]

6. (a) $y = \dot{x} \Rightarrow \dot{y} = \ddot{x}$ **A1**
 $\dot{y} + 3(y) + 1.25x = 0$ **R1**

Note: If no explicit reference is made to $\dot{y} = \ddot{x}$, or equivalent, award **A0R1** if second line is seen.
 If $\frac{dy}{dx}$ used instead of $\frac{dy}{dt}$, award **A0R0**.

$$\dot{y} = -3y - 1.25x$$

AG
[2 marks]

(b) $A = \begin{pmatrix} 0 & 1 \\ -1.25 & -3 \end{pmatrix}$ **A1**
[1 mark]

(c) (i) $\begin{vmatrix} -\lambda & 1 \\ -1.25 & -3-\lambda \end{vmatrix} = 0$ **(M1)**

$$\lambda(\lambda + 3) + 1.25 = 0$$

$$\lambda = -2.5 \quad ; \quad \lambda = -0.5$$

(A1)
A1

continued...

Question 6 continued

$$(ii) \quad \begin{pmatrix} 2.5 & 1 \\ -1.25 & -0.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (M1)$$

$$2.5a + b = 0$$

$$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad A1$$

$$\begin{pmatrix} 0.5 & 1 \\ -1.25 & -2.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0.5a + b = 0$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad A1$$

Note: Award **M1** for a valid attempt to find either eigenvector. Accept equivalent forms of the eigenvectors.
Do not award **FT** for eigenvectors that do not satisfy both rows of the matrix.

[6 marks]

$$(d) \quad \begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-2.5t} \begin{pmatrix} -2 \\ 5 \end{pmatrix} + Be^{-0.5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad M1A1$$

$$t = 0 \Rightarrow x = 8, \dot{x} = y = 0 \quad (M1)$$

$$-2A - 2B = 8$$

$$5A + B = 0 \quad (M1)$$

$$A = 1; B = -5 \quad A1$$

$$x = -2e^{-2.5t} + 10e^{-0.5t} \quad A1$$

Note: Do not award the final **A1** if the answer is given in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-2.5t} \begin{pmatrix} -2 \\ 5 \end{pmatrix} + Be^{-0.5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

[6 marks]

Total: [15 marks]

2. (a) (i) $h(t) = -0.134t + 3.1$ **A1A1**

Note: Award **A1** for an equation in h and t and **A1** for the coefficient -0.134 and constant 3.1 .

- (ii) **EITHER**
the rate of change of height (of water in metres per minute) **A1**

Note: Accept “rate of decrease” or “rate of increase” in place of “rate of change”.

OR
the (average) amount that the height (of the water) decreases each minute **A1**

- (iii) **EITHER**
unreliable to use h on t equation to estimate t **A1**

OR
unreliable to extrapolate from original data **A1**

OR
rate of change (of height) might not remain constant (as the water drains out) **A1**

- (b) (i) $h(t) = 0.002t^2 - 0.174t + 3.2$ **A1**

- (ii) $0.002t^2 - 0.174t + 3.2 = 0$ **(M1)**
26.4 (26.4046...) **A1**

- (iii) **EITHER**
 $(0 \leq) t \leq 26.4$ ($t \leq 26.4046\dots$) **A1**

OR
 $(0 \leq) t \leq 20$ (due to range of original data / interpolation) **A1**

continued...

Question 2 continued

(c) $V = \pi(1)^2 h$ (A1)

EITHER

$$\frac{dV}{dt} = \pi \frac{dh}{dt} \quad \text{M1}$$

OR

attempt to use chain rule M1

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

THEN

$$\frac{dh}{dt} = \frac{1}{\pi} \times -\pi R^2 \sqrt{70560h} \quad \text{A1}$$

$$\frac{dh}{dt} = -R^2 \sqrt{70560h} \quad \text{AG}$$

(d) attempt to separate variables M1

$$\int \frac{1}{\sqrt{70560h}} dh = \int -R^2 dt \quad \text{A1}$$

$$\frac{2\sqrt{h}}{\sqrt{70560}} = -R^2 t + c \quad \text{A1A1}$$

Note: Award **A1** for each correct side of the equation.

$$\sqrt{h} = \frac{\sqrt{70560}}{2} (c - R^2 t) \quad \text{A1}$$

Note: Award the final **A1** for any correct intermediate step that clearly leads to the given equation.

$$h = 17640 (c - R^2 t)^2 \quad \text{AG}$$

(e) $t = 0 \Rightarrow 3.2 = 17640c^2$ (M1)

$$c = 0.0134687... \quad \text{(A1)}$$

substituting $h = 0$ and their non-zero value of c (M1)

$$T = \frac{c}{R^2} = \frac{0.0134687...}{0.023^2}$$

$$= 25.5 \text{ (minutes)} \quad (25.4606...) \quad \text{A1}$$

(f) $h = 0 \Rightarrow c = R^2 t$ (A1)

$$c = 0.023^2 \times 15 \quad (= 0.007935) \quad \text{(A1)}$$

$$t = 0 \Rightarrow h = 17640 (0.023^2 \times 15)^2 \quad \text{(M1)}$$

$$h = 1.11 \text{ (metres)} \quad (1.11068...) \quad \text{A1}$$

continued...

Question 2 continued

- (g) (i) let h be the height of water in the highest container
from parts (d) and (e) we get

$$\frac{dh}{dt} = -35280R^2(0.0134687... - R^2t) \quad \textbf{(M1)(A1)}$$

$$\text{so } \frac{dH}{dt} = 35280R^2(0.0135 - R^2t) - R^2\sqrt{70560H} \quad \textbf{M1A1}$$

$$\left(\frac{dH}{dt} = 18.6631...(0.0134687... - 0.000529t) - 0.000529\sqrt{70560H} \right)$$

$$\left(\frac{dH}{dt} = 0.251367... - 0.0987279... - 0.140518...\sqrt{H} \right)$$

$$\frac{dH}{dt} \approx 0.2514 - 0.009873t - 0.1405\sqrt{H} \quad \textbf{AG}$$

- (ii) evidence of using Euler's method correctly

e.g. $y_1 = 1.05545...$

(A1)

maximum value of $H = 1.45$ (metres) (at 8.5 minutes)

A2

(1.44678... metres)

7. (a) $E = 5(2 \sin t)^2 (= 20 \sin^2 t)$

A1

[1mark]

(b) $\frac{dE}{dt} = 40 \sin t \cos t$

(M1)A1

[2 marks]

(c) $t = 0.126$

(M1)A1

[2 marks]

Total [5 marks]

13. (a) $\frac{dy}{dx} = \frac{16-20}{24-20}$
 $= -1$

M1

A1

[2 marks]

(b) asymptote of trajectory along $\mathbf{r} = k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

M1A1

Note: Award **M1A0** if asymptote along $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

trajectory begins at (8, 10) with negative gradient

A1A1

[4 marks]

Total [6 marks]

14. (a) (i) $a = 33$

A1

(ii) $\frac{1}{\sqrt[3]{0.08}} = 2.32$

M1A1

[3 marks]

(b) volume within outer dome

$$\frac{2}{3}\pi \times 16^3 + \pi \times 16^2 \times 17 = 22\,250.85$$

M1A1

volume within inner dome

$$\pi \int_0^{33} \left(\frac{33-y}{0.08} \right)^{\frac{2}{3}} dy = 3446.92$$

M1A1

$$\text{volume between} = 22\,250.85 - 3446.92 = 18\,803.93 \text{ m}^3$$

A1

[5 marks]

Total [8 marks]

1. (a) $2(8 \times 4 + 3 \times 4 + 3 \times 8)$ **M1**
 $= 136 \text{ (cm}^2\text{)}$ **A1**
[2 marks]

(b) $\sqrt{8^2 + 4^2 + 3^2}$ **M1**
 (AG =) 9.43 (cm) (9.4339..., $\sqrt{89}$) **A1**
[2 marks]

(c) $-2x + 220 = 0$ **M1**
 $x = 110$ **A1**
 110 000 (boxes) **A1**
[3 marks]

(d) $P(x) = \int -2x + 220 \, dx$ **M1**

Note: Award **M1** for evidence of integration.

$P(x) = -x^2 + 220x + c$ **A1A1**

Note: Award **A1** for either $-x^2$ or $220x$ award **A1** for both correct terms and constant of integration.

$1700 = -(20)^2 + 220(20) + c$ **M1**

$c = -2300$

$P(x) = -x^2 + 220x - 2300$ **A1**
[5 marks]

(e) $-x^2 + 220x - 2300 = 0$ **M1**
 $x = 11.005$ **A1**
 11 006 (boxes) **A1**

Note: Award **M1** for their $P(x) = 0$, award **A1** for their correct solution to x .
 Award the final **A1** for expressing their solution to the minimum number of boxes. Do not accept 11 005, the nearest integer, nor 11 000, the answer expressed to 3 significant figures, as these will not satisfy the demand of the question.

[3 marks]

Total [15 marks]

7. (a) $\frac{dv}{dt} = 9.81 - 0.9v$ **M1**
- $\int \frac{1}{9.81 - 0.9v} dv = \int 1 dt$ **M1**
- $-\frac{1}{0.9} \ln(9.81 - 0.9v) = t + c$ **A1**
- $9.81 - 0.9v = Ae^{-0.9t}$ **A1**
- $v = \frac{9.81 - Ae^{-0.9t}}{0.9}$ **A1**
- when $t = 0$, $v = 0$ hence $A = 9.81$ **A1**
- $v = \frac{9.81(1 - e^{-0.9t})}{0.9}$
- $v = 10.9(1 - e^{-0.9t})$ **A1**
- [7 marks]**
- (b) **either** let t tend to infinity, or $\frac{dv}{dt} = 0$ **(M1)**
- $v = 10.9$ **A1**
- [2 marks]**
- (c) $\frac{dx}{dt} = y$ **M1**
- $\frac{dy}{dt} = 9.81 - 0.9y^2$ **A1**
- [2 marks]**
- (d) $x_{n+1} = x_n + 0.2y_n$, $y_{n+1} = y_n + 0.2(9.81 - 0.9(y_n)^2)$ **(M1)(A1)**
- $x = 1.04$, $\frac{dx}{dt} = 3.31$ **(M1)A1**
- [4 marks]**
- (e) 3.3015 **A1**
- [1 mark]**
- (f) $0 = 9.81 - 0.9(v)^2$ **M1**
- $\Rightarrow v = \sqrt{\frac{9.81}{0.9}} = 3.301511\dots (= 3.30)$ **A1**
- [2 marks]**

continued...

Question 7 continued

- (g) the model found the terminal velocity very accurately, so good approximation ***R1***
intermediate values had object exceeding terminal velocity so not
good approximation ***R1***

[2 marks]

Total [20 marks]

2. (a) (i) 2000

(M1)A1(ii) because the value of $\frac{dx}{dt}$ is positive (for $x > 0$)**R1****[3 marks]**(b) (i) substitute $x = 800$, $y = 600$ into both equations**M1**

both equations equal 0

A1

hence an equilibrium point

AG(ii) $x = 0$, $y = 0$ **A1** $x = 2000$, $y = 0$, $x = 0$, $y = 3000$ **M1A1A1**

Note: Award **M1** for an attempt at solving the system provided some values of x and y are found.

[6 marks](c) (i) $\int \frac{1}{x} dx = \int 2 dt$ **M1** $\ln x = 2t + c$ **A1A1**

Note: Award **A1** for RHS, **A1** for LHS.

 $x = e^c e^{2t}$ **M1** $x = Ae^{2t}$ (where $A = e^c$)**AG**(ii) $y = Be^{3t}$ **A1**

Note: Allow any letter for the constant term, including A .

(iii) $x = 15$, $y = 18$ **(M1)A1****[7 marks]***continued...*

Question 2 continued

(d) (i) $x_{n+1} = x_n + 0.2 \frac{x_n}{1000} (2000 - x_n - 2y_n)$

$$y_{n+1} = y_n + 0.2 \frac{y_n}{1000} (3000 - 3x_n - y_n)$$

M1A1

Note: Accept equivalent forms.

(ii) $x = 319, y = 617$

(M1)A1A1

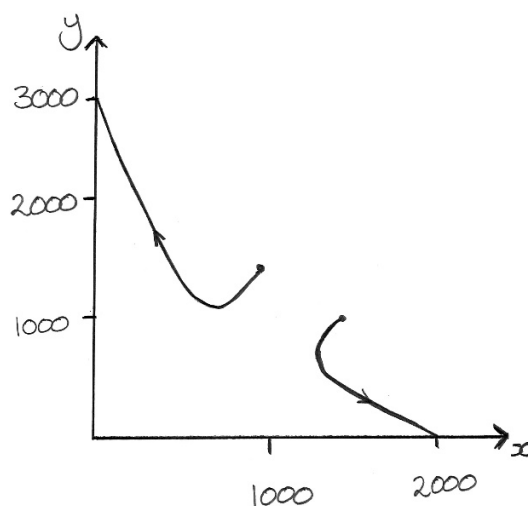
- (iii) number of brown squirrels go down to 0,
black squirrels to a population of 3000

A1

- (iv) number of brown squirrels go to 2000,
number of black squirrels goes down to 0

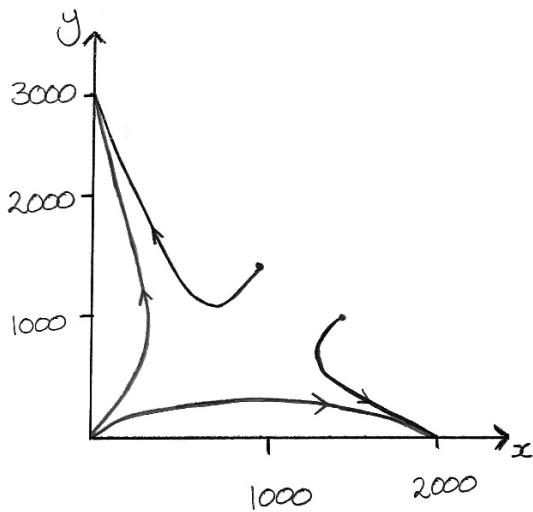
A1**[7 marks]**

- (e) (i) **AND** (ii)

**M1A1A1****[3 marks]***continued...*

Question 2 continued

(f)



A1A1

Note: Award **A1** for a trajectory beginning close to (0, 0) and going to (0, 3000) and **A1** for a trajectory beginning close to (0, 0) and going to (2000, 0) in approximately the correct places.

[2 marks]

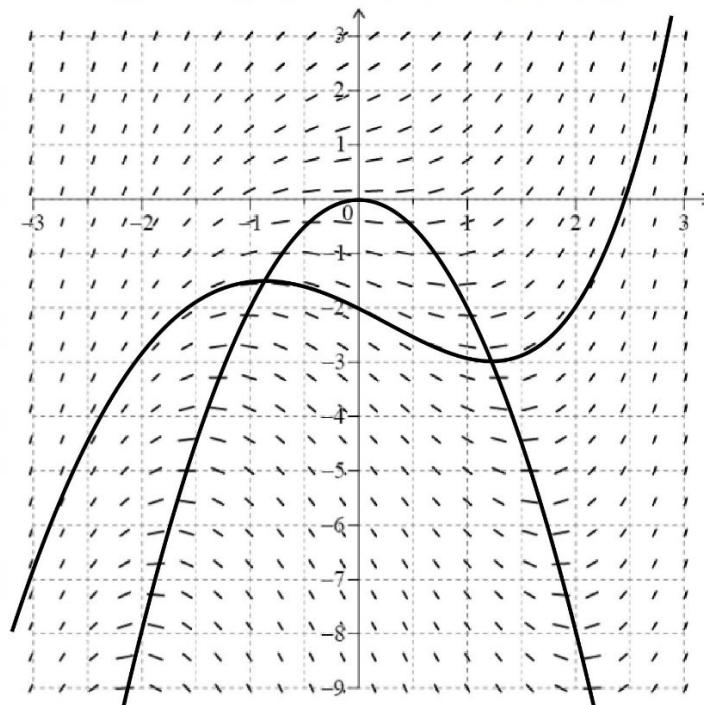
Total [28 marks]

7. (a) (i) $x^2 + \frac{y}{2} = 0$ ($y = -2x^2$) **A1**

(ii) $y = -2x^2$ drawn on diagram (correct shape with a maximum at $(0, 0)$) **A1**

[2 marks]

(b)



correct shape with a local maximum and minimum, passing through $(0, -2)$ **A1**

local maximum and minimum on the graph of $y = -2x^2$ **A1**

[2 marks]

[Total 4 marks]

8. (a) (i) use of product rule

(M1)

$$\frac{dy}{dx} = 2(4 - e^x) + 2x(-e^x)$$

A1

$$= 8 - 2e^x - 2xe^x$$

- (ii) use of product rule

(M1)

$$\frac{d^2y}{dx^2} = -2e^x - 2e^x - 2xe^x$$

A1

$$= -4e^x - 2xe^x$$

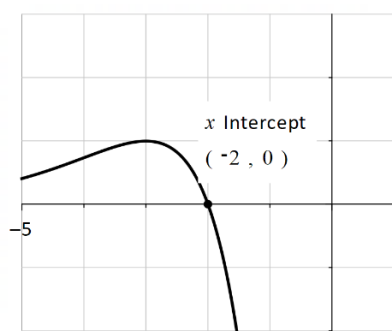
$$= -2(2 + x)e^x$$

[4 marks]

- (b)
- $-2(2 + a)e^a = 0$
- OR sketch of
- $\frac{d^2y}{dx^2}$
- with
- x
- intercept indicated

OR finding the local maximum of $\frac{dy}{dx}$ at $(-2, 8.27)$

(M1)



$$(a =) -2$$

A1

[2 marks]

[Total 6 marks]

14. (a) (i) $\frac{1}{u^2} + \frac{2}{u} + 1$ **A1**

(ii) $\int \left(\frac{1}{(x+2)} + 1 \right)^2 dx$
 $= \int \left(\frac{1}{(x+2)^2} + \frac{2}{x+2} + 1 \right) dx$ **OR** $\int \left(\frac{1}{u^2} + \frac{2}{u} + 1 \right) du$ **(M1)**

$= -\frac{1}{(x+2)} + 2 \ln|x+2| + x (+c)$ **A1A1**

Note: Award **A1** for first expression, **A1** for second two expressions.

Award **A1A0** for a final answer of $= -\frac{1}{u} + 2 \ln(u) + u (+c)$.

[4 marks]

continued...

Question 14 continued

$$(b) \quad \text{volume} = \pi \left[-\frac{1}{(x+2)} + 2 \ln(x+2) + x \right]_0^2 \quad \mathbf{M1}$$

$$= \pi \left(-\frac{1}{4} + 2 \ln(4) + 2 + \frac{1}{2} - 2 \ln 2 \right) \quad \mathbf{A1}$$

$$= \pi \left(\frac{9}{4} + 2 \ln(4) - 2 \ln 2 \right)$$

use of log laws seen, for example **M1**

$$\pi \left(\frac{9}{4} + 4 \ln(2) - 2 \ln 2 \right) \quad \mathbf{OR} \quad \pi \left(\frac{9}{4} + 2 \ln \left(\frac{4}{2} \right) \right)$$

$$= \frac{\pi}{4} (9 + 8 \ln(2)) \quad \mathbf{OR} \quad a = 9, b = 8 \text{ and } c = 2 \quad \mathbf{A1}$$

Note: Other correct integer solutions are possible and should be accepted for example $a = 9$, $b = c = 4$.

[4 marks]
Total [8 marks]

16. (a) use of power rule

(M1)

$$\frac{dW}{dv} = -1.1848v^{-0.84} \quad \text{OR} \quad -1.18v^{-0.84}$$

A1

[2 marks]

$$(b) \quad \frac{dv}{dt} = 5$$

(A1)

$$\frac{dW}{dt} = \frac{dv}{dt} \times \frac{dW}{dv}$$

(M1)

$$\left(\frac{dW}{dt} = -5 \times 1.1848v^{-0.84} \right)$$

when $v = 10$

$$\frac{dW}{dt} = -5 \times 1.1848 \times 10^{-0.84}$$

(M1)

$$-0.856 \quad (-0.856278...)^\circ\text{C min}^{-1}$$

A2

Note: Accept a negative answer communicated in words, “decreasing at a rate of...”.
 Accept a final answer of $-0.852809...^\circ\text{C min}^{-1}$ from use of -1.18 .
 Accept 51.4 (or 51.2) $^\circ\text{C hour}^{-1}$.

[5 marks]

[Total 7 marks]

4. (a) **EITHER**
 attempt to substitute 3, 4 and 7 into area of a trapezoid formula **(M1)**
 $(A =) \frac{1}{2}(7+4)(3)$

OR
 given area expressed as an integral **(M1)**
 $(A =) \int_{-1}^2 (6-x) \, dx$

OR
 attempt to sum area of rectangle and area of triangle **(M1)**
 $(A =) 4 \times 3 + \frac{1}{2}(3)(3)$

THEN
 16.5 (square units) **A1**
[2 marks]

- (b) (i) $(A =) \int_{-1}^2 1.5x^2 - 2.5x + 3 \, dx$ **A1A1**

Note: Award **A1** for the limits $x = -1$, $x = 2$ in correct location. Award **A1** for an integral of the quadratic function, dx must be included. Do not accept “y” in place of the function, given that two equations are in the question.

(ii) 9.75 (square units) **A1**
[3 marks]

- (c) 16.5 – 9.75 **(M1)**
 6.75 (square units) **A1**
[2 marks]
Total [7 marks]

14. $V = \pi \int_0^{10} y^2 \, dx$ OR $\pi \int_0^{10} x^2 \, dy$ (M1)

$$h = 2$$

$$\approx \pi \times \frac{1}{2} \times 2 \times \left((4^2 + 5^2) + 2 \times (6^2 + 8^2 + 7^2 + 3^2) \right) \quad \text{M1A1}$$

$$= 1120 \text{ cm}^3 \quad (1121.548\dots) \quad \text{A1}$$

Note: Do not award the second **M1** if the terms are not squared.

Total [4 marks]

17. (a) use of chain rule (M1)

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

attempt to find $\frac{dy}{dx}$ at $x=1$ (M1)

$$0.2 = 0.04 \times \frac{dx}{dt}$$

$$\left(\frac{dx}{dt} = \right) 5 \text{ m h}^{-1} \quad \text{A1}$$

[3 marks]

- (b) (i) if the position of the snail is (X, Y)

$$\text{from part (a)} \quad \frac{dX}{dt} = \frac{1}{0.04X} \frac{dY}{dt}$$

since speed is 1:

finding modulus of velocity vector and equating to 1 (M1)

$$1 = \sqrt{\left(\frac{\dot{Y}}{0.04X} \right)^2 + \dot{Y}^2} \quad \text{OR} \quad 1 = \sqrt{\dot{X}^2 + 0.0016X^2 \dot{X}^2}$$

$$1 = \dot{Y}^2 \left(\frac{1}{0.0016X^2} + 1 \right) \quad \text{OR} \quad 1 = \dot{X}^2 (1 + 0.0016X^2)$$

$$\dot{Y} = \sqrt{\frac{1}{\frac{1}{0.08Y} + 1}} \quad \text{OR} \quad \dot{X} = \sqrt{\frac{1}{1 + 0.0016X^2}} \quad \text{(A1)}$$

$$\int_{0.02}^2 \sqrt{\frac{1}{0.08Y} + 1} dY = \int_0^T dt \quad \text{OR} \quad \int_1^{10} \sqrt{1 + 0.0016X^2} dX = \int_0^T dt \quad \text{(M1)}$$

$$T = 9.26 \text{ hours} \quad \text{A1}$$

- (ii) EITHER

time for water to reach top is $\frac{2}{0.2} = 10$ hours (seen anywhere) A1

OR

or at time $t = 9.26$, height of water is $0.2 \times 9.26 = 1.852$ A1

THEN

so the water will not reach the snail AG

[5 marks]

Total [8 marks]

4. (a) (i) $y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} + 5\frac{dx}{dt} + 6x = 0$ OR $\frac{dy}{dt} + 5y + 6x = 0$ **M1**

Note: Award **M1** for substituting $\frac{dy}{dt}$ for $\frac{d^2x}{dt^2}$.

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{AG}$$

(ii) $\det \begin{pmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{pmatrix} = 0$ **(M1)**

Note: Award **M1** for an attempt to find eigenvalues. Any indication that $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$ has been used is sufficient for the **(M1)**.

$$-\lambda(-5-\lambda) + 6 = 0 \quad \text{OR} \quad \lambda^2 + 5\lambda + 6 = 0 \quad \text{(A1)}$$

$$\lambda = -2, -3 \quad \text{A1}$$

- (iii) (on a phase portrait the particle approaches (0, 0) as t increases so long term velocity (y) is)
0 **A1**

Note: Only award **A1** for 0 if both eigenvalues in part (a)(ii) are negative. If at least one is positive accept an answer of '*no limit*' or '*infinity*', or in the case of one positive and one negative also accept '*no limit or 0 (depending on initial conditions)*'.

[5 marks]

(b) (i) $y = \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} \quad \text{(A1)}$$

$$\frac{dy}{dt} + 5y + 6x = 3t + 4 \quad \text{A1}$$

- (ii) recognition that $h = 0.1$ in any recurrence formula **(M1)**

$$(t_{n+1} = t_n + 0.1)$$

$$x_{n+1} = x_n + 0.1y_n \quad \text{(A1)}$$

$$y_{n+1} = y_n + 0.1(3t_n + 4 - 5y_n - 6x_n) \quad \text{(A1)}$$

(when $t = 1$,) $x = 0.64402... \approx 0.644$ m **A2**

- (iii) recognizing that y is the velocity
 0.5 m s^{-1} **A1**

[8 marks]

[Total 13 marks]

6. (a) (i) $y = x^{\frac{1}{2}}$ (M1)

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

A1

(ii) gradient at $x = 0.16$ is $\frac{1}{2} \times \frac{1}{\sqrt{0.16}}$ (M1)

$$= 1.25$$

EITHER

$$y - 0.4 = 1.25(x - 0.16)$$

M1

OR

$$0.4 = 1.25(0.16) + b$$

M1

Note: Do not allow working backwards from the given answer.

THEN

hence $y = 1.25x + 0.2$ AG

[4 marks]

(b) $p = 0.45$, $q = 0.4125$ (or 0.413) (accept "(0.45, 0.4125)") A1A1

[2 marks]

(c) (i) $(h(x) =) \frac{1}{2} \sqrt{2(x - 0.2)}$ A2

Note: Award **A1** if only two correct transformations are seen.

(ii) $(a =) 0.28$ A1

(iii) **EITHER**

Correct substitution of their part (b) (or (0.28, 0.2)) into the given expression (M1)

OR

$$\frac{1}{2}(1.25 \times 2(x - 0.2) + 0.2)$$

(M1)

Note: Award **M1** for transforming the equivalent expression for f correctly.

THEN

$(b =) -0.15$ A1

[5 marks]

continued...

Question 6 continued

- (d) (i) recognizing need to add two integrals

(M1)

$$\int_0^{0.16} \sqrt{x} \, dx + \int_{0.16}^{0.5} (1.25x + 0.2) \, dx$$

(A1)

Note: The second integral could be replaced by the formula for the area of a trapezoid $\frac{1}{2} \times 0.34(0.4 + 0.825)$.

$$0.251 \, \text{m}^2 \, (0.250916\dots)$$

A1

- (ii)
- EITHER**

$$\text{area of trapezoid } \frac{1}{2} \times 0.05(0.4125 + 0.825) = 0.0309375$$

(M1)(A1)**OR**

$$\int_{0.45}^{0.5} (8.25x - 3.3) \, dx = 0.0309375$$

(M1)(A1)

Note: If the rounded answer of 0.413 from part (b) is used, the integral is

$$\int_{0.45}^{0.5} (8.24x - 3.295) \, dx = 0.03095 \text{ which would be awarded } \mathbf{(M1)(A1)}.$$

THEN

$$\text{shaded area} = 0.250916\dots - 0.0627292 - 0.0309375$$

(M1)

Note: Award **(M1)** for the subtraction of both 0.0627292... and their area for the trapezoid from their answer to (a)(i).

$$= 0.157 \, \text{m}^2 \, (0.15725)$$

A1**[7 marks]****[Total 18 marks]**

.....

2. (a) attempt to expand given expression **OR** attempt at product rule **(M1)**

$$C = \frac{xk^2}{10} - \frac{3x^3}{1000}$$

$$\frac{dC}{dx} = \frac{k^2}{10} - \frac{9x^2}{1000}$$

M1A1

Note: Award **M1** for power rule correctly applied to at least one term and **A1** for correct answer.

[3 marks]

- (b) equating their $\frac{dC}{dx}$ to zero **(M1)**

$$\frac{k^2}{10} - \frac{9x^2}{1000} = 0$$

$$x^2 = \frac{100k^2}{9}$$

$$x = \frac{10k}{3}$$

(A1)

substituting their x back into given expression

(M1)

$$C_{\max} = \frac{10k}{30} \left(k^2 - \frac{300k^2}{900} \right)$$

$$C_{\max} = \frac{2k^3}{9} (0.222\dots k^3)$$

A1**[4 marks]**

- (c) (i) substituting 20 into given expression and equating to 426 **M1**

$$426 = \frac{20}{10} \left(k^2 - \frac{3}{100} (20)^2 \right)$$

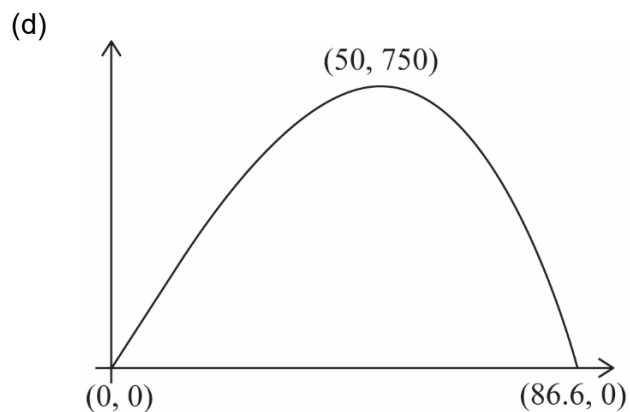
$$k = 15$$

A1

- (ii) 50 **A1**

[3 marks]*continued...*

Question 2 continued

**A1A1A1**

Note: Award **A1** for graph indicating an increasing and then decreasing function (drawn in first quadrant), **A1** for maximum labelled and **A1** for graph drawn for positive x , passing through the origin and 86.6 which is marked on the x -axis or its coordinates are given.

[3 marks]

- (e) setting their expression for C to zero **OR** choosing correct x -intercept on their graph of C

(M1)

$$x_{\max} = 86.6 \quad (86.6025\dots) \text{ litres}$$

A1**[2 marks]****Total [15 marks]**

7. (a) differentiating first equation.

M1

$$\frac{d^2x}{dt^2} = \frac{dy}{dt}$$

substituting in for $\frac{dy}{dt}$

M1

$$= -2x - 3y = -2x - 3\frac{dx}{dt}$$

therefore $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$

AG

Note: The **AG** line must be seen to award the final **M1** mark.

[2 marks]

- (b) the relevant matrix is $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

(M1)

Note: $\begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$ is also possible.

(this has characteristic equation) $-\lambda(-3-\lambda) + 2 = 0$

(A1)

$$\lambda = -1, -2$$

A1

[3 marks]

continued...

Question 7 continued

(c) **EITHER**the general solution is $x = Ae^{-t} + Be^{-2t}$ **M1****Note:** Must have constants, but condone sign error for the **M1**.

$$\text{so } \frac{dx}{dt} = -Ae^{-t} - 2Be^{-2t}$$

M1A1**OR**

attempt to find eigenvectors

(M1)respective eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or any multiple)

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(M1)A1**THEN**

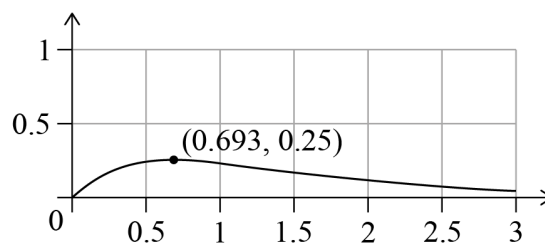
the initial conditions become:

$$0 = A + B$$

$$1 = -A - 2B$$

this is solved by $A = 1$, $B = -1$ **M1**so the solution is $x = e^{-t} - e^{-2t}$ **A1****[5 marks]**

(d)

**A1A1**

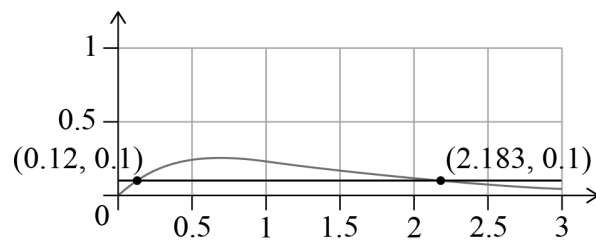
Note: Award **A1** for correct shape (needs to go through origin, have asymptote at $y = 0$ and a single maximum; condone $x < 0$). Award **A1** for correct coordinates of maximum.

[2 marks]

continued...

Question 7 continued

- (e) intersecting graph with
- $y = 0.1$

(M1)

so the time fishing is stopped between 2.1830... and 0.11957...
 $= 2.06(343\dots)$ days

(A1)**A1****[3 marks]**

- (f)
- Any reasonable answer. For example:*

There are greater downsides to allowing fishing when the levels may be dangerous than preventing fishing when the levels are safe.

The concentration of mercury may not be uniform across the river due to natural variation / randomness.

The situation at the power plant might get worse.

Mercury levels are low in water but still may be high in fish.

R1

Note: Award **R1** for a reasonable answer that refers to this specific context (and not a generic response that could apply to *any* model).

[1 mark]**Total [16 marks]**

1. (a) (i) $Q(t) = 3090t - 54000$ (3094.27... t - 54042.3...) **A1A1**

Note: Award at most **A1A0** if answer is not an equation. Award **A1A0** for an answer including either x or y .

[2 marks]

- (ii) 0.755 (0.754741...) **A1**
[1 mark]

- (iii) t is not a random variable **OR** it is not a (bivariate) normal distribution
OR data is not a sample from a population
OR data appears nonlinear
OR r only measures linear correlation **R1**

Note: Do not accept " r is not large enough".

[1 mark]

- (b) (i) attempt to separate variables **(M1)**

$$\int \frac{1}{Q} dQ = \int \beta N dt$$

$$\ln|Q| = \beta Nt + c$$

A1A1A1

Note: Award **A1** for LHS, **A1** for βNt , and **A1** for $+c$.
Award full marks for $Q = e^{\beta Nt + c}$ **OR** $Q = Ae^{\beta Nt}$.
Award **M1A1A1A0** for $Q = e^{\beta Nt}$

[4 marks]

- (ii) attempt at exponential regression **(M1)**
 $Q = 1.15e^{0.292t}$ ($Q = 1.14864...e^{0.292055...t}$) **A1**
OR
attempt at exponential regression **(M1)**
 $Q = 1.15 \times 1.34^t$ ($1.14864... \times 1.33917...^t$) **A1**

Note: Condone answers involving y or x . Condone absence of " $Q =$ "
Award **M1A0** for an incorrect answer in correct format.

[2 marks]

- (iii) 0.999 (0.999431...) **A1**
[1 mark]

- (iv) comparing something to do with
- R^2
- and something to do with
- r

M1**Note:** Examples of where the **M1** should be awarded:

$$R^2 > r$$

$$R > r$$

$$0.999 > 0.755$$

$$0.999 > 0.755^2 \quad (= 0.563)$$

The “correlation coefficient” in the exponential model is larger.

Model B has a larger R^2 Examples of where the **M1** should **not** be awarded:

The exponential model shows better correlation (since not clear how it is being measured)

Model 2 has a better fit

Model 2 is more correlated

an unambiguous comparison between R^2 and r^2 or R and r leading to the conclusion that the model in part (b) is more suitable / better**A1****Note:** Condone candidates claiming that R is the “correlation coefficient” for the non-linear model.**[2 marks]**

- (v) it suggests that there will be more infected computers than the entire population

R1**Note:** Accept any response that recognizes unlimited growth.**[1 mark]**

- (c) $1.15e^{0.292t} = 2.3$ **OR** $1.15 \times 1.34^t = 2.3$ **OR** $t = \frac{\ln 2}{0.292}$ **OR** using the model to find two specific times with values of $Q(t)$ which double
 $t = 2.37$ (days)

M1**A1****Note:** Do not **FT** from a model which is not exponential. Award **M0A0** for an answer of 2.13 which comes from using (10, 20) from the data or any other answer which finds a doubling time from figures given in the table.**[2 marks]**

- (d) an attempt to calculate
- β
- for city X

(M1)

$$\beta = \frac{0.292055...}{2.6 \times 10^6} \quad \text{OR} \quad \beta = \frac{\ln 1.33917...}{2.6 \times 10^6}$$

$$= 1.12328... \times 10^{-7}$$

A1this is larger than 9.64×10^{-8} so the virus spreads more easily in city X**R1****Note:** It is possible to award **M1A0R1**.Condone “so the virus spreads faster in city X” for the final **R1**.**[3 marks]**

(e) $a = 38.3, b = 3086.1$

A1A1**Note:** Award **A1A0** if values are correct but not to 1 dp.**[2 marks]**

(f) (i) $\frac{Q'}{Q} = 0.42228 - 2.5561 \times 10^{-6} Q$

(A1)(A1)**Note:** Award **A1** for each coefficient seen – not necessarily in the equation. Do not penalize seeing in the context of y and x .identifying that the constant is k **OR** that the gradient is $-\frac{k}{L}$ **(M1)**therefore $k = 0.422$ (0.422228...)**A1**

$$\frac{k}{L} = 2.5561 \times 10^{-6}$$

$$L = 165000$$
 (165205)

A1**Note:** Accept a value of L of 164843 from use of 3 sf value of k , or any other value from plausible pre-rounding.
Allow follow-through **within** the question part, from the equation of their line to the final two **A1** marks.**[5 marks]**(ii) recognizing that their L is the eventual number of infected**(M1)**

$$\frac{165205...}{2600000} = 6.35\%$$
 (6.35403...%)

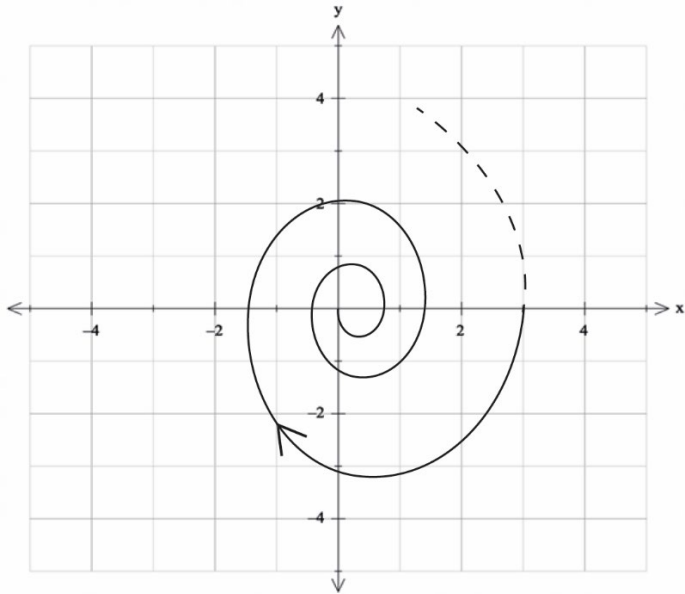
A1**Note:** Accept any final answer consistent with their answer to part (f)(i) unless their L is less than 120146 in which case award at most **M1A0**.**[2 marks]****[Total 28 marks]**

12. (a)

Description	Phase portrait
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	D
$\lambda_1 = 2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	C
$\lambda_1 = -2$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	B

A1A1A1
[3 marks]

(b)



spiral (crossing x -axis at least twice), centre at origin
arrow indicating clockwise, passing through or starting from $(3, 0)$

A1
A1
[2 marks]
Total [5 marks]

14. (a) attempt to use product rule

$$a = 2t^2 \cos(t^2) + \sin(t^2)$$

(M1)

A1

[2 marks]

- (b) graph of a

$$126 \text{ (ms}^{-2}\text{)} \text{ (125.699...)}$$

(M1)

A1

[2 marks]

- (c) attempt at integration by substitution or inspection

$$s = -\frac{1}{2} \cos(t^2) (+c)$$

(M1)

A1

$$(s = 0 \text{ when } t = 0) \Rightarrow c = \frac{1}{2}$$

A1

$$\left(s = -\frac{1}{2} \cos(t^2) + \frac{1}{2} \right)$$

[3 marks]

- (d) $\cos(t^2) \leq 1$

A1

$$-\frac{1}{2} \cos(t^2) \geq -\frac{1}{2}$$

$$\text{so } \frac{1}{2} - \frac{1}{2} \cos(t^2) \geq 0$$

R1

hence the particle never has a negative displacement.

AG

Note: Do not accept reasoning based on a sketch of the graph.

[2 marks]

Total [9 marks]

15. EITHER

$$q_{n+1} = q_n + 0.1 \left(\frac{dq}{dt} \right)_n$$

$$\left(\frac{dq}{dt} \right)_{n+1} = \left(\frac{dq}{dt} \right)_n + 0.1 \left(\frac{d^2q}{dt^2} \right)_n \quad (M1)$$

OR

let $\frac{dq}{dt} = y$

$$q_{n+1} = q_n + 0.1 y_n$$

$$y_{n+1} = y_n + 0.1 \left(\frac{dy}{dt} \right)_n \quad (M1)$$

THEN**EITHER**

$$\frac{dy}{dt} = 200 - 5y - 20q \quad (A1)$$

OR

$$\frac{d^2q}{dt^2} = 200 - 5 \frac{dq}{dt} - 20q \quad (A1)$$

THENevidence of using Euler's method (e.g.) (M1)

0	1	8	140
0.1	1.8	22	54

maximum charge = 12.7 (Coulombs, at $t = 0.7$)**A2**
Note: Award **A0A1** for a final answer of 10.8, from reading the value at $t = 1$.
Total [5 marks]

3. (a) (i) $f'(x) = \frac{-2x}{50} + 2 \left(= \frac{-x}{25} + 2, -0.04x + 2 \right)$ **A1A1**

Note: Award **A1** for each correct term. Award at most **A0A1** if extra terms are seen.

(ii) $0 = \frac{-x}{25} + 2$ **OR** sketch of $f'(x)$ with x -intercept indicated **M1**

$x = 50$ **A1**

$y = 80$ **A1**

$(50, 80)$

Note: Award **M0A0A1** for the coordinate $(50, 80)$ seen either with no working or found from a graph of $f(x)$.

[5 marks]

(b) (i) $\int_0^{70} \frac{-x^2}{50} + 2x + 30 \, dx$ **A1A1**

Note: Award **A1** for a correct integral, **A1** for correct limits in the correct location. Award at most **A0A1** if dx is omitted.

(ii) (Area =) $4710 \, \text{m}^2 \left(4713.33..., \frac{14140}{3} \right)$ **A2**

[4 marks]

(c) (i) $\frac{11.4}{4713.33...} \times 100\%$ **OR** $\left| \frac{4701.93... - 4713.33...}{4713.33...} \right| \times 100\%$ **(M1)**

Note: Award **(M1)** for their correct substitution into the percentage error formula.

$0.242\% \, (0.241867...\%)$ **A1**

Note: Percentage sign is required. Accept $0.242038...\%$ if 4710 is used.

(ii) **EITHER**
reduce the width of the intervals (trapezoids) **A1**

OR
increase the number of intervals (trapezoids) **A1**

Note: Accept equivalent statements. Award **A0** for the ambiguous answer "increase the intervals".

[3 marks]

continued...

Question 3 continued

- (d) (i) width of the square is $70 - x$ **OR** the length of the square is $\frac{-x^2}{50} + 2x + 30$ **(M1)**

Note: Award **(M1)** for $70 - x$ seen anywhere. Accept $\frac{-x^2}{50} + 2x + 30$ but only if this expression is explicitly identified as a dimension of the square.

in term of x , equating the length to the width ED **(M1)**

$$\frac{-x^2}{50} + 2x + 30 = 70 - x$$

$$(x = 14.7920... \text{ or } 135.21)$$

$$(x =) 14.8 \text{ m } (14.7920...) \quad \textbf{A1}$$

Note: Award **M0M0A0** for an unsupported answer of 15. Award at most **M1M0A0** for an approach which leads to $A'(x) = 0$. This will lead to a square base which extends beyond the east boundary of the property. Similar for any solution where F is not on the northern boundary, or GH is not on the east boundary.

(ii) **EITHER**

$$(70 - 14.7920...) ^2 \quad \textbf{(M1)}$$

OR

$$(55.2079...) ^2 \quad \textbf{(M1)}$$

OR

$$\left(\frac{-(14.7920...) ^2}{50} + 2(14.7920...) + 30 \right) ^2 \quad \textbf{(M1)}$$

THEN

$$(\text{Area} =) 3050 \text{ m}^2 \text{ (3047.92...)} \quad \textbf{A1}$$

Note: Follow through from part (d)(i), provided x is between 0 and 70. Award at most **M1A0** if their answer is outside the range of their $[0, 4713.33...]$ from part (b).

[5 marks]
Total [17 marks]

5. (a) $\frac{1}{2}x^3 + 1 = (x-1)^4$ (M1)
 (p =) 2.91 cm (2.91082...) A1

[2 marks]

(b) attempt to make x (or x^2) the subject of $y = \frac{1}{2}x^3 + 1$ (M1)

$x = \sqrt[3]{2(y-1)}$ (or $x^2 = (2(y-1))^{\frac{2}{3}}$) (A1)

(upper limit =) 13.3(315...) (A1)

$V = \int_1^{13.3315...} \pi(2(y-1))^{\frac{2}{3}} dy$ (M1)

Note: Award (M1) for setting up correct integral squaring their expression for x with both correct lower limit and their upper limit, and π .
 Condone omission of dy .

$= 197 \text{ cm}^3$ (196.946...) A1

[5 marks]

(c) $x = y^{\frac{1}{4}} + 1$ (or $x^2 = \left(y^{\frac{1}{4}} + 1\right)^2$) (A1)

$V_2 = \int_0^{13.3315...} \pi(y^{\frac{1}{4}} + 1)^2 dy$ (M1)(A1)

Note: Award (M1) for setting up correct integral squaring their expression for x with their upper limit, and π . Award (A1) for lower limit of 0, dependent on M1. Condone omission of dy .
 If a candidate found an area in part (b), do not award FT for another area calculation seen in part (c).

$= 271.87668...$ (A1)

Note: Accept 271.038... from use of 3sf in the upper limit.

subtracting their volumes (M1)

$271.87668... - 196.946...$

$= 74.9 \text{ cm}^3$ (74.93033...) A1

Note: Accept any answer that rounds to $75 \text{ (cm}^3\text{)}$. If a candidate found an area in part (b), do not award FT for another area calculation seen in part (c).

[6 marks]
 [13 marks]

2. (a) attempt to separate variable **M1**

$$\int \frac{1}{C} dC = \int -k dt$$

$$\ln|C| = -kt (+c) \quad \text{A1}$$

$$C = Ae^{-kt}$$

substituting $t = 0$, $C = d$

$$A = d \quad \text{A1}$$

Note: To award the **A1**, $t = 0$ must be seen.

$$C = de^{-kt} \quad \text{AG} \quad [3 \text{ marks}]$$

- (b) $0.05d = de^{-0.2t}$ **(A1)**
 $15.0 \quad (14.9786\dots)$ **A1**

[2 marks]

- (c) **EITHER**
 first dose: when $t = 2T$ then $C_1 = de^{-0.4T}$ **A1**
 second dose: when $t = T$ then $C_2 = de^{-0.2T}$ **A1**
 third dose: when $t = 0$ then $C_3 = d$ **A1**
 sum the three doses **M1**

Note: A timing of the dose and the expression must be clearly indicated and correct to award **A1**.
 Condone absence of d for **A1A1A1**.

OR

considering values of C before and after a dose **M1**

immediately before the second dose, $C = de^{-0.2T}$

immediately after the second dose, $C = de^{-0.2T} + d$ **A1**

immediately before the third dose, $C = (de^{-0.2T} + d)e^{-0.2T}$ **(A1)**

immediately after the third dose, $C = (de^{-0.2T} + d)e^{-0.2T} + d$ **A1**

THEN

$$\Rightarrow C = d(1 + e^{-0.2T} + e^{-0.4T}) \quad \text{AG}$$

Note: Do not accept any response using the general formula given before part (d).
 Award **M1A1A1A0** if d not included in the final formula.

[4 marks]

(d) $r = e^{-0.2T}$ and $u_1 = d$

A1

correct substitution into geometric series formula

A1

$$C = \frac{d(1 - (e^{-0.2T})^n)}{1 - e^{-0.2T}}$$

$$= d \left(\frac{1 - e^{-0.2nT}}{1 - e^{-0.2T}} \right)$$

AG**[2 marks]**

(e) (i) as $n \rightarrow \infty$, $e^{-0.2nT} \rightarrow 0$ (since $T > 0$)

(M1)

$$H_\infty = \frac{d}{1 - e^{-0.2T}}$$

A1

(ii) **METHOD 1**

$$\lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} H_n \times e^{-0.2T}$$

(M1)(A1)

$$L_\infty = \left(\frac{d}{1 - e^{-0.2T}} \right) e^{-0.2T} \quad \text{OR} \quad = \frac{d}{e^{0.2T} - 1}$$

A1

METHOD 2

L_n occurs immediately before the $(n+1)^{\text{th}}$ dose

(M1)

$$L_n = d \left(\frac{1 - e^{-0.2nT}}{1 - e^{-0.2T}} \right) e^{-0.2T}$$

(A1)

as $n \rightarrow \infty$, $e^{-0.2nT} \rightarrow 0$ (since $T > 0$)

$$L_\infty = \left(\frac{d}{1 - e^{-0.2T}} \right) e^{-0.2T} \quad \text{OR} \quad = \frac{d}{e^{0.2T} - 1}$$

A1**[5 marks]**

(f) (i) **METHOD 1**

$$H_{\infty} - L_{\infty} = \frac{d}{1 - e^{-0.2T}} - \frac{de^{-0.2T}}{1 - e^{-0.2T}} \quad \text{M1}$$

$$= \frac{d(1 - e^{-0.2T})}{1 - e^{-0.2T}} \quad \text{A1}$$

$$= d \quad \text{AG}$$

METHOD 2

$$H_{\infty} - L_{\infty} = \frac{d}{1 - e^{-0.2T}} - \frac{d}{e^{0.2T} - 1} \quad \text{M1}$$

$$= \frac{de^{0.2T} - d - d + de^{-0.2T}}{e^{0.2T} - 1 - e^0 + e^{-0.2T}}$$

$$= \frac{d(e^{0.2T} + e^{-0.2T} - 2)}{e^{0.2T} + e^{-0.2T} - 2} \quad \text{A1}$$

$$= d \quad \text{AG}$$

Note: Award **M1A0** for the use of their formulae from part (e).
Award **M0A0** if at no point $n \rightarrow \infty$ in their response.

(ii) **EITHER**

$$\ln\left(\frac{H_{\infty}}{L_{\infty}}\right) = \ln\left(\frac{\frac{d}{1 - e^{-0.2T}}}{\frac{de^{-0.2T}}{1 - e^{-0.2T}}}\right) \quad \text{M1}$$

$$= \ln(e^{0.2T}) \quad \text{A1}$$

$$= 0.2T \quad \text{A1}$$

OR

$$H_{\infty}e^{-0.2T} = L_{\infty} \quad \text{(M1)}$$

$$\ln\left(\frac{L_{\infty}}{H_{\infty}}\right) = -0.2T \quad \text{A1}$$

$$\ln\left(\frac{L_{\infty}}{H_{\infty}}\right)^{-1} = 0.2T \quad \text{A1}$$

THEN

$$5 \ln\left(\frac{H_{\infty}}{L_{\infty}}\right) = T \quad \text{AG}$$

Note: Award **M1A0** for the use of their incorrect formulae from part (e).
Award **M0A0** if at no point $n \rightarrow \infty$ in their response.

[5 marks]

(g) (i) $d = 0.22$ **A1**

(ii) $T = 7.70$ **A1**

Note: Accept $T = 7.7$.

[2 marks]

(h) $0.06 = 0.22e^{-0.2t}$
 proportion of time = $\frac{6.49641...}{7.702...}$ **(M1)**
 84.3% **A1**

[2 marks]

(i) **EITHER**
 rounding to 8 hours leads to 3 times a day **R1**
OR
 7.7 hours is difficult to schedule **R1**
OR
 a tolerance may be necessary to not be too close to unsafe boundary **R1**
OR
 avoid giving too much of the drug and giving the patient an overdose **R1**

Note: Accept any reasonable argument that engages with the model in a practical sense.
 Do not accept an answer based on part (h) along the lines “it should be given more often”.

[1 mark]

[Total: 26 mark]

4. (a) attempt to substitute $h = 10$ and at least two different values of y into the trapezoidal rule

(M1)

$$\frac{10}{2}((0+0) + 2(3+8+9))$$

$$= 200 \text{ (cm}^2\text{)}$$

A1

[2 marks]

(b) (i) $\int_0^{40} 0.04x^2 - 0.001x^3 \, dx$ OR $\int_0^{40} y \, dx$

A1A1

Note: Award **A1** for a correct integral (including dx), **A1** for correct limits in the correct location.

(ii) $213.33 \text{ (cm}^2\text{)}$

A2

Note: Answer must be given to 2 decimal places to award **A2**. Award **A1A0** for a correct answer given to an incorrect accuracy of at least 3 significant figures, e.g. $213 \text{ (cm}^2\text{)}$.

[4 marks]

- (c) attempt to substitute their parts (a) and (b)(ii) into percentage error formula

(M1)

$$\left| \frac{213.333... - 200}{213.333...} \right| \times 100$$

$$= 6.25(\%) \text{ (6.24999...(\%))}$$

A1

Note: Award **(M1)A0** for a final answer of $-6.25(\%)$ or 0.0625 .

[2 marks]

[Total: 8 marks]

10. (a) $y = x$

A1**[1 mark]**(b) **METHOD 1**equation has the form $y = ax^2 + bx + c$ when $x = 0, y = 0$ so $c = 0$

$$\frac{dy}{dx} = 2ax + b$$

attempt to find the value of b by setting *their* derivative equal to 1 when x is 0 **(M1)**

$$2a(0) + b = 1$$

$$b = 1$$
 (A1)

when $x = -2, y = 0$

$$a = \frac{1}{2} \text{ (and hence } y = \frac{1}{2}x^2 + x \text{)}$$
 A1

METHOD 2equation has the form $y = ax(x + 2)$ **OR** $y = ax^2 + 2ax$ **A1**

$$\frac{dy}{dx} = 2ax + 2a$$

attempt to find the value of a by setting *their* derivative equal to 1 when x is 0 **(M1)**

$$a = \frac{1}{2} \text{ (and hence } y = \frac{1}{2}x^2 + x \text{)}$$
 A1

Note: Writing $y = x(x + 2)$ is incorrect and gains no marks.**[3 marks]***continued...*

Question 10 continued

(c) equation is $y = ax^2 + bx + c$ finding an expression for $\frac{dy}{dx}$ with unknown coefficients (M1)

$$\frac{dy}{dx} = 2ax + b$$

setting up two equations using two points AND/OR one equation using the gradient function (M1)

three correct equations (A1)

$$9a + 3b + c = 3$$

$$36a + 6b + c = 2$$

$$6a + b = 1$$

$$a = -\frac{4}{9}, b = \frac{11}{3}, c = -4 \quad (a = -0.444444..., b = 3.66666..., c = -4) \quad \text{A1}$$

$$(\text{and hence } y = -\frac{4}{9}x^2 + \frac{11}{3}x - 4)$$

[4 marks]

$$(d) \quad f(x) = \begin{cases} \frac{1}{2}x^2 + x & , \quad -2 \leq x < 0 \\ x & , \quad 0 \leq x \leq 3 \\ -\frac{4}{9}x^2 + \frac{11}{3}x - 4 & , \quad 3 < x \leq 6 \end{cases} \quad \text{A1}$$

Note: Condone open or closed endpoints for all intervals.Condone y in place of $f(x)$.Allow **FT** from parts (a), (b) and (c).

[1 mark]

[Total: 9 marks]

13. $\frac{dx}{dt} = y$ (A1)

$\frac{dy}{dt} = -10x - 2y$ (A1)

Note: Writing $\frac{d^2x}{dt^2} = -10x - 2\frac{dx}{dt}$ is a valid approach and should be awarded **A1A1**.

attempt to use the Euler equations shown by finding either a correct x_{n+1} or y_{n+1} (M1)

correct equations for both x_{n+1} and y_{n+1} (A1)

$x_{n+1} = x_n + 0.1(y_n), \quad y_{n+1} = y_n + 0.1(-10x_n - 2y_n)$ (accept equivalent notation)

$(t_{n+1} = t_n + 0.1)$

Note: All of the above marks can be implied by a correct second row in a table **OR** by a correct f_1 and f_2 clearly identified for use in Euler's method formula.

T	x	y
0	0.75	0
0.1	0.75	-0.75
0.2	0.675	-1.35
0.3	0.54	-1.755
0.4	0.3645	-1.944
0.5	0.1701	

so estimate is 0.170 A2

Note: Accept 0.17 rounded to 2 sf.

[Total: 6 marks]

15. METHOD 1 Using the volume formula

volume of a “full” or “half” cylinder (seen anywhere) (A1)

$$\pi \int_2^4 4^2 dy, \quad \pi \times 4^2 \times 2, \quad 32\pi \quad (100.53\dots) \quad \text{OR}$$

$$\pi \int_2^3 4^2 dy, \quad \pi \times 4^2 \times 1, \quad 16\pi \quad (50.265\dots)$$

one correct equation for the diagonal lines (seen anywhere) (A1)

$$y = \frac{1}{2}x \quad \text{or} \quad y = 6 - \frac{1}{2}x$$

attempt to write one equation x in terms of y (M1)

$$x = 2y, \quad x = 12 - 2y$$

EITHER (symmetry plus the volume of the “half” cylinder)

recognition of symmetry between $y = 1$ and $y = 3$ (M1)

$$2\pi \left(\int_1^2 (2y)^2 dy + \int_2^3 4^2 dy \right) \quad (A1)$$

OR (symmetry plus volume of the “full” cylinder)

recognition of symmetry between $y = 1$ and $y = 2$ (M1)

$$2\pi \left(\int_1^2 (2y)^2 dy \right) + \int_2^4 4^2 dy \quad (A1)$$

OR (calculation of separate parts)

(M1)

$$\pi \left(\int_1^2 (2y)^2 dy + \int_2^4 4^2 dy + \int_4^5 (-2y + 12)^2 dy \right) \quad (A1)$$

THEN

$$(\text{volume of the solid} =) 159 \left(159.174\dots, \frac{152\pi}{3} \right) \quad \text{A1}$$

continued...

Question 15 continued**METHOD 2 Geometric approach using cones and cylinders**

volume of a cylinder (seen anywhere)

(A1)

$$\pi \times 4^2 \times 2, 32\pi \text{ (100.53...)} \text{ (a full cylinder) OR}$$

$$\pi \times 4^2 \times 1, 16\pi \text{ (50.265...)} \text{ (a half cylinder)}$$

using volume of cone formula to find the volume of the truncated cone

(M1)

correct expression to find the volume of the truncated cone (seen anywhere)

(A1)

$$\frac{1}{3}(\pi \times 4^2 \times 2 - \pi \times 2^2 \times 1)$$

attempt to find an expression for total volume using symmetry or individual parts

(M1)

correct expression for total volume

(A1)

$$2\left(\frac{1}{3}(\pi 4^2 \times 2 - \pi 2^2 \times 1) + \pi 4^2 \times 1\right) \text{ OR } \frac{1}{3}(\pi 4^2 \times 2 - \pi 2^2 \times 1) + \pi 4^2 \times 2 + \frac{1}{3}(\pi 4^2 \times 2 - \pi 2^2 \times 1)$$

$$\text{(volume of the solid=)} 159 \left(159.174..., \frac{152\pi}{3}\right)$$

A1

Note: There are other valid approaches possible.

[Total: 6 marks]

17. (a) attempt to use the chain rule to set up a related rate (M1)

correct expression A1

$$\frac{dx}{d\theta} = \frac{dx}{dt} \div \frac{d\theta}{dt} \quad \text{OR} \quad \frac{-250}{0.075}$$

$$= -\frac{10000}{3} \quad \text{AG}$$

[2 marks]

- (b) $x(\theta) = \frac{3000}{\tan \theta}$ A1

[1 mark]

- (c) attempt to use chain rule OR quotient rule (M1)

$$\frac{-3000}{\tan^2 \theta \times \cos^2 \theta}, \frac{-3000(\sin \theta(-\sin \theta) - \cos^2 \theta)}{\sin^2 \theta} \quad (A1)$$

$$= -\frac{3000}{\sin^2 \theta} \quad A1$$

[3 marks]

- (d) setting their equation in part (c) equal to the given expression in part (a) (M1)

$$-\frac{3000}{\sin^2 \theta} = -\frac{10000}{3}$$

$$\theta = 1.24904... \quad (A1)$$

$$x(1.24904...) = 1000 \text{ m} \quad A1$$

[3 marks]

[Total: 9 marks]

12. (a) equating volume of sphere formula to 288π

(M1)

$$\frac{4}{3}\pi r^3 = 288\pi$$

$$\Rightarrow r = 6 \text{ (cm)}$$

A1

[2 marks]

- (b) $\frac{dV}{dr} = 4\pi r^2$ (seen anywhere)

(A1)

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$$

(M1)

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$15 = 4\pi \times 6^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{15}{144\pi} \text{ (cm s}^{-1}\text{)} \text{ (0.0332, 0.0331572...)}$$

A1

[3 marks]

Total [5 marks]

16. (a) $(1.04, 0.509) \left((1.03667\dots, 0.509085\dots) \right)$

A1A1

[2 marks]

(b) attempt to make x the subject for either function
 $x = 4y^2$, $x = \cos^{-1} y$

(M1)

A1A1

attempt to use $V = \pi \int x^2 dy$

(M1)

$$V = \pi \int_0^{0.509085\dots} (4y^2)^2 dy + \pi \int_{0.509085\dots}^1 (\cos^{-1} y)^2 dy$$

A1

[5 marks]

(c) $= 1.15 \text{ (units}^3\text{)}$

A2

Note: Do not **FT** from part (b) to part (c).
 Award **A1** for 1.1. with no previous working.

[2 marks]

Total [9 marks]

17. (a) $\frac{dy}{dx}$ is undetermined at $(0, 1)$ **R1**

$$\left(\text{so cannot use } y_n = y_{n-1} + h \left(\frac{x}{(x^2 + 1)(2y - 2)} \right) \right)$$

Note: Accept “undefined”, “indeterminate” or “division by zero” in place of “undetermined”.

[1 mark]

(b) $\int (2y - 2) dy = \int \frac{x}{x^2 + 1} dx$ **M1**

$$y^2 - 2y = \frac{1}{2} \ln(x^2 + 1) + c$$
 A1

substituting $x = 0, y = 1$ **M1**

$$c = -1$$

$$y^2 - 2y + 1 = \frac{1}{2} \ln(x^2 + 1)$$

$$(y - 1)^2 = \frac{1}{2} \ln(x^2 + 1)$$
 A1

$$y - 1 = \sqrt{\frac{1}{2} \ln(x^2 + 1)} \text{ (where positive root required as } y \geq 1)$$

$$y = 1 + \sqrt{\frac{\ln(x^2 + 1)}{2}}$$
 AG

[4 marks]

(c) (when $x = 0.1$) $y = 1.07$ (1.07053...) **A1**

[1 mark]

Total [6 marks]

6. (a) attempt to solve $\det(A - \lambda I) = 0$ (M1)
- $(-0.05 - \lambda)^2 + 25 = 0$ (A1)
- $-0.05 - \lambda = \pm 5i$ (A1)
- $\lambda = -0.05 \pm 5i$ A1

[4 marks]

- (b) (i) spiral A1

- (ii) inwards / towards O A1

[2 marks]

- (c) (i) attempt to substitute $(20, 0)$ into expression for $\frac{dy}{dt}$ (M1)

$$-5(20) - 0.05(0)$$

$$\frac{dy}{dt} = -100 \text{ (ms}^{-1}\text{)} \quad \text{A1}$$

- (ii) $\frac{dx}{dt} = -1$ (A1)

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \quad \text{OR} \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{(M1)}$$

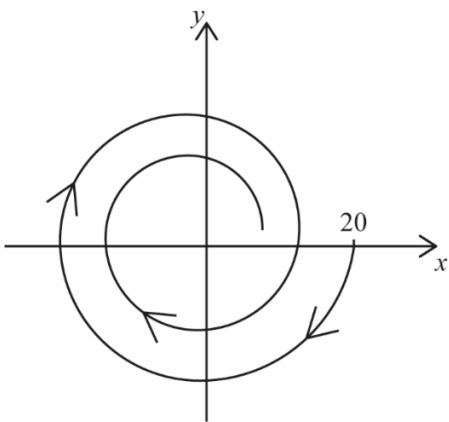
$$(-100 \div -1 =) 100 \quad \text{A1}$$

[5 marks]

continued...

Question 6 continued

(d)



A4

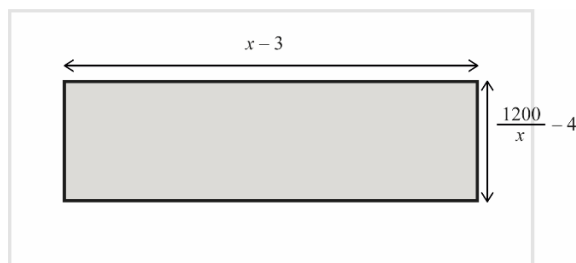
Note: Award **A1** for starting at $(20, 0)$, **A1** for spiral inwards, **A1** for clockwise, **A1** for non-negative gradient at $(20, 0)$.

[4 marks]

[Total: 15 marks]

3. (a)

Note: In methods 1 and 2, full marks are available for candidates who work with a dummy variable, e.g. y , that represents the width of the park and hence is equal to $\frac{1200}{x}$.
The substitution to express an answer in only x may come as late as the final line.

METHOD 1 (finding dimensions of garden)

$$(\text{width of park} =) \frac{1200}{x} \quad \textbf{(A1)}$$

$$(\text{length of garden} =) x - 3, (\text{width of garden} =) \frac{1200}{x} - 4 \quad \textbf{(A1)(A1)}$$

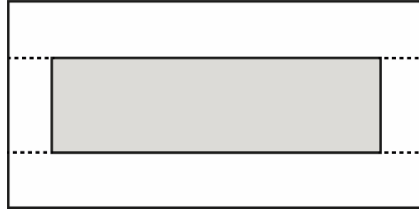
$$A = (x - 3) \times \left(\frac{1200}{x} - 4 \right) \quad \textbf{A1}$$

$$= 1200 - 4x - \frac{3600}{x} + 12 \quad \textbf{A1}$$

$$= 1212 - 4x - \frac{3600}{x} \quad \textbf{AG}$$

continued...

Question 3 continued

METHOD 2 (subtracting the area of the path)

$$\text{width of park} = \frac{1200}{x} \quad (\text{A1})$$

attempt to cut path into 4 (or 8) pieces (M1)

four (or eight) areas of the path expressed in terms of x (A1)

$$A = 1200 - 2x - 2x - 1.5\left(\frac{1200}{x} - 4\right) - 1.5\left(\frac{1200}{x} - 4\right) \quad \text{A1}$$

correct manipulation leading to given result A1

$$= 1212 - 4x - \frac{1800}{x} - \frac{1800}{x}$$

$$= 1212 - 4x - \frac{3600}{x} \quad \text{AG}$$

Note: To award (M1)(A1) without a diagram the division of the park must be clear.

[5 marks]

(b) setting $1212 - 4x - \frac{3600}{x} = 800$ (accept a sketch) (M1)

$$x = 9.64 \text{ (9.64011...)} \text{ (m)} \quad \text{OR} \quad x = 93.4 \text{ (93.3598...)} \text{ (m)} \quad \text{A1}$$

$$\text{(width =) } 124 \text{ (124.479...)} \text{ (m)} \quad \text{A1}$$

$$\text{(width =) } 12.9 \text{ (12.8534...)} \text{ (m)} \quad \text{A1}$$

Note: To award the final A1 both values of x and both values of the width must be seen. Accept 12.8 for second value of width from candidate dividing 1200 by 3 sf value of 93.4.

[4 marks]

(c) $\left(\frac{dA}{dx} =\right) -4 + \frac{3600}{x^2} \quad \text{OR} \quad -4 + 3600x^{-2} \quad \text{A1A1A1}$

Note: Award A1 for -4 , A1 for $+3600$, and A1 for x^{-2} or x^2 in denominator.

[3 marks]

continued...

Question 3 continued

- (d) setting *their* $\frac{dA}{dx}$ equal to 0 **OR** sketch of *their* $\frac{dA}{dx}$ with x -intercept highlighted **M1**

$$(x =) 30 \text{ (m)}$$

A1

Note: To award **A1FT** the candidate's value of x must be within the domain given in the problem ($3 < x < 300$).

[2 marks]

- (e) **EITHER**

evidence of using GDC to find maximum of graph of $A = 1212 - 4x - \frac{3600}{x}$ **(M1)**

OR

substitution of *their* x into A **(M1)**

OR

dividing 1200 by *their* x to find width of park **and** subtracting 3 from *their* x and 4 from the width to find park dimensions **(M1)**

Note: For the last two methods, only follow through if $3 < \text{their } x < 300$.

THEN

$$(A =) 972 \text{ (m}^2\text{)}$$

A1**[2 marks]****Total [16 marks]**

7. (a) $\begin{vmatrix} -4-\lambda & 6 \\ 9 & -1-\lambda \end{vmatrix} = 0$ (M1)

Note: Do not accept $\det(A - \lambda I) = 0$ or similar as evidence of a correct method unless A is explicitly defined to be the given matrix.

$$(-4-\lambda)(-1-\lambda) - 54 = 0$$

$$\lambda = -10, \lambda = 5$$

A1A1

For $\lambda = -10$

$$\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10x \\ -10y \end{pmatrix}$$

(M1)

$$-4x + 6y = -10x$$

$$x + y = 0$$

possible eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or equivalent)

A1

for $\lambda = 5$

$$\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

$$-4x + 6y = 5x$$

$$3x = 2y$$

possible eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (or equivalent)

A1

Note: If both eigenvalues are incorrect then award at most **M1A0A0M1A0A0**.

[6 marks]

(b) attempt to substitute their eigenvalues and eigenvectors equation

(M1)

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-10t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + Be^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

A1

Note: Award at most **(M1)A0** if $\begin{pmatrix} x \\ y \end{pmatrix}$ not seen.

[2 marks]

continued...

Question 7 continued

- (c) At
- $t = 0$
- ,
- $x = 500$
- and
- $y = 125$

$$x = -A + 2B \text{ and } y = A + 3B$$

Solving simultaneously:

(M1)

$$A = -250 \text{ and } B = 125$$

A1

$$\begin{pmatrix} x \\ y \end{pmatrix} = -250e^{-10t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 125e^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Note: Follow through from their eigenvectors.

Accept equivalent values for A and B based on the direction of their eigenvectors and the order of their eigenvalues in the equation.

[2 marks]

- (d) 2:3

A1**[1 mark]**

- (e) attempt to eliminate
- dt
- from the two differential equations

M1

$$\frac{dy}{dx} = \frac{9x - y}{-4x + 6y}$$

substituting initial conditions

(M1)

$$\begin{aligned} &= \frac{9(500) - 125}{-4(500) + 6(125)} \\ &= -3.5 \end{aligned}$$

A1**Note:** Award **M1** for $\frac{dy}{dx} = \frac{-4x + 6y}{9x - y}$.**[3 marks]***continued...*

Question 7 continued

- (f) trajectory or trajectories that are consistent with their eigenvalues

A1
- a trajectory that passes through the point (500, 125) with gradient that is consistent with the response to part (e)

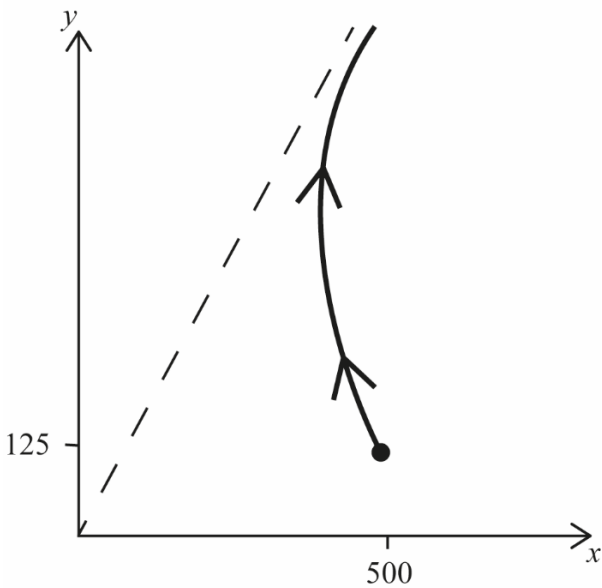
A1
- the diagram contains at least one of their eigenvectors

A1
- (e.g. labelled $y = 1.5x$; $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\lambda = 5$ etc.)
- the trajectory that passes through (125, 500) tends towards an oblique asymptote that corresponds to their eigenvector and the direction is indicated by at least one arrow on the trajectory

A1

Note: For the second **A1**, the point (500, 125) may not be labelled but there should be a point marked on the trajectory that is consistent with these coordinates.

The final **A1** will depend on their eigenvalues. Follow through can be awarded as long as the direction of the trajectory is consistent with the nature of their eigenvalues and eigenvectors.



[4 marks]

Total [18 marks]

1. (a) $C = kd$ (M1)

$0.80 = 0.5k$ OR $\frac{C}{d} = \frac{0.8}{0.5}$ (A1)

$k = 1.6$

$C = 1.6d$ OR $C = \frac{d}{0.625}$ A1

Note: For the final **A1** do not accept $C = \frac{0.8}{0.5}d$ or a correct equation which does not have C as the subject.

[3 marks]

(b) $d = \frac{0.96}{1.6}$ M1

Note: Award **M1** for the substitution of $C = 0.96$ into a correct equation, award **M0** for substitution of $d = 0.6$.

$= 0.6$ AG

[1 mark]

(c) attempt at using Pythagoras M1
 $d^2 = h^2 + (2r)^2$ (A1)

$r^2 = \frac{1}{4}(d^2 - h^2)$ A1

[3 marks]

(d) $V = \pi r^2(2h)$ (A1)

$V = \frac{\pi}{4}(d^2 - h^2)(2h)$ M1

Note: The **M1** is for the substitution of their expression for r^2 into their formula for V .

$V = \frac{\pi}{2}(d^2h - h^3)$ AG

[2 marks]

$$\begin{aligned}
 \text{(e) (i)} \quad V &= \frac{\pi}{2}(d^2h - h^3) \\
 &= \frac{\pi}{2}(0.6^2 \times 0.4 - 0.4^3)
 \end{aligned}
 \quad \textbf{(M1)}$$

Note: The **M1** is for substituting correct values of both d and h in the formula from part (d).

$$= 0.126 \text{ (m}^3\text{)} \left(0.12566\dots, \frac{\pi}{25}, 0.04\pi \right) \quad \textbf{A1}$$

[2 marks]

$$\text{(ii)} \quad V = \frac{\pi}{2}(0.36h - h^3) \quad \textbf{M1}$$

Note: Award **M1** for substitution of $d = 0.6$. This may be seen anywhere.

$$\frac{dV}{dh} = \frac{\pi}{2}(0.36 - 3h^2) = 0 \quad \textbf{A1}$$

Note: Award **A1** for correct differentiation.

$$h^2 = \frac{0.36}{3} \quad (= 0.12) \quad \textbf{A1}$$

Note: Do not award the final **A1** if the working is done with approximate values or if $\sqrt{0.12}$ is substituted into $\frac{dV}{dh}$.

$$h = \sqrt{0.12} \quad \textbf{AG}$$

[3 marks]

$$\begin{aligned}
 \text{(iii)} \quad &\text{substituting } h = \sqrt{0.12} \text{ into equation for } V \textbf{ OR use of graph} \quad \textbf{(M1)} \\
 &V = \frac{\pi}{2}(0.36 - 0.12)\sqrt{0.12}
 \end{aligned}$$

$$0.131 \text{ (m}^3\text{)} \quad (0.130593\dots, 0.0416\pi) \quad \textbf{A1}$$

[2 marks]

– 10 –

M23/5/MATHY/HP3/ENG/TZ1/XX/M

- (f) x -coordinate of S is 0.4 (A1)
 let the y -coordinate be y_S
 attempt at Pythagoras (M1)
 $(y_S + 0.2)^2 = 0.6^2 - 0.4^2$
 $y_S + 0.2 = \sqrt{0.2}$
 $y_S = 0.247$ (0.247213...) (A1)

Note: The (M1) mark can be implied by a y -coordinate of 0.447 or 0.647 seen.

any valid method to find equation (M1)
 e.g. quadratic regression, vertex form, simultaneous equations

Note: Award only if the student has found three points on the curve.

EITHER

$$y = -0.295x^2 + 0.236x + 0.2 \quad \text{A2}$$

$$(y = -0.295081...x^2 + 0.236065...x + 0.2)$$

Note: Award **A1** if one coefficient ($-0.295081...$ or $0.236065...$) is correct or if “ $y =$ ” is missing, **A2** for completely correct equation.
 Award **A1** for $y = -0.294x^2 + 0.235x + 0.2$ obtained from using the 3 sf value of y_S .

OR

$$y = -0.295(x - 0.4)^2 + 0.247 \quad \text{A2}$$

$$(y = -0.295081...(x - 0.4)^2 + 0.247213...)$$

Note: Award **A1** for -0.295 , **A2** for completely correct equation.

[6 marks]

(g) volume $= \pi \int_0^{0.8} (-0.295081...x^2 + 0.236065...x + 0.2)^2 dx$ M1

Note: Award **M1** for the minimum of an integral with the correct limits and their function squared.

$$= 0.135 \text{ (0.135161...)} \quad \text{A1}$$

$$0.135 > 0.131 \quad \text{R1}$$

Note: Award **R1** independently of the previous marks for a correct comparison of their (clearly stated) volume with their answer to part (e)(iii).

Hence the volume is greater than any cylinder volume AG

[3 marks]

(h) Award **A1** for at least one reasonable answer, for example:

the barrel is full of wine when sold
the barrel/stick is constructed with zero thickness
the stick is straight and inflexible
the hole has no diameter
volume of wine is the only important factor in value
aesthetics are not important

Note: Do not accept statements that relate to the barrel having circular cross-section, for example, 'there are no deformities', 'it is perfectly smooth' as these assumptions have already been made with the chosen model.

[1 mark]

[Total 26 marks]

(-115)

9. (a) $p = 1.5; q = 2$

A1A1

[2 marks]

(b) attempt at using chain rule

(M1)

(i) $h'(t) = -\frac{\pi}{4} \sin\left(\frac{\pi}{6}t\right) \left(= -0.785 \sin\left(\frac{\pi}{6}t\right) \right)$

A1

(ii) $h''(t) = -\frac{\pi^2}{24} \cos\left(\frac{\pi}{6}t\right) \left(= -0.411233... \cos\left(\frac{\pi}{6}t\right) \right)$

A1

[3 marks]

(c) (i) attempt to locate points of inflexion or max value of $h'(t)$

(M1)

$h''(t) = -\frac{\pi^2}{24} \cos\left(\frac{\pi}{6}t\right) = 0$ **OR** sketch on graph **OR** $t = 3$ **OR** $\frac{\pi}{6}k = \frac{3\pi}{2}$
 $(k =) 9$

A1

(ii) $(h(9) =) 2 \text{ (m)}$

A1

[3 marks]

[Total 8 marks]

11. (a) (i) **EITHER**

$$(\text{area of } R =) \int_{-1}^1 |x^3 - x| \, dx \quad \mathbf{A1}$$

OR

$$(\text{area of } R =) 2 \times \int_{-1}^0 x^3 - x \, dx \quad \mathbf{OR} \quad (\text{area of } R =) -2 \times \int_0^1 x^3 - x \, dx \quad \mathbf{A1}$$

OR

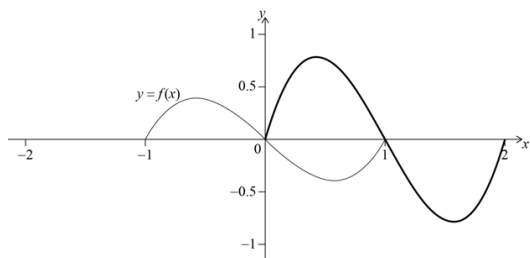
$$(\text{area of } R =) \int_{-1}^0 x^3 - x \, dx - \int_0^1 x^3 - x \, dx \quad \mathbf{A1}$$

(ii) (area of $R =$) 0.5 **A1**

Note: Follow through from part (a)(i) only if answer is greater than zero.

[2 marks]

(b)

**A1A1**

Note: Award **A1** for sketch with correct shape on $[0, 2]$, **A1** for vertical stretch x2. Condone max/min of g extending to 1 / -1.

[2 marks](c) attempt to use $\pi \int y^2 \, dx$ **(M1)**

$$\text{volume} = \pi \int_{-1}^1 (x^3 - x)^2 \, dx \quad \mathbf{(A1)}$$

$$\text{volume} = 0.479 \text{ (cubic units)} \left(= 0.478718\dots, \frac{16\pi}{105} \right) \quad \mathbf{A1}$$

[3 marks]**[Total 7 marks]**

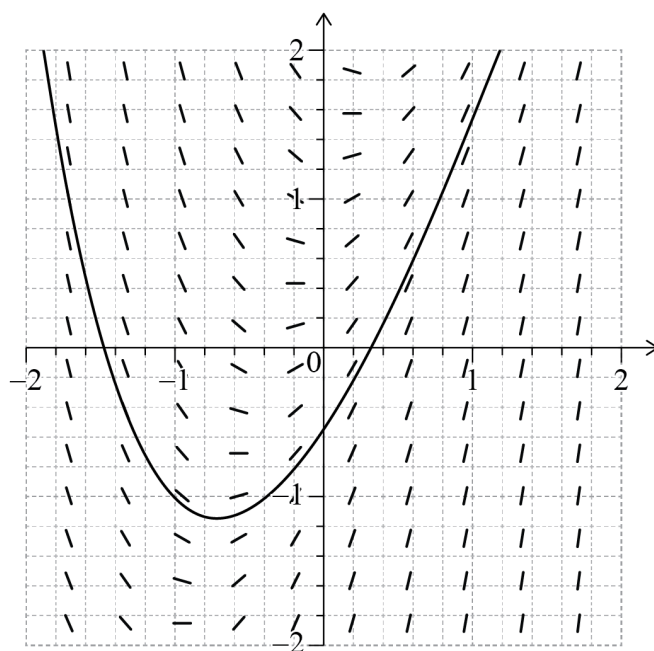
12. (a) gradient $(= -3 + 1 + 1) = -1$
 $y + 1 = -1(x + 1)$
 $x + y + 2 = 0$

A1

A1

[2 marks]

(b)



A1A1

Note: Award **A1** for (approximately) intersecting $(-1, -1)$ and with correct gradient, **A1** for generally plausible shape (e.g. not crossing over LOTS of isoclines).

[2 marks]

[Total 4 marks]

13. (a) attempt to solve $u_1(t) = u_2(t)$ **OR** sketch of two graphs
($T =$) 2

(M1)**A1**

Note: Award **(M1)A0** if additional values of T are seen **OR** if $T = -2$ is their final answer.

[2 marks]

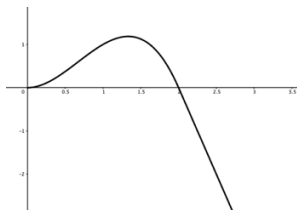
(b) $u_1'(t) = 4t - 3t^2$

A1

$$u_1'(2) = -4 = u_2'(2)$$

R1**[2 marks]**

(c)



recognition of integrating **AND** equating to zero

(M1)

$$\int_0^2 2t^2 - t^3 \, dt + \int_2^k 8 - 4t \, dt = 0$$

(A1)

$$\frac{4}{3} + (8k - 2k^2) - 8 = 0 \quad \text{OR} \quad 1.18350... \left(\frac{6 - \sqrt{6}}{3} \right) \text{ seen}$$

(A1)

Note: Award **(M1)(A1)A0** if integration done correctly but limits are not substituted.

$$(k =) \quad 2.82 \quad \left(= 2.81649..., \frac{6 + \sqrt{6}}{3} \right)$$

A1**[4 marks]****[Total 8 marks]**

4. (a) (i) $h(0) = 0.00623 \text{ (km)} (= 0.00622517)$ **A1**

(ii) this is the height of the nose of the plane (above the runway),
when the plane is on the runway **A1**
[2 marks]

(b) (i) $y = 9.94$ **A1**

Note: Accept $h = 9.94$.

(ii) **EITHER**
this is the height that the (nose of the) plane approaches (but does not reach) **A1**

OR
this is the maximum possible height of the (nose of the) plane **A1**

OR
the (nose of the) plane does not exceed this height **A1**
[2 marks]

(c) **METHOD 1 (chain rule)**

$$h(x) = 10(1 + 150e^{-0.07x})^{-1} - 0.06 \quad (M1)$$

$$\text{find } h'(x) = -10(1 + 150e^{-0.07x})^{-2} \times 150e^{-0.07x} \times -0.07 \quad A1M1A1$$

$$\left(= \frac{105e^{-0.07x}}{(1 + 150e^{-0.07x})^2} \right)$$

Note: Award **A1** for correct first term $(-10(1 + 150e^{-0.07x})^{-2})$, **M1** for attempt to use the chain rule, **A1** for correct use of chain rule $(\times 150e^{-0.07x} \times -0.07)$. Award at most **A1M1A0** if additional terms are seen. The answer is not required to be simplified beyond what is shown in the markscheme.

METHOD 2 (quotient rule)

$$\frac{(1 + 150e^{-0.007x})(0) - 10(150e^{-0.007x} \times -0.007)}{(1 + 150e^{-0.007x})^2} \quad M1A1$$

Note: Award **M1** for attempt to use quotient rule, **A1** for correct use.

$$= \frac{-10(150e^{-0.007x} \times -0.007)}{(1 + 150e^{-0.007x})^2} \quad \left(= \frac{105e^{-0.07x}}{(1 + 150e^{-0.07x})^2} \right) \quad A1A1$$

Note: Award **A1** for correct numerator and **A1** for correct denominator.

(d) evidence of a graph of $h'(x)$

$$\text{maximum at } x = 71.6 \quad (= 71.58051\dots)$$

$$h'(71.58051\dots) = 0.175$$

maximum gradient is less than 0.2

and hence the regulation is being followed

[4 marks]

(M1)

(A1)

A1

A1

[4 marks]

[Total 12 marks]

1. (a) (i) $\frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds}$ A1
 $(v = \frac{ds}{dt})$
 $\frac{dv}{dt} = v \frac{dv}{ds}$ AG
[1 mark]
- (ii) $v \frac{dv}{ds} = g$
 attempt to separate variables M1
 $\int v \, dv = \int g \, ds$
 $\frac{v^2}{2} = gs (+c)$ A1
 using initial conditions (can be done at any point) M1
 $50 = c$
 so $v = \sqrt{2gs + 100}$ A1

Note: Marks are intentionally unimplied to ensure on-syllabus techniques are used.

[4 marks]

- (iii) **EITHER**
 attempt to use their part (a)(ii) to find a value of s when $v = 330$ (M1)
 $330 = \sqrt{2gs + 100}$
 therefore $s = 5551.02...$ A1
 $(5551.02 < 40000)$
 so (the model does predict) he will reach the speed of sound A1
- OR**
 attempt to use their part (a)(ii) to find a value of v when $s = 40000$ (M1)
 $v = \sqrt{2g(40000) + 100}$
 $= 885 \quad (885.49...)$ A1
 $(885 > 330)$
 so (the model does predict) he will reach the speed of sound (before $s = 40000$) A1

Note: For the **OR** method, accept any large s that leads to $v = 330$.
 FT from $\sqrt{2gs}$ gives 885 (885.437...) for v and 5560 (5556.12...) for s
 FT from their v or their s for the final A1, provided M1 is awarded

[3 marks]

- (b) (i) $v = gt (+c)$ OR gradient is a constant (M1)
so the graph should be a straight line A1
[2 marks]
- (ii) the graph is not a straight line / only (approx.) straight for small t , so the model does not appear to be valid R1

Note: Award **R1** for recognising that the graph is non-linear **AND** stating that the model does not appear to be valid

[1 mark]

- (c) (i) $v \frac{dv}{ds} = g - kv^2$ (M1)
separating variables
 $\int \frac{v}{g - kv^2} dv = \int ds$
 $-\frac{1}{2k} \ln(g - kv^2) = s (+c)$ OR $-\frac{1}{2k} \ln|g - kv^2| = s (+c)$ (A1)
rearranging to make v the subject (M1)

Note: Award (M1) for making v the subject of their equation and not just an attempt, or an erroneous equation with v also on the RHS.

$$g - kv^2 = Ae^{-2ks}$$

$$v = \sqrt{\frac{g - Ae^{-2ks}}{k}}$$

applying initial conditions (here or elsewhere) (M1)

$$100 = \frac{g - A}{k}$$

$$A = g - 100k$$

so

$$v = \sqrt{\frac{g - (g - 100k)e^{-2ks}}{k}}$$

A1
[5 marks]

- (ii) $9.672 = 9.8 - 1600k$ A1A1

Note: Award **A1** for correct left-hand side and **A1** for correct right-hand side.

$$k = \frac{9.8 - 9.672}{1600}$$

$$k = 8 \times 10^{-5} \quad \text{AG}$$

Note: Award **A1A0** for $k = 8 \times 10^{-5}$ substituted into the right-hand side of the expression, leading to 9.672.

[2 marks]

(iii) $s \rightarrow \infty, e^{-2ks} \rightarrow 0$ **OR** $\frac{dv}{dt} = 0$ **OR** graph/table (M1)

$(v_{\max} = \sqrt{\frac{g}{k}} =) 350 \text{ (ms}^{-1}\text{)}$ A1

[2 marks]

(iv) upper limit occurs when $s = 40000$ (M1)

Note: The **M1** can be implied by 40000 substituted into their part (c)(i).

349.7 (ms⁻¹) A1

Note: Answer must be to 4 sf.

[2 marks]

(d) $s_{n+1} = s_n + 4000$ (A1)

$v_{n+1} = v_n + 4000 \times \left(\frac{3.98 \times 10^{14}}{v_n (6.41 \times 10^6 - s_n)^2} - (8 \times 10^{-5})v_n \right)$ (M1)(A1)

Note: Award **(M1)** for attempt to use Euler method formula **AND** dividing through by v .

if $v_0 = 10$, then $v_{10} = 361$ (360.658 ...) A1
[4 marks]

- (e) (i) Use a smaller step length R1
OR
 Use a better method such as Runge-Kutta R1
OR
 (Try to) solve the equation exactly R1

[1 mark]

- (ii) Any reasonable response: R1
For example:
 Ignoring parachute / end point of motion / only valid for certain domain.
 Treating Felix as a point object.
 Ignoring weather / wind / air currents.
 Assuming path is directly downwards.
 Assuming perfect measurement of initial speed.

[1 mark]

[Total: 28 marks]

4. attempt to use Euler

$$y_{n+1} = y_n + 0.1 \log(x_n + y_n) \quad (\text{A1})$$

$$y_1 (= 1 + 0.1 \times \log_{10}(1)) = 1 \quad (\text{A1})$$

$$y_2 = 1.004139... \quad (\text{A1})$$

THEN

$$\text{when } x = 2 \quad y(2) \approx 1.61 \text{ (1.60536...)} \quad \text{A1}$$

[Total: 4 marks]

9. (a) attempt to integrate by substitution or inspection

$$4\ln|2x+3|+c \quad \text{OR} \quad 4\ln|x+1.5|+c$$

(M1)**A1A1**

Note: Award **M1** for $\ln(2x+3)$ or $\ln(x+1.5)$, **A1** for the 4 and **A1** for c . The **A** marks can only be awarded if the **M** mark is awarded. Condone absence of modulus signs.

[3 marks]

- (b) recognizing that area is $[4\ln(2x+3)]_0^6$

(M1)

$$= 4\ln(15) - 4\ln(3)$$

(A1)

use of log laws for their expression

(M1)

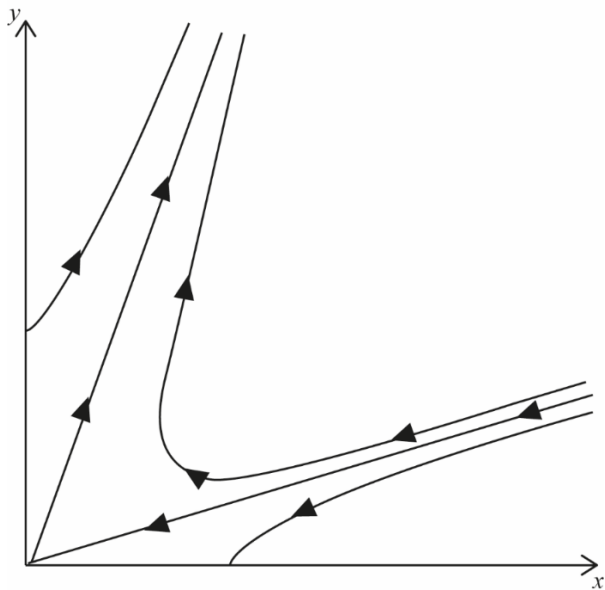
$$= 4\ln(5) \quad (= 2\ln(25) = 1\ln(625))$$

A1

Note: Award **(M1)A0M0A0** for an unsupported final answer of 6.43775...
Award at most **(M1)A1FTM0A0** if their answer from part (a) does not include $\ln$.

[4 marks]**[Total 7 marks]**

15. (a) (i)&(ii)



A1A1A1

Note: Award **A1** for correct directions on eigenvectors, **A1** for correct trajectories, **A1** for correct arrows on trajectories.

[3 marks]

(b) for Y not to die out $y > \frac{1}{3}x$

(R1)

as $x = 252$, $y > 84$

(M1)

(minimum number of new animals is) 25

A1

Note: Award **(R1)(M1)A0** for an unsupported 24.

[3 marks]

[Total 6 marks]

16. attempt to find gradient (M1)

EITHER

gradient of tangent = $-\tan 75^\circ$ ($= -3.73205\dots, -2 - \sqrt{3}$) (A1)(A1)

Note: Award **A1** for negative and **A1** for $\tan 75^\circ$ (or equivalent).

OR

gradient of tangent = $\tan 105^\circ$ ($= -3.73205\dots$) (A2)

THEN

$\frac{dy}{dx} = -5.5 \sin(1.1x)$ (A1)

Note: Award **(A1)** for a labelled sketch of the derivative function.

equating derivative to their gradient (M1)

$-5.5 \sin(1.1x) = -3.73205\dots$ **OR** line on graph

$x = 0.677993\dots$ (A1)

Note: Award **(A1)(M1)A0** for an answer of $x = 38.8$, from calculator being in degrees.

Award **A0M1A0** if " $\frac{d}{dx}(5 \cos(1.1x)) = -3.73205\dots$ " seen, but leading to an incorrect x -value.

height = $5 \cos(1.1 \times 0.677993\dots)$ (M1)

$= 3.67$ (m) (3.67274...) A1

[Total 8 marks]

.....

7. attempt at Euler (M1)

$y_{n+1} = (y_n x_n - 1)0.1 + y_n$ (A1)

x	1	1.1	1.2	1.3	1.4	1.5
y	2	2.1	2.231	2.39872	2.61055...	2.87603...

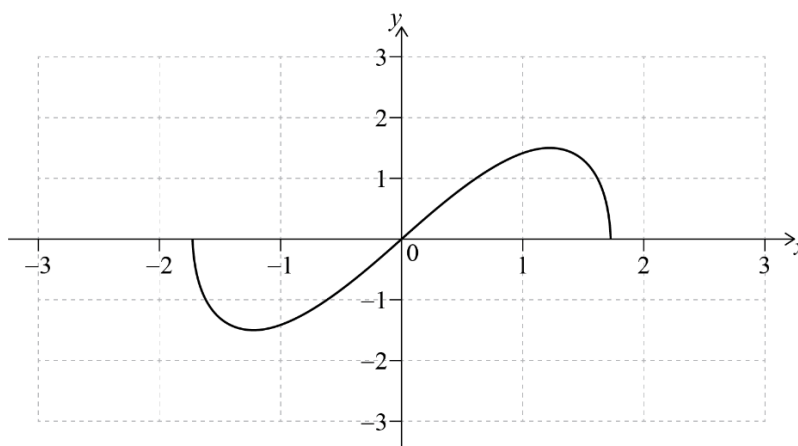
(A1)

Note: Any correct value either as a separate calculation or as part of a table, (including the result).

$y = 2.88$ (2.87603...) A1

[Total: 4 marks]

10. (a)

**A1A1**

Note: Award **A1** smooth curve, through origin, with maximum in 1st quadrant and minimum in 3rd quadrant (approximately correct position).
Award **A1** correct (approximate) position of endpoints and rotational symmetry.
Condone a (small) gap between end of curve and the x -axis.

[2 marks]

(b) (i) $2\pi \int_0^{\sqrt{3}} \left(x\sqrt{3-x^2}\right)^2 dx$ **OR** $\pi \int_{-\sqrt{3}}^{\sqrt{3}} \left(x\sqrt{3-x^2}\right)^2 dx$ (or equivalent) **A1A1**

Note: Award **A1** for integral with correct limits in the correct places, **A1** for completely correct expression;
 dx must be included. Condone a correct answer presented as two integrals, despite the demand of the question: $\pi \left(\int_{-\sqrt{3}}^0 \left(x\sqrt{3-x^2}\right)^2 dx + \int_0^{\sqrt{3}} \left(x\sqrt{3-x^2}\right)^2 dx \right)$.

(ii) $13.1 \left(13.0593..., \frac{12\pi\sqrt{3}}{5} \right)$ **A2**

[4 marks]

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(c) attempt to express $g(x)$ in terms of $f(x)$ **(M1)**

$$g(x) = \frac{1}{2} \left(\frac{x}{2} \sqrt{3 - \left(\frac{x}{2} \right)^2} \right), \quad g(x) = \frac{1}{2} f \left(\frac{x}{2} \right)$$

correct limits for g seen: $-2\sqrt{3}$ and $2\sqrt{3}$ **(A1)**

$$\pi \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(\frac{1}{2} \left(\frac{x}{2} \sqrt{3 - \left(\frac{x}{2} \right)^2} \right) \right)^2 dx \quad \text{OR} \quad \pi \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(\frac{1}{2} f \left(\frac{x}{2} \right) \right)^2 dx \quad \text{OR} \quad 2 \times \frac{1}{2} \times \left(\frac{12\pi\sqrt{3}}{5} \right)$$

$$6.53 \left(6.52967\dots, \frac{6\pi\sqrt{3}}{5} \right)$$

A1**Note:** Some candidates may answer question by transforming their part (b)(ii):

$$2 \times \frac{1}{2} \times \left(\frac{12\pi\sqrt{3}}{5} \right). \text{ This is a valid method and leads to the correct answer.}$$

[3 marks]**[Total: 9 marks]**

11. (a) $y = kx^2 - x$

$$\frac{dy}{dx} = 2kx - 1 \quad (\text{A1})$$

$$m_{\text{tan}} = 1 \quad \text{A1}$$

finding negative reciprocal of their gradient (M1)

$$m_{\text{normal}} = -1$$

$$\text{normal: } y = -\left(x - \frac{1}{k}\right) \quad \text{A1}$$

equating normal line to quadratic curve M1

$$kx^2 - x = -\left(x - \frac{1}{k}\right)$$

correct simplification leading to given result A1

e.g. quadratic formula or difference of two squares

$$kx^2 - \frac{1}{k} = 0$$

$$k^2x^2 - 1 = 0$$

$$(kx - 1)(kx + 1) = 0$$

$$x = -\frac{1}{k}$$

AG

[6 marks]

$$(b) \quad \text{area} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x - \frac{1}{2}\right) - (2x^2 - x) \, dx \quad (\text{M1})(\text{A1})$$

Note: Award **M1** for evidence of subtracting the functions and **A1** for a completely correct integral. (Condone a missing dx.)

$$= 0.333 \left(0.333333..., \frac{1}{3}\right) \quad \text{A1}$$

[3 marks]

[Total: 9 marks]

13. (a) $3 \sin(a)(1 + \cos(a)) = 0$ (M1)
 $a = 3.14$ (π , 3.14159...) A1

[2 marks]

(b) attempt to find the maximum velocity (M1)

e.g. $\frac{dv}{dt} = 0$ OR sketch of v with maximum marked

$v_{\max} = 3.90(\text{ms}^{-1})$ (3.89711...) A1

Note: Accept 3.9 rounded to 2 sf.

[2 marks]

(c) attempt to find integral of speed (condone omission of modulus signs) (M1)

$\int_0^{\pi} |3 \sin(t)(1 + \cos(t))| dt$ OR 6

attempt to substitute into $\text{speed} = \frac{\text{distance}}{\text{time}}$ (M1)

$= \frac{\int_0^{\pi} |3 \sin(t)(1 + \cos(t))| dt}{\pi}$ (A1)

$= 1.91 (\text{ms}^{-1})$ (1.90985..., $\frac{6}{\pi}$) A1

[4 marks]

[Total: 8 marks]

2. (a) heights, 0, 4, 1.75, 3 and 3.75 seen (A2)

Note: Award **A1A0** if **two** of 1.75, 3 or 3.75 are seen.

attempt to use trapezoidal rule formula for their heights (M1)

$$\frac{1}{2} \times 1 \times \{0 + 4 + 2(1.75 + 3 + 3.75)\} \quad (A1)$$

Note: Award **(M1)(A1)** for correctly expressing this as 3 trapezoids and a triangle. The "×1" need not be seen.

$$= 10.5 \text{ (m}^2\text{)} \quad A1$$

[5 marks]

(b) $-\frac{1}{12}x^3 + x^2 + c$ A1A1A1

[3 marks]

(c) $\int_0^4 \left(-\frac{1}{4}x^2 + 2x\right) dx + 1 \times 4 + \frac{1}{2} \times 7 \times 4$ (A1)(M1)(A1)

Note: Award **A1** for correct area of rectangle **OR** triangle, **M1** for substituting correct limits into given integral (may be seen in part (b)), and **A1** for entire expression correct.

$$= 10.6666\dots + 4 + 14$$

$$= 28\frac{2}{3} \text{ (m}^2\text{)} \left(\frac{86}{3}\right) \quad A1$$

Note: The answer must be **exact** for the **A1** to be awarded. For an answer of 28.7 or 28.66 award **(A1)(M1)(A1)A0**.

[4 marks]

(d) (Total area using part (a) =) 28.5 (A1)

$$\text{Percentage error} = \left| \frac{28.5 - 28.6666\dots}{28.6666\dots} \right| \times 100 \quad (M1)$$

Note: if their trapezoid value is incorrect but is used correctly in the percentage error formula, award at most **A0M1A0**. If it is clear from the answer that ×100 has been used, then condone the omission and award the **M** mark.

$$= 0.581 \text{ (\%)} \text{ (0.581395\dots)} \quad A1$$

(accept 0.697 from use of 28.7)

[3 marks]

[Total: 15 marks]

2. (a) attempt to substitute $x = 5$, $y = 5$ into either of the given equations (M1)

Note: Award **M1** for substituting $x = 500$, $y = 500$.

(i) $\frac{dx}{dt} = 2(5) - \frac{(5)(5)}{2}$
 -2.5 (hundred geese per year) / -250 (geese per year) A1
[2 marks]

(ii) $\frac{dy}{dt} = -3(5) + (5)(5)$
 10 (hundred foxes per year) / 1000 (foxes per year) A1
[1 mark]

- (b) (i) **EITHER**

both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ at A R1

OR

if $x = 0$ and $y = 0$ then there are no foxes or geese, so their populations cannot change R1

Note: Award **R0** for just stating "there are no foxes or geese".

[1 mark]

- (ii) attempt to put $\frac{dx}{dt}$ (or $\frac{dy}{dt}$) equal to zero and solve (M1)

$$\frac{dx}{dt} = 0 \Rightarrow x \left(2 - \frac{y}{2} \right) = 0, \quad \frac{dy}{dt} = 0 \Rightarrow y(-3 + x) = 0$$

$p = 3$ (accept $x = 3$) A1

$q = 4$ (accept $y = 4$) A1

[3 marks]

- (c) (i) attempt to solve by separation of variables (M1)

$$\int \frac{dx}{x} = \int 2dt$$

$$\ln x = 2t + c$$

(A1)(A1)

Note: Award **(A1)** for each side correct.

Award **(A1)(A0)** if both sides are integrated correctly but $+c$ is not seen.

Also accept $0.5 \ln x = t + c$ **OR** $0.5 \ln 2x = t + c$.

$x = Ae^{2t}$ (accept $x = e^{2t+c_A}$) A1

[4 marks]

(ii) $y = Be^{-3t}$ (accept $y = e^{-3t+c_B}$) A1

Note: Accept alternative (distinct) labels for their A and B .

[1 mark]

(d) (i) **METHOD 1**attempt to find x^3 and y^2 in terms of t

(e.g. any attempt to substitute their solutions to part (c))

M1

$$x^3 = A^3 e^{6t}, \quad y^2 = B^2 e^{-6t}$$

$$x^3 y^2 = A^3 e^{6t} \times B^2 e^{-6t}$$

A1

$$= A^3 B^2$$

A1

$$= k$$

AG

Note: There will probably be lots of students who assume what is to be proved by introducing a k earlier on. Do not penalize this.

METHOD 2attempting to isolate t **M1**

$$e^{2t} = \frac{x}{A} \quad \text{and} \quad e^{3t} = \frac{B}{y}$$

$$e^{6t} = \frac{x^3}{A^3} \quad \text{OR} \quad e^{6t} = \frac{B^2}{y^2}$$

A1

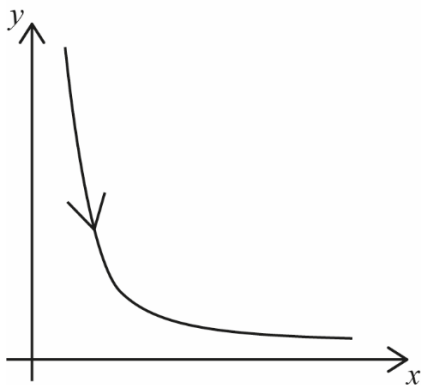
$$\frac{x^3}{A^3} = \frac{B^2}{y^2}$$

A1

$$x^3 y^2 = k$$

AG**[3 marks]**

(ii) $y = \sqrt{\frac{k}{x^3}}$

(A1)**A1A1**

Note: Award **A1** for correct shape (i.e. decreasing, convex and not touching axes), **A1** for correct direction indicated. Ignore any trajectories not in the first quadrant. The first **A1** can only be implied by a correct graph.

[3 marks]

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- (e) attempt to substitute $x = 3 + X$, $y = 4 + Y$ **OR** $x = p + X, y = q + Y$ into $\frac{dx}{dt} = 2x - \frac{xy}{2}$ **M1**

$$\begin{aligned}\dot{X} &= 2(3 + X) - \frac{1}{2}(3 + X)(4 + Y) \\ &= 6 + 2X - \frac{1}{2}(12 + 4X + 3Y + XY) \\ &= 6 + 2X - 6 - 2X - \frac{3}{2}Y - \frac{XY}{2} \\ &= -\frac{3Y}{2} - \frac{XY}{2}\end{aligned}$$

The term in XY can be ignored.

$$= -\frac{3Y}{2} + \dots \approx -\frac{3Y}{2}$$

A1**R1****AG**

Note: Follow through can be awarded for the **M1** and **R1** but not the **A1**.

[3 marks]

(f) $M = \begin{pmatrix} 0 & -\frac{3}{2} \\ 4 & 0 \end{pmatrix}$

A1**[1 mark]**

(g) setting $\begin{vmatrix} -\lambda & -\frac{3}{2} \\ 4 & -\lambda \end{vmatrix} = 0$

(M1)

Note: Setting their determinant to zero can be implied by any correct root seen.

$$\lambda^2 + 6 = 0$$

(A1)

$$\lambda = \pm i\sqrt{6} \quad \text{OR} \quad \pm 2.45i \text{ (} \pm 2.44948\dots i \text{)} \quad \text{OR} \quad \text{noting 2 purely imaginary eigenvalues}$$

A1

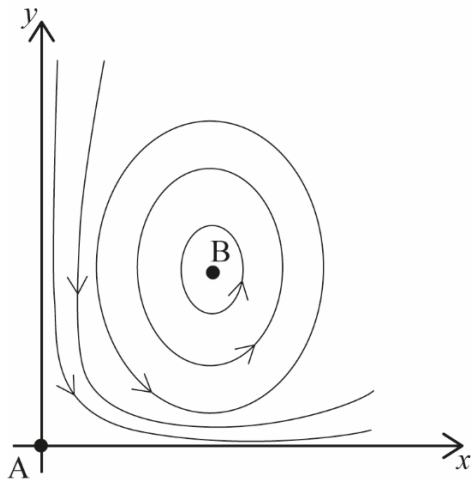
$\Rightarrow$ ellipse (or circle)

R1

Note: Accept “closed loops” or “oval” as descriptions of the trajectories.
Allow follow through for the final **R1** from their eigenvalues as long as they are either both real or complex conjugates. If there are two real eigenvalues the description should be a “node” and if there are two complex eigenvalues the description should be a “focus” or a “spiral”.

[4 marks]

(h)



A1A1A1

Note: Award **A1**: At least one ellipse, oval or closed loop seen. Allow parts of ellipses shown (as in the diagram) as long as at least one complete loop observed.
Award **A1**: Arrows showing correct direction seen on at least one loop. Must be anti-clockwise on a closed loop.
Award **A1**: Ellipses remaining entirely in the 1st quadrant **AND** centre B marked in some way (either directly or via coordinates consistent with their B).

Do not follow through into part 2(h) from part 2(g).

[3 marks]

- (i) The park-keeper is incorrect
The phase portrait shows that the trajectory (starting at (5, 5)) is a closed loop (around the equilibrium point) (despite having a negative initial $\frac{dx}{dt}$ in comparison with large positive $\frac{dy}{dt}$)

A1

R1

Note: Do not award **A1R0**.

[2 marks]

Total [31 marks]

14. (a) attempt to solve $f(x) = 0.25$ (M1)
 0.302 ($0.301928\dots$) A1
[2 marks]
- (b) evidence of using chain rule (multiple of $-4x$ seen with $e^{\langle \text{something} \rangle}$) (M1)
 $f'(x) = -1.2x e^{-2x^2}$ A1
[2 marks]
- (c) $f'(0.302) = -0.302$ ($-0.301928\dots$) (A1)
 finding the negative reciprocal of their gradient (M1)
 gradient of normal $= \frac{-1}{-0.302}$ ($= 3.31$) ($3.312104\dots$)
- recognition that gradient equals $\tan \theta$ **OR** finding the x -intercept of the flagpole and using right-angled triangle to find $\tan \theta$. (M1)
- $\theta = \arctan(3.31)$
- $= 1.28$ ($1.27757\dots$) radians ($= 73.2^\circ$) ($73.1994\dots^\circ$) A1
[4 marks]
[Total 8 marks]

16. (a) 0.781 (0.781124...) (accept $4 - 2\ln(5)$)

A1
[1 mark]

(b) recognizing need to find $\pi \int x^2 \, dy$

(M1)

$$\pi \int_{0.781}^{11.8} x^2 \, dy$$

(A1)

Note: Award **A1** for correct limits and squaring **their** x , seen anywhere.

attempt to make x the subject.

(M1)

$$2 \ln(5 - x) = 4 - y$$

attempt to remove $\ln$ by using exponent base e .

(M1)

$$5 - x = e^{\frac{4-y}{2}}$$

$$x = 5 - e^{\frac{4-y}{2}}$$

(A1)

$$(V =) 631 \text{ (cm}^3\text{)} \text{ (631.071...)}$$

A1

[6 marks]
[Total 7 marks]

17. (a) substituting h and f_1 into formula for x_{n+1} **OR** h and f_2 into y_{n+1} (M1)

Note: **M1** may be awarded for an appropriate table with at least one correct value for F or L , even if there is no evidence of substitution into the Euler equations.

$F_{n+1} = F_n + 4(0.000004F_nL_n - 0.2F_n)$ AND $L_{n+1} = L_n + 4(0.06L_n - 0.00003F_nL_n)$ (A1)

evidence of correct use of Euler's method (A1)

Note: Award **A1** for any correct value of F_{n+1} AND L_{n+1} for $t = 4, 8, \dots$ seen (including this intermediate value presented as a final answer).

t	F	L
0	5000	80000
4	7400	51200
8	7542	18022
12	3683	6037
16	1092	4817
20	303	5342

(i) maximum value of $F = 7542$ (at $t = 8$) A1

(ii) minimum value of $L = 4817$ (at $t = 16$) A1
[5 marks]

(b) (i) they die out (population becomes zero) A1

(ii) Any valid reason R1
E.g. Euler's method is only an approximation
Other factors might change the model over time

[2 marks]
[Total 7 marks]

2. (a) $\frac{1}{2} \times 1 \times ((4 + 0.7) + 2(7.3 + 6.7 + 4 + 1.3))$ (A1)(A1)

$= 21.7 \text{ (21.65) (cm}^2\text{)}$ A1

[3 marks]

(b) $22.1 \text{ (22.08333..., } \frac{265}{12}\text{) (cm}^2\text{)}$ A2

[2 marks]

(c) $600 = \pi r^2 h$ A1

[1 mark]

(d) **METHOD 1** (Substitution of $h = \frac{600}{\pi r^2}$ in $A = 2\pi r^2 + 2\pi r h$)

$A = 2\pi r^2 + 2\pi r h$ (A1)

attempt to isolate h or $\pi r h$ (M1)

$h = \frac{600}{\pi r^2}$ OR $\pi r h = \frac{600}{r}$

correct substitution of their h into correct expression (M1)

$A = 2\pi r^2 + 2\pi r \left(\frac{600}{\pi r^2} \right) \quad \left(A = 2\pi r^2 + \frac{1200}{r} \right)$

$k = 1200$ A1

METHOD 2 (Equating $2\pi r h$ and $\frac{k}{r}$)

$A = 2\pi r^2 + 2\pi r h$ OR $2\pi r h = \frac{k}{r}$ (A1)

attempt to isolate h or $(\pi) r h$ or r or r^2 (M1)

$h = \frac{600}{\pi r^2}$ OR $\pi r h = \frac{600}{r}$ OR $r = \sqrt{\frac{600}{\pi h}}$ OR $r^2 = \frac{600}{\pi h}$

correct substitution of their h or $\pi r h$ or r or r^2 into $2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{k}{r}$ (M1)

$2\pi r \left(\frac{600}{\pi r^2} \right) = \frac{k}{r}$ OR $2 \left(\frac{600}{r} \right) = \frac{k}{r}$ OR $2(600) = k$ OR $2\pi \left(\frac{600}{\pi h} \right) h = k$

$k = 1200$ A1

[4 marks]

continue...

Question 2 continued.

(e) (i) $\frac{dA}{dr} = 4\pi r - 1200r^{-2}$

A1(M1)A1

Note: Award **A1** for $4\pi r$ seen, and **(M1)** for expressing $\frac{1200}{r}$ as $1200r^{-1}$ (can be implied through $\mp \frac{1200}{r^2}$ seen), **A1** for $-1200r^{-2}$. Award at most **A1(M1)A0** if any additional terms are seen.

(ii) $0 = 4\pi r - 1200r^{-2}$ **OR** $\frac{dA}{dr} = 0$

(M1)

$$r = 4.57 \left(4.570781\dots, \sqrt[3]{\frac{300}{\pi}} \right) \text{ (cm)}$$

A1

Note: Award at most **M1A0** if the final answer is in terms of k .

[5 marks]

(f) ($h =$) 9.14 (9.14156.....) (cm)

A1

($C =$) 2π (4.570781.....) ≈ 28.7 (28.7190...)(cm)

A1**EITHER**

the longest dimension of the label (9 cm) is less than both values and hence the label will fit (in any rotation)

R1**OR**

$9 < 9.14$ and $5 < 28.7$

R1

Note: Do not accept an argument based on the comparison of areas.

[3 marks]**[Total 18 marks]**

7. (a) $\frac{dy}{dt} = \frac{d^2x}{dt^2}$

A1

$$\frac{dy}{dt} + ay + bx = 0$$

A1

$$\frac{dy}{dt} = -bx - ay$$

AG**[2 marks]**

(b) $\begin{pmatrix} 0 & 1 \\ -77 & -18 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -7 \begin{pmatrix} x \\ y \end{pmatrix}$ **OR** $\begin{pmatrix} 0 & 1 \\ -77 & -18 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -11 \begin{pmatrix} x \\ y \end{pmatrix}$
 $y = -7x$

(M1)

eigenvector is $\begin{pmatrix} 1 \\ -7 \end{pmatrix}$ (or any multiple)

A1

$$y = -11x$$

eigenvector is $\begin{pmatrix} 1 \\ -11 \end{pmatrix}$ (or any multiple)

A1**[3 marks]**

(c) (i) $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-7t} \begin{pmatrix} 1 \\ -7 \end{pmatrix} + Be^{-11t} \begin{pmatrix} 1 \\ -11 \end{pmatrix}$

(A1)

substitution of initial values

(M1)

two correct equations (not in vector form)

(A1)

$$5 = A + B, 2 = -7A - 11B$$

$$A = \frac{57}{4}, B = -\frac{37}{4}$$

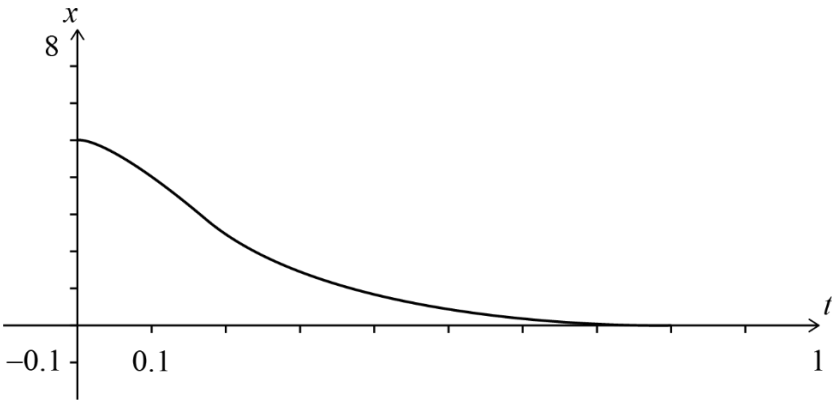
(A1)

$$x = \frac{57}{4}e^{-7t} - \frac{37}{4}e^{-11t} \quad (x = 14.25e^{-7t} - 9.25e^{-11t})$$

A1*continue...*

Question 7 continued.

(ii)



decreasing curve

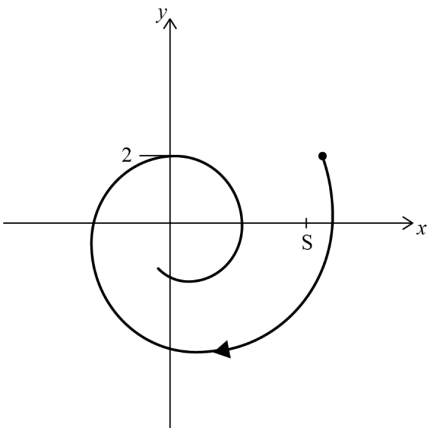
A1

Note: There is a maximum at $t = 0.00496$, but this does not need to be shown.

starting at $x=5$ and asymptote at $x = 0$

A1
[7 marks]

(d) (i)



spiral towards origin
starting at $(5, 2)$

A1
A1

$$\frac{dy}{dt} = -85(5) - 18(2) = -461 < 0$$

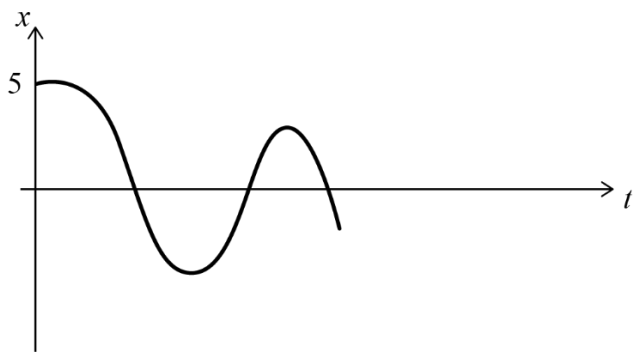
Clockwise spiral

A1

continue...

Question 7 continued.

(ii)



oscillations
starting at $x = 5$, decreasing amplitude

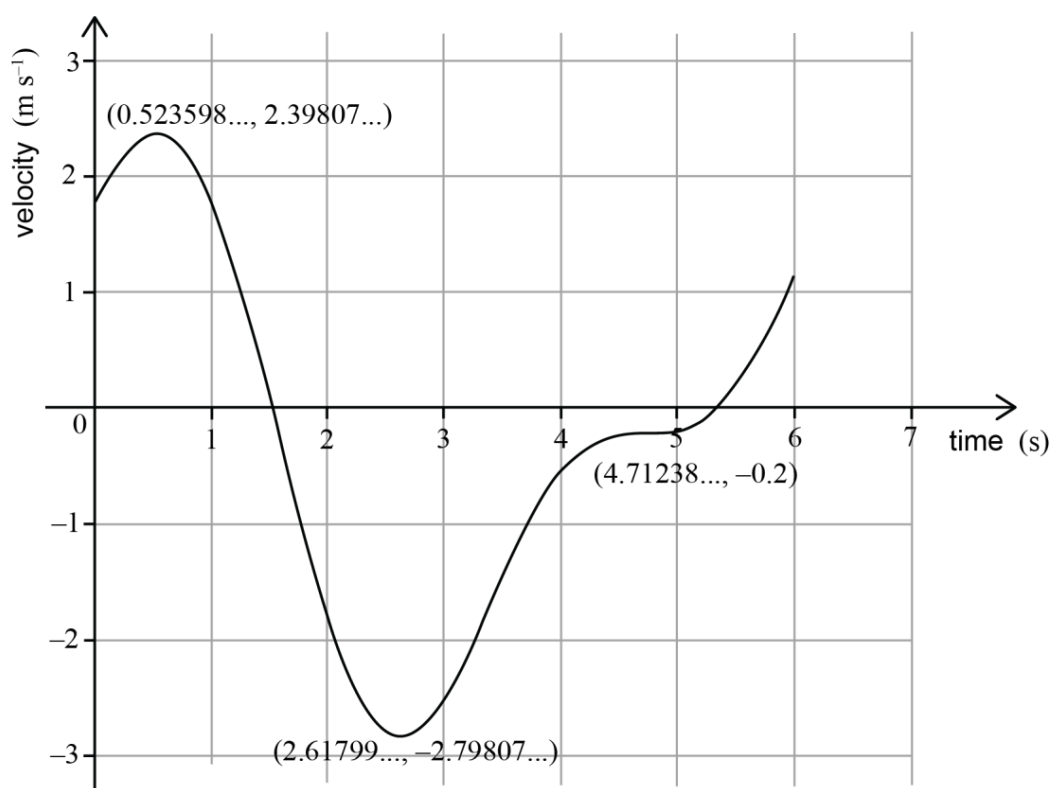
A1
A1
[5 marks]

(e) spring 1 is better as it gives a smoother ride (less oscillations, less bouncy)

R1
[1 mark]
[Total:18 marks]

7. (a) (i) correct general shape **A1**
 curve begins at $(0, 1.8)$ and does not extend beyond $t = 6$ **A1**

Note: Do not award the second **A1** if the first **A1** is not awarded.



(ii) $(0.524, 2.40)$ **A1**

$(2.62, -2.80)$ **A1**

$(4.71, -0.2)$ **A1**

Note: Award **A1** for each correct coordinate. Award marks even if the labels are not shown on the graph near the points.

[5 marks]

(b) (i) **EITHER**

$$\int_0^6 |2 \cos t + \sin 2t - 0.2| dt$$

A1A1

Note: Award **A1** for correct boundaries and use of dt and **A1** for completely correct expression.

OR

$$\left| \int_0^{1.52074..} (2 \cos t + \sin 2t - 0.2) dt \right| + \left| \int_{1.52074..}^{5.31535...} (2 \cos t + \sin 2t - 0.2) dt \right| + \left| \int_{5.31535...}^6 (2 \cos t + \sin 2t - 0.2) dt \right|$$

A1A1

Note: Award **A1** for 1.52074... and 5.31535... as boundaries and **A1** for completely correct expression.
Allow + - + instead of modulus signs

(ii) 7.76 (7.76487...) (m)

A1**[3 marks]****[Total 8 marks]**

8. (a) $f(x) = 0$ seen **OR** sketch of the graph with all three x -intercepts marked **OR** evidence of using sum or product of roots (M1)

(a =) 3 A1
[2 marks]

- (b) evidence of using an integral to find the area (M1)

$$\int f(x), \int_1^3, \text{ etc.}$$

(area =) 4 A1
[2 marks]

- (c) attempt to equate two integrals (M1)

$$\int_1^b f(x)dx = \int_1^b g(x)dx$$

evidence of using their GDC to solve the equation **OR** attempt to expand both sides of the equation into expressions in terms of b (M1)

(b =) 2.57 (2.56949...) A1
[3 marks]

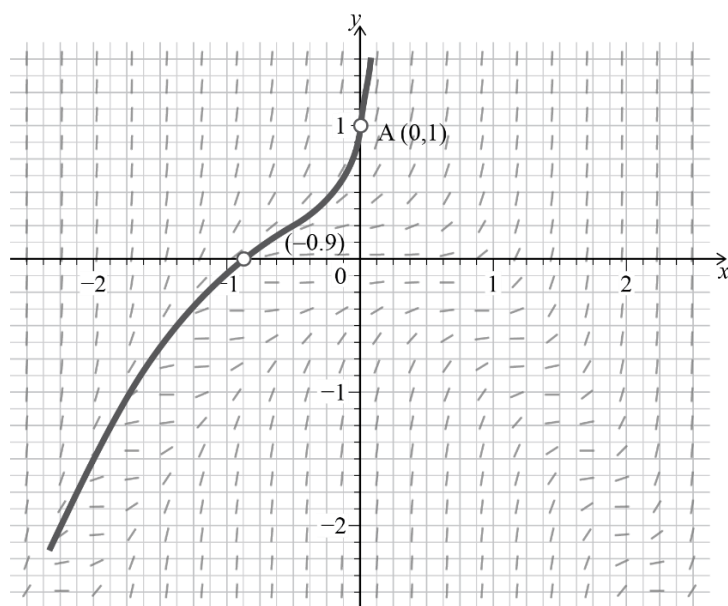
[Total 7 marks]

12. (a) correct shape that goes through point $(0, 1)$

A1

crosses the x -axis is at $x = -0.9$ (± 1 small grid box)

A1



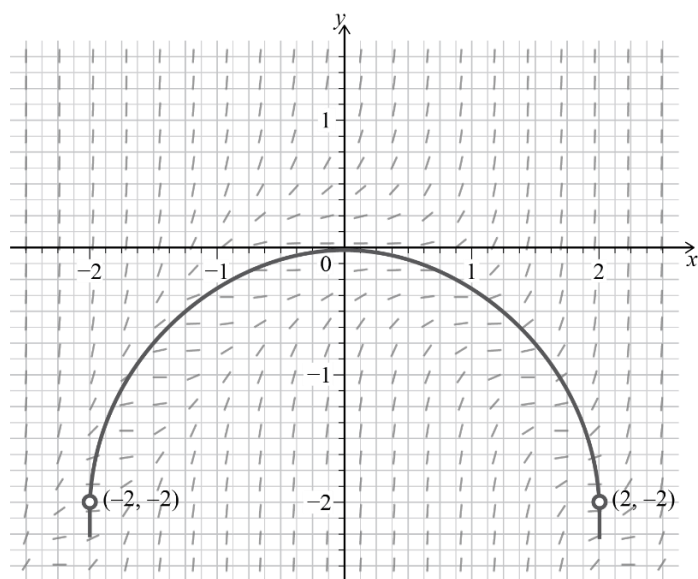
[2 marks]

- (b) correct shape with maximum at $(0, 0)$

A1

smooth curve that goes approximately through points $(-2, -2)$ and $(2, -2)$

A1



[2 marks]

– 20 –

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(c)

EITHERsubstituting $y = ax^2$ into differential equation**(M1)**

$$0 = x^4 + 4x^2(ax^2) + 4(ax^2)^2$$

choosing an appropriate method to solve for a **(M1)**

$$0 = x^4(1 + 4a + 4a^2)$$

$$a = -\frac{1}{2}$$

A1**OR**

$$x^4 + 4x^2y + 4y^2 = (x^2 + 2y)^2$$

(M1)

$$(x^2 + 2y)^2 = 0$$

(M1)

$$y = -\frac{1}{2}x^2 \text{ (so } a = -\frac{1}{2}\text{)}$$

A1**[3 marks]****[Total 7 marks]**

15. (a) (i) $x_{n+1} = x_n + 0.1 \left(\frac{dx}{dt} \right)_n$

$$\left(\frac{dx}{dt} \right)_{n+1} = \left(\frac{dx}{dt} \right)_n + 0.1 \left(\frac{d^2x}{dt^2} \right)_n \quad (M1)$$

$$\frac{d^2x}{dt^2} = 4.9 - 1.2 \frac{dx}{dt} - 4x \quad (A1)$$

$$\left(\frac{dx}{dt} \right)_{n+1} = \left(\frac{dx}{dt} \right)_n + 0.1 \left(4.9 - 1.2 \frac{dx}{dt} - 4x \right)_n \quad (A1)$$

(when $t = 1$ or $n = 10$) $x = 4.36$ (4.36332...) (m) A1

(ii) ($t =$) 2 (seconds) A1

Note: Award **A0** for either $n = 20$ or $t = 20$.

[5 marks]

(b) $x = 3.845$ seen **OR** $x = 3.855$ seen (A1)

substitution into percentage error formula using their value for part (a)(i) and 3.845
or 3.855 or 3.85 (M1)

$$\left| \frac{4.36332... - 3.845}{3.845} \right| \times 100$$

Note: Accept methods using centimetres instead of metres as long as both values use the same unit.

13.5 (13.4803...) (%) A1

[3 marks]

[Total 8 marks]

7. (a) a valid/coherent attempt to use the substitution method (M1)
- $$u = (-)x^2$$
- $$\frac{du}{dx} = (-)2x$$
- (A1)

Note: To award **M1**, both $u =$ and $\frac{du}{dx} =$ must be seen.

integral becomes

$$\frac{1}{2} \int e^{-(u)} du \text{ OR } -\frac{1}{2} \int e^u du$$
 (M1)
$$-\frac{1}{2} e^{-x^2} + c$$
 A1

Note: Award **M1A1M1A0** for a final answer of $-\frac{1}{2}e^{-x^2}$ with c omitted.

Accept integration by inspection, e.g. using $\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$

[4 marks]

- (b) recognition that 0 is the lower limit (A1)
- substituting both 0 and k into previous expression (M1)

$$\int_0^k x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^k$$

$$= \frac{1}{2} - \frac{1}{2} e^{-k^2}$$
 A1

[3 marks]

Total [7 marks]

9. (a) METHOD 1

$$A = xy \quad \text{A1}$$

attempt to use product rule M1

$$\frac{dA}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}$$

$$75 \times (-5) + 40 \times 10$$

$$= 25 \text{ (km}^2\text{/year)} \quad \text{A1}$$

METHOD 2

$$\frac{dx}{dt} = 10, \quad \frac{dy}{dt} = -5$$

$$A = (10t + 75)(40 - 5t) \quad \text{A1}$$

attempt to differentiate M1

$$\frac{dA}{dt} = 10(40 - 5t) - 5(10t + 75) \quad \text{OR} \quad \frac{dA}{dt} = -100t + 25$$

$$\left(\text{at } t = 0, \frac{dA}{dt} \right) = 25 \text{ (km}^2\text{/year)} \quad \text{A1}$$

[3 marks]

(b) increasing since the sign of $\frac{dA}{dt}$ is positive R1

[1 mark]

(c) new x is 85 and new y is 35, therefore new A is 2975 A1

old A was 3000 A1

therefore (a decrease of) 25 (km²) A1

Note: Accept an answer of $-25 \text{ (km}^2\text{)}$

[3 marks]

Total [7 marks]

12. (a) $\frac{dv}{dt} = kv^{1.5}$ **A1**
[1 mark]

(b) using initial condition to find an equation in k **M1**

$$-3 = k \times 8$$

$$k = -\frac{3}{8} (-0.375)$$
 A1

attempt to separate variables **M1**

$$\int v^{-1.5} dv = \int -\frac{3}{8} dt$$

$$2v^{-0.5} = \frac{3}{8}t + c \quad (2v^{-0.5} = 0.375t + c)$$
 A1A1

Note: Allow negative signs on each side.
Award **A1** for each integral

using initial conditions to find constant of integration **M1**

$$c = 1$$
 A1

Note: Might be a different value of c depending on rearrangement.

$$v = \frac{1}{\left(\frac{3}{16}t + \frac{1}{2}\right)^2} \quad \left(= \frac{256}{(3t+8)^2} \right) \quad \left(= \frac{1}{(0.1875t+0.5)^2} \right) \quad \text{OE} \quad \text{A1}$$

[8 marks]
Total [9 marks]

15. (a) attempt to find gradient (M1)
- $$\frac{dy}{dx} = 2x$$
- gradient of tangent is 4
- gradient of normal is $-\frac{1}{4}$ (A1)

equation of normal is $y - 4 = -\frac{1}{4}(x - 2)$ $\left(y = -\frac{1}{4}x + \frac{9}{2}\right)$ OE A1

[3 marks]

- (b) attempt to find gradient M1
- gradient is $\frac{10 - 4}{0 - 2} = -3$ A1

[2 marks]

- (c) METHOD 1

$\theta_1 = \arctan(-3)$ $(\approx -1.249, -71.57^\circ)$ A1

$\theta_2 = \arctan\left(-\frac{1}{4}\right)$ $(\approx -0.2449, -14.04^\circ)$ A1

Note: Award **A1A1** for $\arctan(3) - \arctan\left(\frac{1}{4}\right)$ OR $\arctan(-3) - \arctan\left(-\frac{1}{4}\right)$ seen

OR $\arctan\left(\frac{-3 - \left(-\frac{1}{4}\right)}{1 + (-3) \times \left(-\frac{1}{4}\right)}\right)$ OE.

difference between them is 1.00 (1.00407...) radians (57.5°) $(57.5288...^\circ)$ A1

Note: Accept an obtuse angle between the normal and the ray. 2.14 (2.13752...) radians, 122° (122.471...°).

continued...

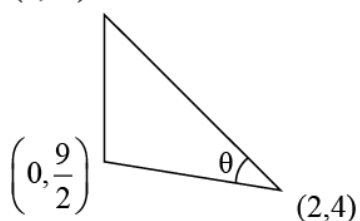
Question 15 continued

METHOD 2

a point on the normal is $\left(0, \frac{9}{2}\right)$

A1

$(0, 10)$



$$\cos \theta = \frac{2^2 + 6^2 + 2^2 + (0.5)^2 - (5.5)^2}{2\sqrt{4^2 + 1^2}\sqrt{1^2 + 3^2}}$$

A1

$$\theta = 1.00 \text{ (1.00407...)} \text{ radians } (57.5^\circ) \text{ (57.5288...}^\circ)$$

A1

Note: Other points on the normal such as $(18, 0)$ may be seen

Note: Accept an obtuse angle between the normal and the ray. $2.14 \text{ (2.13752...)} \text{ radians, } 122^\circ \text{ (122.471...}^\circ\text{)}.$

METHOD 3

writing the ray and normal as vectors

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

A1

the angle between them is given by

$$\cos \theta = \frac{\begin{pmatrix} -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix}}{\sqrt{(-4)^2 + 1^2} \sqrt{(-1)^2 + 3^2}}$$

A1

$$\theta = 1.00 \text{ (1.00407...)} \text{ radians } (57.5^\circ) \text{ (57.5288...}^\circ)$$

A1

Note: Accept an obtuse angle between the normal and the ray. $2.14 \text{ (2.13752...)} \text{ radians, } 122^\circ \text{ (122.471...}^\circ\text{)}.$

[3 marks]
Total [8 marks]

- Q9** (a) attempt to use product rule, two terms with one term correct (M1)
 attempt to use the chain rule to find the derivative of $\sqrt{3x-2}$ (M1)

$$\frac{dy}{dx} = 4(3x-2)^{0.5} + 4x \times \frac{3}{2}(3x-2)^{-0.5} \quad \text{A1}$$

$$\left(= 4(3x-2)^{0.5} + 6x(3x-2)^{-0.5} \right) \left(= \frac{2(9x-4)}{\sqrt{3x-2}} \right)$$

[3 marks]

- (b) substitution of $x = 6$ into their $\frac{dy}{dx}$ (M1)

gradient of tangent = $(16 + 9) = 25$ A1

Note: Follow through for this **A1** from their $\frac{dy}{dx}$.

at $(6, 96)$ tangent is

EITHER

$$y - 96 = 25(x - 6) \quad \text{A1}$$

OR

$$96 = 25 \times 6 + c \quad \text{A1}$$

Note: Do not allow follow through for this **A1**.

THEN

$$y = 25x - 54$$

AG

[3 marks]

Total [6 marks]

- Q11.

(a)

C

A1

[1 mark]

(M1)

(A1)
- (b)

attempting to substitute values into the correct formulae

$$x_{n+1} = x_n + 0.1(3x_n - 2y_n), \quad y_{n+1} = y_n + 0.1(4x_n - y_n)$$
$$(t_{n+1} = t_n + 0.1)$$

any correct value for x_{n+1} and y_{n+1} for $n = 2, 3, \dots$

A1

n	t	x	y
0	5	1	0
1	5.1	1.3	0.4
2	5.2	1.61	0.88
3	5.3	1.917	1.436
4	5.4	2.2049	2.0592

$x = 2.45 \text{ (2.45453...)}$

A1

Note: Accept $x = 2.455$ from rounding to 4 sf.

$y = 2.74 \text{ (2.73524...)}$

A1

[5 marks]

Total [6 marks]

- Q13.** (a) $\frac{dP}{dt} = 0.2(1-P)$
 attempt to separate variables **M1**
 $\int \frac{1}{1-P} dP = \int 0.2 dt$
 attempt to integrate left hand side by substitution or inspection. **(M1)**

Note: The **(M1)** may be awarded if their LHS is $\ln(1-P)$ without the minus.

$$-\ln(1-P) = 0.2t(+c) \quad \text{A1A1}$$

Note: Modulus sign not required as $0 \leq P < 1$.

EITHER

substitution of $P = 0$ when $t = 0$ **M1**

$$-\ln(1-0) = 0.2(0) + c \Rightarrow c = 0 \quad \text{A1}$$

eliminating the natural logarithm **M1**

$$1-P = e^{-0.2t}$$

OR

eliminating the natural logarithm **M1**

$$1-P = e^{-0.2t-c} \text{ or } 1-P = Ae^{-0.2t}$$

substitution of $P = 0$ when $t = 0$ **M1**

$$1 = e^{-c} \Rightarrow c = 0 \text{ OR } 1 = A \quad \text{A1}$$

THEN

$$P = 1 - e^{-0.2t} \quad \text{AG}$$

[7 marks]

- (b) attempt to solve $0.9 = 1 - e^{-0.2t}$ **(M1)**

$$11.5 \text{ (days) } (11.5129\dots) \quad \text{A1}$$

[2 marks]
Total [9 marks]

✓ ✓ ✓ ✓

Q16. (a) $g(x) = 4 \cos\left(\frac{\pi}{8}(x+0.3)\right) - 0.2$ **A1A1**

Note: Accept $g(x) = f(x+0.3) - 0.2$

[2 marks]

(b) 3.57 (3.57262...) **A1**
[1 mark]

(c) (i) use of formula for volume in at least one integral **M1**

Note: In this formula condone one of:
missing or incorrect limits, absence of π , absence of dx for the **M1**.

subtracting two volumes **M1**

$$\frac{\pi}{2} \int_0^4 \left(4 \cos\left(\frac{\pi}{8}x\right)\right)^2 dx - \frac{\pi}{2} \int_0^{3.57262\dots} \left(4 \cos\left(\frac{\pi}{8}(x+0.3)\right) - 0.2\right)^2 dx$$
 A1A1

Note: Award **A1** for each completely correct expression (including the dx).
Award **A1A0** for everything correct except not dividing by two.

(ii) $\frac{\pi}{2}(32 - 23.7729\dots)$
12.9 cm³ (12.9229...) **A1**

Note: Do not award **A1FT** follow through for this part.

[5 marks]
Total [8 marks]

3. (a) evidence of using the values 0,20,40,60,... for their x data **(M1)**

$$f(x) = -0.0989x^2 + 45.9x - 67.8 \quad \textbf{A2}$$

$$(f(x) = -0.0989285...x^2 + 45.925x - 67.7857...)$$

Note: Accept models using full population, $f(x) = -98.9285...x^2 + 45925x - 67785.7...$

Award **(M0)A2** for $f(x) = -0.0989285...x^2 + 421.853...x - 444457$ or $f(x) = -98.9285...x^2 + 421853.5...x - 444457428.6...$ from finding regression without converting years. Award at most **(M1)A1A0** for a correct model with one rounding error or misread. Award at most **(M1)A0A0** for models with multiple rounding errors. Award at most **(M1)A1A0** if the answer is not given in equation form.

[3 marks]

- (b) **EITHER**

R^2 is (very) close to 1 **OR** 0.98843 is (very) close to 1 **R1**

OR

R^2 shows that the model explains 98.843% of the total variation **R1**

THEN

therefore, it supports Ethan's proposal **A1**

Note: Do not award **R0A1**. Accept "Yes" for the A mark as long as reasoning is correct. To award the R1, a comparison to 1 must be seen if not using the percentage method.

[2 marks]

- (c) **EITHER**

the negative value of c is unrealistic as you cannot have a negative population **R1**

OR

the negative value of a is unrealistic because it suggests that the population of Los Angeles will decrease back to 0 at some point in the future / be negative in the future / was negative before 1900

R1
[1 mark]

- (d) (i) $(f'(110) =) 24.2$ (24.1607...) **A1**

Note: Accept 24.1(24.142) from use of 3sf values in model. Accept 400(400.089...) or 400,000 (400,089.28 ...) from use of a model from part (a) where years were not converted.

(recognition that value predicts the) annual growth (in 2010 as)

24,200(24,160.7...) per year **OR** 24.2(24.1607...) thousands per year **A1**

Note: Allow FT within question part for the final A mark for a correct interpretation of their incorrect value for $f'(110)$. Do not follow through for interpreting a value for $f(110)$.

(ii) **EITHER**

population growth is clearly slowing down / only averaged a growth of 10 700 per year between 2000 and 2020

A1

validity questioned since the model's predicted growth per year in 2010 is significantly higher

R1**OR**

total population change between 2000 and 2020 is 214,000 and the predicted change using $f'(110)$ over that time frame 483,200

A1

validity supported since prediction is in the correct order of magnitude / low percentage error for the predicted population of 2020 .

R1

Note: For either of the arguments, the response must include a comparison of their answer for part (d)(i) with specific values from the actual data. FT marks can only be awarded in this question part if value if $15 < f'(110) < 30$. .

Do not award FT marks in this question part for responses that include a model where the years were not converted as their value in (d)(i) would tell them the population rate of change in the year 110.

[4 marks]

- (e) the model predicts the slowing growth rate of the population over the long term / the population is trending towards a carrying capacity

R1
[1 mark]

- (f) (i) attempt to differentiate $g(x)$ using chain/quotient rule

(M1)

$$(g'(x) =) \frac{2800e^{-0.05x}}{(1+14e^{-0.05x})^2}$$

A1A1

- (ii) recognition to find maximum of $g'(x)$

(M1)

(maximum occurs when $x =$) 52.7806...

A1

(therefore, greatest growth rate occurred during the year) 1952

A1

Note: FT within the question part for the final A1 as long as the candidate's given value for x is not a whole number (must be evidence of rounding down to match the context of the problem).

[6 marks]
[Total 17 marks]

6. (a) $\frac{dh}{dt} = k \times \frac{h}{v}$ (A1)

attempt to substitute given values into their equation with a constant of proportionality M1

(When $t = 0$) $-5 = k \times \frac{1350}{27}$ OR $-5 = k \times \frac{1350}{27000}$

$k = -\frac{1}{10}$ A1

$\frac{dh}{dt} = -\frac{h}{10v}$ AG

[3 marks]

(b) $\frac{dh}{dt} = -\frac{h}{10(27 + 0.3t)}$ (A1)

Note: Follow through within question part for all remaining marks for responses that lose the initial A mark here from omitting the 10 $\left(\frac{dh}{dt} = -\frac{h}{27 + 0.3t} \right)$

attempt to separate variables to setup integration (M1)

$\int \frac{1}{h} dh = \int -\frac{1}{10(27 + 0.3t)} dt$ (A1)

EITHER

$\ln h = -\frac{1}{3} \ln(27 + 0.3t) + c$ A1

Note: This A1 can be awarded even if $+c$ is not seen.

$h = A(27 + 0.3t)^{-\frac{1}{3}}$ (A1)

$A = 4050$ (from using $t = 0$ and $h = 1350$) (A1)

$h = 4050(27 + 0.3t)^{-\frac{1}{3}}$ A1

Note: The follow through answer here for those that omit 10 in the first step is.

$h = 79716150(27 + 0.3t)^{-\frac{10}{3}}$

OR

$$\ln h = -\frac{1}{3} \ln(270 + 3t) + c$$

A1

Note: This **A1** can be awarded even if $+c$ is not seen.

$$h = A(270 + 3t)^{-\frac{1}{3}}$$

(A1)

$$A = 4050\sqrt[3]{10} \quad (8725.46\dots) \quad (\text{from using } t = 0 \text{ and } h = 1350)$$

(A1)

$$h = 8730(270 + 3t)^{-\frac{1}{3}}$$

$$\left(h = 8725.46\dots(270 + 3t)^{-\frac{1}{3}}, 4050\sqrt[3]{10}(270 + 3t)^{-\frac{1}{3}} \right)$$

A1**[7 marks]**

- (c) attempt to setup equation or inequality comparing the number of virus particles to the number of healthy cells

(M1)

recognition to multiply expression for v by 1000 **OR** dividing expression for h by 1000 **(M1)**

$$1000(27 + 0.28t) \geq 100(4380(27 + 0.28t)^{-0.36})$$

(A1)

Note: Accept an equation.

$$t = 216 \quad (216.253\dots) \quad (\text{hours})$$

A1

Note: Accept 217 as a contextual answer. Award at most **(M1)(M0)A0A1FT** for an answer of 50100(50136.06...) or 50137 from correct solution of inequality where the candidate has not appreciated the 1000.

[4 marks]**[Total 14 marks]**

7. (a) METHOD 1

attempt to represent area under curve with integral (M1)

$$\int_{0.1}^1 \frac{2}{3} \cos^{-1} \left(\frac{1}{10x} \right) dx \quad (A1)$$

0.780871... seen (A1)

1 – 0.780871...

0.219 (0.219128...) (m²) **A1****METHOD 2**

recognition to split area into three areas (M1)

(1 × (1 – 0.980419...) =) 0.0195807... seen **AND**

(0.1 × 0.980419... =) 0.0980419... seen (A1)

$$\int_{0.1}^1 \left(0.980419... - \frac{2}{3} \cos^{-1} \left(\frac{1}{10x} \right) \right) dx \quad (M1)$$

0.0195807... + 0.0980419... + 0.101505...

0.219 (0.219128...) (m²) **A1****METHOD 3**attempt to rearrange to make x a function of y (M1)

$$x = \frac{1}{10 \cos \left(\frac{3y}{2} \right)}$$

Integral upper bound $\frac{2}{3} \cos^{-1}(0.1) = 0.980419$ seen (A1)attempt to represent area to left of curve with sum of integral **AND** rectangle area (M1)

$$\int_0^{0.980419...} \frac{1}{10 \cos \left(\frac{3y}{2} \right)} dy + 1 \times (1 - 0.980419...)$$

Note: Must see attempt at both an integral and a rectangle area to award **M1A0**.0.219 (0.219128...) (m²)**A1**
[4 marks]

(b) recognition of rectangles forming cylinders when rotated

(M1)**Note:** Evidence of recognition includes:

- the word 'cylinder' written in work
- $\pi r^2 h$ seen anywhere
- $\pi(1)^2$ or $\pi(0.3)^2$ multiplied to anything

attempt to set up integral for middle section

(M1)

Note: Attempt includes any integral that includes π and squaring some form of the given expression, regardless of whether they have attempted to rearrange to solve for x . It is also not necessary to see the dy or proper limits here for the M1.

attempt to rearrange expression to solve for x (possibly seen in part(a))**(M1)**

Note: If this is seen in part (a), it must be seen within an integral here to award the mark.

value of 0.980419... or 0.0195807... seen anywhere (possibly seen in part (a))

A1

correct volume expression seen for one of the cylinders

A1

fully correct expression

A1

$$(V =) \pi(1)^2(0.0196) + \int_0^{0.980} \pi \left(\frac{1}{10 \cos\left(\frac{3y}{2}\right)} \right)^2 dy + \pi(0.3)^2(0.075)$$

$$\left(\pi(1)^2(0.0195807...) + \int_0^{0.980419...} \pi \left(\frac{1}{10 \cos\left(\frac{3y}{2}\right)} \right)^2 dy + \pi(0.3)^2(0.075) \right)$$

[6 marks](c) 0.291 (0.291110...) (m³)**A2**

Note: FT from their expression in part (b) as long as their expression contains three parts and their final value is positive.

[2 marks]**[Total 12 marks]**

2. (a) 0.638 (0.637623...) and 4.10 (4.09811...) **A2A1**

Note: Award **A2** for one correct answer, accept 4.1. Award **A2A0** if coordinate pair is given.

[3 marks]

(b) (i) $\int_{0.638}^{4.1} 1.5^x dx$ OR $\int_{0.638}^{4.1} f(x) dx$

A1A1

Note: **A1** for function and dx , **A1** for limits. Award **A1** for $\int_{0.638}^{4.1} y dx$.

Condone the use of modulus within the integrand.

(ii) 9.81 (9.80788...)(from 3sf)

9.80 (9.79844...) (from full)

A2

Note: Accept 9.8 for **A2**.

(iii) correct expression for area

(A1)

$\int_{0.638}^{4.1} g(x) - f(x) dx$ OR $\int_{0.638}^{4.1} 6 - \frac{3}{x} - 1.5^x dx$

5.38 (5.38290...)

A1

Note: Accept 5.39

[6 marks]

(c) $f'(x) = g'(x)$

(M1)

correct graphs of $f'(x)$ and $g'(x)$

(M1)

Note: Award **M1** if $\frac{3}{x^2}$ seen.

($k =$)1.86 (1.86406...)

A1

[3 marks]

Total [12 marks]

3. (a) $(A =) 2\pi r^2 + 2\pi r h$ A1A1

Note: Award **A1** for either term correct (even if unsimplified) and **A1** for all correct (including simplification and no extra term).

[2 marks]

(b) $V = \frac{2\pi r^3}{3} + \pi r^2 h$ A1A1

$V = \frac{2\pi r^3 + 3\pi r^2 h}{3}$ AG

Note: Award **A1** for one correct term, **A1** for both and completion to required form.

[2 marks]

- (c) 10000 substituted into (b) M1
correct rearrangement with at least one line of intermediate working leading to the given value.

M1

$h = \frac{30000 - 2\pi r^3}{3\pi r^2}$ AG

[2 marks]

- (d) h substituted into their A M1

$(A =) 2\pi r^2 + 2\pi r \left(\frac{30000 - 2\pi r^3}{3\pi r^2} \right)$

further consistent rearrangement that leads to **given** A A1

[2 marks]

(e) $\frac{4\pi r}{3} - \frac{20000}{r^2}$ A1A1A1

Note: Award **A1** for $\frac{4\pi r}{3}$, **A1** for -20000 seen, and **A1** for a correct index for r in second term.

[3 marks]

(f) $\frac{dA}{dr} = 0$ (M1)

EITHER

rearrangement leading to $r^3 = \frac{15000}{\pi}$ (M1)

OR

graph of $\frac{dA}{dr}$ (M1)

THEN

$r = 16.8$ (16.8389...) OR $\sqrt[3]{\frac{15000}{\pi}}$ A1

Note: Accept correct graph of Area function leading to correct minimum value for r for **M1M1A1**.

$h = 0$ A1

[4 marks]

- (g) hemisphere A1

Note: Do not **FT** from incorrect answers to part (f).

[1 mark]

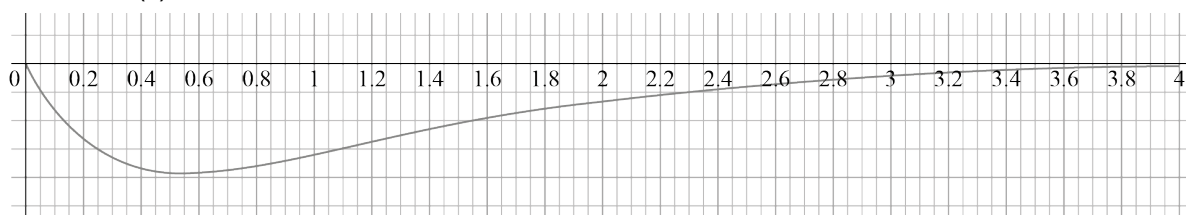
Total [16 marks]

6. (a) associated matrix is $\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$ (A1)
 attempt to find eigenvalues (M1)
 $-\lambda(-4-\lambda)+3=0$
 eigenvalues are -3 and -1 (A1)(A1)
 therefore the general solution for x is $Ae^{-3t} + Be^{-t}$ A1

[5 marks]

- (b) (i) using initial condition for x , $A+B=0$ A1
 attempting to find derivative of general solution (M1)
 $\frac{dx}{dt} = -3Ae^{-3t} - Be^{-t}$
 $-3A - B = -1$ A1
 $A = \frac{1}{2}, B = -\frac{1}{2}$
 $x = \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-t}$ A1

(ii)



any curve starting at 0 and initially decreasing. A1

all else correct (minimum and some indication of POI / asymptote. A1

[6 marks]

- (c) (i) recognition that minimum is relevant to the question (M1)
 0.549 (minutes) (0.549306...) A1
 (ii) any consideration of the y -coordinate of the minimum point (M1)
 0.192 dollars (0.192450...) A1

[4 marks]

continued...

Question 6 continued

(d) attempting to write out the iterative relationship in terms of the variables (M1)

$$x_{n+1} = x_n + 0.1(y_n) \quad \text{span style="float: right;">(A1)}$$

$$y_{n+1} = y_n + 0.1(-4y_n - 3x_n + x_n \sin t_n) \quad \text{span style="float: right;">(A1)}$$

$$t_n = 0.1n$$

using $y_0 = x'_0 = -1$ (A1)

attempt to find x_{10} (M1)

Note: Award the **M1** for $x_1 = -0.1$ and $y_1 = -0.6$ seen as the first iterative stage.

$$x_{10} = -0.172 \text{ } (-0.171795...) \quad \text{span style="float: right;">A1}$$

Note: a common error may be 0.104 ($0.104330...$).

[6 marks]

Total [21 marks]

Q1.

(a) $(d =) 3l$ **A1**

[1 mark]

(b) substitution into volume of cuboid formula $V = l^2 \times 3l (= 375)$ **(M1)**

attempting to solve their $3l^3 = 375$ **(M1)**

$$l^3 = 125 \text{ OR } l = \sqrt[3]{125} \text{ OR } l^3 = \frac{375}{3}$$

$l = 5, d = 15$ **A1**

[3 marks]

(c) finding the area of a square and the area of a (non-square) rectangle 5×5 and 5×15 **(M1)**

$(5 \times 5 + 5 \times 15 + 5 \times 15) \times 2$ **(A1)**

$= 350 \text{cm}^2$ **A1**

Note: The final answer is 350cm^2 . Units are required.

[3 marks]

(d) $\pi r^2 h = 375$ **(A1)**

$h = \frac{375}{\pi r^2}$ **A1**

Note: Award at most **A0A1** for a final answer $h = \frac{V}{\pi r^2}$.

[2 marks]

(e) attempt to substitute their h into the curved surface area $2\pi r h$ **M1**

$A = 2\pi r^2 + 2\pi r \left(\frac{375}{\pi r^2} \right)$ **A1**

$A = 2\pi r^2 + \frac{750}{r}$ **AG**

[2 marks]

(f) $\left(\frac{dA}{dr} \right) 4\pi r - 750r^{-2}$ **A1A1A1**

Note: Award **A1** for $4\pi r$, **A1** for -750 , **A1** for r^{-2} or equivalent. Award at most **A1A1A0** for additional terms seen.

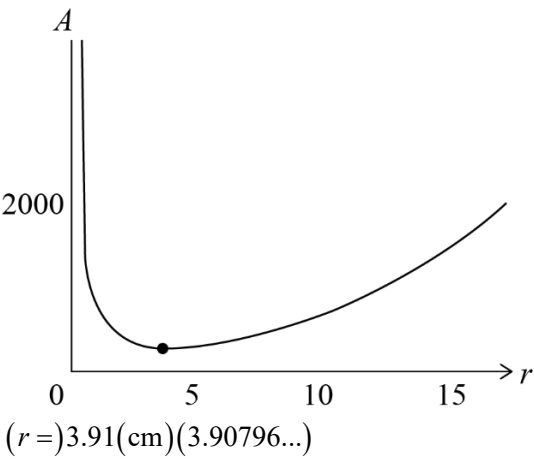
[3 marks]*continued...*

Question 1 continued.

(g) (i)

METHOD 1

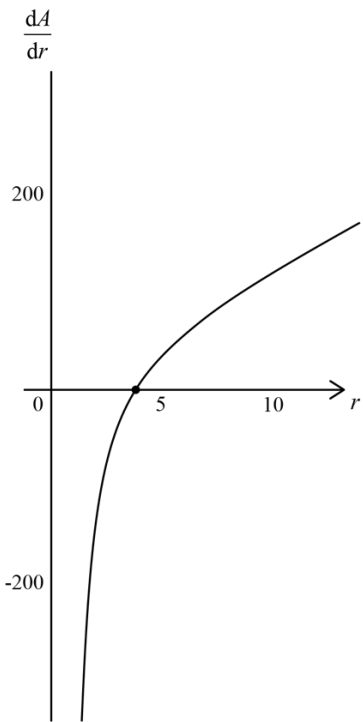
correct sketch with some indication of min point (need not be labelled) **(M1)**



A1

METHOD 2

attempt to solve $4\pi r - \frac{750}{r^2} = 0$, graphically or otherwise **(M1)**



$(r =) 3.91(\text{cm})(3.90796\dots)$

A1

(ii) $288(\text{cm}^2)(287.873\dots)$

A1

[3 marks]

continued...

Question 1 continued.

(h) (i) $\left(\frac{d^2A}{dr^2} = \right) 4\pi + \frac{1500}{r^3}$ **A1**

(ii) (as $r > 0$), $\frac{d^2A}{dr^2} > 0$ **R1**

hence graph is concave-up. **A1**

Note: Do not award **R0A1**. Marks can only be awarded in part (ii) if an expression is seen in part (i).

[3 marks]

Total [20 marks]

Q3.

(a) attempt to substitute into trapezoidal formula (M1)

$$(\text{area of cross-section}) \approx \frac{1.9}{2}(0 + 0 + 2(1.68 + 2.81 + 2.32)) \quad (\text{A1})$$

$$= 12.9 \text{ (m}^2\text{)}(12.939) \quad \text{A1}$$

[3 marks]

(b) attempt to find the volume of water in one second, their (a) $\times 0.3$ (M1)

$$12.939 \times 0.3$$

$$= 3.88(3.8817) \text{ (m}^3\text{)} \quad \text{A1}$$

Note: Accept 3.87 for use of 3sf answer to part (a) 12.9.

[2 marks]

(c) $(2.81 - 1.6 =) 1.21 \text{ (m)}$ A1

[1 mark]

(d) attempt to find maximum eg graphically or $\frac{dy}{dx} = 0$ or using height at $x = 3.8$ (M1)

$$3.07 \text{ (m) OR } 3.066 \text{ (m)} \quad \text{A1}$$

Note: Award **M1A0** for coordinate pair (3.8, 3.066)

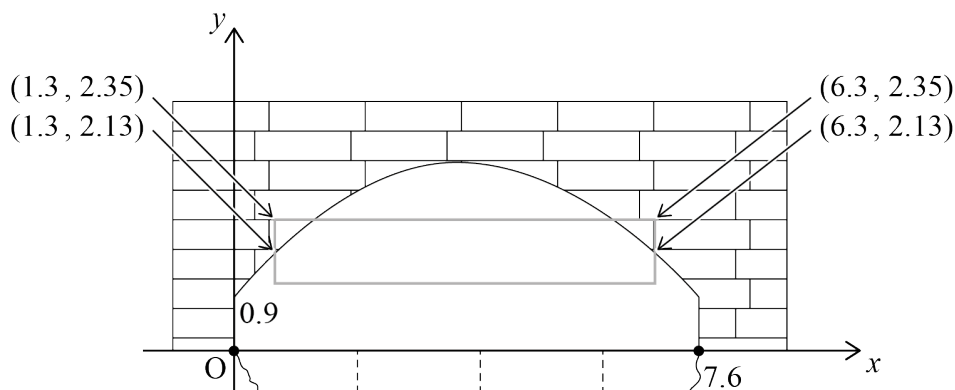
[2 marks]

continued...

Question 3 continued.

(e) **METHOD 1**

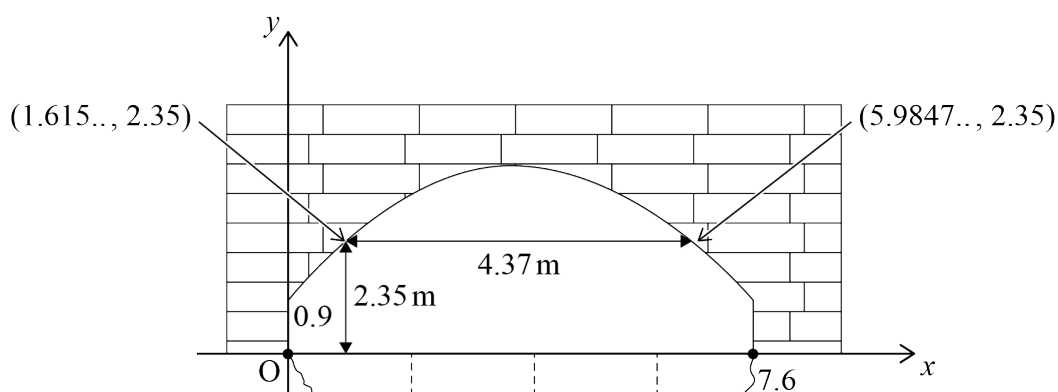
height of bridge < height of boat at
 $x = 1.3$ or $x = 6.3$ (5 m wide boat)

use of $3.8 + 2.5$ or $3.8 - 2.5$ **(M1)**

attempt to find the height of the bridge at either 1.3 or 6.3 m

(M1) $2.13(\text{m})(2.1285)$ **A1** $(2.13 < 2.35)$ the boat cannot pass under the bridge**A1****METHOD 2**

boat is wider (5 m) than the width between
 the arch at $h = 2.35$ m (height of boat)



attempt to find the values of x (1.61520... and 5.98479...) when the bridge is
 2.35 m above the water

(M1)*continued.*

Question 3 continued.

subtract their two values to find the width of bridge at that point at this height **(M1)**

$5.984795... - 1.615204...$ OR $3.8 - 1.615204...$ OR $5.98479... - 3.8$

$4.37(4.36959...)(\text{m})$ OR $2.18 (2.18479...)(\text{m})$ **A1**

$(4.37 < 5)$ OR $(2.18 < 2.5)$ the boat cannot pass under the bridge **A1**

METHOD 3 (see the diagram for method 2)

use of $3.8 + 2.5 (6.3)$ or $3.8 - 2.5 (1.3)$ **(M1)**

attempt to find the values of x when the bridge is 2.35 m above the water **(M1)**

$5.98479... \text{ or } 1.61520... (\text{m})$ **A1**

$6.3 > 5.98...(5.98479...)$ or $1.3 < 1.62 (1.61520...)$ the boat cannot pass under the bridge **A1**

Note: In all 3 methods the final **A1** is dependent on the previous **A1**.

[4 marks]

[Total 12 marks]

7. (a) (i) $\left(B'(x) = \frac{-32}{x^2} + 2\right)$ **OR** $\left(B'(x) = -32x^{-2} + 2\right)$ **A1(M1)A1**

Note: Award **A1** for 2 seen, **(M1)** for expressing $\frac{1}{x}$ as x^{-1} (this can be implied from, either $\frac{1}{x^2}$ or x^{-2} seen in final answer), **A1** for $\frac{-32}{x^2}$. Award at most **A1(M1)A0** if additional terms are seen.

(ii) $B'(x) = 0$ **OR** $0 = \frac{-32}{x^2} + 2$ **OR** sketch of $B'(x)$ with root identified **M1**

Note: It is specified that the student must use their derivative from part (i) to answer this question.

$x = 4$ **(A1)**

Note: FT marks can only be awarded if *their* x -value lies within the given domain of $x \geq 1$.

$(B(x) =) 36$ (minutes) **A1**

[6 marks]

(b) $40 > 20 + \frac{32}{k} + 2k$ **OR** 2 and 8 seen **OR** sketch of $T(n)$ and $T(n) = 40$ **(M1)**

$2 < k < 8$ **OR** $3 \leq k \leq 7$ **OR** 3, 4, 5, 6, 7 **A1**

[2 marks]

Total [8 marks]

10. (a) attempt to use the chain rule or the quotient rule (M1)
 e.g. $(4x + 5)$ seen as part of a product

$$\frac{-6(4x+5)}{(2x^2+5x+4)^2} \quad \text{OR} \quad -6(4x+5)(2x^2+5x+4)^{-2}$$

A1

[2 marks]

- (b) recognize need to find max/min of graph of f' **OR** zero of graph of f'' (M1)
 $-1.63, -0.868$ $(-1.63188..., -0.86811...)$ **A1A1**

[3 marks]

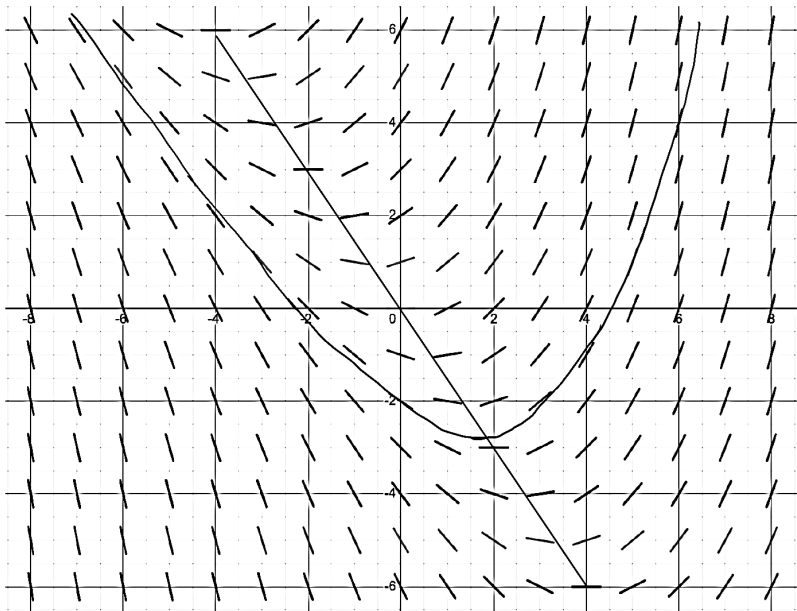
Total [5 marks]

11. (a) (i) $y = -1.5x$ A1

Note: Allow equivalent versions, e.g. $3x + 2y = 0$, $\frac{x}{2} + \frac{y}{3} = 0$.

(ii) straight line from $(-4,6)$ to $(4,-6)$ shown on diagram A1
[2 marks]

(b)



A1A1

Note: Award **A1** for approx. correct shape passing through $(6, 4)$ and **A1** for minimum on line $y = -1.5x$.

[2 marks]
Total [4 marks]

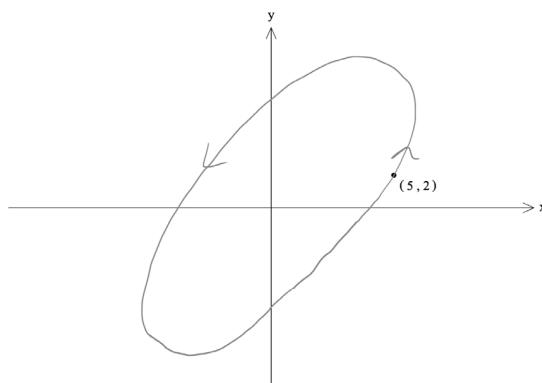
14. (a) **EITHER** $\frac{dx}{dt} = 2.2(5) - 2.6(2)(= 5.8)$ **OR** $\frac{dy}{dt} 2.2(5) - 2.6(2)(= 12.6)$ **(A1)**

recognize need to use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ **OR** $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ **(M1)**

$\left(\frac{dy}{dx} = \right) 2.17 \quad (2.17241\dots)$ **A1**

[3 marks]

(b)



ellipse or circle, centre (0, 0)

A1

anticlockwise

A1

ellipse, with positive gradient at (5, 2)

A1

[3 marks]

Total [6 marks]

15. (a) attempt to find an expression for x^2 or x (M1)
 $x^2 = \sqrt[3]{65536y}$ OR $x = \sqrt[6]{65536y}$ (A1)
 attempt to substitute their expression for x^2 into $\pi \int x^2 dy$ (M1)
 $\pi \int_0^4 \sqrt[3]{65536y} dy$ (A1)
 $603 \text{ (cm}^3\text{)} \text{ (603.185..., 192 } \pi \text{)}$ A1

Note: Award at most (M1)(A1)(M1)(A0)A0 for an answer of 192 from omission of π .

[5 marks]

- (b) recognize need to find the upper limit of an integral (M1)
 attempt to solve $\pi \int_0^h \sqrt[3]{65536y} dy = 250$ (M1)
 $2.07 \text{ (cm)} \text{ (2.06622...)}$ A1

Note: Award at most (M1)(M1)A0 for an answer of 4.88 from omission of π .

[3 marks]

Total [8 marks]

17. (a) 15 (metres)

A1
[1 mark]

(b) (i) recognize need to integrate

(M1)

$$\int_0^{4.5} \frac{\pi^2}{24} \left| \sin\left(\frac{\pi}{2}t\right) \right| dt$$

(A1)

1.12 (1.12387...)

A1

Note: Award at most **(M1)(A1)(A0)** for an unsupported answer of 0.114006... from use of degrees.
Award **(M1)(A0)(A0)** for an unsupported answer of 0.0766792... from omission of absolute value.

(ii) $\frac{dh}{d\theta} = 15\sin\theta$

(A1)

use of chain rule $\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$

(M1)substitute $\theta = 1.12387\dots$ and $t = 4.5$ **(M1)**

$$\frac{dh}{d\theta} = 13.5267\dots, \quad \frac{d\theta}{dt} = 0.290786\dots$$

$$\frac{dh}{dt} = 3.93 \text{ (3.93338...)} \text{ (metres per minute)}$$

A1**[7 marks]**
Total [8 marks]

2. (a) (i) 0.9 A1
- (ii) the rate of change (of the height of the plant is 0.9 cm per day)
3 days after it is planted (accept $t = 3$) A1
[2 marks]
- (b) attempt to solve $\frac{dh}{dt} = 0$ (algebraically or graphically) (M1)
($t =$) 12 (days) A1
[2 marks]
- (c) recognition of the need to integrate (M1)
 $h = 1.2t - 0.05t^2 (+c)$ A1A1
- Note:** Award **A1** for each correct term.
- attempt to use given conditions (2,6.5) to find c (M1)
 $c = 4.3$
($h =$) $1.2t - 0.05t^2 + 4.3$ A1
[5 marks]
- (d) **EITHER**
attempt to find maximum graphically (M1)
- OR**
attempt to substitute their part (b) for t (M1)
- THEN**
 $h = 11.5$ (cm) A1
- NOTE:** If their answer to part (c) is linear or any other function that substantially changes the nature of the question, award at most **M1A0**. Award at most **M1A0** if the x value of their max point (FT from part (c)), contradicts their $x = 12$ found in part (b).
- [2 marks]
- (e) 2 (cm) A1
[1 mark]
- (f) **EITHER**
attempt to write 1.2 as a percentage or percentage increase (M1)
- OR**
attempt to find two values of H and % increase (M1)
- THEN**
($p =$) 20 A1
[2 marks]
- (g) attempt to solve $H(t) = h(t)$ (M1)
 $t = 9.43$ (days) (9.43483...) A1
[2 marks]
- Total [16 marks]**

6. (a) attempt to integrate $12\cos\theta$ with respect to t (M1)
 $(x =) (12\cos\theta)t + 2$ A1A1
 [3 marks]
- (b) attempt to solve $y = 0$ M1
 $t(12\sin\theta - 4.9t) = 0$ A1
 $t = 0$ or $t = \frac{12\sin\theta}{4.9}$
 $t = 0$ is the starting position **OR** it hits the ground when $t > 0$ R1
 so the ball hits the ground at $t = \frac{12\sin\theta}{4.9} = \frac{120\sin\theta}{49}$ AG
 [3 marks]
- (c) attempt to substitute $t = \frac{120\sin\theta}{49}$ into their expression for x . (M1)
 $(x_g =) 12\cos\theta \frac{120\sin\theta}{49} + 2 \left(= \frac{1440}{49} \sin\theta\cos\theta + 2 = (29.3877 \dots) \sin\theta\cos\theta + 2 \right)$ A1
 [2 marks]
- (d) sketch of graph showing maximum (M1)
 $0.785 \ (0.785398\dots, \frac{\pi}{4}, 45^\circ)$ A1

Note: The answer must follow from their expression for x_g .

[2 marks]

- (e) (i) $(x =) (12\cos\theta - 3)t \ (+2)$ A1
 $(x_g =) (12\cos\theta - 3) \frac{120\sin\theta}{49} + 2$ A1
- (ii) sketch of graph showing maximum (M1)
 $0.688 \ (0.688242\dots, 39.433^\circ\dots)$ A1

[4 marks]

Total [14 marks]

2. (a) (i) that the stable/long term population will be the same as before the reduction. **A1**

Note: Condone 'the maximum value / carrying capacity of the population will be 200'.

[1 mark]

- (ii) attempt to substitute into logistic function **(M1)**

$$40 = \frac{200}{1 + Ce^0}$$

$$40 + 40C = 200$$

$$C = 4$$

A1
[2 marks]

- (iii) correct substitution into their logistic function **(A1)**

$$70 = \frac{200}{1 + 4e^{-5k}}$$

$$k = 0.153 \left(0.153451..., -\frac{1}{5} \ln \left(\frac{130}{280} \right) \right)$$

A1
[2 marks]

- (iv) correct substitution into their logistic function **(A1)**

$$x = \frac{200}{1 + 4e^{-10 \times 0.153451...}} (= 107.397....)$$

$$x = 107$$

A1
[2 marks]

continued...

Question 2 continued

(b) (i) $\frac{dx}{dt} = 0$

A1**[1 mark]**

(ii) the population is steady/stable (at 200) OR this (200) is the equilibrium population

A1

Note: Accept phrases such as ‘the rate of change of wolf population is zero’. Do not accept 200 is the maximum. Do not follow through from an incorrect (b)(i).

[1 mark](iii) **EITHER**

$$\frac{1}{\ln 200 - \ln x} \times \text{an attempt to differentiate } \ln 200 - \ln x$$

M1

$$f'(x) = \frac{1}{\ln 200 - \ln x} \times \left(-\frac{1}{x}\right)$$

A1**OR**

$$\frac{1}{\ln\left(\frac{200}{x}\right)} \times \frac{x}{200} \times -\frac{200}{x^2}$$

M1A1

Note: Award **M1** if there is a product of three expressions and two are correct.

$$= \frac{-1}{x \ln\left(\frac{200}{x}\right)}$$

AG**[2 marks]***continued...*

Question 2 continued

(iv) attempt to separate variables

M1

$$\int \frac{-1}{x \ln\left(\frac{200}{x}\right)} dx = \int -a dt$$

A1

Note: Award **A1** also for $\int \frac{1}{x \left(\ln\left(\frac{200}{x}\right) \right)} dx = \int a dt$.

$$\ln(\ln 200 - \ln x) = -at (+c)$$

A1A1

Note: Award **A1** for LHS and **A1** for RHS.

$$\ln 200 - \ln x = e^{-at+c} \text{ or } \ln 200 - \ln x = Ae^{-at}, (A = e^c)$$

A1

$$\ln x = \ln 200 - Ae^{-at}$$

AG
[5 marks](v) correct substitution $\ln 40 = \ln 200 - Ae^{-0}$ **(A1)**

$$A = \ln(5)$$

A1

$$(p = 5)$$

Note: Award **A1A0** for an unsupported 1.61 (1.60943...).

[2 marks]*continued...*

Question 2 continued

(vi) correct substitution $\ln 70 = \ln 200 - \ln 5 \times e^{-5a}$

A1

$a = 0.0854528\dots$

A1**Note:** Award **A1** if their unrounded answer has at least four significant figures and roundsto 0.0855 OR if an exact value equivalent to $0.2 \ln \left(\frac{\ln 5}{\ln \left(\frac{200}{70} \right)} \right)$ is seen.

$a = 0.0855$ to 3 significant figures

AG
[2 marks]

(vii) correct substitution $\ln x = \ln 200 - \ln 5 \times e^{-10 \times 0.0854528\dots}$

(A1)

100.839...

(A1)**Note:** Accept 100.871... from use of 3sf 0.0855.

101 to the nearest whole number

A1
[3 marks]

(c) both are overestimates OR the Gompertz is closer to the true value

A1**[1 mark]****Note:** Follow through on their answers for 2(a)(iv) and 2 (b)(vii) if both positive values and these values are compared with 85.*continued...*

Question 2 continued

- (d) (i) attempt to use Euler

(M1)

Note: Condone maximum of one error from missing x_n + at start, or incorrect/missing subscripts, absence of 0.5.

$$x_{n+1} = x_n + 0.5 \left(0.0855x_n \ln \left(\frac{200}{x_n} \right) - 0.03x_n \right)$$

(A1)

evidence of correct use of Euler's method

(A1)

Note: Award **(A1)** for any correct value of x_{n+1} for $n = 2, 3, \dots$ seen (including this intermediate value presented as a final answer).

after 10 years the population will be 89 wolves (89.42803...)

A1**[4 marks]**

- (ii) correctly compare their result to the actual number of wolves or to the other models or both. Allow even if this is implicit, for example, it is more accurate / realistic than the other models

A1**[1 mark]****Total [29 marks]**